Problems for the UML / Junior Seminar

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1 General Problems

Problem 1: Rank of a Curve. Building on investigations from last year, can one see the rank of the elliptic curve by investigating (smoothed) sums of a_p and a_p^2? Let $\lambda_p = \frac{a_p}{\sqrt{p}}$. Similar to the proof of the Prime Number Theorem we can show

$$\sum_{\rho} \frac{x^\rho}{\rho} = -\sum_{p \leq x} \lambda_p \log p - \sum_{p \leq \sqrt{x}} \lambda_p^2 \log p + 2x^{\frac{1}{2}} + small, \tag{1.1}$$

where ρ runs thru the zeros of the elliptic curve. Let r_a be the multiplicity of the zero at the critical point $s = 1$. We can show the above implies

$$\frac{1}{x^{\frac{1}{2}}} \sum_{p \leq x} \lambda_p \log p = (1 - 2r_a) + O\Big(\sum_{\rho \neq 1} \frac{x^{\rho - \frac{1}{2}}}{\rho}\Big). \tag{1.2}$$

Instead of having a sharp cut-off, we could instead study a smoothed version of the above. If we assume the Generalized Riemann Hypothesis, we might expect the sum to oscillate around $1 - 2r_a$.

Problem 2: The Parity Conjecture. We can analytically continue the L-function of an elliptic curve, and we get a functional equation $\Lambda(s, E) = \epsilon_E \Lambda(2 - s, E)$, where $\epsilon_E = \pm 1$. The Parity Conjecture asserts that (in the limit) $\epsilon_E = 1$ and $\epsilon_E = -1$ are equally likely. Alternatively, one could see if the Parity Conjecture still holds if, instead of considering the family of all elliptic curves, we instead consider a one-parameter family $y^2 = x^3 + A(t)x + B(t)$.

Problem 3: Conductors. $\Lambda(s,E) = (2\pi)^{-s} N^{\frac{s}{2}} \Gamma(s) L(s)$, where N is the conductor, and has the same prime factors as the discriminant. Study how the conductor varies in a family (say the family of all elliptic curves, or maybe a one-parameter family). In many applications (for example, trying to bound the average rank or investigating density functions) what enters our equations is not N but $\log N$. For example, consider the family $\mathcal{F}_N = \{E_{a,b} : y^2 = x^3 + ax + b, a \in [N^2, 2N^2], b \in [N^3, 2N^3]\}$. Then $\Delta_{a,b}$ is of size N^6 for all curves in this family. How does $\log N_{a,b}$ vary?

Problem 4: Excess Rank. Consider one-parameter families of elliptic curves. By construction, we can often force points of infinite order on our curves. Let us assume we have a family where we have forced r points on each curve E_t. By the Birch and Swinnerton-Dyer Conjecture, the geometric rank of the group of rational points should equal the order of vanishing of the associated L-function at the critical point. Assuming the Parity Conjecture, we might expect half the curves to have rank r, and half to have rank $r+1$. (The functional equation being odd forces another zero). Previous investigations (see, for example, Fermigier) for many curves with small values for t have shown that this is not the case; ie, that for small t, for many families about 32% have rank r, 48% have rank $r+1$, 18% have rank $r+2$, and 2% have rank $r+3$. Is this a general phenomenon that persists in the limit, or the result of small numbers?

Problem 5: Elliptic Surfaces. Let $\mathcal{F}$ be a parametrized family of Elliptic Curves. For a curve $E \in \mathcal{F}$, denote the curve by $y^2 = f_E(x)$. For a prime p and a curve $E \in \mathcal{F}$, let

$$a_E(p) = \sum_{x=0}^{p-1} \left(\frac{f_E(x)}{p}\right) = O(\sqrt{p}) \tag{1.3}$$

Silverman and Rose have shown that, if our family is parametrized by a variable t, summing $a_E(p) = a_{E_t}(p)$ is related to the rank over $\mathbf{Q}(T)$. Specifically, let $\mathcal{E}$ be the elliptic surface $y^2 + a_1(T)xy + a_3(T) = x^3 + a_2(T)x + a_6(T)$,

$a_i(T) \in \mathbf{Z}[T]$, $\Delta(T) \neq 0$. (Note: in this case, our family F will often be $\{E_t : t \in [N, 2N]\}$. Let $A_{\mathcal{E}}(p) = \frac{1}{p}\sum_{t=0}^{p-1} a_{E_t}(p)$. (Note: $A_{\mathcal{E}}(p)$ is bounded independently of p). Silverman and Rosen show (assuming Tate's conjecture for Elliptic Surfaces, including the weak non-vanishing assumption that $L_2(\mathcal{E}, s)$ does not vanish on the line $Re(s) = 2$)

$$\lim_{X \to \infty} \sum_{p \leq X} -A_{\mathcal{E}}(p) \log p = \textbf{rank } \mathcal{E}(\mathbf{Q}(T)). \tag{1.4}$$

For curves of the form $y^2 = x^3 + A(t)x + B(t)$, we can often prove the necessary assumptions. It is possible to cleverly chose families so that $A_{\mathcal{E}}(p)$ is constant. One can also (by numerical experimentation) find families where $A_{\mathcal{E}}(p)$ varies, but is an integer (and always in some range $[-N, M]$). Since we are dividing by p, it is not at all immediate that we should get an integer, and in fact, there are plenty of curves where we don't. Could these be the coefficients of an interesting form (say, something associated to Dirichlet L-series, or Galois representations, ...)?

Problem 6: Torsion Groups. For any elliptic curve, we can calculate its torsion group; moreover, only a few groups may appear as the result. In the family of all elliptic curves, what are the relative frequencies of each group? If we specialize to interesting one-parameter families, what are the relative frequencies here?

Problem 7: Points of Low Height. For any elliptic curve, we can try and find points of small height. How does the height of the lowest point (lowest infinite point?) vary in a family? Do curves in a given family with a low point have higher than predicted rank?

Problem 8: Birch and Swinnerton-Dyer Conjecture. In its simplest form, the B-SD Conjecture states the geometric rank of an elliptic curve should equal the analytic rank (the order of vanishing of the associated L-function at the critical point). For small rank curves, numerically verify.

Problem 9: Sato-Tate. We know that $|a_p| \leq 2\sqrt{p}$; we can therefore write a_p as $2\cos\theta_p$. How does θ_p vary with p? Fix a one-parameter family. Write $a_{E_t}(p)$ as $2\cos\theta_{E_t}(p)$. For large primes, how do the $\theta_{E_t}(p)$ vary?

Problem 10: Lowest Zero. We expect the zeros of an elliptic curve to follow the GUE distribution. To see this may be beyond the computational ability of the UML (to see the convergence may require more computations than we can do on our machines; many such investigations use large amounts of time on supercomputers). We can, however, look at the first few zeros near the critical point. If we are (again) looking at a family, we can study the distribution of the lowest (or second lowest) zero. We may want to normalize with respect to the logarithm of the conductor.