Birth, Growth, Death, and Chaos

Part 2: Dynamical systems and chaos

We continue to study the iterated mapping or dynamical system

$$P_{n+1} = [1 + r(1 - P_n)]\, P_n \quad (c = 1 \text{ or } P \text{ rescaled}) \tag{1}$$

in its own right, even though, as we discussed briefly before, this simple model is too naïve for real biological or physical systems. Let's take the "extreme" value $r = 3$, or

$$P_{n+1} = 4P_n - 3P_n^2 \ ,$$

for which the graph just "fills the box" (see Figure 1).

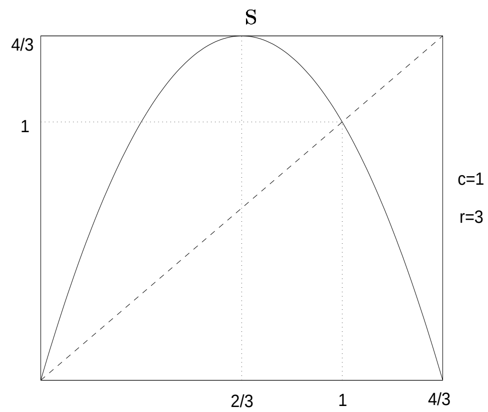

Figure 1: The logistic map with $r = 3$.

You will try this case in the lab, and observe that almost all solutions are **chaotic**, in the sense that their behavior is, in a precise sense, equivalent to the outcome of tossing a fair coin.

Here's an indication of how one proves this assertion, in the course of which we'll explain the notion of "chaos" a little better.

Changing Variables

When discussing the "qualitative" properties of orbits, we are interested in the behavior of orbits near fixed points or periodic orbits, and qualities like stability, more than in the exact values of the subsequent populations. Such qualitative properties get preserved by changes of variables, as explained below.

Suppose you are the "supplies manager" of a lab that is doing experiments with a population of fish that can be modeled fairly well with a logistic equation, and where the fish need to be fed with an expensive and highly perishable food. Then you want to take into account the size of the population expected every week, so that you have enough food on hand each week (otherwise the experiment would be hopelessly skewed) but not a large excess (because that would be a waste). As is often the case, the unit price of the food you buy depends on the quantity; for instance, assume that up to 200 fish, you spend 5 cents per fish every week; for every fish after the first 200, you have to spend only 2.5 cents per fish every week. It follows that the total sum of money you spend each week is not simply proportional to the size of the population that week. For instance, where you would spend \$5 to feed 100 fish, it would cost only \$12.50 to feed 300, or even only \$15 to feed 400 fish; see Figure 2

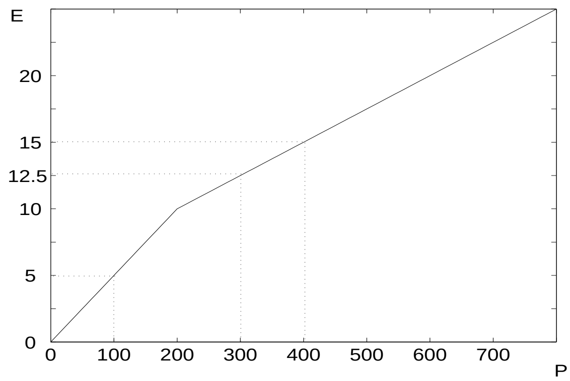

Figure 2: Weekly expenditure E to pay for the food of P fish.

In fact, you could keep track of the changing weekly expenditures, just like the biologists keep track of the fish population: you could map E_n (expenditure in week n) as a function of n, the week number; you could also graph E_{n+1} as a function of E_n. The "behavior" of E_n as a function of n should "shadow" that of P_n: if P_n remains fixed, then so does E_n; if P_n cycles between two values, then so does E_n; if P_n takes on many different values, then we expect E_n to do so. What would the graph of E_{n+1} versus E_n look like? We can compute E_{n+1} from E_n in the following steps:

- if we spend E_n, then that is because there is a corresponding number of fish P_n, which can be read from the graph in Figure 2. (For instance if $E_n = \$17.5$, then $P_n = 500$.)
- given P_n, we compute $P_{n+1} = P_n[1 + r(1 - \frac{P_n}{c})]$. (For instance, if $c = 600$, and $r = 3$, then $P_n = 500$ gives $P_{n+1} = 500 \times [1 + 3(1 - \frac{500}{600})] = 750$.)
- we then know P_{n+1}, so we can again read from Figure 2 what E_{n+1} is. (For $P_{n+1} = 750$, we get $E_{n+1} = \$23.75$. Note that in our example, $P_{n+1} = 1.5 \times P_n$, yet $E_{n+1} \neq 1.5 \times E_n$.)

In this way, one can plot the graph of E_{n+1} as a function of E_n, given in Figure 3. It is,

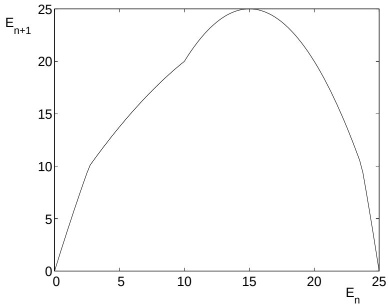

Figure 3: The graph sharing the dynamics governing E_n; Figure 4 shows how this graph is constructed (it is the mirror image of the E_n versus E_{n+1} graph in the lower right of Figure 4).

as expected, not the same graph as that for P_{n+1} as a function of P_n; in particular it has a "kink" that reflects the "breakpoint" in Figure 2, and it is lopsided, reflecting the change in unit price for larger quantities (illustrated in Figure 2). Figure 4 gives a geometrical

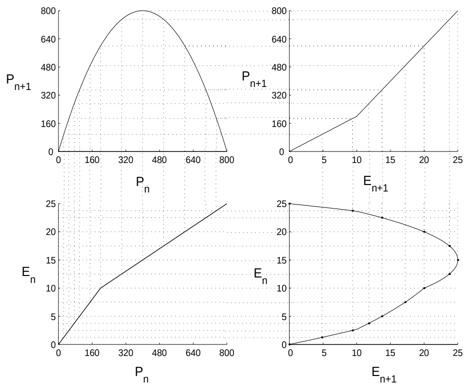

Figure 4: How to translate the graph of P_{n+1} versus P_n into a graph of E_n versus E_{n+1}.

picture of how the E_{n+1} versus E_n graph is constructed: starting from E_n on the vertical axis of the lower right graph, it finds P_n (lower left), then P_{n+1} (upper right), which then is used to determine the (E_n, E_{n+1}) couple on the lower right graph.

Despite their differences, the graphs of P_{n+1} versus P_n and E_{n+1} versus E_n are qualitatively the same: if the fixed point in the population graph is stable, then it will be stable also in the expenditure graph and vice versa; if these fixed points are unstable, but the population has a stable cycle with period 2, then so has the expenditure graph. The two evolutions are really the same phenomenon, observed from two different vantage points: the "fish counting biologist" and the "pennies counting accountant", and that is why they **have to** have similar behavior.

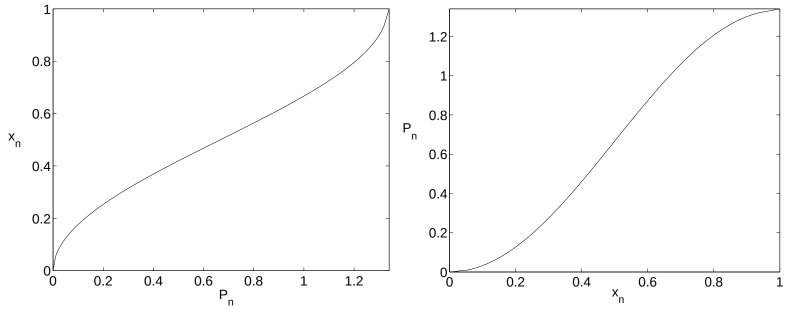

Figure 5: x_n versus P_n on the left; its inverse P_n versus x_n on the right.

A particularly useful change of variables

The discussion above showed that a "change of variables" can transform the P_{n+1} versus P_n graph into another graph that still qualitatively describes the "same" evolution. For the logistic equation with $r = 3$ there is a particularly useful change of variables that will help us to understand the chaotic behavior of the dynamics of the population. Because we are no longer thinking of the new variable as an expenditure E_n, we will just label it by x_n.

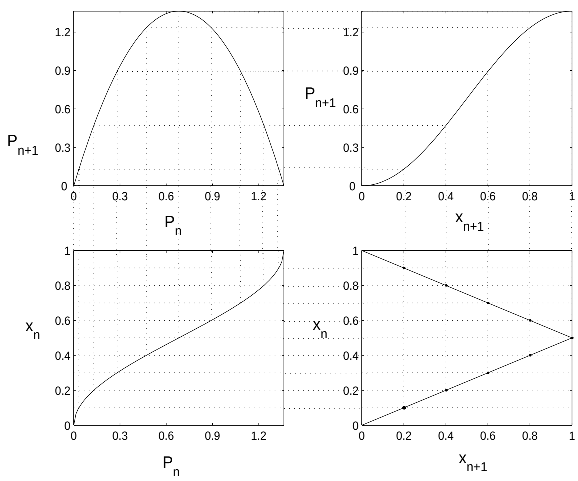

Figure 6: How to translate the graph of P_{n+1} versus P_n into a graph of x_n versus x_{n+1}.

Figure 5 shows a proposed x_n versus P_n, and its inverse P_n versus x_n; if we use it to construct a map x_{n+1} versus x_n along the steps outlined above, as shown in Figure 6, then we obtain Figure 7. Figure 7 corresponds to the dynamical evolution given by

$$x_{n+1} = \begin{cases} 2x_n & \text{for } 0 \le x_n \le \frac{1}{2} \\ 2(1 - x_n) & \text{for } \frac{1}{2} < x_n \le 1 \end{cases}$$

This transformed mapping is called the **tent** map, and it is **piecewise linear**. For the tent map we can easily find special orbits. For instance, if $x_0 = 0.4$, then $x_1 = 2 \times 0.4 = 0.8$,

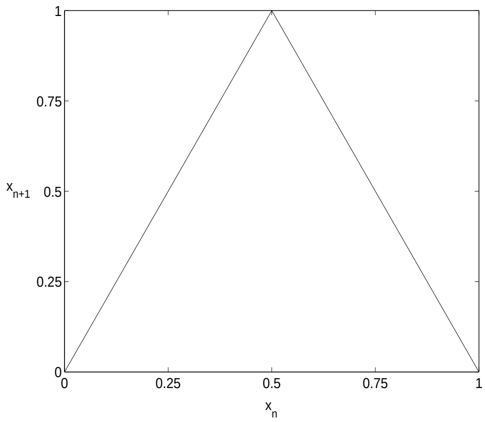

Figure 7: The tent map.

$x_2 = 2(1-0.8) = 0.4$, and thus $x_{2l} = 0.4$, $x_{2l+1} = 0.8$ for all l; this cycle of period 2 then corresponds to $P_0 = \frac{4}{3}\sin(0.2\pi)^2 \approx 0.460655$, $P_1 = \frac{4}{3}\sin(0.4\pi)^2 \approx 1.206011$. You can check on the lab that this gives a cycle of period 2 for $r = 3$ logistic case. (You should use lots of decimals in P_0 when you try this out, and even then you should expect that the periodic behavior in your simulation won't last forever, because you won't have the exactly right P_0, and the cycle is unstable.) You can find other cycles with other periods, that need not even be multiples of 2: $x_0 = \frac{2}{7}$ gives $x_1 = 2 \times \frac{2}{7} = \frac{4}{7}$, $x_2 = 2(1-\frac{4}{7}) = \frac{6}{7}$, $x_3 = 2(1-\frac{6}{7}) = \frac{2}{7}$, so that we have a cycle with period 3. This corresponds to the following cycle with period 3 for the $r = 3$ logistic map: $P_0 \approx 0.251007$, $P_1 \approx 0.815014$, $P_3 \approx 1.267313$; this is again an unstable cycle. We can understand, by using the correspondence with the tent map, that any cycle for the $r = 3$ logistic map must be unstable, by the following argument. To assess the stability of a cycle of period L for the tent map, with $f(x_0) = x_1, f(x_1) = x_2, \ldots, f(x_{L-1}) = x_L, f(x_L) = x_0$, we must compute $|f'(x_0)f'(x_1)\ldots f'(x_L)|$, combining the slopes of f at the different x_i; we mentioned earlier that the cycle is stable only when this product has absolute value less than 1. Since the slope at any point x is always 2 or -2, the product of slopes always exceeds 1, so that the cycle must be unstable. It follows that any cycle for the tent map is unstable, and therefore that any cycle for the $r = 3$ logistic map must also be unstable, since the two maps are related to each other via the change of variables.

We are interested not only in the existence and stability of cycles, but also in understanding the reasons for the sensitivity on initial conditions. It turns out that it is helpful to first consider another piecewise linear map, called the doubling map. We will come back later to the tent map and the logistic map.

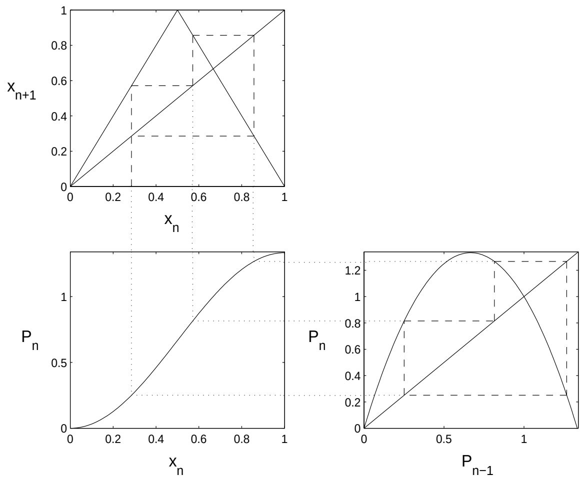

Figure 8: Transferring a cycle of period 3 for the tent map to the corresponding cycle of period 3 for the logistic map with $r = 3$.

Explaining the striking effect of this particular coordinate change

The striking form of Figure 7 is explained by a little trigonometry. We have $P_{n+1} = 4P_n - 3P_n^2$. The graph of x_n versus P_n, given in Figure 5, corresponds to the formula $x_n = \frac{2}{\pi}\sin^{-1}\left(\sqrt{\frac{3P_n}{4}}\right)$, which can be rewritten as

$$P_n = \frac{4}{3}\left[\sin\left(\frac{\pi}{2}x_n\right)\right]^2, \tag{2}$$

where x_n ranges between 0 and 1. It follows that we can write an explicit formula for x_{n+1} versus x_n: $x_{n+1} = \frac{2}{\pi}\sin^{-1}(u_{n+1})$ where

$$\begin{aligned} u_{n+1} &= \sqrt{\frac{3P_{n+1}}{4}} = \sqrt{\frac{3}{4}(4P_n - 3P_n^2)} \\ &= \sqrt{4 \times \frac{3}{4}P_n(1 - \frac{3}{4}P_n)} = 2u_n\sqrt{1 - u_n^2}, \end{aligned}$$

using the shorthand $u_n = \sqrt{\frac{3}{4}P_n}$. Since, according to (2), $u_n = \sin(\frac{\pi}{2}x_n)$, we obtain

$$\begin{aligned} \sin(\frac{\pi}{2}x_{n+1}) &= u_{n+1} = 2\sin(\frac{\pi}{2}x_n)\sqrt{1 - \sin(\frac{\pi}{2}x_n)^2} \\ &= 2\sin(\frac{\pi}{2}x_n)\cos(\frac{\pi}{2}x_n) = \sin(\pi x_n), \end{aligned} \tag{3}$$

where we have used the trigonometric formula $\sin(2y) = 2\sin(y)\cos(y)$. Our computation (3) suggests that we have thus

$$\frac{\pi}{2}x_{n+1} = \pi x_n \quad \text{or} \quad x_{n+1} = 2x_n. \tag{4}$$

Note however that we have defined x_n such that it always lies between 0 and 1. It follows that the rule (4) can only hold if $x_n \leq \frac{1}{2}$. If $x_n > \frac{1}{2}$ then we can use the useful trigonometric equation $\sin(\pi - y) = \sin(y)$ to deduce $\sin(\frac{\pi}{2}x_{n+1}) = \sin(\pi(1 - x_n))$, which suggests, for $x_n > \frac{1}{2}$,

$$\frac{\pi}{2}x_{n+1} = \pi(1 - x_n) \quad \text{or} \quad x_{n+1} = 2(1 - x_n).$$

In summary we have

$$x_{n+1} = \begin{cases} 2x_n & \text{for } 0 \leq x_n \leq \frac{1}{2} \\ 2(1 - x_n) & \text{for } \frac{1}{2} < x_n \leq 1 \end{cases}$$

This gives exactly the graph in Figure 7! This trick was first proposed in 1946 by two famous mathematicians, J. Vol Neumann and S. Ulam.

The doubling map

Here is a brief description of a simple mathematical machine where iterating a function generates chaos: iteration of the doubling map.

The doubling map amounts to doubling a number between 0 and 1 and discarding the integer part, so that, for example,

$$0.32 \rightarrow 0.64 \text{ (nothing to discard)}$$
$$0.64 \rightarrow 1.28 \rightarrow 0.28 \text{ (drop the 1).}$$

The notes to follow summarize some properties of iterations of the doubling map, showing that it is effectively "as chaotic as tossing a fair coin." Indeed, if one picks an initial condition "at random," with probability one it will be an irrational number whose binary expansion $a_1a_2a_3\ldots a_n\ldots$ will never repeat, and as we shall see, this implies that its orbit under iteration of f will dance about "chaotically" in the interval $0 \leq x \leq 1$, visiting every little segment with equal probability.

Let's look at this machinery in some more detail. Take a number x_0 between 0 and 1, double it, and subtract the integer part to produce the output x_1. This is just another use of modular arithmetic, as in the cryptography and error correction units. Repeat *ad infinitum.* Calling the operation "$x \mapsto 2x$ modulo 1," "$x \mapsto f(x)$," we thus iterate the rule:

$$x_{n+1} = f(x_n) \; , \tag{5}$$

given the initial datum x_0. The beginning of an orbit of (5) might look like

$$0.3653 \mapsto 0.7306 \mapsto 1.4612 = 0.4612 \mapsto 0.9224 \mapsto 1.8448 = 0.8448 \mapsto \ldots.$$

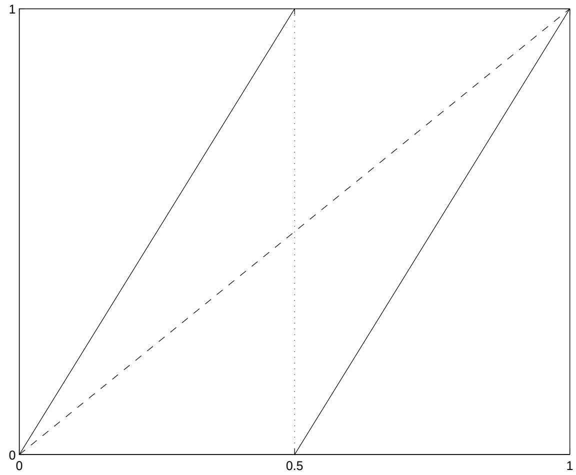

Figure 9: The doubling map.

Binary representations and the doubling map

To understand this dynamical system better, we shall represent the number in the interval in binary form rather than in decimal form. We now introduce the construction of this representation and state some of its properties. Given $x_0 \geq 0$, with $x_0 \leq 1$, we can define numbers $a_1, a_2, a_3, \ldots$, all of which are either 0 or 1, as follows:

- if $x_0 \geq \frac{1}{2}$, then $a_1 = 1$; if $x_0 < \frac{1}{2}$, then $a_1 = 0$;
- we compute $x_1 = 2x_0 - a_1 = f(x_0)$;
- if $x_1 \geq \frac{1}{2}$, then $a_2 = 1$; if $x_1 < \frac{1}{2}$, then $a_2 = 0$;
- we compute $x_2 = 2x_1 - a_2$;
- and so on...

Note that this is very similar to the determination of decimal digits. For instance, we write $\frac{1}{13} = .07692307...$, where the different digits are determined as follows: $y_0 = \frac{1}{13}$ satisfies $\mathbf{0} \leq 10y_0 < 1$; then $y_1 = 10y_0 - 0 = \frac{10}{13}$ satisfies $\mathbf{7} \leq 10y_1 < 8$; $y_2 = 10y_1 - 7 = \frac{9}{13}$ satisfies $\mathbf{6} \leq 10y_2 < 7$; $y_3 = 10y_2 - 6 = \frac{12}{13}$ satisfies $\mathbf{9} \leq 10y_3 < 10$; and so on: one by one, we find the successive digits in the **decimal** expansion of $\frac{1}{13}$ (see Figure 10). By contrast the **binary** expansion of $\frac{1}{13}$ is given by multiplying by 2 instead of 10: $x_0 = \frac{1}{13}$ satisfies $\mathbf{0} \leq 2x_0 \leq 1$; $x_1 = 2x_0 = \frac{2}{13}$ satisfies $\mathbf{0} \leq 2x_1 < 1$; $x_2 = 2x_1 = \frac{4}{13}$ satisfies $\mathbf{0} \leq 2x_2 < 1$; $x_3 = 2x_2 = \frac{8}{13}$ satisfies $\mathbf{1} \leq 2x_3 < 2$; $x_4 = 2x_3 - 1 = \frac{3}{13}$ satisfies $\mathbf{0} \leq 2x_4 < 1$; $x_5 = 2x_4 = \frac{6}{13}$ satisfies $\mathbf{0} \leq 2x_5 < 1$; $x_6 = 2x_5 = \frac{12}{13}$ satisfies $\mathbf{1} \leq 2x_6 < 2$; and so on, leading to the expansion .000100111011... for the binary representation of $\frac{1}{13}$ (see Figure 11).

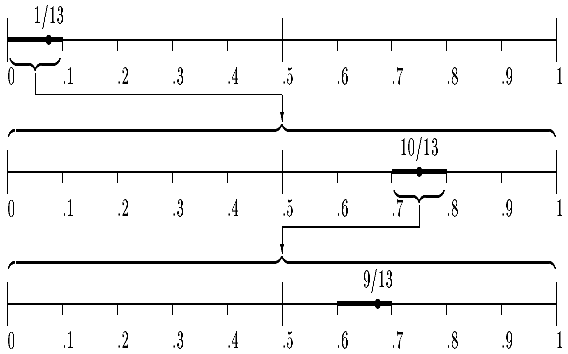

Figure 10: the decimal expansion: zooming in by successive factors of 10 gives us the decimal expansion of a number. Here $\frac{1}{13} = .076....$

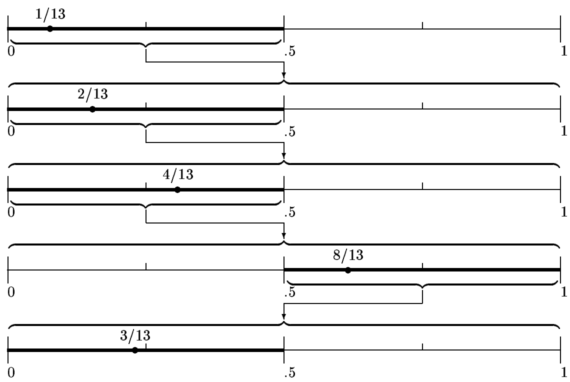

Figure 11: the binary expansion: zooming in by successive factors of 2 gives us the binary expansion of a number. Here $\frac{1}{13} = .0010....$

The decimal expansion of $\frac{1}{13}$ tells us that

$$\frac{1}{13} = 7 \times 10^{-2} + 6 \times 10^{-3} + 9 \times 10^{-4} + 2 \times 10^{-5} + 3 \times 10^{-6} + 0 \times 10^{-7} + 7 \times 10^{-8} + \dots$$

where the dots ... represent a remainder that is less than 10^{-8}; similarly the binary expansion tells us that

$$\begin{aligned} \frac{1}{13} \quad = \quad & 0 \times \frac{1}{2} + 0 \times \frac{1}{4} + 0 \times \frac{1}{8} + \frac{1}{16} + 0 \times \frac{1}{32} + 0 \times \frac{1}{64} + \frac{1}{128} \\ & + \frac{1}{256} + \frac{1}{512} + 0 \times \frac{1}{1024} + \frac{1}{2048} + \frac{1}{4096} + \dots \end{aligned}$$

where the ... now represent a remainder that is less than $2^{-12} = \frac{1}{4096}$. (All this is a little different from the binary arithmetic of the cryptography unit, but it's closely related: there we were working with integers, and now we introduce numbers smaller than 1.) For a general number x_0, we obtain $a_1, a_2, a_3, a_4, \ldots$ as given above and we have

$$\begin{aligned} x_0 &= \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots + \frac{a_k}{2^k} + \cdots \\ &= \sum_{j=1}^{\infty} \frac{a_j}{2^j}, \end{aligned}$$

where the $\sum$ symbol indicates that we look at the limit of the "partial sums" $\frac{a_1}{2}$, $\frac{a_1}{2} + \frac{a_2}{4}, \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8}, \ldots$ which get closer and closer to x_0. We can thus also start from any possible choice of a_j, either 0 or 1, and determine the corresponding value of x_0. For instance, if $a_{2l} = 1$, $a_{2l+1} = 0$ for all $l \geq 1$, corresponding to the representation .01010101..., then

$$\begin{aligned} x_0 &= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots \\ &= \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^k} + \cdots \end{aligned} \tag{6}$$

We can compute all the partial sums:

$$\begin{aligned} \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^k} &= \frac{1}{4}(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^{k-1}}) \\ &= \frac{1}{4} \cdot \frac{1 - (\frac{1}{4})^k}{1 - \frac{1}{4}} = \frac{1 - (\frac{1}{4})^k}{3}; \end{aligned}$$

as k becomes larger and larger, this gets closer and closer to $\frac{1}{3}$, so that $x_0 = \frac{1}{3}$. In the problem set, you will see another way to obtain this same result.

The correspondence is not unique, but we can remedy this by identifying pairs of sequences whose heads coincide and whose tails are $1000\ldots0\ldots$ and $0111\ldots1\ldots$ respectively; for instance the two sequences .10000... and .01111... represent the same number:

$$\begin{aligned} .10000\ldots &= \frac{1}{2} \\ .01111\ldots &= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots \\ &= \frac{1}{4}(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots) \\ &= \frac{1}{4} \times 2 = \frac{1}{2}. \end{aligned}$$

Similarly, for any choice of $a1, a_2, \ldots, a_N$, the sequences $.a_1a_2a_3\ldots a_N10000\ldots$ and $.a_1a_2a_3\ldots a_N01111\ldots$ represent the same number.

Iterating the doubling map and symbolic dynamics

If $x_0 = .a_1a_2a_3a_4...$, then we have:

- if $x_0 < \frac{1}{2}$, $a_1 = 0$, and

$$\begin{aligned} f(x_0) = 2x_0 &= 2(\frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \cdots) \\ &= \frac{a_2}{2} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \cdots) \\ &= .a_2a_3a_4a_5... \end{aligned}$$

- if $x_0 \geq \frac{1}{2}$, $a_1 = 1$, and

$$\begin{aligned} f(x_0) = 2x_0 - 1 &= 2(\frac{1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \cdots) - 1 \\ &= \frac{a_2}{2} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \cdots) \\ &= .a_2a_3a_4a_5... \end{aligned}$$

In **both** cases, $f(.a_1a_2a_3a_4...) = .a_2a_3a_4...$: the doubling map consists of simply dropping the first binary expansion bit.

It follows that the binary representation is very convenient for writing the successive iterates of the doubling map: starting at x_0, we see that the **orbit** $\{x_0, x_1, \ldots, x_k, \ldots\}$ of the point x_0 is given by [1]:

$$x_k = f^k(x_0) = \sum_{j=k+1}^{\infty} \frac{a_j}{2^{j-k}} \ .$$

Iteration of the map successively reveals elements of the binary sequence further and further to the right. After k iterations, the binary representation of x_k begins with the symbol a_k and so the state lies in the left or right half of the interval $0 \leq x \leq 1$ if $a_k = 0$ or 1 respectively. The initial condition x_0, via its binary representation, **completely determines** all the future behavior of the orbit. The sequence of a_k gives a "symbolic dynamics": indicating the successive left and right halves of $[0, 1]$ into which $f^k(x_0)$ will land. Conversely, just saying whether the successive $f^k(x_0)$ will be in the left or right half of $[0, 1]$ determines the a_k and thus x_0.

So far so good. However, in each iteration the leading symbol is removed and **some information is lost**. If we knew the initial condition x_0 to infinite accuracy (we had all the a_k's

[1] The notation $f(x)$ means "the value of the function at the point x," so iterating a function is $x_1 = f(x_0)$; $x_2 = f(x_1) = f(f(x_0)), \ldots x_3 = f(f(f(x_0)))$. To save space, we will write $f(f(x_0))$ as $f^2(x_0)$, etc. The superscript 2 means "second iterate," **not the square!**

for k between 1 and infinity), this would not matter, for at each step we would still know the current state x_k exactly. But if we can measure or store only, say, the first N binary places $(a_1, a_2, \ldots, a_N)$, then, after N iterations, we cannot even say whether the state x_{N+1} lies above or below $\frac{1}{2}$! Moreover, even if two initial conditions differ only at the Nth binary place, so that they lie within distance $\frac{1}{2^{N-1}}$ of one another, after N iterations they fall on opposite sides of $\frac{1}{2}$ and thereafter behave essentially independently. This is referred to as **sensitive dependence on initial conditions**: the expansive doubling dynamics amplifies small errors at the same time as the *modulo 1* operation prevents orbits escaping from $[0, 1]$.

The symbolic binary description allows much more. As pointed above to every infinite sequence of 0's and 1's there corresponds a point in $[0, 1]$, and *vice versa* (up to the non-uniqueness described in the gray box). This implies that, given any random sequence (generated by tossing a coin and assigning Heads = 0, Tails = 1, for instance), there is an initial state x_0 such that the orbit $\{f^k(x_0)\}$ realizes exactly that sequence. Hence our system has infinitely many such essentially random orbits. There are also infinitely many periodic orbits, corresponding to sequences such as 001001001... All these periodic (or eventually-periodic) orbits correspond to rational numbers, i.e. to fractions of integers. For instance,

$$.001100110011... = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^{11}} + \frac{1}{2^{12}} + \ldots$$

Since

$$\begin{aligned} & \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^8} + \ldots + \frac{1}{2^{4k-1}} + \frac{1}{2^{4k}} \\ = \ & \Big(\frac{1}{2^3} + \frac{1}{2^4}\Big)\Big(1 + \frac{1}{2^4} + \frac{1}{2^8} + \ldots + \frac{1}{2^{4(k-1)}}\Big) \\ = \ & \Big(\frac{1}{8} + \frac{1}{16}\Big)\Big(1 + \frac{1}{16} + \frac{1}{16^2} + \ldots + \frac{1}{16^{(k-1)}}\Big) \\ = \ & \frac{3}{16}\Big(\frac{1 - \frac{1}{16^k}}{1 - \frac{1}{16}}\Big) \\ = \ & \frac{3}{16}\Big(\frac{1 - 1/2^{4k}}{15/16}\Big) \end{aligned}$$

where $\frac{1}{2^{4k}}$ becomes negligibly small as k grows, and we obtain .001100110011... = $\frac{3}{15}$. The periodic orbit containing $\frac{3}{15}$ is given by $x_0 = \frac{3}{15}, x_1 = \frac{6}{15}, x_2 = \frac{12}{15}, x_3 = \frac{9}{15}, x_4 = x_0$. Note that we could also have taken $\frac{6}{15} = .01100110011\ldots$ $(or \frac{12}{15}, or \frac{9}{15})$ as a starting point, to find the same periodic orbit. However, if you produce a sequence $a_1, a_2, a_3, \ldots$ by tossing a fair coin, then the probability that you would end up with a periodic sequence is very small as long as you look at long, finite sequences, and it becomes smaller and smaller as the length of the sequence increases; for an infinite sequence, the probability that such a fair Heads/Tails sequence is periodic, equals zero.

Counting numbers

Rational numbers (numbers that can be written as fractions, or numbers with a periodic decimal or binary expansion) are, as was just argued, much less "probable" than irrational numbers (all the other real numbers, such as π, $\sqrt{2}$, ...). In fact there are much fewer rational numbers than irrational numbers; you could count the rational numbers (if you were thus inclined), but not the irrationals.
What does it mean to "count" an infinite set? It means that you could imagine a scheme to label them all by integers, making it possible to enumerate them exhaustively in one (possibly unfinishing) sequence.
The simplest example is given by the positive integers: just listing them in their natural order

$$1, 2, 3, 4, 5, 6, \ldots$$

provides an enumeration. The set of all integers is countable as well, even though it does not have a smallest number to start counting from. You can enumerate them exhaustively as follows:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

The set of all positive fractions is countable: the following trick, due to 19th century mathematician Georg Cantor, gives a way to enumerate them. First, make a big table of all possible fractions, where their numerator indicates the row number, and the denominator the column number, as follows:

$$\begin{array}{cccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Clearly, every possible positive fraction will be somewhere in this table. Next, in Figure 12 we show how any such table can be naturally enumerated, starting from the upper left corner. It follows that the set of all fractions, and thus the set of all rationals, is countable. With all these clever tricks, you may well wonder whether not all numbers are countable! The answer is no: there are far too many irrational numbers to make any organized enumeration possible. We can use binary expansions to see this.
Suppose we could enumerate all the real numbers between 0 and 1, that is there would be a way of listing them in a long sequence, $z_1, z_2, z_3, z_4, \ldots$. For each of these z_n, we can find the associated binary sequence for which

$$z_n = \sum_{k=1}^{\infty} a_{n;k} 2^{-k}.$$

Let us now manufacture a new number z between 0 and 1 as follows:

$$z = \sum_{k=1}^{\infty} (1 - a_{k;k}) 2^{-k},$$

that is, for its k-th binary entry we look at the k-th entry of z_k, and we take the opposite (0 if $a_{k;k} = 1$, 1 if $a_{k;k} = 0$). Since, for each k, z differs from z_k (it has at least one different digit!), it follows that z is <u>not</u> in the list $z_1, z_2, z_3, z_4, \ldots$. But this list was supposed to be exhaustive! The contradiction means that our original assumption was false: the numbers between 0 and 1 cannot be enumerated.

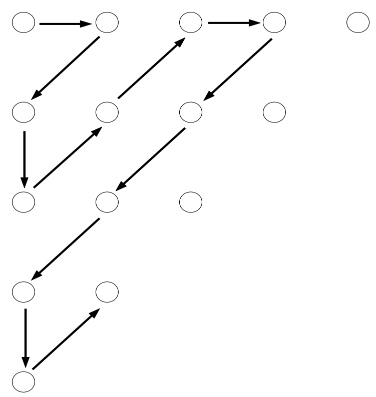

Figure 12: Enumeration of the table in the gray box above.

One can also use the binary representation to show that a **dense orbit** exists. Consider the sequence $\mathbf{a}^*$ formed by concatenating all possible sequences of lengths $1, 2, 3, \ldots$ end to end:

$$\mathbf{a}^* = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001 \ldots.$$

As one iterates and drops leading symbols, every possible subsequence of any given length appears at the head. This implies that the orbit of the point x^* corresponding to $\mathbf{a}^*$ contains points which approximate, to any desired accuracy, **every** point in the interval $0 \leq x \leq 1$. Therefore $f^k(x^*)$ is a dense orbit. It is a kind of infinite thread which stitches all the different orbits together in an inseparable web.

The technical definition of chaos requires just these properties: sensitive dependence, density of period orbits, and a dense orbit (not all of them are independent).

Back to the tent map and the logistic equation

We have now introduced three different maps: our original logistic map, its equivalent (through a change of variable) the tent map and the also piecewise linear but even simpler doubling map. For the doubling map we saw that we could characterize a point x_0 by whether its orbit $x_1, x_2, x_3, \ldots$ would end up to the left or to the right of $\frac{1}{2}$ in successive steps; in that case the corresponding binary sequence (0 for left, 1 for right) was exactly the binary expansion of x_0. We can look at a similar symbolic dynamics for the other two maps. In that case, the symbolic dynamics won't give the binary expansion; we'll see what it does instead. Figure13 shows the three maps and a view of their symbolic dynamics.

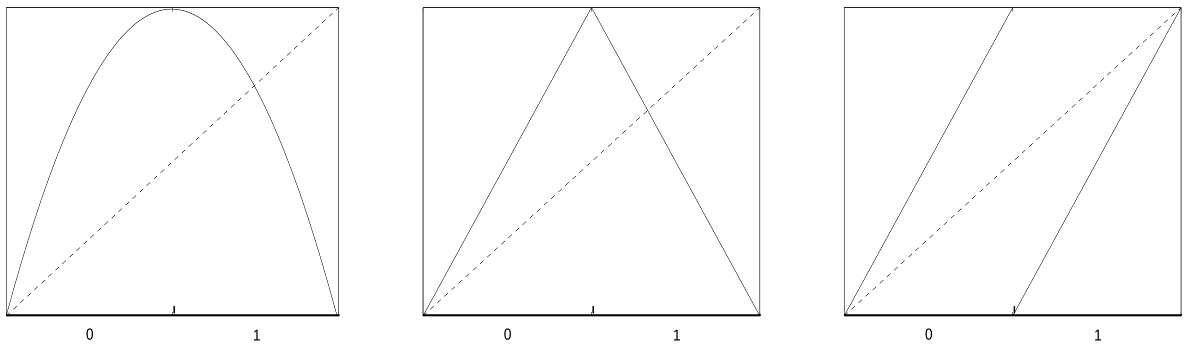

Figure 13: The three maps.

We assign a semi-infinite sequence $\{a_j\}_{j=1}^{\infty} = a_1 a_2 a_3 \ldots a_k \ldots$ to each point (initial condition) x according to the rule

$$a_k = \begin{cases} 0 \text{ if } f^k(x) \text{ lies in the left half of } [0,1]; \\ 1 \text{ if } f^k(x) \text{ lies in the right half of } [0,1]. \end{cases}$$

The sequence is the **itinerary of** x. It tells where the orbit of x goes. For the doubling map—it is EXACTLY the binary expansion of x, as we have seen. For the tent map, it's a bit more complicated. Compare the two pictures in Figure 14, which illustrate the intervals of length $\frac{1}{8}$, identified by three symbols each, specifying the start of the itineraries of all points within them.

Can you figure out the general pattern, or a rule that will generate it? This is the renaming coordinate change referred to before.

For the logistic map, these little intervals are of different lengths, reflecting the nonlinear nature of the map. Here are the four intervals 00, 01, 10, and 11 (see Figure 15). Check it out using the staircase orbit construction that we learned in Part 1 of the lecture notes.

The sizes differ, but the ordering is identical to that for the (piecewise-) linear tent map. We "pull back" the regularly spaced intervals for the tent to the logistic map using the change of coordinates described earlier in Section 4.3.1.

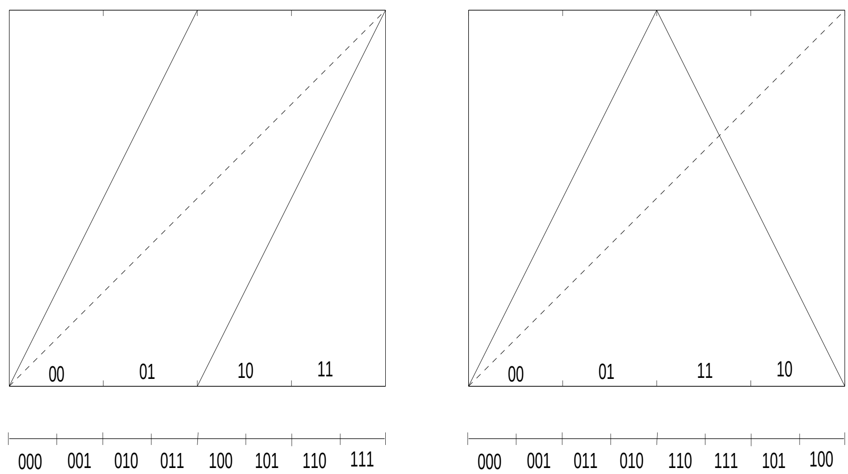

Figure 14: Naming subintervals by symbol strings.

The statistics of chaos

The same kind of argument explains the histogram that you will produce in the lab, showing where typical solutions go as one iterates the map for $r = 3$. We first argue that, for the tent map (or, indeed, for the doubling map), this corresponding histogram is a horizontal straight line: the constant function. This means that every region of size δ, say, has an equal chance of being visited by a typical solution. To see this, consider the images of a little interval of initial conditions of length δ. The first image $f(\delta)$ has length 2δ, the second, $f^2(\delta)$, has length $4\delta, \ldots$, and the kth, $f^k(\delta)$, has length $2^k\delta$. See Figure 16.

The lengths are uniformly stretched, so that, for an evenly distributed group of initial conditions in any (small) region, there's an equal chance of finding orbits anywhere in the whole interval, provided you wait long enough.

Transforming this flat histogram back via the change of coordinates, we get the function $h(x) = \frac{1}{\pi\sqrt{x(1-x)}}$, where $x = 1$ is the "top" of the box. Compare this with the histogram you get on the computer, after suitable rescaling (see Figure 17).

Here is an alternative, more "computational" method of seeing that segments of equal lengths have equal chances of being visited by typical orbits, which draws on the probability theory we saw in Unit 3. See Figure 18.

Each segment of length $\frac{1}{2^k}$ is specified by a binary string of length k. Each initial point is specified by an infinite binary string, so, for a "typical" initial point, the probability that a given entry is 0 (or 1) is exactly $P(a_j = 0) = \frac{1}{2}, P(a_j = 1) = \frac{1}{2}$. Hence, the probability of

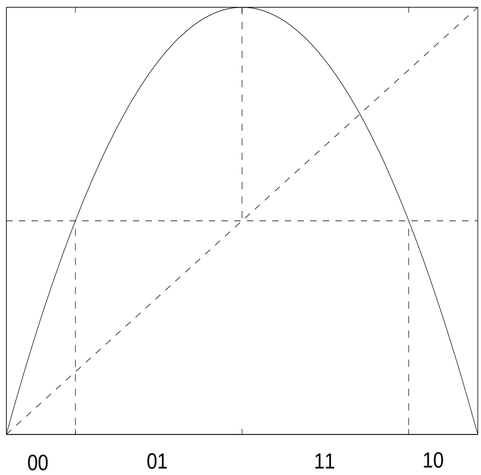

Figure 15: Subintervals for the logistic map.

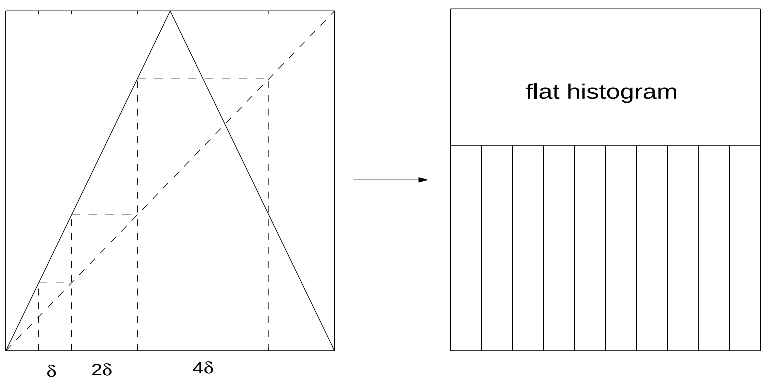

Figure 16: A histogram for solutions of the tent map.

any given string of length k appearing in a typical string is

$$P(a_1)P(a_2)\dots P(a_k) = \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdots\cdot\frac{1}{2}}_{k \text{ times}} = \frac{1}{2^k}\ ,$$

because each entry may be regarded as an independent event. The sequence $a_0\ a_1\ a_2\dots$ describes the itinerary of the point, or the subintervals it visits (recall Figure 14). So the probability that, at any given step, the solution is in any given one of the 2^k intervals of length $\frac{1}{2^k}$ is exactly $\frac{1}{2^k}$.

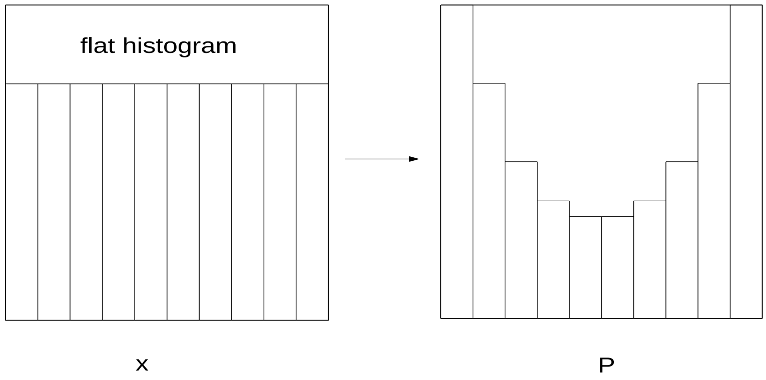

Figure 17: Transforming to the histogram for the logistic map.

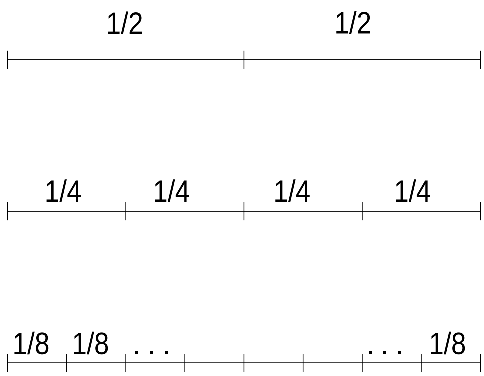

Figure 18: Probabilities and lengths of subintervals.