Birth, Growth, Death, and Chaos

Also see "For All Practical Purposes" (FAPP), Chapter 18.

This is a unit about DYNAMICAL SYSTEMS, mathematical models for phenomena that change over time. We'll start with some simple, everyday examples, which we'll use to establish some notation and conventions, and to explain our graphical and geometric presentation. We will then go on to examples that are still very simple, but that show complex behavior similar to what is observed in true populations.

Understanding such dynamical systems is important to gain insight, for instance, in how fish populations can be abundant one year, but much smaller the next year, or in how epidemics come up and then disappear for a while, only to ravage again a few years later. We often want to have a good understanding of how human intervention can change these cycles, or what type of intervention is safest or most prudent: in the examples above, we may want to know how much fish can be harvested without causing overfishing disruptions, or what measures are the most efficient to combat illness outbreaks. It turns out that even when left to themselves, these biological systems can have very complex, even chaotic behavior – this does of course complicate the task of understanding the effect of human measures! In this unit, we shall see how such complex behaviors can arise in even very simple models.

We see here a different use of mathematics from what we've seen in some other parts of the course. Rather than using mathematics to design schemes for data compression or encryption, which can be viewed as making the world fit **our** mathematical constructions, we are here trying to derive a mathematical **model** of (part of) the " natural" world. We have to construct something that makes mathematical sense and is reasonably realistic/accurate when applied to describe/predict the phenomena modeled.

Note: In this module there will be occasional sidebars that point to slightly more advanced mathematics. They will be indicated by a box with a gray background. You can skip them if you wish, but they are instructive if you have had some calculus (and remember it ;-)).

Part 1: Building and understanding models

1 Investments and interest

1.1 You bury your money

Here, with P \$ to start with, you have the same amount, P, at the end of every year. Graphically:

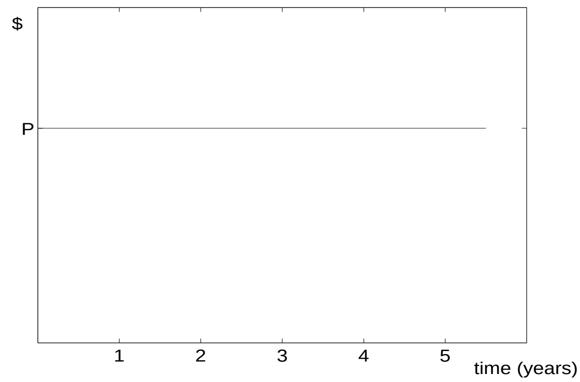


Figure 1: You bury your money!

Very boring. No bank with this policy would survive long.

1.2 Simple interest

You invest with return at a rate r/year. If the bank quotes an interest rate of 5%, $r = \frac{5}{100} = .05$. (The growth rate is normalized so that $r = 1$ corresponds to 100%.) Suppose you withdraw the interest each year, so that the principal, P remains at the same constant level as before. The total amount of money, consisting of the money in the bank plus the accumulated interest that you put under your mattress each year, however, is growing. After one year, you have $P + r \times P = P(1 + r)$, after two years $P + r \times P + r \times P = P(1 + 2r)$, after n years $P + n \times r \times P = P(1 + nr)$. Plotting this total, we get the graph of a line sloping upward:

Note that the effect is the same, whether the bank adds to interest at each year's end $(P + rP)$ or in two six-month installments $P + \frac{r}{2}P + \frac{r}{2}P$ (assuming you withdraw the inter-

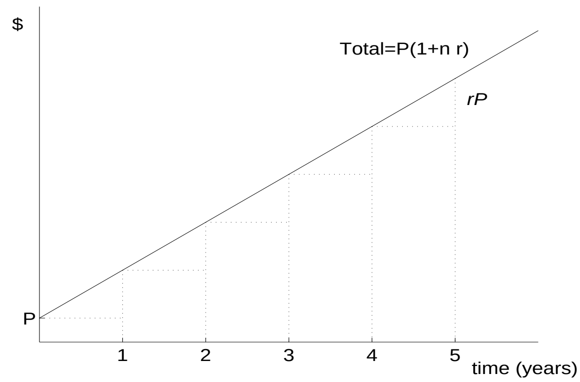
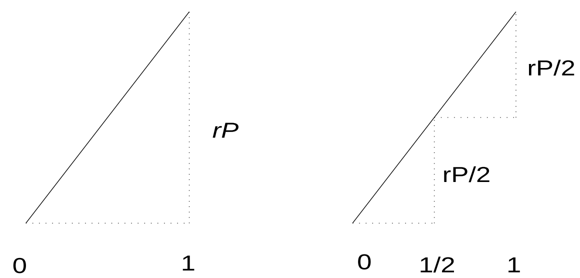


Figure 2: Simple interest.

est as it is awarded), or in twelve one-month installments. The Principal, P, on which each interest payment is based, is not changing!



1.3 Compound interest

Now you leave your money in the bank—you reinvest and interest is compounded. If compounded annually, you have, starting with P_0 (principal at time 0):

time	total
0	P_0
1	$P_1 = P_0 + rP_0 = P_0(1+r) =$ new principal
2	$P_2 = P_0(1+r) + r \times P_0(1+r) = P_0(1+r)(1+r) = P_0(1+r)^2$
3	$P_3 = P_0(1+r)^2 \times (1+r) = P_0(1+r)^3$
4	$P_4 = P_0(1+r)^4$
$\vdots$	
n	$P_n = P_0(1+r)^n$

This is **geometric** or **exponential** growth. It is much faster "in the end." Graphically,

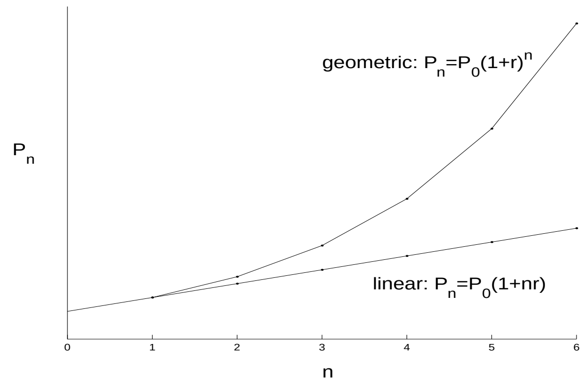


Figure 3: Comparison of simple and compound interest.

Note, however, that the year to year relationship is a linear one:

$$P_{n+1} = (1+r)P_n \ .$$

The cumulative effect of applying this year after year leads to the nonlinear exponential growth.

Example: Invest \$1000 at 5%, $r = \frac{5}{100} = .05$

Saving method	formula	Total amount in			
		year 0	year 1	year 10	year 100
Bury it	P_0	\$1000	\$1000	\$1000	\$1000
Simple interest	$P_0(1+nr)$	\$1000	\$1050	\$1500	\$6000
Compounded Yearly	$P_0(1+r)^n$	\$1000	\$1050	\$1628.89	\$131,501.26

e.g., at year ten: simple interest leads to a total of $1000(1 + 10 \times 0.05) = 1500$
compound interest leads to a total of $1000(1 + 0.05)^{10} = 1628.89$

Since compounding increases the principal, the number of times per year interest is paid affects the outcome. For instance, an annual rate r compounded gives after one year,

$$P_0\left(1+\frac{r}{4}\right)\left(1+\frac{r}{4}\right)\left(1+\frac{r}{4}\right)\left(1+\frac{r}{4}\right)=P_0\left(1+\frac{r}{4}\right)^4 .$$

After n years you obtain: $P_0\left(1+\frac{r}{4}\right)^{4n}$. If the same interest rate is compounded monthly $P_0\left(1+\frac{r}{12}\right)^{12n}$, or daily: $P_0\left(1+\frac{r}{365}\right)^{365n}$. The effective annual rate is:

$$\begin{aligned}&(1+r)=1.05\rightarrow 5\%\\&\left(1+\tfrac{r}{4}\right)^4=1.05095\rightarrow 5.095\%\\&\left(1+\tfrac{r}{12}\right)^{12}=1.05116\rightarrow 5.116\%\\&\left(1+\tfrac{r}{365}\right)^{365}=1.05127\rightarrow 5.127\%\end{aligned}$$

More frequent compounding gives better rates. For Bill Gates, this can make a big difference!

Digression: What happens if we compound hourly, minutely, secondly? Mathematically, we ask what the limiting behavior is of

$$\left(1+\frac{r}{s}\right)^s \quad (s = \text{number of compounding periods/year}),$$

as s becomes larger and larger? (In mathematical language, "as s approaches infinity?") Now, for fixed r, $\frac{r}{s}\rightarrow 0$ so $\left(1+\frac{r}{s}\right)\rightarrow 1$, and of course $1\times 1\times 1\times 1$ no matter how many times we multiply 1 with itself. But until s is truly infinite, $1+\frac{r}{s}>1$, and multiplying any number bigger than 1 by itself gives a result **bigger** than the original number, so in the limit, after infinitely many increases, we expect $(1+\text{a bit})\times(1+\text{a bit})\times(1+\text{a bit})\times\cdots\rightarrow\infty$. Is this a paradox?

Not really. In the limit we have 1^∞: an "indeterminate form." One can use techniques from calculus to show that the limit:

$$\text{limit}_{s\rightarrow\infty}\left(1+\frac{r}{s}\right)^s=e^r ,$$

where e is a "magic number," a universal constant—the natural logarithm base $e=2.71828182\ldots$ (like $\pi=3.14159265\ldots$ it is an irrational number whose decimal representation goes on forever without repeating).

Daily compounding is already pretty close to this limit:

$$\text{for r=0.05 we have}\quad\left(1+\frac{r}{365}\right)^{365}=\left(1+\frac{0.05}{365}\right)^{365}=1.051267496\ldots ,$$

compared with $e^r=e^{0.05}=1.051271096\ldots$. For more information, see Eli Maor "*e*: The Story of a Number," Princeton University Press, 1994, and pieces of chapter 17 of FAPP.

2 Some interesting mathematical observations about geometrically growing (or decreasing) quantities

With compound interest, we found that the total amount of money P_n after n years is given by $P_n = P_0(1+r)^n$, or, if we rewrite $1+r$ as a, $P_n = P_0 a^n$. This was under the assumption that you put in P_0 at the start (year "zero"), and then never withdraw or deposit any money. Suppose now that you have a very disciplined saving strategy, under which you save every year an amount S, which you deposit into your account. How much will you have after n years? We have to adjust the formula: now we have

$$P_n = (1+r)P_{n-1} + S.$$

Can we still write a formula that lets us compute P_n straight from P_0, without intermediate steps? Let's try:

$$\begin{aligned}
P_1 &= (1+r)P_0 + S, \qquad (1)\\
P_2 &= (1+r)P_1 + S = (1+r)^2 P_0 + (1+r)S + S,\\
P_3 &= (1+r)P_2 + S = (1+r)^3 P_0 + (1+r)^2 S + (1+r)S + S,\\
&\vdots
\end{aligned}$$

so that

$$P_n = (1+r)^n P_0 + [(1+r)^{n-1} + (1+r)^{n-2} + \ldots + (1+r)^2 + (1+r) + 1]S.$$

It would be convenient if we had a simple formula for

$$(1+r)^{n-1} + (1+r)^{n-2} + \ldots + (1+r)^2 + (1+r) + 1,$$

or, with $a = (1+r)$, for

$$a^{n-1} + a^{n-2} + \ldots + a + 1.$$

There is a simple trick that allows us to write such a formula. If we multiply the whole sum by $(a-1)$, we obtain

$$\begin{aligned}
&(a-1)(a^{n-1} + a^{n-2} + \ldots + a^2 + a + 1) \qquad (2)\\
&= (a-1)a^{n-1} + (a-1)a^{n-2} + \ldots + (a-1)a + (a-1)\\
&= a^n - a^{n-1} + a^{n-1} - a^{n-2} + \ldots + a^2 - a + a - 1\\
&= a^n - 1 \qquad \text{(all the other terms cancel).}
\end{aligned}$$

It follows that

$$a^{n-1} + a^{n-2} + \ldots + a + 1 = \frac{a^n - 1}{a - 1}.$$

We can apply this to our bank account problem:

$$\begin{aligned}
P_n &= (1+r)^n P_0 + [(1+r)^{n-1} + (1+r)^{n-2} + \ldots + (1+r)^2 + (1+r) + 1]S \qquad (3)\\
&= (1+r)^n P_0 + \frac{(1+r)^n - 1}{(1+r) - 1} S = (1+r)^n P_0 + \frac{(1+r)^n - 1}{r} S.
\end{aligned}$$

This is the simple formula we wanted. Note that you can also rewrite it as

$$P_n = (1+r)^n[P_0 + \frac{S}{r}] - \frac{S}{r};$$

it is as if your original capital P_0 had been augmented by $\frac{1}{r}S$ (which equals to $20S$ if $r = 0.05$) before compounding, and then "adjusted" again by subtracting $\frac{1}{r}S$ afterward.

The formula (4) is very useful in many instances when you deal with geometric growth. It works just as well for geometric decrease, that is, the formula still holds if $0 < a < 1$. In that case it is more customary to change the sign of both the numerator and the denominator of the fraction (which does not affect the fraction), and to write

$$1 + a + a^2 + \ldots + a^{n-2} + a^{n-1} = \frac{1-a^n}{1-a}.$$

We can now observe another interesting fact:

$$\frac{1}{1-a} - (1 + a + a^2 + \ldots + a^{n-2} + a^{n-1}) = \frac{a^n}{1-a};$$

if $a < 1$, then we know that a^n becomes smaller and smaller as n grows; in fact by choosing n sufficiently large, a^n can be made as tiny as you like. In mathematical terms, a^n tends to zero as n tends to infinity. As n grows, the number of terms in $1 + a + a^2 + \ldots + a^{n-1}$ grows as well, and this formula is telling us that, as n grows, the sum $1 + a + a^2 + \ldots + a^{n-1}$ tends to $\frac{1}{1-a}$ (because the difference tends to 0).

This is the reason why the unending decimal fraction $0.999999\ldots$ equals to 1. We have

$$\begin{aligned} 0.999999\ldots &= \frac{9}{10} \times 1.111111\ldots \qquad (4) \\ &= \frac{9}{10} \times [1 + 0.1 + (0.1)^2 + (0.1)^3 + (0.1)^4 + \ldots] \\ &= \frac{9}{10} \times \frac{1}{1-0.1} = \frac{9}{10} \times \frac{1}{0.9} = \frac{9}{10} \times \frac{1}{\frac{9}{10}} = 1. \end{aligned}$$

3 Population growth

Building on these models of interest accumulation, we will discuss simple models of population dynamics.

3.1 No limits to growth

First suppose our population is living in an "infinite" environment—no limitations on food, no overcrowding. To build such a model, we start very simply and then improve. We denote by P_n = the population (of ants, bacteria, whatever) at generation or time n, and by r the difference birthrate $-$ death rate = net rate of increase (or decrease, if deaths exceed births and $r < 0$). Starting with P_0 we have

$$P_1 = P_0(1+r), P_2 = P_1(1+r) = P_0(1+r)^2, \ldots, P_n = P_0(1+r)^n,$$

just like compound interest. We'll focus on the "generation to generation" rule:

$$P_{n+1} = (1+r)P_n \tag{1}$$

So, as we have seen, with $r > 0$ the population grows exponentially: this is the famous conclusion of **Malthus**. With $r < 0$, it decreases to zero.

There is a problem here. Suppose we start with $P_0 = 1000$ bugs and $r = 0.05$, then $P_1 = 1000(1+0.05) = 1050$ but $P_2 = 1102.5$; even though we can't have living fractional bugs! This underlines the fact that we are dealing with a **model** based on statistical birth/death rates. (It is similar to the statement that an average US families has 2.3 children.) With large populations, we lose very little by rounding to the nearest integer.

Now (1) is an example of a **dynamical system**. Given the **state** (here, the population) at "time" n, it predicts, in a deterministic way, the state at time $n+1$, which can then be fed back in to the same rule, to get the states at $n+2, n+3, \ldots$, etc. In mathematical language, we **iterate** the relationship or mapping defined by (1). Sometimes we write

$$P \mapsto (1+r)P ,$$

or "P maps to $(1+r)P$" to specify the transformation rule. The rule (1) is just our compound interest rule again. Compare it with the simple interest rule

$$P_{n+1} = P_n + rP_0 \quad \text{or} \quad P \mapsto P + rP_0.$$

3.2 Limits to growth: the logistic equation

Now suppose food resources are finite. We introduce the idea of a **carrying capacity**, c = the maximum sustainable population. Instead of a fixed growth rate r, we suppose that, at very small populations, the growth rate is almost unchanged at r, but as P increases, the growth rate decreases until it hits zero for $P = c$ and thereafter becomes negative (deaths exceed births when overcrowding occurs). The simplest mathematical function for such a changing rate is

$$r\left(1 - \frac{P}{c}\right) ,$$

so we modify our rule (1) to read:

$$P_{n+1} = \left[1 + r\left(1 - \frac{P_n}{c}\right)\right] P_n \ . \tag{2}$$

This is called the **logistic mapping**.

Mathematically, (2) is a much more complicated dynamical system than (1). (1) is **linear**, the "output" P_{n+1} is a linear function of the input P_n—if you double P_n, you double P_{n+1}, etc. (2) is **nonlinear**, P_{n+1} depends on P_n^2 as well as P_n, and this makes all the difference. What difference does it make? Well—it's hard to say immediately, because the dynamical system (2) is no longer "soluble in closed form" in the way that (1) is. We can iterate (compound) the simple linear relationship $P_{n+1} = (1+r)P_n$ to get $P_1 = (1+r)P_0, \ldots, P_n = (1+r)^n P_0$, predicting P_n in terms of P_0 all in one shot (recall page 3). There's no "shortcut" formula in the case of (2):

$$P_1 = \left[1 + r\left(1 - \tfrac{P_0}{c}\right)\right] P_0 \ ,$$

$$P_2 = \left[1 + r\left(1 - \frac{\left\{\left[1+r\left(1-\frac{P_0}{c}\right)\right]P_0\right\}}{c}\right)\right] \left\{\left[1 + r\left(1 - \tfrac{P_0}{c}\right)\right] P_0\right\} \ ,$$

etc. You have to calculate step by step.

Of course, this is easy on a computer—and we'll do that in the lab, but it's interesting first to develop a geometrical interpretation of **iterating a mapping**.

Iterating the mapping via the geometric "staircase method"

The function $\left(1 + r\left(1 - \frac{P}{c}\right)\right) P$ defines a **reproduction curve**. Plotting $\left(1 + r\left(1 - \frac{P}{c}\right)\right) P$ vs. P gives a neat way of predicting populations year to year (see next page). For small P, $1 - \frac{P}{c} \approx 1$ and the graph is nearly linear, so growth occurs much as in the linear model (1). Before plotting it, we sometimes "normalize" the population P by measuring it units of c, the carrying capacity, so $P = 1$ replaces $P = c$ (we can multiply by c afterwards to get real numbers)[1]. Here's the graph, drawn for a value $r > 0$ (see Figure 4).

To find P_1, given P_0, go up to the graph from P_0 on the abscissa (horizontal axis) to get the value P_1, which you read by going across to the ordinate (vertical axis). See the figure above. Then we must "inject" P_1 into the mapping again, by copying its value onto the abscissa,

[1]Letting $P_n = cq_n$ in (2), we get $cq_{n+1} = [1 + r(1 - q_n)]\, cq_n$ or simply, canceling c:

$$q_{n+1} = [1 + r(1 - q_n)]\, q_n \tag{2a}$$

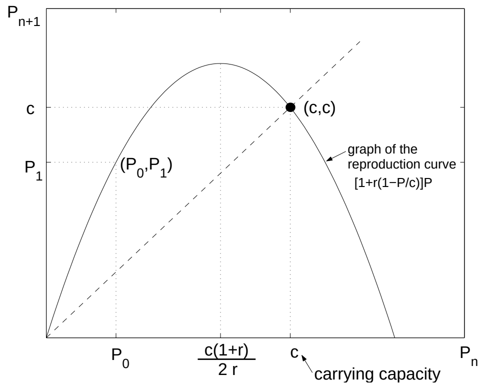


Figure 4: Iterating the logistic map.

going up and so on (and on and on). This is achieved geometrically by going from the graph at the point with coordinates (P_0, P_1) to the diagonal (P_1, P_1) and then up (or down) to the graph at (P_1, P_2), back horizontally to the diagonal (P_2, P_2) and so on. See Figure 5. One "follows the staircase" (see Figure 5). This will be automated for you in the lab.

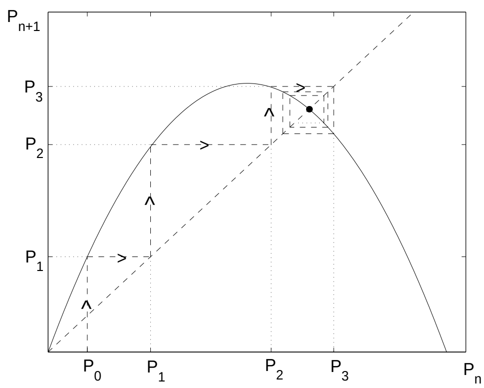


Figure 5: Iterating the logistic map, continued.

Here it appears that the population converges on c, the carrying capacity. What happens in the case below (see Figure 6), for which $r < 0$, if we start from $P_0 < c$? Why? (For $P_0 > c$, this model is not very reasonable – see Figure 6.)

In each case, we can also make a graph of P_n vs. n; Figure 7 shows the following population time series or histories.

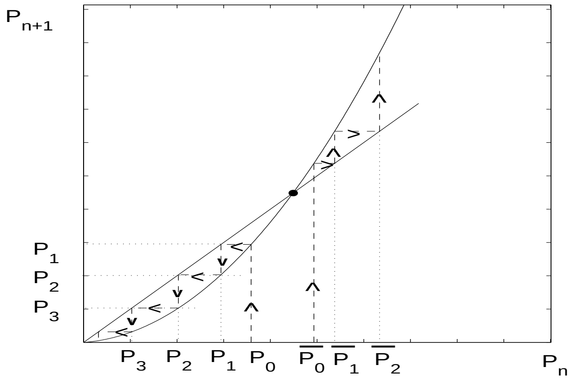


Figure 6: More iterations, $r < 0$.

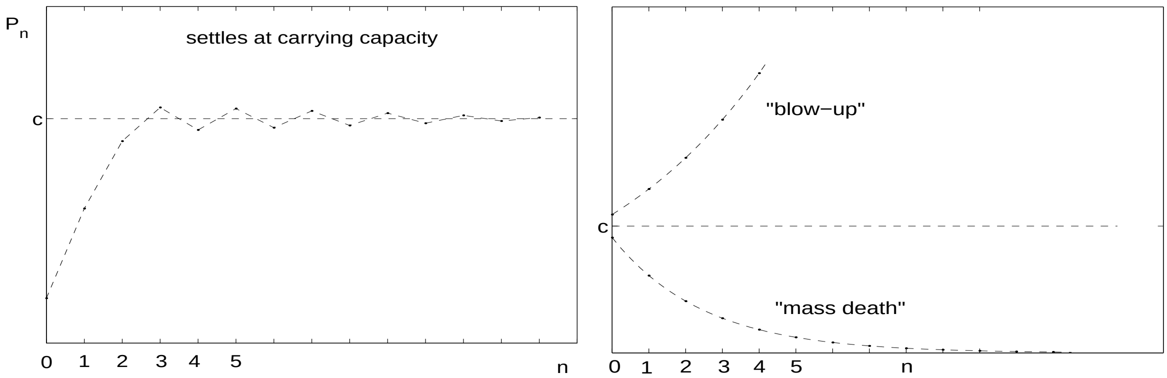


Figure 7: Population growth and decay for $r > 0$ and $r < 0$.

3.3 Fixed points and stability

To understand this behavior it's useful to think of **equilibrium** values or **fixed points**: population values which remain constant year to year. To find these, we set $P_{n+1} = P_n = P$ to get, from (2):

$$P = \left[1 + r\left(1 - \frac{P}{c}\right)\right] P \ . \tag{3}$$

Obviously $P = 0$ is a solution, regardless of the values of r and c; it corresponds to the situation where there is no-one to die or have babies. There's another solution: $P = c$, the carrying capacity. Figures 5 and 6, for $r > 0$ and $r < 0$, show that the dynamical behavior is very different for these two cases. Figure 5 illustrates a case with $0 < r < 2$; for any initial population $P_0 > 0$ in the allowable range, we find that P_n converges on c as n grows. For $r > 2$ a more complex behavior occurs; we will come back to this below. The case $r < 0$ is shown in Figure 6. Starting with an initial population P_0 less than c, one finds that P_n tends to 0; on the other hand, starting with $P_0 > c$, P_n tends to infinity!! Only if P_0 is exactly equal to c does it remain there forever.

To summarize:

For $2 > r > 0$,	$P = 0$ is an unstable equilibrium; neighboring orbits escape,
	$P = c$ is a stable equilibrium;
For $r < 0$,	$P = 0$ is a stable equilibrium: neighboring orbits are "trapped",
	$P = c$ is an unstable equilibrium.

These conclusions for $P = 0$ make sense biologically. Also, ignoring the growth-limiting term for a moment, for small P, $r\left(1 - \frac{P}{c}\right) \approx r$, and we recover the linear Malthusian model (See Figure **??**)

$$P_{n+1} = (1 + r)P_n \ .$$

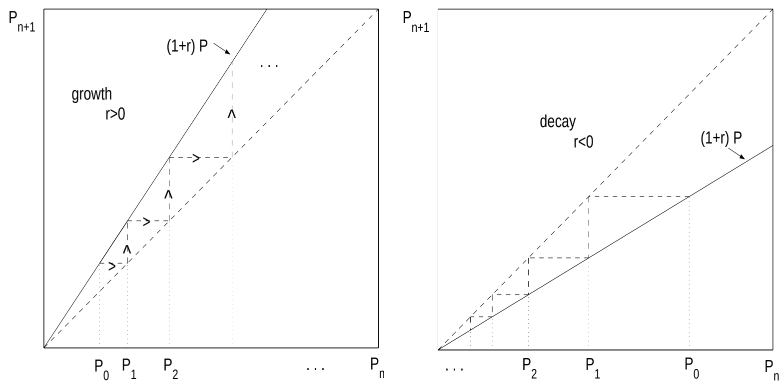


Figure 8: Exponential growth and decay by iteration.

These pictures show the graphical representations of the solution $P_n = (1 + r)^n P_0$. For $r > 0$ P_n keeps growing since births exceed deaths; for $r < 0$ P_n keeps shrinking, since deaths exceed births, and tends to zero. Geometrically, the **slope** of the line $(1 + r)P$ at $P = 0$ determines the stability: If the graph is steeper than the diagonal, the population keeps growing, with ever larger increases, which is very unstable; if the graph is less steep than the diagonal, then the population shrinks to zero, with smaller changes in every generation, which is a stable behavior (although sad for the ever dwindling population.). For more general **nonlinear** models (including the logistic model (see Figure 9)) the slope plays a similar role in stability. The graph is now curved, and to gauge the slope near a particular P, we will consider tangent lines. The slopes of these tangent lines can be computed via calculus.

So: How steep the tangent line is at a fixed point P determines the stability of the fixed point P for the iterated map $x \mapsto f(x)$ $(P = f(P))$. This steepness is measured by a number,

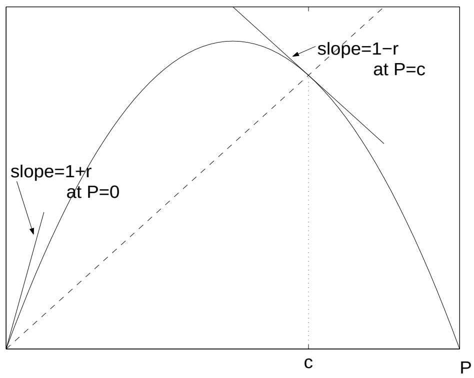


Figure 9: The slope at a fixed point.

called slope. A diagonal has slope 1; slopes less than 1 indicate lines that are less steep than the diagonal; slopes larger than 1 characterize lines that are steeper than the diagonal. In summary:

$|\text{Slope at } P| > 1 \Rightarrow P$ is unstable—nearby solutions escape;

$|\text{Slope at } P| < 1 \Rightarrow P$ is stable—nearby solutions are trapped/attracted.

For the fixed point $P = c$ of the logistic map, the slope of the tangent line is $(1 - r)$. So if $|1 - r| < 1$, solutions approach $P = c$ while if $|1 - r| > 1$, solutions move away from $P = c$:

$$\begin{array}{ll} |1-r| < 1 & \Rightarrow c \text{ is a stable fixed point;} \\ |1-r| > 1 & \Rightarrow c \text{ is unstable.} \end{array}$$

Q. Why is only the absolute value $|1 - r|$ important for stability? What difference does the sign of $(1 - r)$ make?

Slopes of tangent lines can be computed by using calculus: The derivative of a function at a point = slope of tangent line = rate of change of the function. See Figure 10.
To define the derivative, which we write $\frac{df}{dx}$ or f', we take the limit as $\Delta x \to 0$ of a sequence of ratios: $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$. The slopes of the secant lines approach that of the tangent line. We get better and better approximations as $\Delta x \to 0$.
Warning! Derivatives aren't always defined, e.g., at "corners."

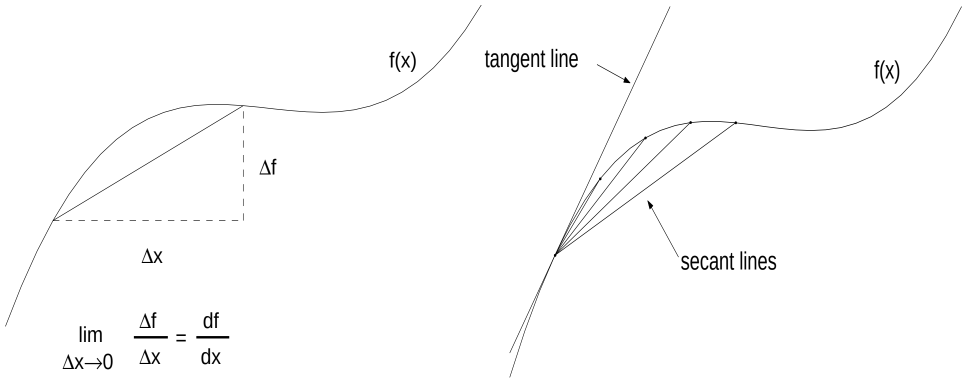


Figure 10: Defining the slope (tangent line).

3.4 Cycles and their stability

We saw that a value for P that satisfies $f(P) = P$ is a fixed point, in the sense that if $P_n = P$, then all later populations equal P as well: $P_{n+l} = P$ for all $l \geq 0$.

If $r > 2$, then we can have other interesting patterns. It is then possible to find q_1, q_2 so that $f(q_1) = q_2$ and $f(q_2) = q_1$. It follows that if $P_n = q_1$, then $P_{n+1} = q_2$, $P_{n+2} = q_1$, and so on: $P_{n+2l} = q_1$, $P_{n+2l+1} = q_2$ for all $l \geq 0$. The population now alternates between these two values. This is called a cycle; because the cycle visits only two values, it is a cycle of length 2. In dynamical systems terms it is also called a periodic orbit, with period 2. (The term "orbit" harks back to one of the earliest dynamical systems ever studied – the planets orbiting the sun! A periodic orbit is an orbit that starts all over again and again, every time after a fixed length of time, called the period. Because $P_{n+2} = P_n$ in our case, the period is 2. Figure 11 shows the logistic map with the two values q_1 and q_2.

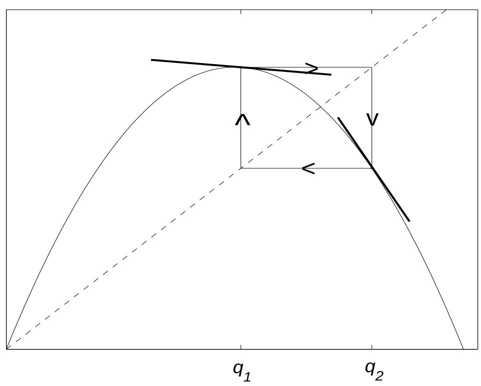


Figure 11: A period 2 orbit.

We can again discuss stability for such a cycle. In the case of a fixed point, we said q was a stable fixed point if

- q is a fixed point
- if P_n is close to q, then the subsequent P_{n+l} also are close, and in fact even closer to q, with the distance between $Pn + l$ and q tending to zero.

In the case of a cycle or a periodic orbit with period 2, we say the cycle is stable if

- q_1, q_2 constitute a cycle: $f(q_1) = q_2$ and $f(q_2) = q_1$.
- if P_n is close to q_1, then P_{n+1} is close to q_2, and the subsequent $Pn + l$ are also close and in fact ever closer to q_1 (if l is even) or q_2 (if l is odd), with the difference tending to zero

The stability of a cycle is again determined by the slopes of the tangent lines: if $|f'(q_1) \cdot f'(q_2)| < 1$, then the cycle is stable. If $|f'(q_1) \cdot f'(q_2)| > 1$, then the cycle is unstable: populations that are near to (but not exactly equal to) q_1 or q_2 will, in subsequent generations, deviate more and more from q_1 and q_2.

Similarly, one can also have cycles with longer periods. For some values of r, one can find q_1, q_2, q_3, q_4 such that $f(q_1) = q_2$, $f(q_2) = q_3$, $f(q_3) = q_4$, and $f(q_4) = q_1$. In this case we have a periodic orbit of period 4. Again, this period can be stable ("attracting" nearby populations) or unstable (populations that start nearby deviate more and more from the cycle), depending on the value of $|f'(q_1) \cdot f'(q_2) \cdot f'(q_3) \cdot f'(q_4)|$: if it is < 1, then the periodic orbit is stable; if it is > 1, then the periodic orbit is unstable.

In the lab, you'll explore the properties of the logistic map; you can try increasing r in small steps from 0, say 0.25, 0.5, 0.75, ..., 2.0, 2.1, 2.2, ..., 2.9, 3.0, and you will look at some r-ranges in more detail. As r increases, you will see that you first encounter a stable fixed point behavior, attracting all populations, corresponding to $|1 - r| < 1$, which means that $r < 2$; when r crosses the value 2, we obtain $|1 - r| > 1$ for $r > 2$, so that the fixed point becomes unstable, but at the same time a stable period-2 orbit appears. This is called a "bifurcation": you can recognize it on Figure 12, which shows, for every value of r, the values of P, where you have a stable fixed point, or a stable periodic orbit. (The number of values above a given r indicates the period of the orbit, which visits all these values in turn.) At $r = 2.449$ we see a new bifurcation: the period-2 orbit becomes unstable, and a stable period-4 orbit appears. For increasing r, more and more bifurcations appear, which you will explore in the lab. The behavior of the orbits becomes more and more complicated, until we hit $r = 2.6$. For $r > 2.6$ chaos reigns –that is for Part 2 of this module. We finish Part 1 with a few comments:

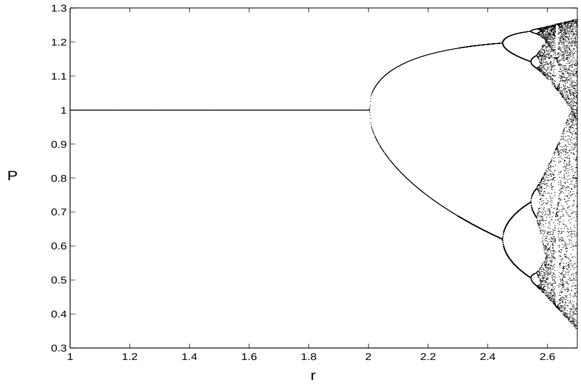


Figure 12: The bifurcation diagram for the logistic map.

1. From the graphs for $r < 0$ in Figures 6 and 8 we see that the model doesn't make much biological sense for $r < 0$: the population grows with ever increasing speed. Why? Hint: Think about the growth rate term $g(P) = r\left(1 - \frac{P}{c}\right)$. For $r > 0$ g starts positive, but becomes negative for $P > c$; corresponding to growth if the population is small, but decline of the population numbers if it exceeds the capacity of the system. For $r < 0$, the whole "capacity" term does not make sense anymore: the equation would correspond to more and more resources being added all the time!

   Moral: A mathematical model may be reasonable in some ranges, but may be unreasonable in others!

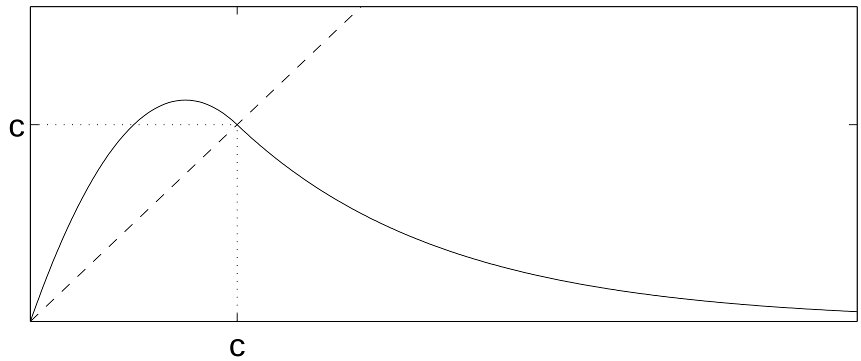


Figure 13: An alternative population model.

2. In our graphs of P_{n+1} as a function of P_n, we have an inverted parabola for the logistic map. This parabola has a maximum for $P_n = \frac{c(1+r)}{2r}$, where $P_{n+1} = \frac{c(1+r)^2}{4r}$. If this value exceeds $\frac{c(1+r)}{r}$, then the staircase construction will lead to $P_{n+2} < 0$, which clearly does not make sense either. It follows that we have to restrict ourselves to $\frac{1}{4}(1+r) \leq 1$,

or $r \leq 3$, at least as long as we consider the logistic map. This aspect of the model can be remedied by defining a new functional relationship with a graph like Figure 13. The formula $P_{n+1} = \alpha P_n e^{-\beta P_n}$ provides an example. This is again closer to the fish population maps we saw in class!