Lecture by John F. Nash Jr.

An Interesting Equation

The equation that we have discovered is a 4th order covariant tensor partial differential equation applicable to the metric tensor of a space-time. It is simplest in form if written with the use of the Einstein G-tensor. Then it can take the form

$$\Box\, G^{ab} + G^{ps}\left(2R_{p\ \ s}^{\ a\ \ b} - \frac{1}{2}g^{ab}R_{ps}\right) = 0$$

And this equation is formally divergence free in the same way that the Einstein vacuum equation (PROVIDED that that is written with the G tensor set equal to zero rather than with the Ricci tensor set equal to zero) is divergence free.

The equation is derivable as the result of the calculation of the Euler- Lagrange equations corresponding to a Lagrangian, namely the Lagrangian below:

$$\int\int\int\int \sqrt{-g}(2R^{ps}R_{ps} - R^2)dx^1\, dx^2\, dx^3\, dx^4$$

And now we can remark that indeed the general class of Lagrangians including ours (written above) has previously caught the attention of theorists and has been considered to be of interest in connection with attempts to develop theories of quantum gravity.

But my observation was, from my reading in the area, that those who had been broadly interested in this general family of Lagrangians had never focused effectively on the specific choice. (Our specific choice involves a specific comparative weighting of the term quadratic in the scalar curvature and that quadratic in the Ricci tensor.)

It was a quite different route that led us to the specific form of Lagrangian that leads to our vacuum equation. More that 10 years ago I sought to find a natural equation suitable for what I thought of as a Yukawa-like gravitational field where the factor of $1/r$ in the decay of the potential at large distances from a source would be combined with another factor representing a very slow decay of exponential character. These potentials, in simpler non-relativistic contexts, can relate to the Klein-Gordon equation.

The original ideas that I had were later reconsidered, but I found that the equation with the Yukawa-like aspect eliminated was of interest, as it seemed to me, as an alternative vacuum equation.

And the most interesting aspect of the equation may be that it seems to provide for or enable a wider variety of gravitational waves.

It is well known that standard GR has the consequence that any gravitational waves are "transverse" waves and the structure of the tensor vacuum equations of GR makes it so that compressional waves are as if forbidden.

In classical seismology there have been recognized both compressional and transverse forms of earthquake waves and these have been called, respectively, P waves and S waves because the P waves tended to arrive first at a station at some distance from the earthquake.

In electromagnetic theory Maxwells equations, which can be naturally generalized to the covariant context of a space-time, do not of themselves exclude non-transverse waves. However our experience is that electromagnetic waves, as observed, are transverse waves.

And also, as far as they can be observed, and as far as could be expected from understandable sources, gravitational waves are transverse waves.

But since the equations describing electromagnetic waves do admit of the consistent possibility of non-transverse waves one is thus given a clue to wonder about the possibility of equations formally admitting the possibility of compressional gravitational waves.

Wave-Like Form of the Scalar Equation

It was discovered only recently by me that the scalar equation naturally derived from the tensor equation for vacuum, particularly in the case of 4 space-time dimensions, has a form extremely suggestive of waves. The scalar derived equation can be obtained by formally contracting the general vacuum equation with the metric tensor. This results at first in an equation involving G (the scalar derived from the Einstein tensor) and the Ricci tensor and the scalar curvature R. And G, being the scalar trace of the Einstein tensor, can be expressed in term of R but this expression involves the number of dimensions, n. So we get as the scalar equation derived from the original vacuum equation this result:

$$\Box R + \left(\frac{n-4}{2n-4}\right)(2R^{ps}R_{ps} - R^2) = 0$$

And now two things are notable about the form of this resulting scalar equation: (1): If $n = 2$ there is a singularity and this simply corresponds to the fact that the Einstein G-tensor is identically vanishing if $n = 2$, so there isnt any derived scalar equation of this type for two dimensions. (2): For $n = 4$ we find the nice surprise that the scalar equation entirely simplifies and then asserts simply that the scalar curvature satisfies the wave operator (which is a dAlembertian if we think in terms of $3 + 1$ dimensions).

So the scalar equation is

$$\Box R = 0 \quad \text{PROVIDED that} \quad n = 4.$$

And further remarks that can be made are that it turns out, in 4 dimensions, that this scalar equation is satisfied by any vacuum solution of the Einstein (tensor) equation for space- time (with any value of the “cosmological constant” λ).

Related to this observation about the derived scalar equation in 4 dimensions, it also happens that (specifically in 4 dimensions) that our general vacuum equation is satisfied by any Einstein space or therefore by any solution of Einstein's tensor equation with a cosmological constant.

Input from Gravitating Matter/Energy

What we have not understood, so far, in the study of this 4th order tensor partial differential equation for space-time, is how to appropriately link it to the gravitation of matter (or of matter and energy, in view of the relativistic inter- convertibility of these).

We can get some insight from the history of the analogous issue in the original development of GR. Originally Einstein had only the tensor equation for vacuum but with that he was able, effectively, to study the gravitational field of the sun (amounting, in precise form, to a Schwarzschild solution). And that was sufficient to consider the gravitational deflection of light rays passing near the sun and the expected perturbation effect on the motion of the planet Mercury.

Then, later in the year 1915, D. Hilbert and Einstein, acting separately but at almost the same time, each published a theory for general relativity with the gravitation of matter described by an "energy- momentum tensor". These ideas were entirely equivalent but were derived differently by Hilbert and Einstein.

As I see it their answer derives naturally from the fact that the standard energy-momentum tensor T^{ab} has the property, assuming expected inertial motion and conservative evolution of sources such as electromagnetic fields, that it is "divergence free" which means, as a tensor relation, that

$$T^{ab}{}_{;b} = 0 \qquad \left(\text{or} \;\; T_{ab;}{}^{b} = 0\right)$$

And the vacuum equation of Einstein can be converted to an equation in terms of the G-tensor of Einstein so that then it involves a tensor which is, constitutionally, inherently divergence-free so that it matches the tensor describing matter or energy-momentum (matter and energy).

And this property of being formally divergence-free naturally results whenever a tensor equation is derived from a Lagrangian defined from a covariantly defined scalar integral.

We have made some studies looking for the possibility of forming an appropriate "input term" which could be put on one side of an equation where on the other side, equated to it, would be the 4th order expression that is equated to zero in our vacuum equation. We have not been able to find a proper answer simply in this fashion but the question of the appropriate "input" relation for matter or "energy momentum" has its natural subtleties and it seems quite possible that a good theoretical concept can be found.

We found an expression derivable from a T^{ab} tensor describing energy momentum that WOULD have necessarily the correct matching properties modulo the context of a flat vacuum (so that all curvature tensors would be vanishing). (There were some other reasons also behind the consideration of this expression.)

$$I^{ab} = \Box\, T^{ab} - T^{as}{}_{;s}{}^{b} - T^{sb}{}_{;s}{}^{a} + g^{ab} T^{ps}{}_{;ps}$$

This input expression, if the space-time is flat, is divergence-free even if the T^{ab} tensor is NOT divergence free (and thus not like the standard concept in GR of inertially moving matter). And since our vacuum equation does allow there to be non-transverse waves, it seems logical, actually, to look for a concept of input from matter or energy-momentum that would NOT depend on the concept of the conservative motion of the matter. This would be analogous, in relation to Maxwell's equations, to input from electrical charges that would not be required to be at all conserved with regard to their evolution in time.

What we have found (which is of some formal mathematical interest but does not solve the problem of finding a good input theory) is that a term can be added to the four terms of I^{ab} as described above so that the result is then automatically divergence-free provided merely that the space-time has either a vanishing or a constant Ricci curvature tensor. Thus if the space-time is as it would be as a solution of Einstein's vacuum relations then this condition is satisfied even if the vacuum equation is of the more general form involving a "cosmological constant".

With this additive term the input expression would be

$$\overset{*}{I}{}^{ab} = \Box\, T^{ab} - T^{as}{}_{;s}{}^{b} - T^{sb}{}_{;s}{}^{a} + g^{ab}T^{ps}{}_{;ps} \ - 2T^{ps}R^{a}{}_{ps}{}^{b}$$

but this expression does not actually have the matching property, in terms of being always divergence-free, if the space-time that underlies the presence of gravitating matter would be either (1): perturbed by the presence of matter or (2): perturbed by the presence of non-transverse gravitational waves (of the sort that are allowable with our vacuum equation).

Existing Ideas of the Literature on Quantum Gravity

I remember reading, some time ago, in a few places, that a Lagrangian having metric curvature terms in quadratic combinations would give rise to a "renormalizable" theory while the regular Lagrangian (which involves simply R) of the standard GR theory does not give a "renormalizable" theory. And this was considered by the writers to be very important in connection with attempts to form a theory of quantum gravitation.

But I don't myself understand either renormalization or the general theory of quantization. (To me it seems like "quantum theory" is in a sense like a traditional herbal medicine used by "witch doctors". We don't REALLY understand what is happening, what the ultimate truth really is, but we have a "cook book" of procedures and rituals that can be used to obtain useful and practical calculations (independent of fundamental truth).)

So it seems of some interest, indeed, that a tensor equation of fourth order may have some possible eventual connection with some conceivable theoretical concept of quantized gravity but it is for me needed to find other reasons for justifying the study of an equation such as that given here.

Added Remarks or also General Debate

Since this lecture is concerned with a topic that does not involve studies lying within the areas of the speaker's greatest recognized professional expertise and since it is considered as "of interest" (but not necessarily of permanent value (when the future judges)) it seems appropriate not to fill the hour simply with a lecture presentation but instead to save some time during which questions and answers or suggestions and debate may occur.

There may persons in the room here at Penn State who are scholars or physicists expert in the areas of gravitation and general relativity and some of them may have very relevant ideas or information which could be communicated.

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