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THE AGENCIES METHOD FOR MODELING COALITIONS AND COOPERATION IN GAMES

JOHN F. NASH, JR.
Department of Mathematics
Princeton University, USA
kjfnj@math.princeton.edu

The idea leading to this study originated some time ago when I talked at a gathering of high school graduates at a summer science camp. I spoke about the theme of "the evolution of cooperation" (in Nature) and about how that topic was amenable to studies involving Game Theory (which, more frequently, has been used in research in economics).

After that event I was stimulated to think of the possibility of modeling cooperation in games through actions of acceptance in which one player could simply accept the "agency" of another player or of an existing coalition of players.

The action of acceptance would have the form of being entirely cooperative, as if "altruistic", and not at all competitive, but there was also the idea that the game would be studied under circumstances of repetition and that every player would have the possibility of reacting in a non-cooperative fashion to any undesirable pattern of behavior of any another player. Thus the game studied would be analogous to the repeated games of "Prisoner's Dilemma" variety that have been studied in theoretical biology.

These studies of "PD" (or "Prisoner's Dilemma") games have revealed the paradoxical possibility of the natural evolution of cooperative behavior when the interacting organisms or species are presumed only to be endowed with self-interested motivations, thus motivations of a non-cooperative type.

Games in Theory and Games Played by Humans

I feel, personally, that the study of experimental games is the proper route of travel for finding "the ultimate truth" in relation to games as played by human players. But in practical game theory the players can be corporations or states; so the problem of usefully analyzing a game does not, in a practical sense, reduce to a problem only of the analysis of human behavior.

It is apparent that actual human behavior is guided by complex human instincts promoting cooperation among individuals and that moreover there are also the various cultural influences acting to modify individual human behavior and at least often to influence the behavior of humans toward enhanced cooperativeness.

Our study has the character of an experiment, but rather than working directly with human subjects we computationally discover the evolutionarily stable behavior of a triad of bargaining or negotiative players. And these players are, as far as the experimental science is concerned, equivalent to a set of three robots. So whether or not the experiment can be carried out successfully becomes simply a matter of the mathematics.

These computations are found to be "heavy" so that our research could not have been done in the earlier days of game theory, like in the 50's, because of the inadequacy of the computing resources then. (And for the future we envision the feasibility of the study of much more complicated models for 4, 5, or more players, with many more distinct strategy parameters being involved.)

Demands and Acceptance Probabilities in the Case of Two Players

We first worked out the function of players' "demands" controlling their "probabilities of acceptance" (in a repeated game context) for the case of games of a simple bargaining type of two players. We present an explanation of this to prepare for and facilitate explaining the modeling structure for three (or more) players.

Originally, in our first trials of the new ideas, we studied a model bargaining problem where the set of accessible possibilities was enclosed by a parametrically described algebraic curve (forming the Pareto boundary). This was arranged so as to have a natural bargaining solution point at $(u1 = 1/2, u2 = 1/2)$ (referring to the players' utility functions). The total bargaining problem was asymmetric, but modulo the theory of localized determination of the solution point, it was such that $(u1, u2) = (1/2, 1/2)$ should be the compromise bargain.

(We were surprised, however, when we found that if we used (as described below) different "epsilon numbers" (see below) for the players that that difference would unbalance the model's selection of a bargaining solution (!). Later, thinking about it, we realized that the use of appropriately matching epsilon numbers for the players could be naturally justified. In a game problem with transferable utility (like with our studies for 3 players) this amounts to using THE SAME epsilon number for all players.)

For Player 1 we let $d1$ stand for his demand (number) and $e1$ for his epsilon-number. The "epsilons" make the "reactive" behavior of a player depend smoothly on the numbers that the players choose as parameters of strategy so that we can obtain the system of equations to be solved for the equilibria by differentiating a player's expected payoff function with respect to each of the strategy parameters that that player controls. Then his "acceptance rate" $a1$ is defined in terms of these numbers plus also the data number $u1b2$ which is "the amount of utility that Player 1 is given by the Player 2 when Player 2 has become the agent for both of the players (and has selected a point on the Pareto boundary)" (and this data is observable by Player 1 simply through the known history of Player 2's behavior in the repeated game).

We need a specific rule of relationship between $d1$ and $a1$ and this is (was) specified by the relations:

$$A1 = \mathrm{Exp}[(u1b2 - d1)/e1], \quad \text{and} \quad a1 = A1/(1 + A1).$$

(Which formulae have the effect that $A1$ is positive and that $a1$ is like a positive probability, between 0 and 1.)

In a completely dual fashion, for Player 2:

$$A2 = \mathrm{Exp}[(u2b1 - d2)/e2], \quad \text{and} \quad a2 = A2/(1 + A2).$$

As we remarked already, we discovered from the calculations that we needed to use $e1 = e2$ if we wanted desirable results! (But it seems that this can be justified as "impartial" if we consider another means for introducing probabilistic uncertainty affecting the consequences of demands; in particular, if the uncertainty resulted from "fuzziness" about the knowledge of the precise location of the Pareto boundary then that version of ignorance would affect the players in an impartial fashion.)

In the case of two players the players would simultaneously vote, with each player voting either to accept the other as the general agent or voting, in effect, for himself/herself instead. Then our first idea was to apply an "election rule" declaring that an election was void if both of the players voted for accepting (the other player) and only effective if only one player made a voting choice of acceptance. So then we specified that the election should be repeated with a certain probability (say probability $(1-e4)$) whenever both players had voted acceptance votes (and if that retrying process ultimately failed then the players finally were given the null reward $\{0, 0\}$ (in utilities) for failure to cooperate!).

This complicated the "payoff formula" somewhat but the VECTOR of payoffs, $\{PP1, PP2\}$, was ultimately calculable as functionally dependent on $a1$, $a2$, $u1b2$, and $u2b1$. (In this listing the utility amounts were regarded as resulting from STRATEGY CHOICES by the players where $P1$ would actually choose $\{u1b1$, $u2b1\}$ as a point chosen BY $P1$ (!) on the Pareto boundary curve. So, from the curve, $u1b1$ could be interpreted as a function of $u2b1$, with $P1$ interpreted as simply choosing $u2b1$ strategically.)

The vector $\{PP1, PP2\}$ becomes a function of $a1, a2, u1b2, u2b1, u1b1$, and $u2b2$. And this reduces to the 4 quantities first listed because of $u1b1$ and $u2b2$ being determined by the Pareto curve.

And $a1$ is a function of $d1$ and $u1b2$ with $a2$ similarly controlled by $d2$ and $u2b1$.

So, ultimately, we arrive at 4 equations in four variables for the conditions of equilibrium. These derive from the partial derivatives of the payoff function for a player taken with respect to the parameters describing his strategic options.

Thus there are the partial derivatives of $PP1$ with respect to $d1$ and with respect to $u2b1$ and these are both to vanish. Then there are two dual equations derived from $PP2$.

(Later we learned that differentiating $PP1$ with respect to $a1$ directly, rather than viewing $a1$ as a function of $d1$, would give a simplified (but equivalent) version of the $d1$-associated equation.)

Graphics and Illustrative Data

We interrupt the explanation, in the main text, of the mathematics of the model design to give descriptions relating to the six "Figures" which are placed later just before the references. These "Figures" illustrate the results achieved by the "experimental mathematics" work, that is, by our calculations developed in the project work.

Figure 1 shows how the "Pareto efficiency" of equilibrium solutions varies as the parameter $e3$ (which modulates the smoothness of the variation of an "acceptance probability", like $a1f23$, as a function of the associated "demand parameter", like $d1f23$. The calculations to find the points for the graph were carried through by A. K. (Alexander Kontorovich) for the case where $e3$ varies. $e4$ and $e5$ are fixed at 1/4, and the game is entirely symmetric, with each two-person coalition assigned the value of 1/5 (by the "characteristic function" defining this simple CF game). The curve is quite smooth and when $e3$ decreases to 1/20 or smaller we see that the total payoff increases to 0.9 or more, or the Pareto efficiency is 90% or better.

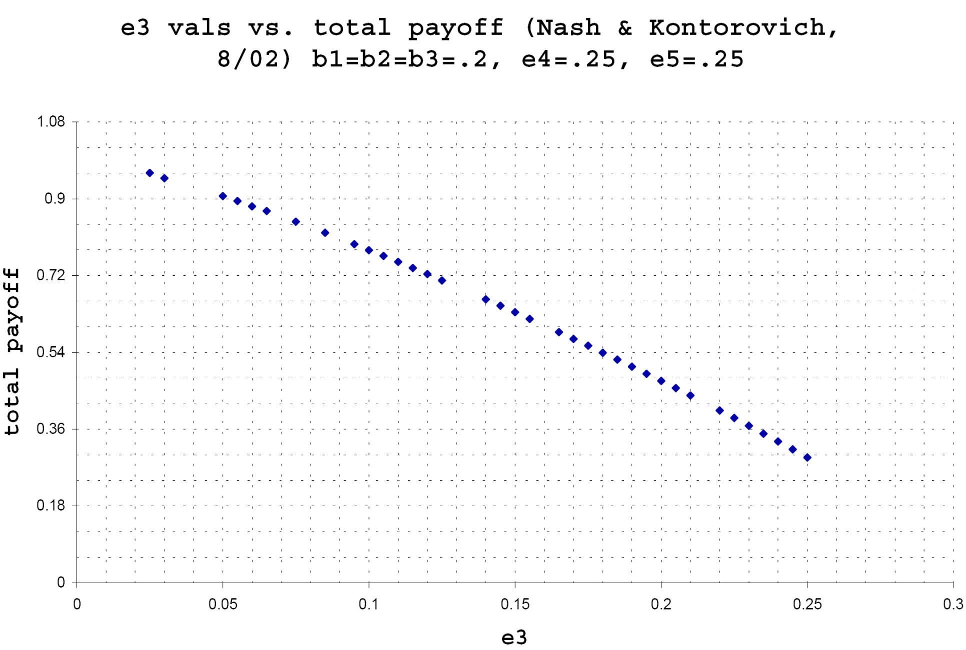

Fig. 1.

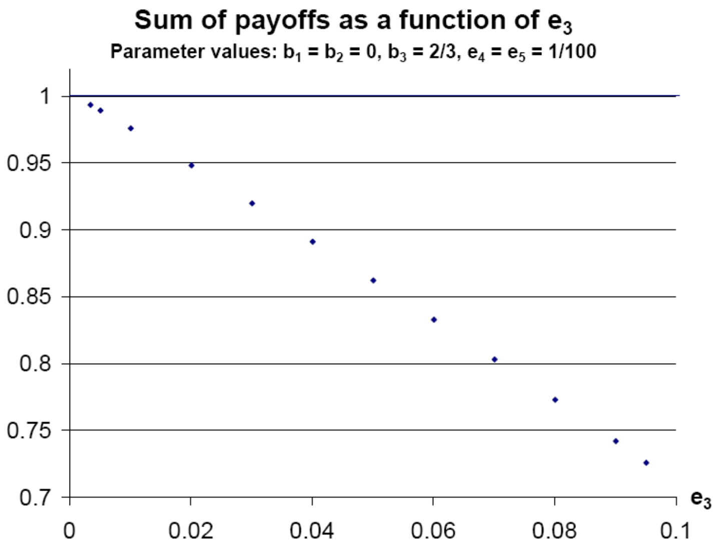

Fig. 2.

Figure 2 is similar or parallel to Fig. 1 but was worked out later, by S. L. (Sebastian Ludmer). This graph is derived from a game with only a symmetry between $P1$ and $P2$, and these are the "favored" players since, regarding the coalitions, $v(1,2) = 2/3$, while the other two-player coalitions have no value. And $e3$ varies while $e4$ and $e5$ are both set arbitrarily at $1/100$. The same pattern (of improving efficiency) appears on this graph it is easy to see that the ASYMPTOTIC level of the payoff-obtaining efficiency of the players at equilibrium is 100%.

Figure 3 shows three graphs relating to three versions of "payoff prediction" according to three different sources. First there is the classical "Shapley value", which becomes simply a line in this Figure, graphed in blue, second is the "prediction" derivable from the nucleolus, graphed in green, and third is the result of equilibria based on our model, graphed in pink.

For each individual graph, of the three, the quantity plotted in relation to the vertical axis is an "imbalance" measure that evaluates the extent to which the two more favored players, $P1$ and $P2$, get more payoff than $P3$. This is calculated as Imbalance $= p1 + p2 - 2 \times p3$ with the notation being that pk is the payoff received by player Pk, for $k = 1, 2, 3$. Players $P1$ and $P2$ are favored and symmetrically situated in this game where $v(1,2) = b3$ and the other two coalitions of two players are without any payoff value. The "imbalance" is charted as the function with values according to the vertical scale while the value of $b3 = v(1,2)$ relates to the horizontal axis and scale. Figure 3 was derived from calculations and graphics work of S. L.

You will notice on Fig. 3 that the data presentation ends for $b3 > 0.7$ (roughly). What happened, actually, was that as we continued the finding of a family of solutions to that level of $b3 = v(1,2)$ we found that player $P3$ began to use values

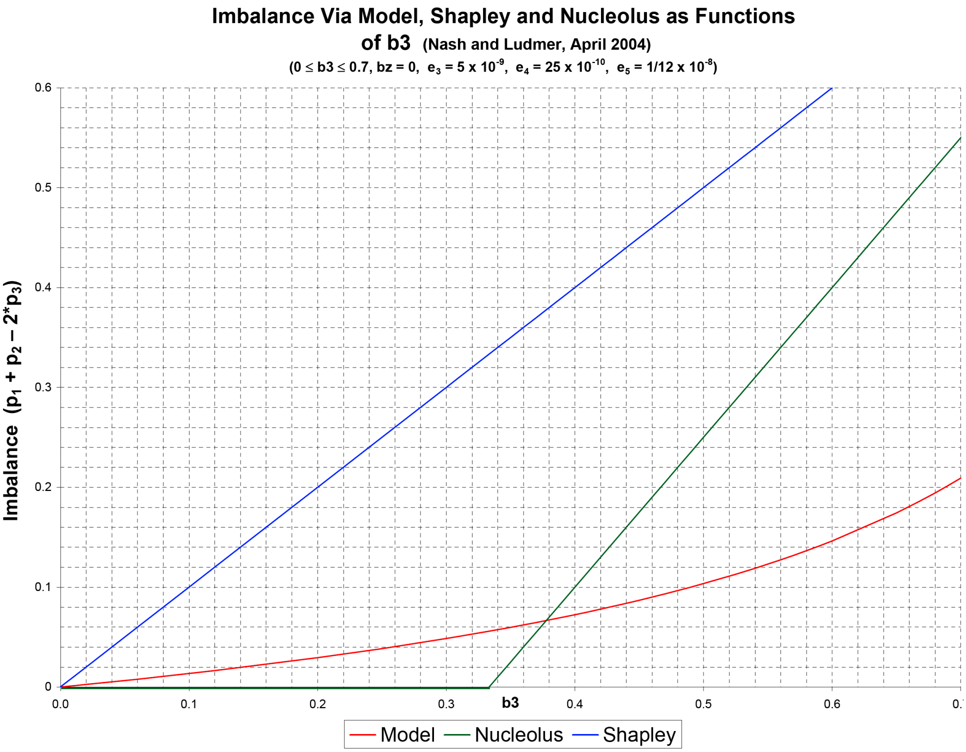

Fig. 3.

of $a3f1 = a3f2$ that were NEGATIVE (in the mathematical solutions of the system of 21 equations). But this is impossible (!) in the interpretation, since $a3f1 = a3f2$ is a probability. So it is natural to start, at that level of $b3$, to consider a modified system of equations corresponding to a game model where $P3$ simply lacks the options corresponding to $a3f1$ and $a3f2$.

We did that, but then, almost immediately, further complications seemed to arise, from other parameters that might go negative.

So we did not find what seemed like a properly acceptable modeling for the cases of $b3$ having really large values. But, in relation to the concept of "pro-cooperative games" that we discuss as a major topic below, it seems natural that these games with only $b3 = v(1, 2)$ being large should have the characteristics (not yet scientifically defined) of "pro-cooperative" games while if all of $b1$, $b2$, and $b3$ are, say, larger than $4/5$ then the game situation is really one where "Two's company, three's a crowd."

Figure 4 shows a dual perspective, again for symmetric games. Here bz is the quantity linked to the horizontal axis. And this is $bz = b1 = b2 = v(1, 3) = v(2, 3)$ and $b3 = v(1, 2) = 0$ here. The "epsilon" parameters are set at the fixed values of $e3 = 5 \times 10^{-9}$, $e4 = 25 \times 10^{-10}$, and $e5 = 1/12 \times 10^{-8}$. And for these circumstances

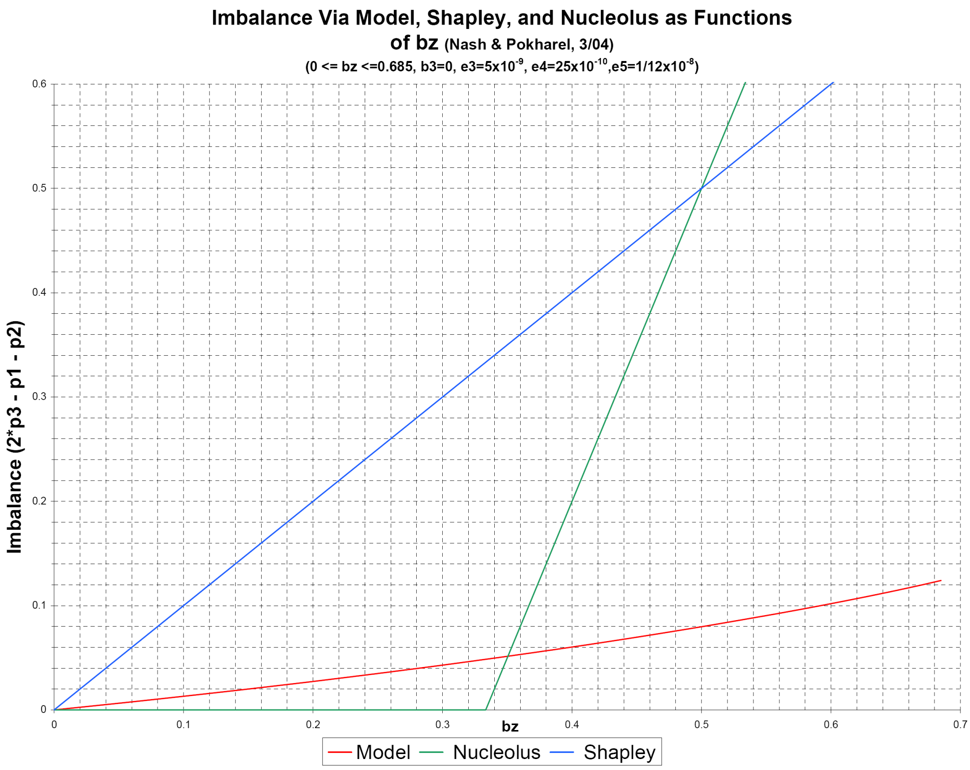

Fig. 4.

$P3$ is the favored player. So we measure the extent of the advantage of $P3$ through an "imbalance" measure defined as $2 \times p3 - p1 - p2$ where $p1, p2$, and $p3$ the payoffs gained by $P1, P2$, and $P3$.

We should remark that the Imbalance curve might rise more rapidly with increasing bz if we had a game of fully transferable utility (properly modeled). In effect, in coalition with either $P1$ or $P2$ the stronger player, $P3$, might be able to demand a "Lion's share" of the payoff received from a two-player coalition (if the "grand coalition" of all three players failed to form). The game rules in our studied modeling led to a payoff function where the payoff to any two-player coalition was always split evenly between the two members.

So, comparatively, in case of the graphs of Fig. 3, since $P1$ and $P2$ are symmetrically situated in the game, it is natural for them to divide their payoff equally when their payoff comes only from the resources of $v(1, 2)$. And $v(1, 2) = b3$, which is the horizontal axis variable in the charts of Fig. 3.

Again, like with Fig. 3, the blue graph derives from the Shapley value, the green from the nucleolus, and the pink from the results from our model (with the specific choices of $e3, e4$, and $e5$). It seems plausible that if the modeling allowed the agent representing a final coalition of only two players to executively choose the splitting of that payoff among the two members that then $P3$ would likely achieve a more favored status. Thus the result should be that the graph of the model results would turn upwards more, with increasing bz. (The graphs of Fig. 4, and the calculations of equilibrium solutions needed for the points, were prepared by A.P. (Atul Pokharel).)

(In relation to both Figs. 3 and 4 we should remark that the limited extent of the horizontal range, with $b3$ or bz increasing only to the level of about 0.7 derived from mathematical difficulties. With, for example, $b3$ increasing beyond about 0.7, a phenomenon found was that the solutions found came to have $a3f12$ and $a3f21$ becoming negative. But these are probabilities so that negative values are forbidden. The situation is perfectly natural, game-theoretically; it seems to become unprofitable for $P3$ to make any use of his/her option to ACCEPT the leader of a coalition of $P1$ and $P2$ (so that $P3$ would join in a grand coalition led by that agent). (We made some attempts to continue a pathway of solutions onward to further increasing values of $b3$ on the basis of modified equations based on $a3f12 = a3f21 = 0$, but this effort quickly failed because of other parameters moving to become zero and the general situation becoming too complex.)

Later we began to realize that there are games which could be called "pro-cooperative" and which are similar to bargaining problems of two parties. And, on the other hand, for three players, there are games where the old folk saying (in English) of "Two's company, three's a crowd." becomes fitting. In those Cases a single central equilibrium corresponding to sometimes the formation any one of all of the two-player coalitions and sometimes either of the others would be actually an UNSTABLE equilibrium concept. And really, for whatever reasons might be found, a final coalition of only two players could be the big winner.

Figure 5 shows some of the numerical results for a solution where $b3 = 1/3$ and $b1$ and $b2$ are zero. This is a symmetric game (with $P1$ and $P2$ situated the same). The values found for all of the parameters are listed but as 21 numbers since for each variable, like $a1f3$ there is a dual like $a2f3$. The coincidence of a "market clearing" phenomenon that we found observationally can be observed in noting that the numbers $u1b2r13 = u2b1r2 = y10$ and $u1b23r1 = u2b13r2 = y14$ are the same (and this is confirmed by calculations out to ∼50 decimal places).

At the level of $b3 = 1/3$ the nucleolus has not yet begun its pattern of having an imbalance in favor of players $P1$ and $P2$ (who are favored by the CF values in this example). We actually calculated the values for the listed parameters to a much higher level of accuracy but the figure is prepared for a lecture presentation.

Figure 6 is very much like Fig. 5 and also concerns an example where only $b3$ is non-zero. Again the coincidence of the values of $u1b2r13, u2b1r23, u1b23r1$ and $u2b13r2$ can be noted.

Details of the Modeling for Three Players

When there are three players instead of two we need to arrange to have two successive stages for "acceptance votes" where any one player could vote to accept the

Parameters:

$bz = 0 \qquad b3 = \frac{1}{3} \qquad e3 = \frac{1}{200616} \qquad e4 = \frac{1}{235464} \qquad e5 = \frac{1}{1182264}$

Values of the Solution Variables:

$a1f2 = a2f1 = 0.05735701$

$a1f3 = a2f3 = 0.0608687$

$a3f1 = a3f2 = 0.06634303$

$a12 = a21 = 0.07704447$

$a13 = a23 = 0.10034813$

$a31 = a32 = 0.10340788$

$af12 = af21 = 0.05574581$

$af23 = af13 = 0.10034813$

$af32 = af31 = 0.10034004$

$u2b1r23 = u1b2r13 = 0.34082523$

$u1b23r1 = u2b13r2 = 0.34082523$

$u3b2r31 = u3b1r32 = 0.31834698$

$u1b3r12 = u2b3r21 = 0.34082543$

$u3b12r3 = u3b21r3 = 0.31834728$

$u2b31r2 = u1b32r1 = 0.34082512$

$u2b12r3 = u1b21r3 = 0.34082208$

$u3b23r1 = u3b13r2 = 0.31834426$

$u3b1r23 = u3b2r13 = 0.31834426$

$u1b31r2 = u2b32r1 = 0.34082261$

$u1b3r21 = u2b3r12 = 0.34082214$

$u2b1r32 = u1b2r31 = 0.34082262$

Normalized Payoffs:

$u1 = 0.34082615 \approx \frac{91}{267}$

$u2 = 0.34082615 \approx \frac{91}{267}$

$u3 = 0.31834768 \approx \frac{85}{267}$

Shapley Value: $\left\{\frac{7}{18}, \frac{7}{18}, \frac{2}{9}\right\}$

Nucleolus: $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$

Fig. 5.

(unconstrained!) agency function of any other player. This principle continues to apply to players who had already themselves become agents. So two steps of coalescence of this sort result necessarily in the achievement of the "grand coalition" in the form that all three of the players are represented by one of them who, as the agent acting for the other two, can access all the resources of the grand coalition (which are simply $v(1, 2, 3)$ since we simplify by considering a "CF game" that is DEFINED by the characteristic function given for it).

Parameters:

$bz = 0$ $b3 = \frac{2}{3}$ $e3 = \frac{1}{100000}$ $e4 = \frac{1}{100000}$ $e5 = \frac{1}{100000}$

Values of the Solution Variables:

$a1f2 = a2f1 = 0.05079737$

$a1f3 = a2f3 = 0.06756534$

$a3f1 = a3f2 = 0.09153050$

$a12 = a21 = 0.23178182$

$a13 = a23 = 0.37583832$

$a31 = a32 = 0.40645201$

$af12 = af21 = 0.05717150$

$af23 = af13 = 0.37583832$

$af32 = af31 = 0.37454125$

$u2b1r23 = u1b2r13 = 0.36377224$

$u1b23r1 = u2b13r2 = 0.36377224$

$u3b2r31 = u3b1r32 = 0.27244676$

$u1b3r12 = u2b3r21 = 0.36377476$

$u3b12r3 = u3b21r3 = 0.27245019$

$u2b31r2 = u1b32r1 = 0.36377132$

$u2b12r3 = u1b21r3 = 0.36376368$

$u3b23r1 = u3b13r2 = 0.27244249$

$u3b1r23 = u3b2r13 = 0.27244249$

$u1b31r2 = u2b32r1 = 0.36376858$

$u1b3r21 = u2b3r12 = 0.36376470$

$u2b1r32 = u1b2r31 = 0.36376470$

Normalized Payoffs:

$u1 = 0.36376691 \approx \frac{4}{11}$

$u2 = 0.36376691 \approx \frac{4}{11}$

$u3 = 0.27244197 \approx \frac{3}{11}$

Shapley Value: $\left\{\frac{4}{9}, \frac{4}{9}, \frac{1}{9}\right\}$

Nucleolus: $\left\{\frac{5}{12}, \frac{5}{12}, \frac{1}{6}\right\}$

Fig. 6.

For the specific modeling we simplify further by having $v(1, 2, 3) = 1$ and $v(1) = v(2) = v(3) = 0$ and we call (for convenience with Mathematica, etc.) the values of the two-player coalitions by the names $b3 = v(1, 2)$, $b2 = v(1, 3)$, and $b1 = v(2, 3)$. These three numbers, $b1, b2$, and $b3$ define the games of the family we studied. We finally obtained graphs illustrating how the calculated payoffs (to the players, based on our model) would vary, as $b3$ (or $b1$ and $b2$) would vary, compared with similar graphs for the Shapley value and the nucleolus (which are calculable for any CF game).

At the first stage of elections (in which every player is both a candidate to become an agent and also a voter capable of electing some other player to become his authorized representative agent) there are six possible votes of acceptance and we described the probabilities for each of these by the parameter symbols $a1f2, a1f3, a2f1, a2f3, a3f1$, and $a3f2$. ($a3f2$, for example, is the probability of the action of $P3$ (Player 3) to vote for $P2$ (which is to vote to accept $P2$ as his elected agent).)

These probabilities, like all of the voting probabilities in the model, need to be related to demands, as we will explain.

After the first stage of elections is complete and one agency has been elected (Note that this requires some precision of the election rules that we need to specify.) then there remains one "solo player" and one coalition of two players of which one of the two (like a strong committee chairman) has been elected to be the empowered agent acting for both of them.

Then for the second stage of elections there are 12 numbers that describe the probabilities of "acceptance votes" (but only two of these numbers are truly relevant corresponding to each of the six possible ways in which an agency had been elected as the first stage of agency elections). These numbers are $a12f3$ and $a3f12$, $a13f2$ and $a2f13, a21f3$ and $a3f21, a23f1$ and $a1f23, a31f2$ and $a2f31$, and $a32f1$ and $a1f32$.

Thus $a12f3$ is the probability of a vote by $P1$ representing the coalition, led by $P1$, of $P1$ and $P2$, voting for his (and his coalition's) acceptance of $P3$ as the final agent (and thus as effective leader, finally, of the grand coalition). Alternatively $a3f12$ is the probability of a vote by $P3$, as a solo player, to accept the leadership of the (1, 2) coalition (as led by $P1$) to become also effective as his enabled agency and thus to access the resources of the grand coalition.

With the election process we need rules that specify simple outcomes (eliminating tie vote complications, etc.) so what we used was that if in any election there was more than one vote of acceptance that then a random event would select just one of those (two or three) acceptances to become the effective vote. This convention suggested the naturalness of allowing an election to be repeated when none of the voting players had voted for an acceptance.

The convention of repeating failed elections seemed to be a very favorable idea. (It also seems to favor some of our projected refinements or extensions of the modeling, as we explain later.) So, as variable parameters affecting the model structure, we introduced "epsilons" called $e4$ and $e5$ where the probability of repeating a failed election AT THE FIRST STAGE OF AGENCY ELECTIONS would be $(1 - e4)$ (this is expected to be a "high probability") and the similar probability applying in the event of election failures at the second stage would be $(1 - e5)$.

(We will say more below about the benefits of having elections that are typically repeated when no party votes.)

Besides the set (presented above) of 18 numbers describing the probabilities for votes of acceptance there is a set of 24 numbers that describe how the players choose (this is a strategy choice!) to allocate utility among themselves and these numbers are linked with the 12 differentiable possibilities for how some individual player happened to be elected to become the final agent. These numbers are $\{u2b1r23, u3b1r23\}$, $\{u2b1r32, u3b1r32\}$, $\{u2b12r3, u3b12r3\}$, $\{u2b13r2, u3b13r2\}$, $\{u1b2r13, u3b2r13\}$, $\{u1b2r31, u3b2r31\}$, $\{u1b21r3, u3b21r3\}$, $\{u1b23r1, u3b23r1\}$, $\{u1b3r12, u2b3r12\}$, $\{u1b3r21, u2b3r21\}$, $\{u1b31r2, u2b31r2\}$, and $\{u1b32r1, u2b32r1\}$.

The notational pattern is that, e.g., $u1b2r31$ represents "the quota of utility allocated to Player 1 by Player 2 in situations where Player 2 was elected as final agent by the coalition of Players 3 and 1 when this coalition was led by Player 3". The "hidden allocations" are like $u2b2r31$ and these must be non-negative. $u2b2r31$ would be the amount that Player 2 would allocate to himself in this situation. Of course $u2b2r31 = 1 - u1b2r31 - u3b2r31$ because the resources, $v(1, 2, 3)$, of the grand coalition are simply +1.

Another generic case is like $u3b13r2$ where the final agent (Player 1 here) was previously the agent in control of a coalition (coalition (1, 3) here) and he allocates $u3b13r2$ to Player 3 and $u2b13r2$ to Player 2.

So these $uxbxrxx$ and $uxbxxrx$ numbers must all lie between zero and +1.

Relations of Demands and Acceptance Probabilities

For most of the cases, in the modeling of the games of three players, the relations between the acceptance probabilities and the controlling "demands" (which demands are parameters that are strategy choices of the players) are natural extensions of the comparable relations for two player games (and this is simply because MOST of these numbers actually relate to "second stage" elections where the field is reduced to just two voters and two candidates!).

Thus we specify that $a12f3$ is to be controlled by a "demand" $d12f3$ which is made by Player 1, who is the competent voter in the situation (which is that (1, 2) is led by Player 1 and that $P3$ is "solo"). The formulae controlling the relation mathematically are

$$a12f3 = A12f3/(1 + A12f3)$$

(with)

$$A12f3 = Exp[(u1b3r12 - d12f3)/e3].$$

This is actually EXACTLY LIKE the relation used for a bargaining game of two players. ($u1b3r12$ corresponds to $u1b2$ there.) But here the perspective is thus: "Player 1 is leading (1, 2) and considering whether or not to accept $P3$ as the final agent, so in relation to this he looks at the utility payoff, $u1b3r12$, that he WOULD BE ALLOCATED by Player 3 in the event of that (effective) acceptance and he compares that number with his demand $d12f3$ and this comparison (modulated by

$e3$) controls Player 1's probability of voting to accept Player 3 in the situation". (Note incidentally that Player 1 here appears as acting entirely in his selfish interest and disregarding any concerns of the (represented) Player 2 (!).)

Similarly we specify, for $a2f13$ as typical, that

$$a2f13 = A2f13/(1 + A2f13)$$

(with)

$$A2f13 = \text{Exp}[(u2b13r2 - d2f13)/e3].$$

So for the 12 acceptance probabilities relating to the possibilities for votes at the second stage of elections there are linked 12 demand numbers, as described above.

But for the first stage of elections the version of modeling (perhaps not optimal) that we happened to use had three demands that controlled the six a-numbers $a1f2$, $a1f3$, etc. that applied to that stage of the process of elections. This was because we only allowed that a player should choose a single demand number so that $d1$, $d2$, and $d3$ were these choices. Then each player's choice of his "demand" controlled BOTH of his probabilities for voting for acceptance (of one or another of the two other players).

Thus a relationship between $d1$ and the pair of $a1f2$ and $a1f3$ was created so that Player 1's (strategy) choice of $d1$ modulated his BEHAVIOR (as described by $a1f2$ and $a1f3$. In this relationship we used calculated utility expectation measures that we called $q12$ and $q13$. Here, to illustrate, $q12$ is "the expectation" of the average receipt of utility, by $P1$, conditional on the assumption that $P1$ has achieved acceptance of $P2$ (at the first stage of elections) so that the coalition (2, 1), led by Player 2 is formed to (enter into) play at the second stage of elections". This quantity $q12$ happens to be calculable entirely from $e3$, $e5$, and the quantities that describe the behavior of players $P2$ and $P3$.

The governing formulae relating $d1$ to $a1f2$ and $a1f3$ are then these:

$$a1f2 = A1f2/(1 + A1f2 + A1f3) \quad \text{and} \quad a1f3 = A1f3/(1 + A1f2 + A1f3)$$

(with)

$$A1f2 = \text{Exp}[(q12 - d1)/e3] \quad \text{and} \quad A1f3 = \text{Exp}[(q13 - d1)/e3].$$

The structure is that $A1f2$ is a non-negative number which is large or small depending on how the rewards to be expected by $P1$ when $P1$ would manage to accept the agency offered by $P2$ compare with $d1$ while $A1f3$ similarly depends on the prospects if $P1$ becomes an acceptor of $P3$. Then the formulae (on the first of the lines of equations just above) give the definitions or constructions of $a1f2$ and $a1f3$ such that these can be the probabilities of exclusive events (since either $P1$ can vote for accepting $P2$, or $P1$ can vote for $P3$ (similarly), or $P1$ can simply decline to make any vote for an acceptance).

(The expressions derivable for $q12$ and $q13$ are not very long and they are dual under symmetry of $P2$ and $P3$, so for illustration,

$$q12 = ((1 - a21f3) \times (1 - a3f21) \times b3 \times e5 + 2 \times a21f3 \times u1b3r21 + a3f21 \times ((2 - a21f3) \times u1b21r3 - a21f3 \times u1b3r21))/ (2 \times (1 - (1 - a21f3) \times (1 - a3f21) \times (1 - e5)))$$

and $q13$ is dual to this.)

We can also remark that a technical point of detail enters into the actual calculation of the formula above: If it happens (which has probability $e5$ at each trial) that a second stage election failed after Player 1 had accepted Player 2 then in that case our rule was that the players $P1$ and $P2$ were to be each given a payoff of $b3/2$ while $P3$ is to be given zero (and this instance of the playing of the repeated game is then complete). (Thus technically our example game is an NTU game but we can still use Shapley value and the nucleolus because of generalizations of those.) (It would have made the players' strategies more complex to allow the agent for a two-player coalition to allocate at his discretion the limited resources of that coalition and it would presumably be irrelevant for a game symmetric between $P1$ and $P2$ with only the (1, 2) coalition having resources.)

The Equations for the Equilibrium Solutions

From the quantities above, not including the demand numbers, the patterns of the actual (steady) behavior of the three players is fully described. These numbers, 18 a-numbers and 24 u-numbers, or 42 parameters in all, therefore describe the directly observed behavior of the players.

So we can compute the (moderately lengthy) terms of a vector payoff function, say $\{PP1, PP2, PP3\}$ describing the payoff consequences to the players of their behavior with the terms being rational fractional expressions in these 42 quantities plus also a dependence on $e4$ and $e5$ (because of the effects of the chances of repeating failed elections).

It is, however, the d-numbers and the u-numbers (but not the a-numbers) that are officially the strategic choices of the players. For the proper set of equilibrium equations we need to work with these. This involves, in principle, the substitution for all of the a-numbers of the expressions derivable for them as functions of the d-numbers and the u-numbers. So suppose that the completion of these substitutions would give us an expanded vector payoff function $\{PP1du, PP2du, PP3du\}$ in which all appearances of a-numbers have been replaced by the expressions that describe their reactive varying (as the players vary their behaviors in reaction to the observed actions of the other players). Then the set of $24 + 15 = 39$ equilibrium equations for the strategic u-numbers and d-numbers are derivable by taking, for each of the strategic parameters, the partial derivative, with respect to it, of the payoff function of the player who is the controller of that strategy parameter.

Actually, however, we found, starting with the study of two-player cases, that we could take a modified route of derivation and arrive at simplified, yet equivalent equations. But, skipping all the details, the PI worked with the project assistant Alexander Kontorovich in a program where the two workers independently derived the equations so that the results could provide good confirmations. Methods were also developed, working within Mathematica, that could exploit the symmetries of the game (with $b1, b2$ and $b3$ as symmetric symbols) and these gave both cross-checking benefits and also made it possible to derive multiple variant equations from one good calculation. In particular, the 24 equations associated with u-numbers, because of the 3-factorial (3!) symmetry, became 4 groups of six and only one of each group really needed direct calculation.

In the end, with the chosen simplifications, we transformed into equations NOT INCLUDING any of the d-numbers but including all the a-numbers. This needed the additional inclusion of three equations of the sort of an equation linking $a1f2$ and $a1f3$, which both depend on $d1$, which is being eliminated from the simplified equations. Thus there are 42 equations, involving as variables the a-numbers and the u-numbers, for a general game.

When the game has symmetries the equation set can be much reduced. If $b1 = b2 = b3$ then all of the two player coalitions have the same strength and then we can look for solutions involving the same behavior for all players. Then the equations reduced to merely 7 in number (and this was a good basis for finding the first solutions!). If merely $b1 = b2$ then the coalitions (1, 3) and (2, 3) have the same strength and we can look for solutions with $P1$ and $P2$ in symmetrically patterned behavior. This leads to a reduction to 21 equations, and we did most of our work on calculations with these 21 equations since that level of symmetry was enough to yield differentiation among the various value concepts that could be compared.

The Methods for Finding Solutions and Calculating Data Points

This work has in two ways an experimental character. First, the actual design of a model is like a matter of artistic discretion, and it is simply an ATTEMPT to provide for the possibility of naturally reactive behavior so that the phenomenon of "the evolution of cooperation" may occur and may be revealed through the actual calculation of equilibrium solutions.

Whatever choices we make at first, with regard to how the players are to reactively behave, there is, a priori, the possibility that some other design might have each player (or an individual player) behaving more effectively in terms of effectively inducing desirably cooperative behavior on the part of the other players. Our present work, in its nature, does not attempt to find the ultimately ideal form of reactive behavior (of an individual player) so that, with that behavior, that the resulting equilibrium in the context of a repeated game of interactive behavior would be optimized as far as the interests of that player are concerned. In principle, we feel, the issue of the optimization of the form of the reactive behavior pattern of an individual player is what would be done in Nature by selective evolution. In game-theoretical studies the parallel achievement might be realized by comparison of alternative models. (It is certainly rather straightforward to compare various programs, say for playing chess or Go by simply letting the programs compete in playing the game.)

Initially the variable quantities that describe the behavior of the 3 players in the repeated game were (or are) specified by descriptive names, such as $d2, a1f3, d12f3, a31f2, a2f31, u1b2r13$, etc. And there were 42 of these variables, although there are only 39 dimensions given by the "strategy parameters". (This is related to the circumstance that we used the a-quantities and the u-quantities for the 42 equations, and for this eliminated the d-quantities.) (Thus, for example, both $a1f2$ and $a1f3$ relate to and depend functionally on $d1$ so that when $d1$ is replaced by them in the equations, with $d2$ and $d3$ similarly replaced, we then have 3 more variables in the equations than would be in equations with d-quantities rather than a-quantities.) (The a-variables refer to acceptance probabilities, describing behavior, while the b-variables describe strategic choices of a type called "demands".)

We ultimately substituted a shortened notation for the variables that we mentioned above which describe the strategic choices and the resulting behavior of the players. We put $x1$ for $a1f2$, then $x7$ for $a12$ (which is short for $a12f3$) $x13$ for $af12$ (which is short for $a3f12$), then $x19$ for $u1b3r12$, and finally $x42$ for $u2b1r32$. (Note that there are fully 24 of the $uxbxrxx$ and $uxbxxrx$ variety of variables, which describe utility allocation choices made by individual players and which apply to the circumstances when that player becomes the "final agent" or "general agent" and is enabled to allocate, from the resources (which are $+1$ in transferable utility) of the "grand coalition", to the other players (with the remainder retained by the allocating player).)

In our computational studies there were only a very few examples, without any symmetry of the players, that were solved for the equilibrium (a presumably appropriate "central" equilibrium with all variables of the $aifj$, $aijfk$, and $aifjk$ varieties being non-zero).

Most of the QUANTITY of our work on the calculation of solutions in this project of research was done on finding solutions for games which had players $P1$ and $P2$ situated symmetrically. Also, in these cases, we studied particularly either games with $b1 = v(2,3) = 0$ and $b2 = v(1,3) = 0$ or games with $b1 = b2 = bz$ (bz is just a notation for $b1$ and $b2$ here) and $b3 = v(1,2) = 0$. And from these two sub-categories of these three person games we ultimately derived enough data so that we could plot smooth graphs describing how the equilibrium solutions varied as the data of $b1, b2$, and $b3$ would vary in these two particular fashions.

And for the work which was specifically concerned with games having $P1$ and $P2$ situated symmetrically it was very natural to simplify further the notation so that we introduced y-variables ranging $y1, y2, \ldots, y21$ which represented all the data of the variables $x1, \ldots, x42$ when $P1$ and $P2$ are in a bilaterally symmetric situation.

Methods Used for Improving Approximations to Solutions

Working either with 42 equations for $x1, x2, \ldots, x42$ or with 21 equations for $y1, y2, \ldots, y21$, we used Mathematica (on Linux) as the general framework for calculations. After initially obtaining one or a few good numerical solutions then others could be obtained by the use of what are called, mathematically, "homotopy methods", and this was essentially a matter of always simply finding a new solution that would be numerically in close approximation to a known solution that had been previously found.

So the chart of Fig. 1 illustrates the use of data that was obtained in this fashion when, of the defining parameters in the equations, only $e3$ was varying and the solutions were being obtained for fully symmetric games with $b1 = b2 = b3 = 1/5$. And we can remark that FULL symmetry reduces the number of descriptive variables needed down to 7 (from 42 in general or 21 with two players being symmetrically situated in the game).

The similar chart of Fig. 2 illustrates the use of data describing solutions for games with only a symmetry of the situations of Players $P1$ and $P2$. (This chart was prepared by S. L. (Sebastian Ludmer).)

Actually, the first numerical solution found was found by A. K. (Alexander Kontorovich) who used a computer "amoeba" method of automatic searching to get close to it. This was a matter of searching in 7 dimensions. And after a few initial solutions were known all the others that were obtained were found using the "homotopy" procedure and moving gradually from one solution to another along a trail of neighboring points, in 7 or 21 or 42 dimensions.

We developed a series of Mathematica programs for improving the quality of an initially given approximate solution of a system of simultaneous equations. (Versions of these programs and of associated computational data developed in the work on this research project will be made available in a reference web site that we are preparing to be available to readers of this publication.) These programs work by modifying the variables (e.g.: $y1, \ldots, y21$) of an approximate solution so that the values of the equations, after the substitution of the array of numerical values of the descriptive variables will be smaller. And for this purpose of improvement we used the measure of quality formed simply by the sum of the squares of the numerical values of all the equations of the system (as they would be calculated from the substitution into them of the 7, 21, or 42 numerical values for the variables).

In connection with this paper, an Internet access connection will be provided (at least for a time) to files accessible on my "home page" at the URL location of: "http://www.math.princeton.edu/jfnj/texts_and_graphics/AGENCIES_and_COOPERATIVE_GAMES/pubfiles "

Observed Market Clearing Phenomena

Relatively late in the period of the work on the calculation of numerical solutions we found, empirically (by observation), that some of the parameter values in calculated solutions were coming out the same. This was first noticed, but not understood, when we initially solved for solutions of entirely symmetric games (where $b1 = b2 = b3$). We found that of 4 distinct quantities describing a player's choice for the allocation of utility (if the player would become the "final agent") that two of these quantities were very nearly the same (according to the numerical calculations done with many decimal places of accuracy using Mathematica).

It turns out that IF the game is generally non-symmetric (like with $b1 = 1/7, b2 = 1/6, b3 = 1/5$ (for a case for which we computed the solution very precisely)) then that there are no coincident values of any of the 42 unknown quantities that are solved for to find the equilibrium. But on the other hand, if there is a symmetry of Player 1 and Player 2 then it always works out (for cases where we can find solutions with all parameters having non-extreme values) that there are at least two coincident parameter values. Specifically, $y10 = y14$ always. These symbols were defined to represent, for $y10$, either $u1b2r13$ or $u2b1r23$, and for $y14$ either $u1b23r1$ or $u2b13r2$ (where these alternative meanings involve simply the permutation of the symmetrically situated players 1 and 2 (who are symmetrically situated if $b1 = b2$)).

Then an inferable consequence of $y10 = y14$ is that $u2b1r23 = u2b13r2$ so that the BEHAVIOR of Player 1 is such that the amount of utility that he will allocate to Player 2 becomes INDEPENDENT of whether $P1$ was elected by $P2$ after $P2$ was elected by $P3$ or whether $P1$ was elected by $P3$ first and then by $P2$. (So if $P1$ is the final agent and if it was $P2$ that supplied the final vote electing him/her to this position then he gives the same payoff amount to $P2$ in each of these cases.)

This suggests the economic concept of a "market price" which is associated with the "market clearing" concept.

Further discovered coincidences are found if there is symmetry of players 1 and 2 with $b3 = 0$ and $b1 = b2$. Then we find that $y5$, corresponding to both $a13$ and $a23$, comes out to be numerically the same as $y8$, corresponding to both $af13$ and $af23$. (This is an equality of acceptance probabilities rather than of amounts of utility allocated.) And also in these cases we find $y17 = y19$ which has the effect, in particular, that $u3b13r2 = u3b2r13$, which can be said in words as "If the 2-coalition '13' led by Player 1 has formed at the first step of elections then the amount finally allocated to Player 3 will be independent of which agency is elected at the second stage of elections".

But it should be noted that the equalities of $y5$ and $y8$ and of $y17$ and $y19$ are found when $b3 = 0$ but not when $b3 > 0$ (with $b1 = b2$).

Reluctant Acceptance Behavior

It was only as a consequence of actually working on the details of the research project that we discovered the apparent desirability of allowing the players to find the sort of an equilibrium in which they would only rarely, comparatively, vote to accept the agency function of another player.

The players MUST be sometimes accepting, in a global sense, or they would never be gaining any of the benefits specified for the coalitions by the characteristic function.

It is obvious enough that the acceptance action is quasi-altruistic, since the agent accepted is not at all constrained to consider properly the interests of a player accepting him/her EXCEPT through the structure of the repeated game context AND through the reactive behavior of the players built into the model structure. A sometimes accepting player will also be "DEMANDING" to be rewarded by his chosen standard (of benefits) in relation to the average utility consequences, for him/her of any particular type of acceptance vote.

So we found that simply providing a rule for the probable repetition of failed acceptance elections caused the calculable equilibria to shift, in line with the highness of the probability of election repetitions, so that the same sort of efficiency of getting close to the Pareto boundary would be attained with lower probabilities for accepting behavior, in the voting, whenever the probability of repeating a failed election would be improved.

Thus the players could become as if wise negotiators waiting patiently for the other sides to make concessions!

There was another advantage found with arranging for "asymptotically perfectly reluctant accepting" and this was that this idea seemed to remove what otherwise would appear as an arbitrary rule for the elections, the rule that if more than one voter voted, then a single voting action, chosen at random, would be certified and made effective.

Pro-Cooperative Games and Evaluations of Games

The forthcoming book of E. Maskin, which expands on his Presidential Address to the Econometric Society, has a theme that connects with our idea of "Pro-Cooperative Games". This is the theme of "externalities" as realistic considerations that are not included in the formal description of a game (say as a "CF game" in particular) and which COULD act, for example, to (effectively) prevent the formation of the grand coalition.

We began to see that in our games studied by our modeling method (with agencies) that if the strengths of all of the 2-player coalitions were quite large (and comparable to $v = (1, 2, 3) = +1$) then that it would be quite reasonable, in a repeated game context, for there to be various stable equilibria. Thus any two of the players could be seen as being able to "learn" that they are natural allies and then, through an alliance, gain the lion's share of all the possible benefits from the game.

The concept of a "pro-cooperative game" would be that of games where such an alliance of two players would not be able to thus benefit them. Then these games would be more properly suitable for being assigned a "value" (by whatever means of evaluation would ultimately be found and accepted).

As we remarked above in the text commentary on Fig. 3 we found complications when we pursued the calculations of a type of main equilibrium solution from when $b3$ was small (or zero) out through a continuation path until $b3$ became approximately 0.7 in value. Then $a3f12 = a3f21$ become vanishing. (Or $y7$, among the 21 variables describing a game with symmetry between $P1$ and $P2$, becomes vanishing.) (In our actually used notation for Mathematica we shortened $a3f12$ to $af12$ and to $x13$ while $a3f21$ was $af21$ or $x15$. And under the presumption of symmetry between $P1$ and $P2$ then both $x13$ and $x15$ became $y7$ as 42 variables became only 21 variables, modulo the symmetry.)

If there were only the vanishing of $y7$ the path of solutions might have been reasonably continued through higher values of $b3$.

It seems reasonable that with only $b3 = v(1,\ 2)$ being non-zero that the game should continue to be "pro-cooperative", so that some sort of a cooperative evaluation theory could be reasonable. And we could set up and study a modified system of equations with $y7$ (which is $a3f12$ and $a3f21$) set equal to zero and with a condition simply to the effect that $P3$ should not find it profitable to restore an abandoned pattern of sometimes accepting an elected coalition of the two other players.

We tried this, but what was found was that VERY SOON, as $b3$ was increased, there appeared to be other parameters, besides $y7$, moving to take exceptional values (like zero). So this made the effort to continue the pathway of solutions begin to seem questionable. So here further study seems needed or appropriate and also if there are refined or varied forms of modeling the game of evolutionarily stable reactive cooperation (deriving from non-cooperative foundations) then there could be different results found in the case of moving to values of $b3$ close to $+1$ (with $b1$ and $b2$ at least comparatively small).

An Attempt to Study Variants of Modeling with Attorney-Agents

I (or we) spent more than a year trying to find a good variant modeling which would have the game parties (the players) moving into modes of cooperation with the assistance of some species of supportive agents that would guide the process for the originally concerned players. This seemed like a natural idea because of the observable parallels of the cooperative behavior of humans in their societies. But in the end this effort seemed to fail (at least as attempted). Part of the problem was that it seemed unnatural for a player to be "reluctant" to accept the inter-mediation of an attorney agent if that agent would be at all well suited for its function.

(We thought of the attorney-agents as being robotic in function. And in the special case of two-player games we had indeed earlier found that the same type of a cooperative game outcome COULD be realized with a model in which the two primary players would need to BOTH accept the agency of a specified robotic attorney-agent to achieve cooperation (and to gain access to the utility available under the condition of cooperation).)

We failed to find what seemed like a good modeling involving robotically functioning attorney agents for the case of three-player games. There was the complication of the STAGES of the coalescence of the players and ultimately it became apparently TOO BENEFICIAL for two of the original players to achieve representation by an attorney-agent. So the players could not, in an equilibrium of their playing, behave as if "reluctant" to accept an agency.

And of course, retrospectively, it seems pretty clear that our scheme for "robotic" attorney agents, designed so as to work to benefit the interests of the original players, did not correspond to "real life", in human societies, where it is their competition AMONG THEMSELVES that drives attorneys (or lawyers) to work on behalf of the true interests of their clients!

Prospective Model Improvements or Refinements

Because the whole concept of our idea of modeling the attainment of cooperation by the players in a 3-party game situation in terms of a process involving a sequence of elections of agencies was inspired by thinking of the analogy to evolution in Nature, it is logical to consider that when evolution has already arrived at SOME DEGREE of success in improving the cooperativeness of the behavior of players (that initially were entirely independent in their interests, and with regard to their utility functions); because of this, we should consequently realize that a found modeling that favors effectively cooperative behavior by the players in a form that derives from their independently motivated actions (like actions in a non-cooperative game) may not PERFECTLY model the NATURAL possibilities for the attainment of cooperative behavior by the route of a form of evolution. This is because evolution in Nature is generally viewed as an ONGOING process and thus it cannot be expected, presumably, at any particular time, to have arrived at final perfection.

So, in effect, deriving from this consideration, if we found a first model, for a game of three players defined as a CF game by a specified characteristic function, in which stable effective cooperation was realized with the players playing in a non-cooperative game of procedures for actually achieving cooperation; then, from the general viewpoint of evolutionary theory, we DO NOT KNOW, from the apparent success of the model, that we have found an ultimately PERFECT form of modeling for the natural process of cooperation.

Therefore, in a logical or philosophical sense, we should think that we don't know that a specific model (which allows for a natural mode of cooperative behavior in a repeated game context) is perfect and gives the final answer until we have made some further explorations.

In Nature an example is given by lichen species. In a complex case there will be combined (1) a fungus, (2) a green alga species, and (3) a cyanobacterial species, with all of the three contributing distinct and essential functions.

Over a period of time, like, say, 100 million years, a complex lichen variety existing today might naturally evolve into a form exhibiting changes in the two or three component species and changes in how the components would effectively cooperate.

The analogy to this is that a found formula (as it were) for the cooperation of three payers in a three-player game context might not be perfect (and a final answer) if a better formula might appear as a natural evolution of a search for reasonable models (that naturally enable cooperation in repeated game contexts).

Thus, in principle, game theorists working with this sort of modeling, where cooperation results from a repeated game equilibrium, need to consider to what extent they can justifiably think that perfection has been achieved with regard to how each player's behavior can be considered to optimize in relation to his/her interaction with the others.

Because the project research has already exhibited results that compare very interestingly with the analogous results (in terms of predicting game payoffs) that are derivable from the Shapley value, from the nucleolus, from models of the random proposers type, or also with the results of an experimental study, we can wonder if variations in detail of our modeling would affect these comparisons in one direction or another.

Possible or Likely Model Variations for Improvement

One quite simple idea is that if by election a coalition of two players has been formed, with one of them elected as the agent authorized to act for both, then that it is not apparently in the interests of those two players to give information unnecessarily to the third player, so that the identity of the agent-leader who was elected in the formation of that coalition of two players may as well or better be kept secret. Then we find that very nice reductions of the quantity of the strategic data necessary for the players are a consequence. For example, the a-numbers of the types illustrated by $a1f23$ and $a1f32$ would need to become coincident, with the same applying to the related strategically chosen d-numbers and $d1f23$ and $d1f32$, simply because $P1$ WOULD NOT KNOW which of $P2$ and $P3$ had been elected to be (as it were) the chairman of the committee formed by the two of them.

Furthermore, $u2b1r23$ and $u2b1r32$ would likewise need to be the same number (if we assume that $P1$ does not become informed in relation to which of $P2$ and $P3$ had been the chairman (or leader-agent) of the earlier formed committee/coalition of $P2$ and $P3$).

Thus with secret coalitions, where the fact that the first stage of agency elections has succeeded would be known to all but where the remaining solo player would not be advised of WHO had elected WHOM in the formation of the coalition of the others; this change would reduce the total number of strategic parameters needed from 39 to 30. (And this with probably no loss of good representation of the interactions of the players' interdependent interests.)

And there are two other areas where it seems that the modeling can be refined. First, we have actually studied a model game AS IF it were a CF game DEFINED by a characteristic function (as in Von Neumann and Morgenstern) and with the privilege of transferable utility being provided for the players. But actually the payoff function was calculated on the basis that if the grand coalition is not formed but a coalition of two players did form then that that two-player coalition is awarded the value, $b1$, $b2$, or $b3$, appropriate for it, BUT that this amount is divided EQUALLY between the two players in that coalition. (So if, for example, the coalition $\{1, 3\}$ associated with $b2 = v(1,\ 3)$ formed but $\{1, 2, 3\}$ did not form then our applied rule is simply that each of $P1$ and $P3$ receives the payoff of $b2/2$.)

And another area where our modeling had an arguably arbitrary simplification concerns the demand strategies of players voting in the first stage of the elections. Player $P1$, for example, chooses strategically the demand number $d1$ and from this his behavior actions in voting, $a1f2$ and $a1f3$, are determined.

Instead of having only $d1$, $d2$, and $d3$ we could rather have $d1f2, d1f3, d2f1, d2f3, d3f1$, and $d3f2$. And then each action (voting) probability $aifj$ would be associated with its own demand strategy, so that $d3f2$ would control $a3f2$, etc.

This model change, by itself, would add three variables. And the change with having secret leader-agents (like secret committee chairmanships) would take away 9 variables. With both of these changes there would be 36 variables and equations.

Additionally, particularly for the relatively simpler 3-person games we could generalize from CF games (games defined by a characteristic function) to games of the more general type defined by a partition function, such as were studied, originally by Lucas and Thrall, and by Myerson in 1977. This generalization would seem to involve adding very little extra length to the formulae describing the payoff vector function or the derived formulae for the equilibrium equations (for games of 3 players).

Study Possibilities for Games of Four Players

There are types of CF games of 4 players that are essentially different from games of three or two players, yet which, because of symmetries of the game, would involve a much smaller number of strategy parameters to be determined than would be needed for a plausible model for a general sort of 4-person CF game.

The players could have two types and then have complete symmetry in relation to the game structure except for the difference of types. Thus, if A and B were the types then the characteristic function data would be determined by $v(A) = v(B) = 0, v(A, A), v(A, B), v(B, B), v(A, A, B), v(A, B, B)$, and $v(A, A, B, B) = 1$. (This varies in five dimensions.)

Or another variety of very simple 4-person game for computational study would be, instead, a 4-person game where one player is different and three are isomorphic in form. Then, as a CF game it would be described, in terms of players of types R (for regular) and X (for exceptional), by $v(R) = v(X) = 0, v(R, R), v(R, X), v(R, R, R), v(R, R, X)$, and $v(R, R, R, X) = 1$. (And this varies in only 4 dimensions.)

Comparisons with Random Proposers Modeling

Several references in the Bibliography can be viewed as works that relate to the "Nash program" (which was the suggestion that the study of cooperative games should, somehow, be reduced to that of non-cooperative games).

We also have references to studies of games of more than two parties that are based on a method of modeling with "random proposers". These studies have themselves cited the influence of earlier studies, by Rubinstein *et al.*, of "alternating offers" models that have been somewhat successful for studies bargaining games of two parties.

With the "random proposers" models the mathematical calculations necessary to find equilibria are not so difficult as to limit the results available to numerical approximations (as they were found to be with our modeling based on elected agencies) and some really nice results have been obtained in the sense that relatively simple relations to the nucleolus and the Shapley value have appeared in the outcomes.

But what is the truth (if there is any truth!!)?

We suspect, actually, (and this can be viewed as a private opinion) that the "random proposers" modeling simplifies the bargaining and negotiating context by removing an element of the "free enterprise" type. So while nice results are deduced, in terms of mathematical simplicity, they may be only approximate results, in some sense.

With our modeling scheme, with "agencies", it is if all players are always proposers but only occasionally does a player become an ACCEPTOR (of the proposal of another player). So the element of "free enterprise" possibly enters as the players, on their own initiatives, select and decide upon which proposals to accept.

Researchers studying the "random proposals" models have observed that when a player becomes a "proposer" that this seems to give a differential advantage to him. Of course using random assignments, among the players, evens out the advantages, but that is not the same as a "free enterprise" process in its basic nature.

We feel that a truly ideal concept, in relation to the study of games, is to achieve the capacity to give valuable appraisals of a game situation to the players or prospective players of the game.

An "arbitration scheme" (in the words of Luce and Raiffa) could be developed on the basis of a theory that seemed helpful toward game appraisals. But we wish to remark that, in principle, there is some risk of building an "arbitration formula" into a modeling procedure used in studying games. So the arbitration formula cannot be validly DERIVED if it is already inserted as an assumption in the modeling!

Connections with Experimental Games Research

There has been a little of experimental games investigation that has some relations with our studies (of cooperative three-person games) in which we have used a specific model structure (with "demands" and "acceptance probabilities"). The experiments were carried out in the lab of Prof. Axel Ockenfels in Cologne with him being assisted, in one way or another, by Prof. R. Selten, Prof. R. Nagel, and myself. It is expected that a report on these experiments will be published separately.

In the design for the experiments the human game-players being observed were free to think however they wished, about their strategies of action, but coalitions, or "agencies" were formed like in our model. And the payoff rules, in relation to the final coalition or set of coalitions and players, were the same. (So the players did not have any need to think about choosing numbers to be their "demands".)

The basic design of the experimental routine, as a "flow chart" was made by Prof. Selten and he also chose a set of 10 example games to be studied.

It was difficult to compare directly with the computable results from our specific model for computations because for most of the games the model solution could not be easily calculated (since the two-player coalitions had too much strength) (or they were not enough like "pro-cooperative games").

However the experimental results DID seem to confirm, in a general way, the pattern of experiments leading to "more egalitarian" outcomes than would be suggested by notable mathematical indicator concepts like the Shapley value or the nucleolus.

Relevant Existing Literature

The recent work of Abreu and Pearce notably involves, like our modeling scheme, the study of the repeated games context. In this context they are able to deal with both the cooperative and the competitive aspects of a situation of bargaining that is not of the simplest variety.

Harsanyi was an early pioneer explorer in the search for theoretical understanding of cooperative games including, particularly, games of the NTU category.

Rubinstein's 1982 paper in Econometrica influenced various later papers connecting bargaining and offers and acceptances.

Others, for example Gul (1989), Osborne and Rubinstein (1990), Montero-Garcia (1998), Seidmann and Winter (1998), Ferreira (1999), and Ray and Vohra (1999), in a general sense, look at coalitions from a "dynamical" viewpoint, understanding that the participants in a game-like situation must act appropriately for coalitions to actually form.

The papers of Baron and Ferejohn, of Okada, and of Gomes have made use of the "random proposers modeling" and are good sources in relation to that idea (which has antecedents going back to Rubinstein and Ståhl).

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The work of the project assistants was particularly important, first, for developing and double-checking the actual concrete system or systems of equations to

be solved for the equilibrium solutions. And this work was mostly done within the framework of Mathematica and programs running within "sessions" of Mathematica.

On some of the graphic charts there are references indicating which one of the three project assistants happened to have prepared that chart.

In addition I want to mention that I was very substantially assisted, in preparing the proposal (for the research project) (which was submitted to the Economics division of the NSF foundation), by the aid and guidance of Professor Avinash Dixit of Princeton.

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