NOTES ON TOPOLOGICAL STABILITY

by

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These notes are part of the first chapter of a series of lectures given by the author in the spring of 1970. The ultimate aim of these notes will be to prove the theorem that the set of topologically stable mappings form a dense subset of $C^\infty(N, P)$ for any finite dimensional manifolds N and P where N is compact. The first chapter is a study of the Thom-Whitney theory of stratified sets and stratified mappings. The connection of the material in these notes with the theorem on the density of topologically stable mappings appears in § 11, where we give Thom's second isotopy lemma. This result gives sufficient conditions for two mappings to be topologically equivalent.

§1. Condition a. We begin by introducing some notions that are due to Whitney ([5] and [6]).

Let μ be a positive number or ∞, which will be fixed throughout this chapter. By "smooth" we will mean differentiable of class C^{μ}.

Let M be a smooth (i.e., C^{μ}) n-manifold without boundary. By a smooth (i.e., C^{μ}) submanifold of M, we will mean a subset X of M such that for every $x \in X$ there exists a coordinate chart (φ, U) of class C^{μ} such that $x \in U$ and $\varphi(X \cap U) = \mathbb{R}^k \cap \varphi(U)$, for a suitable coordinate plane $\mathbb{R}^k$ in $\mathbb{R}^n$. In the definition of submanifold, we do not assume that X is closed. However, it follows from the definition of submanifold that X is locally closed i.e., each point in S has a neighborhood U in M such that $X \cap U$ is closed in U.

If X is an r-dimensional submanifold of M and $x \in X$, then the tangent space TX_x of X at x is a point in the Grassmannian bundle of r-planes in TM_x. In what follows "convergence" means convergence in the standard topology on this bundle.

Let X and Y be smooth submanifolds of M and let $y \in Y$. Set $r = \dim X$.

DEFINITION 1.1. We say the pair (X, Y) satisfies condition a at y if the following holds. Given any sequence x_i of points in X such that $x_i \to y$ and TX_{x_i} converges to some r-plane $\tau \subseteq TM_y$, we have $TY_y \subseteq \tau$.

Example 1.2. (Whitney [6]). Let x, y, z denote coordinates for $\mathbb{C}^3$. Let Y be the z-axis and let X be the set $\{zx^2 - y^2 = 0\}$ with the z-axis deleted. (In Figure 1, we have sketched the intersection of X with $\mathbb{R}^3$.) Then X and Y are complex analytic submanifolds of $\mathbb{C}^3$. It is easily seen that (X, Y) satisfies condition a at all points of Y except the origin, and that it does not satisfy condition a there.

We will say that the pair (X, Y) satisfies condition a if it satisfies condition a at every point of Y.

In Example 1.2, the pair (X, Y) does not satisfy condition a. If we set $Z = \{0\}$ and $Y' = Y - Z$, then the pairs (X, Y'), (X, Z), and (Y', Z) do satisfy condition a.

§ 2. Condition b. We will begin by defining Whitney's condition b for submanifolds of $\mathbb{R}^n$. Then we extend this definition to submanifolds of an arbitrary manifold, using the definition in $\mathbb{R}^n$. We will also show that condition b implies condition a.

If $x, y \in \mathbb{R}^n$ and $x \neq y$, then the secant $\overline{xy}$ will denote the line in $\mathbb{R}^n$ which is parallel to the line joining x and y and passes through the origin. For any $x \in \mathbb{R}^n$ we identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ in the standard way.

Let X, Y be (smooth) submanifolds of $\mathbb{R}^n$. Let $y \in Y$. Let $r = \dim X$.

DEFINITION 2.1. We say that the pair (X, Y) satisfies condition b at y if the following holds. Let x_i be a sequence of points in X, converging to y and y_i a sequence of points in Y, also converging to y. Suppose TX_{x_i} converges to some r-plane $\tau \subseteq \mathbb{R}^n$ and that $x_i \neq y_i$ for all i and the secants $\overline{x_i y_i}$ converge (in projective space P^{n-1}) to some line $\ell \subseteq \mathbb{R}^n$. Then $\ell \subseteq \tau$.

Let (X', Y') be a second pair of submanifolds of $\mathbb{R}^n$, and let $y' \in Y'$.

LEMMA 2.2. Suppose there exist open neighborhoods U and U' of y and y' in $\mathbb{R}^n$ and a (smooth) diffeomorphism $\varphi : U \to U'$ such that $\varphi(U \cap X) = U' \cap X'$, $\varphi(U \cap Y) = U' \cap Y'$ and $\varphi(y) = y'$. Then (X, Y) satisfies condition b at y if and only if (X', Y') satisfies condition b at y'.

Proof: Obvious.

DEFINITION 2.2. Let M be a manifold and X, Y submanifolds. Let $y \in Y$. We say that (X, Y) satisfies condition b at y if for some coordinate chart (φ, U) about y, we have that the pair $(\varphi(U \cap X), \varphi(U \cap Y))$ satisfies condition b at $\varphi(y)$.

In view of Lemma 2.2, if (X, Y) satisfies condition b at y, then for every coordinate chart (φ, U) about y, we have that $(\varphi(U \cap X), \varphi(U \cap Y))$ satisfies condition b at y.

For the rest of this section, let M be a manifold and X and Y submanifolds and let $y \in Y$.

PROPOSITION 2.4. If (X, Y) satisfies condition b at y then it satisfies condition a at y.

Proof: Since both conditions a and b are purely local, we may suppose that X and Y are submanifolds of $\mathbb{R}^n$. Let x_i be a sequence of points in X such that $x_i \to y$ and $TX_{x_i} \to \tau$, for some $\tau \subseteq T\mathbb{R}^n_y = \mathbb{R}^n$. We must show that $TY_y \subseteq \tau$. Suppose otherwise.

Then there exists a line $\ell \subseteq \mathbb{R}^n$, passing through the origin, such that $\ell \subseteq TY_y$ but $\ell \not\subseteq \tau$. Since $\ell \subseteq TY_y$, we can choose a sequence of points $y_i \in Y$ such that $y_i \neq x_i$, $y_i \to y$ and $\widehat{y_i x_i} \to \ell$. But since $\ell \not\subseteq \tau$, this contradicts condition b. Q.E.D.

We say (X, Y) satisfies condition b if it satisfies condition b at every point $y \in Y$.

Example 2.5. Let X be the spiral in $\mathbb{R}^2$ defined by the condition that the tangent of X makes a constant angle with the radial vector, and let Y be the origin. In polar coordinates, this spiral is given by $r - \beta\theta =$ constant. Then the pair (X, Y) does not satisfy condition b. For, by definition, the angle α between the line TX_x and the secant $\widehat{0x}$ is independent of x. If $x_i \in X$ is a sequence converging to 0, then the tangents TX_{x_i} converge to a line $\tau \subseteq \mathbb{R}^2$, and $\widehat{0x_i}$ converges to a line ℓ, which makes an angle α with τ.

Example 2.6. (Whitney [6]). Let x, y, z be coordinates for $\mathbb{C}^3$. Let Y be the z-axis. Let X be the set $\{y^2 + x^3 - z^2x^2 = 0\}$ with the z-axis deleted. (In Figure 2 we have sketched the intersection of X with $\mathbb{R}^3$.) It is easily seen that the pair (X, Y) satisfies condition a, and the pair (X, Y) satisfies condition b at all points of Y except the origin and that it does not satisfy condition b there.

PROPOSITION 2.5. Suppose $y \in \overline{X - Y}$ and (X, Y) satisfies condition b at y. Then $\dim Y < \dim X$.

Proof: It is enough to consider the case when $M = \mathbb{R}^m$. Since $y \in \overline{X - Y}$, there exists a sequence x_i in $X - Y$ which converges to y. By the compactness of the Grassmannian, we may suppose, by passing to a subsequence if necessary, that TX_{x_i} converges to an r plane $\tau \subseteq \mathbb{R}^m$ (where $r = \dim X$). Since condition b implies condition a (Proposition 2.4), $TY_y \subseteq \tau$. For i sufficiently large, there is a point y_i on Y which minimizes the distance to x_i. By passing to a subsequence if necessary, we may suppose the secants $x_i y_i$ converge to a line $\ell \subseteq \mathbb{R}^n$. Since y_i minimizes the distance to x_i, the secant $y_i x_i$ is orthogonal to TY_{y_i}; hence ℓ is orthogonal to TY_y. Since (X, Y) satisfies condition b at y, we have $\ell \subseteq \tau$. We have shown $TY_y + \ell \subseteq \tau$ and ℓ is orthogonal to TY_y; hence $\dim X = \dim \tau > \dim TY_y = \dim Y$. Q.E.D.

§3. Blowing up. In the next section, we will give an intrinsic formulation of condition b which will be useful later on. This formulation depends on the notion of blowing up a manifold along a submanifold, which we define in this section.

Let N be a manifold and U a closed submanifold. By the manifold $B_U N$ obtained by blowing up N along U, we will mean the manifold defined in the following way. As a set $B_U N$ is the disjoint union $(N - U) \cup P\eta_U$, where $P\eta_U$ denotes the projective normal bundle of U in N.

By the natural projection $\pi : B_U N \to N$, we mean the mapping defined by letting $\pi | P\eta_U$ be the projection of $P\eta_U$ on U and letting $\pi | N - U$ be the inclusion of $N - U$ into N.

To define the differentiable structure on $B_U N$, we first consider the case when N is open in $\mathbb{R}^n$ and $U = \mathbb{R}^r \cap N$, where $\mathbb{R}^r$ is the coordinate plane defined by the vanishing of the last $n - r$ coordinates. Then we have a mapping $\alpha : B_U N \to \mathbb{R}^n \times \mathbb{R}P^{n-r-1}$ defined as follows. First, $\alpha | P\eta_U$ is the standard identification of $P\eta_U$ with $U \times \mathbb{R}P^{n-r-1} \subseteq \mathbb{R}^n \times \mathbb{R}P^{n-r-1}$. Secondly, if $x = (x_1, \cdots, x_n) \in \mathbb{R}^n - \mathbb{R}^r$, then $\alpha(x) = (x, \beta(x))$, where $\beta(x)$ is the point in $\mathbb{R}P^{n-r-1}$ with homogeneous coordinates $(x_{r+1}, \cdots, x_n)$.

It is easily verified that $\alpha[B_U N]$ is a C^∞ submanifold of $\mathbb{R}^n \times \mathbb{R}P^{n-r-1}$ as follows. Let $(x_1, \cdots, x_n)$ denote the coordinates of $\mathbb{R}^n$. Let $X_{r+1}, \cdots, X_n$ denote the homogeneous coordinates for $\mathbb{R}P^{n-r-1}$. For $r+1 \le i \le n$, let Z_i denote the subset of $\mathbb{R}P^{n-r-1}$ defined by $X_i \ne 0$, and let X_{ji} be the real valued function $X_{ji} = X_j/X_i$ on Z_i. Then the intersection of $\alpha[B_U N]$ with $N \times Z_i$ is the set defined by

$$x_j = X_{ji} x_i \qquad r+1 \le j \le n, \quad j \ne i \quad .$$

Therefore $\alpha[B_U N]$ is a submanifold of $\mathbb{R}^n \times \mathbb{R}P^{n-r-1}$.

Since the mapping α is injective, we may define a manifold structure on $B_U N$ by pulling back the manifold structure on $\alpha[B_U N]$.

Now, let N' be a second open subset of $\mathbb{R}^n$, let $U' = \mathbb{R}^r \cap N'$, and let $\varphi : (N, U) \to (N', U')$ be a C^μ diffeomorphism. Let $\varphi_* : B_U N \to B_{U'} N'$ be the induced mapping, defined by letting $\varphi_* | P\eta_U : P\eta_U \to P\eta_{U'}$ be the mapping induced by the differential, and letting $\varphi_* | N - U : N - U \to N' - U'$ be the restriction of φ. Then φ_* is a diffeomorphism of class $C^{\mu-1}$.

To show this, we first observe that φ_* is a bijection and $(\varphi_*)^{-1} = (\varphi^{-1})_*$. Therefore, it suffices to show that φ_* is of class $C^{\mu-1}$. To show this, it is enough to show that $x_i \circ \varphi_*$ is of class $C^{\mu-1}$, $1 \le i \le n$, that $(\varphi_*^{-1})(Z_i)$ is open, $r+1 \le i \le n$, and that $X_{ji} \circ \varphi_*$ is of class $C^{\mu-1}$ for $r+1 \le j \le n$ and $j \ne i$. Since

$$x_i \circ \varphi_* = x_i \circ \varphi \circ \pi \quad ,$$

where $\pi : B_U N \to N$ is the natural projection, the first statement is obvious.

To prove the remaining two statements, we set $\varphi_i = x_i \circ \varphi$ and observe that there exist functions $\phi_{i\alpha}$ of class $C^{\mu-1}$, for $r+1 \le i$, $\alpha \le n$, such that

$$(*) \qquad \varphi_i = \sum_{\alpha=r+1}^{n} x_\alpha \phi_{i\alpha} \quad .$$

This is proved as follows. Since for $r+1 \le i \le n$, we have that φ_i vanishes on $U = N \cap \mathbb{R}^r$, we get that

$$\varphi_i(x_1, \cdots, x_n) = \int_0^1 \frac{d}{dt} \varphi_i(x_1, \cdots, x_r, tx_{r+1}, \cdots, tx_n) dt$$

$$= \sum_{\alpha=r+1}^{n} x_\alpha \int_0^1 \frac{\partial \varphi_i}{\partial x_\alpha}(x_1, \cdots, x_r, tx_{r+1}, \cdots, tx_n) dt$$

so that $*$ holds, where

$$\phi_{i\alpha} = \int_0^1 \frac{\partial \varphi_i}{\partial x_\alpha}(x_1, \cdots, x_r, tx_{r+1}, \cdots, tx_n) dt \quad .$$

In view of $(*)$, $\varphi_*^{-1}(Z_i) \cap Z_k$ is the subset of Z_k defined by

$$\sum_{\alpha=r+1}^{n} X_{\alpha k} \phi_{i\alpha} \ne 0 \quad ,$$

and hence is open. It follows that $\varphi_*^{-1} Z_i$ is open. It also follows from

*) that

$$x_{ji} \circ \varphi_{*} = \frac{\Sigma^{n}_{\alpha=r+1} x_{\alpha k} \phi_{j\alpha}}{\Sigma^{n}_{\alpha=r+1} x_{\alpha k} \phi_{i\alpha}}$$

on $\varphi_{*}^{-1}(Z_i) \cap Z_k$, and hence is of class $C^{\mu-1}$ there.

This completes the proof that φ_{*} is a diffeomorphism of class $C^{\mu-1}$.

Now we return to the general situation where N is a manifold, and U is a closed submanifold, both of class C^{μ}. In view of what we have just done, we can construct a differentiable structure on the part of $B_U N$ which lies above any coordinate patch, and the differentiable structures above different coordinate patches are $C^{\mu-1}$ compatible. Thus, we obtain the structure of a manifold of class $C^{\mu-1}$ on $B_U N$.

Note that the natural projection $\pi : B_U N \to N$ is differentiable of class $C^{\mu-1}$.

Since we have defined a structure of a manifold of class $C^{\mu-1}$ on $B_U N$, we have also defined a topology on $B_U N$. In the local case, when $N = \mathbb{R}^n$ and $U = \mathbb{R}^r$, this topology may be described more directly. Let $\{x_i\}$ be a sequence of points in $\mathbb{R}^n - \mathbb{R}^r$, and suppose $x_i \to x \in \mathbb{R}^r$. Let $\ell \in \mathbb{R}P^{n-r-1}$, so that (x, ℓ) is a member of $B_U N$,

if we identify $B_U N$ with the subset $\alpha[B_U N]$ of $\mathbb{R}^n \times P\mathbb{R}^{n-r}$, as above. Then it is easily seen that x_i converges (in $B_U N$) to (x, ℓ) if and only if the secants $x_i x_i'$ converge to ℓ, where x_i' denotes the projection of x_i on $\mathbb{R}^r$.

This suggests that it should be possible to reformulate condition b in terms of "blowing up". We do this in the next section.

§4. An intrinsic formulation of condition b. Let N be a smooth manifold. Let Δ_N denote the diagonal in N^2. By the fat square of N, we will mean the manifold $F(N)$ obtained by blowing up N^2 along Δ_N.

The normal bundle η of Δ_N in N^2 can be identified with the tangent bundle TN in a canonical way, as follows. If $x \in \Delta_N$, then by definition

$$\eta_x = (TN_x \oplus TN_x)/\text{diagonal} \quad .$$

The mapping of $TN_x \oplus TN_x$ into TN_x which sends $v \oplus w$ to $v - w$ induces an isomorphism of η_x with TN_x. We use this isomorphism to identify η_x with TN_x.

From this identification and the definition of the process of blowing up a manifold along a submanifold, it follows that

$$F(N) = PT(N) \cup (N^2 - \Delta_N) \quad \text{(disjoint union)}$$

where $PT(N)$ denotes the projective tangent bundle of N. Thus, points of $F(N)$ are of two kinds: pairs (x, y) with $x, y \in N$ and $x \neq y$ and tangent directions on N.

It follows from the previous section that $F(N)$ is a manifold of class $C^{\mu-1}$.

Roughly speaking, a sequence $\{(x_i, y_i)\}$ of points in $N^2 - \Delta_N$ converges to a tangent direction ℓ on N if the sequences $\{x_i\}$ and $\{y_i\}$ converge to the same point x in N and the direction from x_i to y_i converges to ℓ. In the case $N = \mathbb{R}^n$, this can be made precise: $\{(x_i, y_i)\}$ converges to $(x, \ell) \in \mathbb{R}^n \times \mathbb{R}P^{n-1}$ if both $\{x_i\}$ and $\{y_i\}$ converge to x, and the secants $x_i y_i$ converge to ℓ.

Now let X and Y be smooth submanifolds of N and let $y \in Y$. Suppose Y is closed. In view of the previous paragraph, we obtain the following result.

PROPOSITION 4.1. The pair (X, Y) satisfies condition b at y if and only if the following condition holds. Let $\{x_i\}$ be any sequence of points in X and $\{y_i\}$ any sequence of points in Y such that $x_i \neq y_i$. Suppose $\{x_i\} \to y$, $\{y_i\} \to y$, $\{(x_i, y_i)\}$ converges to a line $\ell \subseteq PTN_y$, and $\{TX_{x_i}\}$ converges (in the Grassmannian of r planes in TN, where $r = \dim X$) to an r-plane $\tau \subseteq TN_y$. Then $\ell \subseteq \tau$.

§5. Whitney pre-stratifications. Let M be a smooth (i.e., C^{μ}) manifold without boundary. Let S be a subset of M. By a pre-stratification $\mathcal{S}$ of S, we will mean a cover of S by pairwise disjoint smooth submanifolds of M, which lie in S. We will say that $\mathcal{S}$ is locally finite if each point of M has a neighborhood which meets at most finitely many strata. We say $\mathcal{S}$ satisfies the condition of the frontier if for each stratum X of $\mathcal{S}$ its frontier $(\bar{X} - X) \cap S$ is a union of strata.

We will say $\mathcal{S}$ is a Whitney pre-stratification if it is locally finite, satisfies the condition of the frontier, and (X, Y) satisfies condition b for any pair (X, Y) of strata of $\mathcal{S}$.

Let $\mathcal{S}$ be a Whitney pre-stratification of a subset S of a manifold M. Suppose X and Y are strata. We write $Y < X$ if Y is in the frontier of X. In view of Proposition 2.5, if $Y < X$ then $\dim Y < \dim X$. It follows easily that the relation "$<$" defines a partial order on $\mathcal{S}$.

Remark. Let M be a manifold, S a closed subset of M, and $\mathcal{S}$ a Whitney pre-stratification of S. Let x and x' be two points in the same connected component of a stratum of $\mathcal{S}$. Then there exists a homeomorphism h of M onto itself which preserves S and $\mathcal{S}$ such that $h(x) = x'$. This follows from Thom's theory [4] and we will prove it below. In the case $\mathcal{S}$ has only two strata, it is quite easy to prove by an argument due to Thom [4 , p.242].

We sketch Thom's argument for the two strata case here. The only non-trivial case is when the two strata satisfy $X < Y$ and the two points x and x' are in X. In this case X is closed and $X = \bar{Y} = Y \cup X$.

For simplicity, we will suppose that M is compact, though it is not difficult to modify the argument to make it work in the case M is non-compact.

Let N be a small tubular neighborhood of X in M, let $\pi : N \to X$ be a smooth retraction, and let ρ be a smooth function on M such that $\rho \geq 0$, $X = \{\rho = 0\}$, and at a point $x \in X$, ρ is non-degenerate on the normal plane to X in the sense that the Hessian matrix of ρ at x has rank equal to the codimension of X.

Now let x and x' be two points in the same connected component of X. Let v_X be a smooth vector field on X such that the trajectory of v starting at x arrives at x' at time $t = 1$.

For $\epsilon > 0$ sufficiently small, the subset $M_\epsilon = \{\rho = \epsilon\}$ of N is compact, and $\pi : M_\epsilon \to X$ is a submersion. Furthermore, $Y_\epsilon = M_\epsilon \cap Y$ is compact, and it follows from condition b that $\pi : Y_\epsilon \to X$ is a submersion for ϵ sufficiently small. It follows easily that there is a vector field v on $M - X$ and an $\epsilon_1 > 0$ such that v is tangent along Y, and the following hold.

$$(*)\qquad v\rho(m) = 0$$
$$(**)\qquad \pi_e v(m) = v_X(\pi m) \qquad \text{if } \begin{cases} m \in M - X \\ \text{and } \rho(m) < \epsilon_1 \end{cases}.$$

From * and the compactness of M, it follows that the trajectory of v starting at any point of $M - X$ is defined for all time. Hence v generates a one-parameter group $\{h_t^0, t \in \mathbb{R}\}$ of diffeomorphisms of $M - X$. Clearly v_X generates a one-parameter group $\{h_t^X : t \in \mathbb{R}\}$ of diffeomorphisms of X. Let $h_t : M \to M$ be defined by $h_t|M - X = h_t^0$ and $h_t|X = h_t^X$. It follows from (*) and (**) that $h_t^X \pi(m) = \pi h_t^0(m)$ if $m \in M - X$ and $\rho(m) < \epsilon_1$. Hence h_t is a homeomorphism of M. Clearly h_t preserves X, and furthermore h_t preserves Y, since v is tangent along Y. Finally $h_1(x) = x'$ since trajectory of v_X starting at x arrives at x' at time $t = 1$.

Thus $h = h_1$ is the required homeomorphism of M.

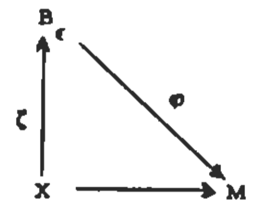
§6. Tubular neighborhoods. In this section, we define the notion of a tubular neighborhood of a submanifold of a manifold, and prove an existence and uniqueness theorem for tubular neighborhoods. Our existence and uniqueness theorem is slightly more general than the standard one (cf., Lang [2]). The method of proof we use was suggested to us by A. Ogus.

We recall that a vector bundle E over a smooth manifold M is said to be smooth if the coordinate transition functions which define E are smooth functions. By a smooth inner product on a vector bundle E, we will mean a rule which assigns to each fiber E_u of E an inner product $(\ ,\ )_u$ on E_u and which has the following property: If U is any open set in M and s_1, s_2 are two smooth sections of E above U then the mapping $u \to (s_1(u), s_2(u))_u$ is smooth. From now on, we will assume all vector bundles and inner products on vector bundles are smooth, unless the contrary is explicitly stated. By a (smooth) inner product bundle, we mean a pair consisting of a (smooth) vector bundle E and a (smooth) inner product on E.

If $\pi : E \to M$ is an inner product bundle over a manifold, and ϵ is a positive function on M, then the open ϵ-ball bundle B_ϵ of E will be defined as the set of e in E such that $\|e\| < \epsilon(\pi e)$, where $\|e\|$ is defined as $(e, e)^{1/2}$.

Let M be a manifold and X a submanifold.

DEFINITION. A <u>tubular neighborhood</u> T of X in M is a triple (E, ϵ, φ), where $\pi : E \to X$ is an inner product bundle, ϵ is a positive smooth function on X, and φ is a diffeomorphism of B_ϵ onto an open subset of M which commutes with the zero section ζ of E:



We set $|T| = \varphi(B_\epsilon)$. By the <u>projection associated to</u> T, we mean the mapping $\pi_T = \pi \circ \varphi^{-1} : |T| \to X$. By the <u>tubular function associated to</u> T, we mean the non-negative real valued function

$$\rho_T = \rho \circ \varphi^{-1} : |T| \longrightarrow \mathbb{R} \qquad \text{where } \rho(e) = \|e\|^2 \text{ for all } e \in |T| ,$$

It follows from these definitions that π_T is a retraction of $|T|$ on X, i.e., the composition

$$X \xrightarrow{\text{inclusion}} |T| \xrightarrow{\pi_T} X$$

is the identity. Also, X is the 0-set of ρ_T, the differential of ρ_T vanishes only on X, and (in the case $\mu \geq 2$) at a point $x \in X$, ρ_T is non-degenerate on the normal plane to X in the sense that the Hessian matrix of ρ at x has rank equal to the co-dimension of X.

If U is a subset of X, the <u>restriction</u> $T|U$ of T to U is defined as $(E|U, \epsilon|U, \varphi|U)$.

If $T = (E, \epsilon, \varphi)$ and $T' = (E', \epsilon', \varphi')$ are two tubular neighborhoods of X in M, an inner product bundle isomorphism $\phi : E \to E'$ will be said to be an <u>isomorphism</u> of T with T' if there exists a positive continuous function ϵ'' on X such that $\epsilon'' \leq \min(\epsilon, \epsilon')$ and $\varphi' \circ \phi | B_{\epsilon''} = \varphi | B_{\epsilon''}$. Note that if this holds, then $\pi_T | \varphi B_{\epsilon''} = \pi_{T'} | \varphi B_{\epsilon''}$ and $\rho_T | \varphi B_{\epsilon''} = \rho_{T'} | \varphi B_{\epsilon''}$. We say T and T' are <u>isomorphic</u> and write $T \sim T'$ if there exists an isomorphism from T to T'.

A smooth mapping $f : M \to P$ will be said to be a <u>submersion</u> if $df : TM_x \to TP_{f(x)}$ is onto for each $x \in M$.

Throughout the rest of this section, let $f : M \to P$ be a smooth mapping, and X a submanifold of M.

A tubular neighborhood T of X in M will be said to be <u>compatible with</u> f if $f \circ \pi_T = f \,|\, |T|$. A mapping h of M into itself will be said to be <u>compatible with</u> f if $f \circ h = f$. A homotopy $H : M \times I \to M$ of M into itself will be said to be <u>compatible with</u> f if $f \circ H_t = f$ for all $t \in I$ $(= [0, 1])$. By an <u>isotopy</u> of M, we will mean a smooth mapping

$H : M \times I \to M$ such that $H_0 = \mathrm{id} : M \to M$ and $H_t : M \to M$ is a diffeomorphism for all $t \in I$. If h is a diffeomorphism of M into itself, the support of h will mean the closure of $\{x \in M : h(x) \neq x\}$. Likewise, if $H : M \times I \to M$ is an isotopy, the support of H will mean the closure of $\{x \in M : t \in I,\ H(x,t) \neq x\}$.

If M' is a second manifold and X' is a submanifold of M', and $h : (M, X) \to (M', X')$ is a diffeomorphism, then for any tubular neighborhood $T = (E, \epsilon, \varphi)$ of X we define a tubular neighborhood $h_* T$ of X' by $h_* T = ((h^{-1})^* E,\ \epsilon \circ h^{-1},\ h \circ \varphi)$.

We will begin by stating and proving a uniqueness theorem for tubular neighborhoods, and then we will derive an existence theorem from the uniqueness theorem. This procedure of deducing the existence theorem from the uniqueness theorem was suggested to us by A. Ogus.

The simplest uniqueness theorem for tubular neighborhoods states that if X is closed and T_0 and T_1 are tubular neighborhoods of X in M, then there exists a diffeomorphism h of M onto itself which leaves X point-wise fixed such that $h_* T_0 \sim T_1$. Moreover, h can be chosen so that there is an isotopy H of M with $h_1 = H$ which leaves X point wise fixed. We can generalize this result in various ways.

First, under the hypothesis that T_0 and T_1 are compatible with f and $f|X$ is a submersion, we can choose h and H to be compatible with f. Secondly, if $T_0|U \sim T_1|U$ for some open set U in X, and Z is a closed subset of M such that $Z \cap X \subseteq U$, then we can choose h and H to leave Z point-wise fixed.

The following proposition implies these statements, and has some other wrinkles as well. We will use it in its full generality.

PROPOSITION 6.1 (Uniqueness of tubular neighborhoods). Suppose the submanifold X of M is closed, and $f|X : X \to P$ is a submersion. Let U be an open subset of X, let U' and V' be closed subsets of X, let V be an open subset of M, and suppose $U' \subseteq U$ and $V' \subseteq V$. (See Figure 3.) Let T_0 and T_1 be tubular neighborhoods of X in M which are compatible with f and suppose there is an isomorphism $\phi_0 : T_0|U \to T_1|U$. Then there is an isotopy $H : M \times I \to M$, compatible with f, leaving X point-wise fixed, and with support in V, such that $h_* T_0|V' \cup U' \sim T_1|V' \cup U'$, where $h = H_1$. Moreover, if N is any neighborhood of the diagonal in $M \times M$, we can choose H such that $(H_t(x), x) \in N$ for any $t \in I$ and $x \in M$. Also, we can choose H so that there is an isomorphism $\phi : h_* T_0|V' \cup U' \to T_1|V' \cup U'$ such that $\phi|U' = \phi_0|U'$.

Proof. Let $m = \dim M$, $c = \operatorname{cod} X$, and $p = \dim P$. For $k < m$, let $\mathbb{R}^k$ be embedded as $\mathbb{R}^k \times O_{m-k}$ in $\mathbb{R}^m$. We will say that we are in the local case when V' is compact and there exists a diffeomorphism Φ of M onto an open subset of $\mathbb{R}^m$, such that $\Phi(X) = \mathbb{R}^{m-c} \cap \Phi(M)$, and a diffeomorphism Ψ of P onto an open subset of $\mathbb{R}^p$ such that the following diagram commutes, where π is given by $\pi(x_1, \cdots, x_m) = (x_1, \cdots, x_p)$

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & \mathbb{R}^m \\ \downarrow f & & \downarrow \pi \\ P & \xrightarrow{\Psi} & \mathbb{R}^p \end{array} \quad .$$

There are two steps in the proof:

Step 1. Reduction to the local case. From the hypothesis that $f|X$ is a submersion, it follows that for each $x \in X$ there exists an open neighborhood W_x of x in M, a diffeomorphism Φ_x of W_x onto an open subset of $\mathbb{R}^m$ such that $\Phi(W_x \cap X) = \Phi(W_x) \cap \mathbb{R}^{m-c}$, and a diffeomorphism Ψ_x of $f(W_x)$ onto an open subset of $\mathbb{R}^p$ such that the following diagram commutes

$$\begin{array}{ccc} W_x & \xrightarrow{\Phi_x} & \mathbb{R}^m \\ \downarrow f & & \downarrow \pi \\ f(W_x) & \xrightarrow{\Psi_x} & \mathbb{R}^p \end{array} \quad .$$

Furthermore, we may suppose each W_x is relatively compact, and that

$$(*) \quad \begin{array}{l} W_x \cap V' \neq \emptyset \Rightarrow W_x \subseteq V \\ W_x \cap U' \neq \emptyset \Rightarrow W_x \cap X \subseteq U \end{array} \quad .$$

Then $\{M - X\} \cup \{W_x\}$ is a cover of M, so that there exists a locally finite refinement of it, which we may take to be of the form $\{M - X\} \cup \{W_i\}$, where each W_i is contained in $W_{x(i)}$ for some $x_i \in X$. Since M has a countable basis for its topology, the collection $\{W_i\}$ is countable. Now we discard all W_i for which $W_i \cap U' \neq \emptyset$ or $W_i \cap V' = \emptyset$, and we index the remaining W_i's by the positive integers. Then we have $V' \subseteq U \cup W_1 \cup V_2 \cup \cdots$, and $W_i \subseteq V$ for all i, by $*$.

We can choose closed sets $W_i' \subseteq W_i \cap X$ such that $V' \subseteq U \cup W_1' \cup W_2' \cup \cdots$. Since $W_i' \subseteq W_{x(i)}$, and the latter is relatively compact, it follows that W_i' is compact.

Now we construct by induction a sequence $H^0, H^1, H^2, \cdots$ of isotopies of M into itself and sequence $\phi_0, \phi_1, \phi_2, \cdots$ of isomorphisms of tubular neighborhoods. We let H^0 be defined by H_t^0 = identity , $0 \le t \le 1$, and let ϕ_0 be as given in the statement of the proposition.

For the inductive step, we suppose that $H^0, H^1, \cdots, H^{i-1}$ and $\phi_0, \cdots, \phi_{i-1}$ have been constructed, are compatible with f and leave X point-wise fixed. We let G^j be the isotopy of M defined by $G_t^j = H_t^j \circ H_t^{j-1} \circ \cdots \circ H_t^0$. We set $g^j = G_1^j$. We let $U_j = U \cup W_1 \cup \cdots \cup W_j$ and suppose $\operatorname{supp} G^{i-1} \subsetneq U_{i-1} \cap V$. Furthermore, we suppose $(G_t^{i-1}(x), x) \in N$ for all $x \in M$ and $t \in [0,1]$, and that ϕ_{i-1} is an isomorphism of tubular neighborhoods $g_*^{i-1} T_0 | \overline{U}_{i-1}^* \to T_1 | \overline{U}_{i-1}^*$, where U_{i-1}^* is an open neighhorhood of $U' \cup W_1' \cup \cdots \cup W_{i-1}'$ in X.

Then it follows from the local case of the proposition that H^i and ϕ_i can be chosen so that the conditions of the induction are satisfied. For, let W_i^0 be an open subset of W_i such that $W_i' \subseteq W_i^0$ and W_i^0 is relatively compact in W_i, and let U_i^* be an open neighborhood of $U' \cup W_1' \cup \cdots \cup W_i'$ in X whose closure lies in $U_{i-1}^* \cup W_i^0$. From the local case, it follows that we can construct an isotopy H^i of W_i, compatible with f, leaving $X \cap W^i$ point-wise fixed, and with support in W_i^0 such that $h_*^i g_*^{i-1} T_0 | \overline{U}_i^* \cap W_i \sim T_1 | \overline{U}_i^* \cap W_i$, where $h^i = H_1^i$. (This is because $g_*^{i-1} T_0 | \overline{U}_{i-1}^* \cap W_i \sim T_1 | \overline{U}_{i-1}^* \cap W_i$ and $U_{i-1}^* \subset W^0$.) Moreover, we may

choose H^i so that H_t^i is arbitrarily close to the identity for all t, and so there is an isomorphism

$$\phi_i : h_*^i g_*^{i-1} T_0 | \overline{U}_i^* \cap W_i \longrightarrow T_1 | \overline{U}_i^* \cap W_i$$

such that

$$\phi_i | \overline{U}_i^* \cap W_i \cap \overline{U}_{i-1}^* = \phi_{i-1} | \overline{U}_i^* \cap W_i \cap \overline{U}_{i-1}^* \quad .$$

Since $\operatorname{supp} H^i$ is in a compact subset of W_i, we may extend H^i to an isotopy of M whose support lies in W_i. Likewise, we may extend ϕ_i to all of U_i^* by letting $\phi_i | U_{i-1}^* = \phi_{i-1} | U_{i-1}^*$. Then H^i and ϕ_i satisfy the conditions of the induction.

Now if it is true that the sequence $G_t^i(x)$ is eventually constant in a neighborhood of any point $x \in M$, we can set

$$H_t(x) = \lim_{i \to \infty} G_t^i(x)$$

and

$$\phi(x) = \lim_{i \to \infty} \phi_i(x)$$

(since the latter is eventually constant in a neighborhood of any point). If we choose N so that the projection $\pi_2 : \overline{N} \to M$ is proper (where π_2 denotes the projection on the second factor), then it is easily seen that the sequence $G_t^i(x)$ is eventually constant in a neighborhood of any point $x \in M$, and that H and ϕ have the required properties.

This completes the reduction to the local case.

Proof in the local case. Let $T_0 = (E_0, \epsilon_0, \varphi_0)$ and $T_1 = (E_1, \epsilon_1, \varphi_1)$. We will first construct an isomorphism $\phi : E_0 \to E_1$ of inner product bundles which extends $\phi_0 | U'$, and then construct the isotopy H to have the required properties.

The tubular neighborhood T_i $(i = 0, 1)$ gives a natural identification α_i of E_i with the normal bundle ν_X of X in M. Explicitly, if $x \in X$, the restriction of α_i to the fiber $E_{i,x}$ is the composition

$$E_{i,x} = T(E_{i,x})_0 \xrightarrow{d\varphi_i} TM_x \xrightarrow{\text{proj.}} TM_x | TX_x = \nu_{X,x} \quad .$$

Let $\beta = \alpha_1^{-1}\alpha_0 : E_0 \to E_1$. We may consider β as a section of $\mathrm{Iso}(E_0, E_1)$, where the latter is the bundle whose fiber over x is the space of isomorphisms of $E_{0,x}$ into $E_{1,x}$. In general, β will not be of class C^μ, only of class $C^{\mu-1}$; however, we may approximate β arbitrarily closely on any compact subset of X by a section β_1 of class C^μ.

To construct ϕ, we will need the following well known lemma in linear algebra.

LEMMA. Let V and W be vector spaces, provided with inner products i and j. Let $L : V \to W$ be a vector space isomorphism. Then there exists a unique positive definite self-adjoint linear mapping $H : W \to W$ such that $H \circ L : V \to V$ preserves inner products.

Remark 1. It is easily seen that this is equivalent to the assertion that any invertible matrix L of real numbers has a unique decomposition $L = H^{-1}U$ where H is a positive definite symmetric matrix and U is an orthogonal matrix.

Remark 2. Similarly, it is easily verified that there exists a unique positive definite self-adjoint linear mapping $H_1 : V \to V$ such that $L \circ H_1 : V \to W$ preserves inner products, and that $H_1 = L^{-1}HL$.

Proof of the lemma. Existence. Let $e_1, \cdots, e_n$ be an orthonormal basis for V, and let $A = (\alpha_{ij})$ be the matrix given by $\alpha_{ij} = (Le_i, Le_j)_j$. Then α_{ij} is symmetric and positive definite. It follows from the spectral theorem for symmetric positive definite matrices that we may choose the basis $e_1, \cdots, e_n$ so that (α_{ij}) is a diagonal matrix: $\alpha_{ij} = \lambda_i \delta_{ij}$ (where δ_{ij} is the Kronecher delta symbol). Let $f_i = L(e_i)/\sqrt{\lambda_i}$. Then $f_1, \cdots, f_n$ is an orthonormal basis of V. Let $H : W \to W$ be given by $H(f_i) = f_i/\sqrt{\lambda_i}$. Then H has the required properties.

Uniqueness. If there were two, H and H', we would have that $U = (HL) \circ (H'L)^{-1}$ is orthogonal. Then $UH'L = HL$ so $UH' = H$. Taking adjoints, we then obtain $H'U^{-1} = H$ so that $H'^2 = H'U^{-1}UH' = H^2$

This implies $H' = H$, since a positive definite self-adjoint mapping has only one positive definite self-adjoint square root. Q. E. D.

Now we return to the proof of the uniqueness of tubular neighborhoods. For each $x \in X$, let η_x be the unique self-adjoint positive definite linear automorphism of $E_{1,x}$ such that $\phi_x = \eta_x \circ \beta_{1,x} : E_{0,x} \to E_{1,x}$ preserves inner products. Clearly, $\phi = \{\phi_x\}$ is a smooth isomorphism of E_0 into E_1, and it preserves inner products. From the fact that η_x is positive definite and self-adjoint it follows that $(1 - t)$ identity $+ t\eta_x$ is an automorphism of $E_{1,x}$ for $0 \subseteq t \subseteq 1$. Hence if β_1 is chosen sufficiently close to β, it follows that

$$(1 - t)\beta + t\phi : E_0 \longrightarrow E_1$$

is an isomorphism for $0 \leq t \leq 1$. Moreover, if we choose β_1 so that $\beta_1 = \beta$ in a neighborhood of U' (which we may do since $\beta | U = \phi_0$ by definition of β), then η = identity in a neighborhood of U_1, so that $\phi | U' = \phi_0 | U'$.

Since we are in the local case, we may suppose without loss of generality that M is open in $\mathbb{R}^m$, P is open in $\mathbb{R}^p$, $X = \mathbb{R}^{m-c} \cap M$, and $f = \pi | M$. It is easily seen that there exists a neighborhood V_1 of V' in V such that for all $m \in V_1$, we have that

$$g_t(m) = \varphi_1 \circ ((1 - t)\beta + t\phi) \circ \varphi_0^{-1}(m)$$

is defined. Since $V' \subseteq X$, we have $g_t | V'$ = inclusion. Since V' is compact there exists an open neighborhood V_2 of V' in V_1 such that $g_s(V_2) \subseteq g_t(V_1)$ for $0 \leq s$, $t \leq 1$. Let ρ be a C^∞ function on M which is identically 1 in a neighborhood of V' and which has compact support $\subseteq V_2$. Let $G_{s,t} : M \to M$ be defined by

$$G_{s,t}(m) = (1 - \rho(m))m + \rho(m) g_t g_s^{-1}(m) \quad , \quad m \in V_2$$

$$G_{s,t}(m) = m \qquad m \in M - V_2 \quad .$$

Then $G_{s,t}$ is a smooth mapping for $0 \leq s$, $t \leq 1$, and it depends smoothly on s and t. Since $G_{t,t}$ = identity and there is a compact set which contains the support of $G_{s,t}$ for all s and t, it follows that there exists $\delta > 0$ such that $G_{s,t}$ is a diffeomorphism for $|s - t| < \delta$. Let n be a positive integer such that $\frac{1}{n} < \delta$ and set

$$H_t = G_{0\frac{t}{n}} \circ G_{\frac{t}{n},\frac{2t}{n}} \circ G_{\frac{n-1}{n}t,t} \quad .$$

Then H_t is an isotopy of M into M, and it follows from the definition of H that $H_1 = g_1$ in sufficiently small neighborhood of V'. Also, it follows from the definitions that g_t and H_t is the identity in a sufficiently small neighborhood of U' for all t. Thus $H_1 = g_1$ in a sufficiently small neighborhood of $U' \cup V'$. Clearly $\operatorname{supp} H \subseteq V_2 \subseteq V$.

Furthermore, $H_1 \circ \varphi_0 = g_1 \circ \varphi_0 = \varphi_1 \circ \phi$ in a sufficiently small neighborhood of $U' \cup V'$. Thus ϕ is an isomorphism of $H_{1*}T_0 | U' \cup V'$ with $T_1 | U' \cup V'$.

It is clear from the construction that H is compatible with f and leaves X point-wise fixed. Finally, by choosing the function ρ used in the construction of G to have support in a very small neighborhood of V', we may arrange for H_t to be as close to the identity (in the compact-open topology) as we like. Q. E. D.

Now we state and prove the existence theorem for tubular neighborhoods.

PROPOSITION 6.2. Suppose $f|X : X \to P$ is a submersion. Let U be an open subset of X and let T_0 be a tubular neighborhood of U in X. Let U' be a subset of U which is closed in X. Then there exists a tubular neighborhood T of X in M such that $T|U' \sim T_0|U'$.

Proof. It is enough to consider the case when X is closed in M. For, in general, there is an open subset M_0 in M such that X is a closed subset of M_0, since X is locally closed in M. Clearly a tubular neighborhood of X in M_0 is a tubular neighborhood of X in M.

The local case of this proposition is trivial.

To prove the proposition in general, we take a locally finite family $\{W_i\}$ of open sets in M having the following properties:

(a) For each i, there is a coordinate chart $\varphi_i : W_i \to \mathbb{R}^n$ such that $\varphi_i(W_i \cap X) = \varphi_i(W_i) \cap \mathbb{R}^{n-c}$ (where $c = \operatorname{cod} X$) and such that there is a coordinate chart $\phi_i : f(W_i) \to \mathbb{R}^p$ such that the following diagram commutes

$$\begin{array}{ccc} W_i & \xrightarrow{\varphi_i} & \mathbb{R}^n \\ \downarrow f & & \downarrow \pi \\ f(W_i) & \xrightarrow{\phi_i} & \mathbb{R}^p \end{array}$$

(b) each $\overline{W_i}$ is compact, and

(c) $\{W_i \cap X\}$ is a cover of X.

Furthermore, we can choose closed sets $W_i' \subseteq W_i$ such that $\{W_i'\}$ is a cover of X. Since M has a countable basis for its topology, the family $\{W_i\}$ is countable. We will suppose that it is indexed by the positive integers. For each positive integer we let $U_i = U \cup W_1 \cup \cdots \cup W_i$ and $U_i' = U' \cup W_1' \cup \cdots \cup W_i'$. We let $U_0 = U$ and $U_0' = U'$.

Now we construct by induction on i an open neighborhood U_i'' of U_i' in X and a tubular neighborhood T_i of U_i''. We take T_0 as given. For the inductive step, we suppose U''_{i-1} and T_{i-1} have been constructed. We let U''_i be any open neighborhood of U'_i in X which is relatively compact in $W_i \cup U''_{i-1}$.

Since $U''_i \subseteq W_i - U'_{i-1}$, there exist open sets A and B in U''_i such that $U''_i = A \cup B$, $\bar{A} \subseteq W - U'_{i-1}$ and $\bar{B} \subseteq U''_{i-1}$. Since the existence theorem for tubular neighborboods is true in the local case, we may choose a tubular neighborhood T'_i of $W_i \cap X$ in W_i. Then we have two tubular neighborhoods of $U''_{i-1} \cap W_i \cap X$ in M, namely the restrictions of T'_i and T_{i-1}. Since $A \cap B$ is relatively compact in $(U''_i - U'_{i-1}) \cap V_i \cap X$, we may find a diffeomorphism h of M onto itself leaving X pointwise fixed such that $h_* T_{i-1} A \cap B \sim T'_i | A \cap B$. Furthermore, we may suppose h is compatible with f and h is the identity outside an arbitrarily small neighborhood of $\overline{A \cap B}$; in particular, that h is the identity in a neighborhood of U'_{i-1}. Since $h_* T_{i-1} | A \cap B \sim T'_i | A \cap B$ there is a tubular neighborhood T_i of $U''_i = A \cup B$ in M such that $T_i | A \sim T'_i | A$ and $T_i | B \sim h_* T_{i-1} | B$. Clearly T_i is compatible with f.

Furthermore, $T_i \sim T_{i-1}$ in a neighborhood of U'_{i-1}. It follows easily that there is a tubular neighborhood T of X in M such that $T \sim T_i$ in a neighborhood of U'_i for all i, and that this tubular neighborhood is compatible with f. Q. E. D.

§7. Control data. Throughout this section, let M be a manifold and $\mathcal{S}$ a Whitney pre-stratification of a subset S of M.

Suppose that for each stratum X of $\mathcal{S}$ we are given a tubular neighborhood T_X of X in M. Let $\pi_X : |T_X| \to X$ denote the projection associated to T_X and $\rho_X : |T_X| \to \mathbb{R}$ the tubular function associated to T_X.

DEFINITION. *The family $\{T_X\}$ of tubular neighborhoods will be called control data for $\mathcal{S}$ if the following commutation relations are satisfied: if X and Y are strata and $X < Y$, then*

$$\pi_X \pi_Y(m) = \pi_X(m)$$

$$\rho_X \pi_Y(m) = \rho_X(m)$$

for all m such that both sides of the equation are defined, i.e., all $m \in |T_X| \cap |T_Y|$ such that $\pi_Y(m) \in |T_X|$.

If f maps M into P, then the family $\{T_X\}$ will be said to be *compatible with* f if for all $X \in \mathcal{S}$ and all $m \in |T_X|$, we have $f\pi_X(m) = f(m)$.

PROPOSITION 7.1. *If $f : M \to P$ is a submersion, then there exists a family $\{T_X\}$ of control data for $\mathcal{S}$ which is compatible with f.*

For the proof of the proposition, we will need Lemma 7.3 below. The proof of Lemma 7.3 depends on Lemma 7.2, which says (roughly speaking) that every tubular neighborhood is locally like a standard example.

DEFINITION. By the standard tubular neighborhood $T_{m,c}$ of $\mathbb{R}^{m-c} \times 0_c$ in $\mathbb{R}^m$, we mean the triple (E, ϵ, φ), where E is the trivial bundle over $\mathbb{R}^{m-c}$ with fiber $\mathbb{R}^c$ (provided with its standard inner product), $\epsilon = 1$, and $\varphi : B_\epsilon \to \mathbb{R}^m$ is the restriction map of the identification mapping $\mathbb{R}^{m-c} \times \mathbb{R}^c \to \mathbb{R}^m$.

More generally if U is open in $\mathbb{R}^{m-c}$, the standard tubular neighborhood of U in $\mathbb{R}^m$ will mean $T_{m,c}|U$.

LEMMA 7.2. If X is a submanifold of M, T_X is a tubular neighborhood of X, and $x \in X$, then there exists a coordinate chart $\varphi : U \to \mathbb{R}^m$, where U is open in M and $x \in U$, such that $\varphi(X \cap U) = \varphi(U) \cap \mathbb{R}^{m-c}$ (where $c = \operatorname{cod} X$) and such that

$$\varphi_*(T_X | X \cap U) \sim T_{m,c} | \varphi(X \cap U) \quad .$$

Proof. Immediate from the definitions.

If $T = (E, \epsilon, \varphi)$ is a tubular neighborhood of X in M and ϵ' is any smooth positive function on X, we let $|T|_{\epsilon'} = \varphi(B_\epsilon \cap \overline{B_{\epsilon'}})$, $|T|^0_{\epsilon'} = \varphi(B_\epsilon \cap B_{\epsilon'})$ and $\partial |T|_{\epsilon'} = \varphi(B_\epsilon \cap S_{\epsilon'})$ where $S_{\epsilon'}$ is the ϵ' sphere bundle in E, i.e., $S_{\epsilon'} = \{v \in E : \|v\| = \epsilon'(\pi(v))\}$ where $\pi : E \to X$ denotes the projection. Clearly $|T|_{\epsilon'}$ is a smooth manifold with boundary $\partial |T|_{\epsilon'}$ and interior $|T|^0_{\epsilon'}$. We will say ϵ' is admissible if $\epsilon' < \epsilon$. In this case the tubular retraction $\pi_T : |T|_{\epsilon'} \to X$ is a proper mapping

LEMMA 7.3. Let X and Y be disjoint submanifolds of M such that the pair (Y, X) satisfies condition b. Let T be a tubular neighborhood of X in M. Then there exists a positive smooth function ϵ' on X such that the mapping

$$(\rho_T, \pi_T) : Y \cap |T|^0_{\epsilon'} \longrightarrow \mathbb{R} \times X$$

is a submersion.

Proof. Let Σ be the set of $y \in |T|$ such that the rank of the mapping

$$(\rho_T, \pi_T) : Y \cap |T| \longrightarrow \mathbb{R} \times X$$

at y is $< \dim(\mathbb{R} \times X)$. The lemma is equivalent to the assertion that for any $x \in X$ there exists a neighborhood N of x in M such that $N \cap \Sigma = \emptyset$. Since this is a purely local statement, it follows from Lemma 7.2 that it is enough to prove the proposition when $M = \mathbb{R}^m$, $X = \mathbb{R}^{m-c} \times 0_c$, and T is the standard tubular neighborhood $T_{m,c}$

of $\mathbb{R}^{m-c}$ in $\mathbb{R}^m$. In this case π_T is the orthogonal projection of $\mathbb{R}^m$ on $\mathbb{R}^{m-c}$, and ρ_T is the function which is given by $\rho(y) = \text{dist.}(y, \mathbb{R}^{m-c})^2$.

Let $y \in |T| - \mathbb{R}^{m-c}$. The kernel of the differential of (π_T, ρ_T) at y is the orthogonal complement of $(\mathbb{R}^{m-c} \times 0_c) \oplus \widehat{y\pi_T(y)}$ in $\mathbb{R}^m$. The hypothesis that condition b is satisfied implies that for y near $\mathbb{R}^{m-c}$, $(\mathbb{R}^{m-c} \times 0_c) \oplus \widehat{y\pi_T(y)}$ is close in the Grassmannian of $m-c+1$ planes in m space to a $m-c+1$ plane which lies in TY_y. Hence for y near enough to $\mathbb{R}^{m-c}$, we have that TY_y is transversal to the kernel of the differential of (π_T, ρ_T) at y, so that $(\pi_T, \rho_T)|Y$ is a submersion at y, i.e., $y \notin \Sigma$. Q.E.D.

<u>Proof of Proposition</u> 7.1. Let $\mathcal{S}_k$ denote the family of strata of $\mathcal{S}$ of dimension $\leq k$, and let S_k denote the union of all strata in $\mathcal{S}_k$. We will show by induction on k that the proposition is true for $\mathcal{S}_k$ and S_k in place of $\mathcal{S}$ and S.

For the inductive step, we suppose that for each stratum X of dimension $< k$, we are given a tubular neighborhood T_X of X, and this family of tubular neighborhoods satisfies the commutation relations.

By shrinking the T_X if necessary, we may suppose that if X and Y are strata of dimension $< k$ which are not comparable (i.e., neither $Y < X$ nor $X < Y$ holds), then $|T_X| \cap |T_Y| = \emptyset$. To construct the T_X on the strata of dimension k, we may do it one stratum at a time, since there are no commutation relations to be satisfied among the strata of the same dimension. Let X be a stratum of dimension k.

We construct the tubular neighborhoods T_X in two steps, as follows. For each $\ell \leq k$, we let U_ℓ denote the union of all $|T_Y|$ for $Y < X$ and $\dim Y \geq \ell$. We let $X_\ell = U_\ell \cap X$. In the first step, we construct a tubular neighborhood T_ℓ of X_ℓ by decreasing induction on ℓ. In the inductive step, we will shrink various $|T_Y|$, but this is permitted, since we do it only a finite number of times. Then in the second step, we extend T_0 to a tubular neighborhood T_X of X.

<u>First step</u>. For $\ell = k$, we have $X_k = \emptyset$, so there is nothing to construct. For the inductive step, we suppose that $T_{\ell+1}$ has been constructed and that the following special cases of the commutation relations are satisfied: if $Y < X$, $\dim Y \geq \ell+1$, $m \in |T_{\ell+1}| \cap |T_Y|$ and $\pi_{\ell+1}(m) \in |T_Y|$, where $\pi_{\ell+1} = \pi_{T_{\ell+1}}$, then

$$(*_{\ell+1}) \qquad \begin{aligned} \rho_Y \pi_{\ell+1}(m) &= \rho_Y(m) \\ \pi_Y \pi_{\ell+1}(m) &= \pi_Y(m) \end{aligned}$$

By replacing $T_{\ell+1}$ with a smaller tubular neighborhood if necessary, we may suppose that for $m \in |T_{\ell+1}|$ there is $Z < X$ with $\dim Z > \ell$ such that $m \in |T_Z|$ and $\pi_{\ell+1}(m) \in |T_Z|$.

To construct T_ℓ it is enough to construct T_ℓ on $|T_Y| \cap X$ for each stratum $Y < X$ of dimension ℓ seperately, since if Y and Y' are two strata of dimension ℓ, we have $|T_Y| \cap |T_{Y'}| = \emptyset$, since Y and Y' are not comparable.

Thus, we wish to construct a tubular neighborhood $T_{X,Y}$ of $|T_Y| \cap X$ whose restriction to $|T_Y| \cap X_{\ell+1}$ is isomorphic to the restriction of $T_{\ell+1}$, such that the following commutation relation is satisfied: if $m \in |T_{X,Y}| \cap |T_Y|$ and $\pi_{X,Y}(m) \in |T_Y|$, where $\pi_{X,Y} = \pi_{T_{X,Y}}$, then

$$\rho_Y \pi_{X,Y}(m) = \rho_Y(m)$$

$$\pi_Y \pi_{X,Y}(m) = \pi_Y(m) \quad .$$

By shrinking $|T_Y|$ if necessary, we may arrange that if $m \in |T_{\ell+1}| \cap |T_Y|$ and $\pi_{\ell+1}(m) \in |T_Y|$, then this commutation relation is already satisfied (with $\pi_{\ell+1}$ in place of $\pi_{X,Y}$) for the following reason. Since $m \in |T_{\ell+1}|$, there exists $Z < X$ with $\dim Z > \ell$, $m \in |T_Z|$ and $\pi_{\ell+1}(m) \in |T_Z|$. Since $\pi_{\ell+1}(m) \in |T_Y| \cap |T_Z|$, the last named space is not empty; hence Y and Z are comparable, and by dimension restrictions $Y < Z$. Therefore

$$\rho_Y \pi_{\ell+1}(m) = \rho_Y \pi_Z \pi_{\ell+1}(m) = \rho_Y \pi_Z(m) = \rho_Y(m)$$

$$\pi_Y \pi_{\ell+1}(m) = \pi_Y \pi_Z \pi_{\ell+1}(m) = \pi_Y \pi_Z(m) = \pi_Y(m) \quad .$$

(We may have to shrink $|T_Y|$ to guarantee that these equalities hold for all $m \in |T_{\ell+1}| \cap |T_Y|$.)

Furthermore, by shrinking T_Y further if necessary, we may suppose that

$$(\rho_Y, \pi_Y) : |T_Y| \cap X \longrightarrow \mathbb{R} \times Y$$

is a submersion. The commutation relation that we must verify is precisely the condition that $T_{X,Y}$ be compatible with the mapping $(\rho_Y, \pi_Y) : |T_Y| \cap X_{\ell+1} \to \mathbb{R} \times Y$. Therefore from the generalized tubular neighborhood theorem, we get that if $X^0_{\ell+1}$ is an open subset of X whose closure lies in $X_{\ell+1}$, then there exists $T_{X,Y}$ which satisfies the commutation relations and whose restriction to $|T_Y| \cap X^0_{\ell+1}$ is isomorphic to the restriction of $T_{\ell+1}$. Now we replace T_Z for $Z < X$ by smaller tubular neighborhoods T'_Z such that $X'_{\ell+1} \subseteq X^0_{\ell+1}$, where $X'_{\ell+1}$ is defined analogously to $X_{\ell+1}$, but with T'_Z in place of T_Z. Then $T_{X,Y}$ has the required properties.

This completes the first step: we conclude that there exists a tubular neighborhood T_0 of X_0 satisfying $(*_0)$ for any $Y < X$.

Second step. From $(*_0)$, it follows that we may assume that T_0 is compatible with f. For, by replacing T_0 with a smaller tubular neighborhood if necessary, we may assume that if $m \in |T_0|$, then for some $Y < X$, we have $m \in |T_Y|$ and $\pi_0(m) \in |T_Y|$. Then

$$f\pi_0(m) = f\pi_Y\pi_0(m) = f\pi_Y(m) = f(m) \quad .$$

Since T_0 is compatible with f, we may extend a suitable restriction of T_0 to a tubular neighborhood T of X which is compatible with f, by the generalised tubular neighborhood theorem. Then, by replacing the T_Y with possibly smaller tubular neighborhoods (as in Step 1), we get that the compatibility conditions are satisfied.

This completes the construction of T_X, and therefore also completes the proof of the proposition.

§8. Abstract pre-stratified sets. If V is a closed subset of a manifold M which admits a Whitney pre-stratification (in the sense defined in Section 5) then we can find control data for this pre-stratification by the previous section. This provides V with considerable structure. The purpose of this section is to axiomatise the sort of structure which occurs. We depart only slightly from Thom's notion of abstract stratified set ([3] and [4]).

DEFINITION 1. An abstract pre-stratified set is a triple $(V, \mathcal{S}, \mathcal{J})$ satisfying the following axioms, A1 - A9.

(A1) V is a Hausdorff, locally compact topological space with a countable basis for its topology.

This axiom implies that V is metrizable. For, since V is locally compact, it is regular, so the metrizability of V follows from Urysohn metrization theorem (Kelly [1]). Since V is metrizable, every subset X of V is normal (in the sense that any two disjoint closed subsets of X can be separated by open sets). We will often use this fact without explicit mention.

(A2) $\mathcal{S}$ is a family of locally closed subsets of V, such that V is the disjoint union of the members of $\mathcal{S}$.

The members of $\mathcal{S}$ will be called the strata of V.

(A3) Each stratum of V is a topological manifold (in the induced topology), provided with a smoothness structure (of class C^μ).

(A4) The family $\mathcal{S}$ is locally finite.

(A5) The family $\mathcal{S}$ satisfies the axiom of the frontier: if $X, Y \in \mathcal{S}$ and $Y \cap \overline{X} \neq \emptyset$, then $Y \subseteq \overline{X}$.

If $Y \subseteq \overline{X}$ and $Y \neq X$, we write $Y < X$. This relation is obviously transitive: $Z < Y$ and $Y < X$ imply $Z < X$.

(A6) $\mathcal{J}$ is a triple $\{\{T_X\}, \{\pi_X\}, \{\rho_X\}\}$, where for each $X \in \mathcal{S}$, T_X is an open neighborhood of X in V, π_X is a continuous retraction of T_X onto X, and $\rho_X : X \to [0, \infty)$ is a continuous function.

We will call T_X the tubular neighborhood of X (with respect to the given structure of a pre-stratified set on V), π_X the local retraction of T_X onto X and ρ_X the tubular function of X.

(A7) $X = \{v \in T_X : \rho_X(v) = 0\}$.

If X and Y are any strata, we let $T_{X,Y} = T_X \cap Y$, $\pi_{X,Y} = \pi_X | T_{X,Y}$, and $\rho_{X,Y} = \rho_X | T_{X,Y}$. Then $\pi_{X,Y}$ is a mapping of $T_{X,Y}$ into X and $\rho_{X,Y}$ is a mapping of $T_{X,Y}$ into $(0, \infty)$. Of course, $T_{X,Y}$ may be empty, in which case these are the empty mappings.

(A8) For any strata X and Y the mapping

$$(\pi_{X,Y}, \rho_{X,Y}) : T_{X,Y} \longrightarrow X \times (0, \infty)$$

is a smooth submersion. This implies $\dim X < \dim Y$ when $T_{X,Y} \neq \emptyset$.

(A9) For any strata X, Y, and Z, we have

$$\pi_{X,Y}\pi_{Y,Z}(v) = \pi_{X,Z}(v)$$

$$\rho_{X,Y}\pi_{Y,Z}(v) = \rho_{X,Z}(v)$$

whenever both sides of this equation are defined, i.e., whenever $v \in T_{Y,Z}$ and $\pi_{Y,Z}(v) \in T_{X,Y}$.

DEFINITION 2. We say that two stratified sets $\{V, \mathcal{S}, \mathcal{J}\}$ and $\{V', \mathcal{S}', \mathcal{J}'\}$ are equivalent if the following conditions hold.

(a). $V = V'$, $\mathcal{S} = \mathcal{S}'$, and for each stratum X of $\mathcal{S} = \mathcal{S}'$, the two smoothness structures on X given by the two stratifications are the same.

(b). If $\mathcal{J} = \{\{T_X\}, \{\pi_X\}, \{\rho_X\}\}$ and $\mathcal{J}' = \{\{T'_X\}, \{\pi'_X\}, \{\rho'_X\}\}$, then for each stratum X, there exists a neighborhood T''_X of X in

$T_X \cap T'_X$ such that $\rho_X | T''_X = \rho'_X | T''_X$ and $\pi_X | T''_X = \pi'_X | T''_X$.

From the normality of arbitrary subsets of a stratified set, it follows that any (abstract) pre-stratified set is equivalent to one which satisfies the following conditions

(A10) If X, Y are strata and $T_{X,Y} \neq \emptyset$, then $X < Y$.

(A11) If X, Y are strata and $T_X \cap T_Y \neq \emptyset$, then X and Y are comparable, i.e., one of the following holds: $X < Y$, $Y < X$, or $X = Y$.

From (A10), it follows that $X < Y$ if and only if $T_{X,Y} \neq \emptyset$, and from (A11) that X and Y are comparable if and only if $T_X \cap T_Y \neq \emptyset$.

Note that from (A8) it follows that the relation $X < Y$ defines a partial order on $\mathcal{S}$. It is enough to verify $X < Y$ and $Y < X$ do not hold simultaneously. But (A8) implies $X < Y \Rightarrow \dim X < \dim Y$.

As an example of an (abstract) pre-stratified set, let V be a subset of a manifold M and suppose V admits a Whitney pre-stratification $\mathcal{S}$, and let $\{T'_X\}$ be a family of control data for $\mathcal{S}$, and let $\pi_X : T'_X \to X$ and $\rho_X : T'_X \to (0, \infty)$. Set $\mathcal{T} = \{T_X\}$. Then $(V, \mathcal{S}, \mathcal{T})$ is an abstract pre-stratified set. In this way, we associate with any system of control data for a Whitney pre-stratified set V, a structure of an abstract pre-stratified set on V.

Hence it follows from Proposition 7.1 that any Whitney pre-stratified set admits the structure of an abstract pre-stratified set.

If $(V, \mathcal{S}, \mathcal{T})$ is a pre-stratified set, V' is any topological space, and $\varphi : V' \to V$ is a homeomorphism, then the structure of a stratified set on V "pulls back" in an obvious way to give a structure $(V', \varphi^*\mathcal{S}, \varphi^*\mathcal{T})$ of a stratified set on V'.

If $(V', \mathcal{S}', \mathcal{T}')$ and $(V, \mathcal{S}, \mathcal{T})$ are abstract pre-stratified sets, then a homeomorphism $\varphi : V' \to V$ is said to be an <u>isomorphism</u> of stratified sets if $(V', \mathcal{S}', \mathcal{T}')$ is equivalent to $(V', \varphi^*\mathcal{S}', \varphi^*\mathcal{T}')$.

The uniqueness result that we will prove below implies the following: if $(V, \mathcal{S}, \mathcal{T})$ is a Whitney pre-stratified set, and $\mathcal{T}$ and $\mathcal{T}'$ are two systems of control data, then the abstract pre-stratified sets $(V, \mathcal{S}, \mathcal{T})$ and $(V, \mathcal{S}, \mathcal{T})$ are isomorphic.

§9. Controlled vector fields. Throughout this section, we let $(V, \mathcal{S}, \mathcal{T})$ be an (abstract) pre-stratified set. We suppose $\mu \geq 2$.

DEFINITION. By a stratified vector field η on V, we mean a collection $\{\eta_X : X \in \mathcal{S}\}$, where for each stratum X, we have that η_X is a smooth vector field on X.

By smooth vector field we mean a vector field of class $C^{\mu-1}$.

Let $\mathcal{T} = \{\{T_X\}, \{\pi_X\}, \{\rho_X\}\}$, and for two strata X and Y, let $T_{X,Y}$, $\pi_{X,Y}$, and $\rho_{X,Y}$ be defined as in the previous section.

DEFINITION. A stratified vector field η on V will be said to be controlled (by $\mathcal{T}$) if the following control conditions are satisfied: for any stratum Y there exists a neighborhood T'_Y of Y in T_Y such that for any second stratum $X > Y$ and any $v \in T'_Y \cap X$, we have

(9.1-a) $$\eta_X \rho_{Y,X}(v) = 0$$

(9.1-b) $$(\pi_{Y,X})_* \eta_X(v) = \eta_Y(\pi_{Y,X}(v)) \quad .$$

DEFINITION. If P is a smooth manifold and $f : V \to P$ is a continuous mapping, we will say that f is a controlled submersion if the following conditions are satisfied.

(1) $f|X : X \to P$ is a smooth submersion, for each stratum X of V.

(2). For any stratum X, there is a neighborhood T'_X of X in T_X such that $f(v) = f\pi_X(v)$ for all $v \in T'_X$.

Note that both the notions that we have just introduced depend only on the equivalence class of the pre-stratified set $(V, \mathcal{S}, \mathcal{T})$, i.e., if $(V, \mathcal{S}, \mathcal{T}')$ is a pre-stratified set which is equivalent to $(V, \mathcal{S}, \mathcal{T})$, then a controlled vector field (or controlled submersion) with respect to one of these pre-stratified sets is the same as a controlled vector field (or controlled submersion) with respect to the other.

PROPOSITION 9.1. If $f : V \to P$ is a controlled submersion, then for any smooth vector field ζ on P, there is a controlled vector field η on V such that $f_* \eta(v) = \zeta(f(v))$ for all $v \in V$.

Proof. By induction on the dimension of V (where the dimension of V is defined to be the supremum of the dimensions of the strata of V). By the k skeleton V_k of V, we will mean the union of all strata of V of dimension $\leq k$. Clearly V_k has the structure of a stratified set, where the strata of V_k are the strata of V which lie in V_k, the tubular neighborhoods are the intersections with V_k of the tubular neighborhoods (in V) of strata in V_k and the local retractions and tubular function on V_k are the restrictions of the local retractions and tubular functions on V.

In the case $\dim V = 0$, the statement of the proposition is trivial. Hence, by induction, it is enough to show that if the proposition is true whenever $\dim V \leq k$ then it is true when $\dim V = k+1$. Thus, we may (and do) assume that $\dim V = k+1$ and that there is a controlled vector field η_k on V_k such that $f_* \eta_k(v) = \zeta(f(v))$ for all $v \in V_k$. We will show that there exists a controlled vector field η on V which extends η_k such that $f_* \eta(v) = \zeta(f(v))$ for all $v \in V$.

To construct η, it is enough to construct η_X separately for each stratum X of V such that $\dim X = k+1$, because the condition that a vector field be controlled involves only strata Y, X such that $Y < X$.

Since by the induction assumption η_k is controlled, we can choose neighborhoods T^1_Y of Y in T_Y (one for each stratum $Y \subsetneq V_k$) such that if $Y < Z$ are strata, then the control conditions (9.1) are satisfied (with Z in place of X) for $v \in T^1_Y \cap Z$. By the assumption that f is controlled, we may choose the neighborhoods T^1_Y such that $f(v) = f\pi_Y(v)$ for all $v \in T^1_Y$.

It is easily seen that we may choose neighborhoods T^2_Y of Y in T^1_Y (one for each stratum $Y \subseteq V_k$) such that the following holds: if $Y < Z$ are strata in V_k then

$$\pi_Z(T^2_Y \cap T^2_Z) \subseteq T^1_Y \quad .$$

We can furthermore choose the T^2_Y so that T^2_Y is closed in $V - \partial Y$ (where ∂Y denotes the frontier of Y), since $V - \partial Y$ is metrisable and therefore normal, and Y is closed in $V - \partial Y$. Finally, we can choose the T^2_Y so that if Y is not comparable to Z, then $T^2_Y \cap T^2_Z = \emptyset$.

Now consider the following conditions on a vector field η_X on X:

(9.2-a_Y). The control condition (9.1) is satisfied for any $v \in T^2_Y \cap X$.

(9.2 - b). $f_* \eta_X(v) = \zeta(f(v))$ for all $v \in X$.

We claim that there is a vector field η_X on X satisfying (9.2-b) and (9.2-a_Y) for all strata $Y < X$. To prove this claim will clearly be enough to prove the proposition.

Consider a point $v \in X$. The set S_v of strata $Y < X$ such that $v \in T^2_Y$ is totally ordered by inclusion, since if Y and Z are not comparable then $T^2_Y \cap T^2_Z = \emptyset$. If S_v is not empty, then there is a largest member $Y = Y_v$.

Suppose for the moment this is the case and (9.2-a_Y) holds at v. Then (9.2-a_Z) holds for all $Z \in S_v$. For either $Z = Y$ or $Z < Y$. In the latter case $\pi_Y(v) \in T^1_Z$ (by the choice of the T^2_Y's). Then

$$\eta_X \rho_{Z,X}(v) = \eta_X \rho_{Z,Y}\pi_{Y,X}(v) = 0$$

and

$$\begin{aligned}(\pi_{Z,X})_*\eta_X(v) &= (\pi_{Z,Y})_*(\pi_{Y,X})_*\eta_X(v)\\ &= (\pi_{Z,Y})_*\eta_Y(\pi_{Y,X}(v))\\ &= \eta_Z(\pi_{Z,Y}\pi_{Y,X}(v))\\ &= \eta_Z(\pi_{Z,X}(v)) \quad .\end{aligned}$$

Thus (9.2-a_Z) holds at v for all $Z \in \mathcal{S}_v$. Furthermore

$$\begin{aligned}f_*\eta_X(v) &= (f \circ \pi_{Y,X})_*\eta_X(v)\\ &= f_*\eta_Y(\pi_{Y,X}(v))\\ &= \zeta(f(v)) \quad .\end{aligned}$$

Thus (9.2-b) holds at v.

This shows that to construct η_X satisfying (9.2-b) and (9.2-a_Y) for all $Y < X$, it is enough to construct η_X satisfying (9.2-a_{Y_v}) at v for all $v \in X$ for which $\mathcal{S}_v$ is non-empty, and satisfying (9.2-b) at v for all $v \in X$ for which $\mathcal{S}_v$ is empty. Clearly, we can construct a vector field η_X in a neighborhood of each point v in X satisfying the appropriate condition (9.2-a_{Y_v}) or (9.2-b). Since the set of vectors satisfying the appropriate condition in TX_v is convex, we may construct η_X globally by means of a partition of unity. Q. E. D.

§10. One parameter groups. Let V be a topological space. By a one-parameter group of homeomorphisms of V, we mean a continuous mapping $\alpha : \mathbb{R} \times V \to V$ such that $\alpha_{t+s}(v) = \alpha_t\alpha_s(v)$ for all $t, s \in \mathbb{R}$ and all $v \in V$. Now suppose V is a stratified set $(V, \mathcal{S}, \mathfrak{J})$ and α preserves each stratum. If η is a stratified vector field on V, we say that η generates α if the following condition is satisfied. For any $v \in V$, the mapping $t \to \alpha_t(v)$ of $\mathbb{R}$ into V is C^1 (as a mapping into the stratum which contains v) and

$$\frac{d}{dt}(\alpha_t(v))\Big|_{t=0} = \eta(v) \quad .$$

Note that this implies

$$\frac{d}{dt}(\alpha_t(v)) = \eta(\alpha_t(v)) \quad , \qquad t \in \mathbb{R} \quad .$$

It is well known that any C^1 vector field on a compact manifold without boundary generates a unique one-parameter group (see, e.g., [2, p. 66]). It is also known that to extend this result to non-compact manifolds, we must generalize the notion of one parameter group.

DEFINITION. Let V be a locally compact space. A local one-parameter group (on V) is a pair (J, α), where J is an open subset of $\mathbb{R} \times V$ and $\alpha : J \to V$ is a continuous mapping such that the following hold.

(a). $0 \times V \subseteq J$.

(b). If $v \in V$, then the set $J_v = J \cap (\mathbb{R} \times v) \subseteq \mathbb{R}$ is an open interval (a_v, b_v) .

(c). If $v \in V$, and t, s and $t+s$ are in (a_v, b_v) then $\alpha(t+s, v) = \alpha(t, \alpha(s, v))$.

(d). For any $v \in V$ and any compact set $K \subseteq V$, there exists $\epsilon > 0$ such that $\alpha(v, t) \notin K$ if $t \in (a_v, a_v + \epsilon) \cup (b_v - \epsilon, b_v)$.

From now on in this section, we suppose $(V, \mathcal{S}, \mathfrak{J})$ is an (abstract) pre-stratified set, and η is a stratified vector field on V .

DEFINITION. If (J, α) is a local one-parameter group (on V) , we say η generates α if the following conditions a - c are satisfied.

(a). Each stratum X of V is invariant under α , i.e., $\alpha[J \cap (\mathbb{R} \times X)] \subseteq X$.

(b). For each $v \in V$, the mapping $t \to \alpha(t, v)$ of (a_v, b_v) into the stratum which contains v is C^1 .

(c). For any $v \in V$, we have

$$\frac{d}{dt}\alpha(t, v)\Big|_{t=0} = \eta(v) \quad .$$

Since α is a one-parameter group, condition c is equivalent to:

(c′). For any $(t, v) \in J$, we have

$$\frac{d}{dt}\alpha(t, v) = \eta(\alpha(t, v)) \quad .$$

This generalises the ordinary notion of what it means for a vector field to generate a local one-parameter group.

Since $(V, \mathcal{S}, \mathfrak{J})$ is a pre-stratified set, it makes sense to talk of a controlled vector field on V (Section 5).

PROPOSITION 10.1. If η is a controlled vector field on V then η generates a unique local one-parameter group (J, α) .

Proof. For each stratum X , the restriction η_X of η to X is a smooth vector field on X (by the definition of stratified vector field); hence η_X generates a smooth local one-parameter group (J_X, α_X) of diffeomorphisms of X , by a standard result in differential geometry [2, IV, §2] . Let (J, α) be defined by

$$J = \bigcup_{X \in \mathcal{S}} J_X \qquad \alpha = \bigcup_{X \in \mathcal{S}} \alpha_X \quad .$$

We assert that (J, α) is a local one-parameter group generated by η .

It is clear that a, b, and c in the definition of local one-parameter group hold, and that if α is a local one-parameter group, then it is generated by v. Uniqueness is obvious. All that remains to be verified is that J is open, α is continuous, and d holds.

We begin by showing that d holds. If not, there exists $v \in V$ and a compact set K in V such that $\alpha(t, v) \in K$ for values of t arbitrarily close to a_v or b_v. We may suppose that $\alpha(t, v) \in K$ for values of t arbitrarily close to b_v; the other case is treated similarly. Then there exists a sequence $\{t_i\}$, converging to b_v from below, such that $y = \lim \alpha_v(t_i)$ exists and lies in K. Let X (resp. Y) denote the stratum of V which contains v (resp. y).

If $X = Y$, we get a contradiction to the fact that α_X is a one-parameter group. Otherwise $Y < X$. For large i, $\rho_{Y,X}(\alpha_v(t_i))$ and $\pi_{Y,X}(\alpha_v(t_i))$ are defined, and the control conditions are satisfied for $m_i = \alpha_v(t_i)$.

Thus, by taking i sufficiently large, we may suppose that there exists $\epsilon > t - t_i$ such that $[0, \epsilon] \subseteq J_{y_i}$, where $y_i = \pi_{Y,X}(m_i)$, and if T_Y is the tubular neighborhood of Y, π_Y is the local retraction of T_Y onto Y and ρ_Y is the tubular function of Y, then $\rho_{Y,X}(m_i) < \epsilon_Y$ on $\alpha_{y_i}([0, \epsilon])$ and the control conditions for the pair Y, X are satisfied for $m \in \{\rho_{Y,X} = \rho_{Y,X}(m_i)\} \cap \pi^{-1}_{Y,X}(\alpha_{y_i}([0, \epsilon]) \cap X$. Since

$\{\rho_{Y,X} = \rho_{Y,X}(m_i)\} \cap \pi^{-1}_{Y,X}(\alpha_{y_i}[0, \epsilon])$ is compact (because $\rho_{Y,X}(m_i) < \epsilon_Y$ on $\alpha_{y_i}([0, \epsilon]))$, and α_v stays in X (by definition), it follows from the control conditions that

$$\alpha_v(t_i + s) \in \{\rho_{Y,X} = \rho_{Y,X}(m_i)\} \cap \pi^{-1}_{Y,X}(\alpha_{y_i}(s)) \cap X \quad \text{for} \quad 0 \le s \le \epsilon .$$

But this contradicts the hypothesis that $\alpha_v(t_j) \to y$ as $j \to \infty$. This contradiction proves d.

Now let $(t, v) \in J$. We will show that J is a neighborhood of (t, v) and α is continuous at (t, v). We will suppose $t \ge 0$; the other case is treated similarly. As before, let X be the stratum which contains v. Since α_X is a local one-parameter group, there is a compact neighborhood U of v in X and an $\epsilon > 0$ such that $[-\epsilon, t + \epsilon] \times U \subseteq J$. Let T_X denote the tubular neighborhood of X, π_X the local retraction of T_X on X, and ρ_X the tubular function of X. Since $\alpha_X([-\epsilon, t + \epsilon] \times U)$ is compact, we may choose an $\epsilon_1 > 0$ such that the following hold:

(a). Let $\Sigma = \{y \in T_X : \rho_X(y) \le \epsilon_1$ and $\pi_X(y) \in \alpha_X([-\epsilon, t + \epsilon] \times U)\}$. Then Σ is compact.

(b). If $y \in \Sigma$, then the control conditions for the pair X, Y hold at y, where Y is the stratum which contains y.

Clearly, the set Γ_0 of $y \in T_X$ such that $\rho_X(y) \le \epsilon_1$ and $\pi_X(y) \in U$ is a neighborhood of v in V. If $y \in \Sigma_0$, it follows from the control conditions that

$$\rho_X(\alpha_y(s)) = \rho_X(y)$$

$$\pi_X(\alpha_y(s)) = \alpha_{\pi_X(y)}(s)$$

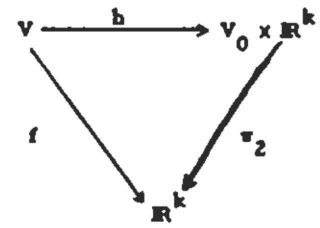
for all $s \in J_y$ such that $\alpha_y(s') \in \Sigma$ for $0 \le s' < s$. From these facts and d, it follows that $[-\epsilon, t + \epsilon] \times \Sigma_0 \subseteq J$; thus J contains a neighborhood of (t, v).

The argument that we have just given shows that if $(t', y) \in [t - \epsilon, t + \epsilon] \times \Sigma_0$, then $y' = \alpha(t', y) \in T_X$, $\rho_X(y') \le \epsilon_1$, and $\pi_X(y') = \alpha(t', \pi_X(y))$. Hence for an arbitrarily small neighborhood of $\alpha(t, x)$ we may choose $\epsilon > 0$ and a neighborhood Σ_1. Hence α is continuous at (t, v). Q. E. D.

COROLLARY 10.2. *Let* P *be a manifold, and* $f : V \to P$ *be a proper, controlled submersion. Then* f *is a locally trivial fibration.*

Proof. It is enough to consider the case when $P = \mathbb{R}^k$ and show in this case that there is a homeomorphism $h : V \to V_0 \times \mathbb{R}^k$, where V_0 denotes the fiber of V over 0, such that the following diagram commutes:

Diagram 10.1



where π_2 denotes the projection on the second factor.

Consider the coordinate vector fields $\partial_1, \cdots, \partial_k$ on $\mathbb{R}^k$. By Proposition 10.1, for each i, $1 \le i \le k$, there is a controlled vector field $\tilde{\partial}_i$ on V such that

$$f_* \tilde{\partial}_i(v) = \partial_i(f(v)) \quad , \qquad v \in V \quad .$$

By Proposition 10.1, each $\tilde{\partial}_i$ generates a local one-parameter group (J_i, α_i). Clearly $f(\alpha_i(t, v)) = f(v) + (0, \cdots, 0, t, 0, \cdots, 0)$, where the non-vanishing entry is in the ith place. Then from the assumption that f is proper and condition d in the definition of one parameter group, it follows that $J_i = \mathbb{R} \times V$. Let h be given by

$$h(v) = (\alpha_1(-t_1, \alpha_2(-t_2, \cdots, \alpha_k(-t_k, v) \cdots)), f(v))$$

where we set $f(v) = (t_1, \cdots, t_k)$. It is easily seen that h maps V into $V_0 \times \mathbb{R}^k$ and that Diagram 10.1 commutes. Let $\bar{h} : V_0 \times \mathbb{R}^k \to V$ be defined by

$$\bar{h}(v,(t_1,\cdots,t_k)) = \alpha_k(t_k,\cdots,\alpha_2(t_2,\alpha_1(t_1,v)))\quad .$$

From the fact that the α_i's are one-parameter groups, it follows that $h\bar{h} = \bar{h}h$ = identity. Hence h is a homeomorphism, as required. Q. E. D.

Note that V_0 has a natural structure of a pre-stratified set $(V_0, \mathcal{S}_0, \mathcal{J}_0)$, where $\mathcal{S}_0$ and $\mathcal{J}_0$ are defined as follows. $\mathcal{S}_0$ is the collection $\{X \cap V_0 : X \in \mathcal{S}\}$. If $X \in \mathcal{S}$ and $X_0 = X \cap V_0$ is the corresponding member of $\mathcal{S}_0$, then we let $T_{X_0} = T_X \cap V_0$, $\pi_{X_0} = \pi_X | T_{X_0}$ and $\rho_{X_0} = \rho_X | T_{X_0}$. Note that π_{X_0} maps T_{X_0} into X_0 because f is a controlled submersion. We let $\mathcal{J}_0$ be the triple $\{\{T_{X_0}\}, \{\pi_{X_0}\}, \{\rho_{X_0}\}\}$.

Furthermore $V_0 \times \mathbb{R}^k$ has a structure of a pre-stratified set (defined in an obvious way).

COROLLARY 10.3. If h is constructed as in the proof of Corollary 10.2, then h is an isomorphism of pre-stratified sets.

Proof. Immediate from the construction of h. (See the end of Section 8 for the definition of isomorphism.)

COROLLARY 10.4. Let M be a manifold, let X be a closed subset of M and let $\mathcal{S}$ be a Whitney pre-stratification of S. Let X and Y be strata with $X < Y$. Let W be a submanifold of M which meets X transversally. Then $X \cap W \subseteq \overline{Y \cap W}$.

Proof. Let $x \in X \cap W$. To show $x \in \overline{Y \cap W}$, it is enough to consider what happens in a neighborhood of x. By replacing M with a sufficiently small open neighborhood of x, we may suppose that X is connected and closed, and there exists a tubular neighborhood T_X of X in M such that $W \cap T_X = \pi_X^{-1}(W \cap X)$, where $\pi_X : T_X \to X$ is the projection associated to T_X. From Lemma 7.3, it follows that by choosing T_X sufficiently small, we may suppose that there exists $\epsilon > 0$ such that $\rho_X < \epsilon$ on T_X, where ρ_X is the tubular function associated to T_X, where $(\rho_X, \pi_X) : T_X \to [0,\epsilon) \times X$ is proper, and where for each stratum Z of $\mathcal{S}$, the mapping

$$(\rho_X, \pi_X)|Z : Z \cap T_X \longrightarrow (0,\epsilon) \times X$$

is a submersion.

Let $\mathcal{S}' = \{Z \cap (T_X - X) : Z \in \mathcal{S}\}$. Then $\mathcal{S}'$ is a Whitney pre-stratification of $S \cap (T_X - X)$. By Proposition 10.1, there is a family of control data $\mathcal{T}$ for $\mathcal{S}$ which is compatible with (ρ_X, π_X). Then $(S \cap (T_X - X), \mathcal{S}', \mathcal{J}')$ is an abstract pre-stratified set and (ρ_X, π_X) is a controlled submersion. Hence by Corollary 10.2, $S \cap (T_X - X)$ is a locally trivial bundle over $(0,\epsilon) \times X$, and by Corollary 10.3, the local trivializations respect the stratification.

It follows that any stratum of $\mathcal{S}'$ (e.g., $Y \cap (T_X - X)$) intersects each fiber of (ρ_X, π_X). In particular $\emptyset \neq Y \cap (\rho_X, \pi_X)^{-1}(\epsilon', X) \subseteq Y \cap W$ for $0 < \epsilon' < \epsilon$. It follows that $x \in \overline{Y \cap W}$. Q.E.D.

The next corollary says that a pre-stratification which satisfies all the conditions of a Whitney pre-stratification except the condition of the frontier also satisfies the condition of the frontier, provided that its strata are connected.

COROLLARY 10.5. Let M be a manifold and $\mathcal{S}$ be a locally finite pre-stratification of a closed subset V of M whose strata are connected such that any pair of strata satisfy condition b. Then $\mathcal{S}$ is a Whitney pre-stratification.

Proof. It suffices to show that the condition of the frontier holds. Suppose X and Y are strata and $Y \cap \overline{X} \neq \emptyset$. The proof of Corollary 10.4 shows that $Y \cap \overline{X}$ is open in Y. Since $Y \cap \overline{X}$ is clearly closed in Y, and Y is connected, this proves $Y \subseteq \overline{X}$.

The proof of Corollary 10.4 also shows:

COROLLARY 10.6. Let M be a manifold, $\mathcal{S}$ a Whitney pre-stratification of M, X a stratum of M, and T_X a tubular neighborhood of X in M such that for any stratum Z of $\mathcal{S}$, the

mapping $(\rho_X, \pi_X) : (|T_X| - X) \cap Z \longrightarrow X_\epsilon$ is a submersion, where $T_X = (E, \varphi, \epsilon)$ and $X_\epsilon = \{t, x \in \mathbb{R} \times X : 0 < t < \epsilon(x)\}$. Then the bundle $(|T_X| - X, (\rho_X, \pi_X), X_\epsilon)$ is locally trivial and the local trivializations can be chosen to respect the stratification.

§11. The isotopy lemmas of Thom. In this section, we will state Thom's first and second isotopy lemmas. We will prove the first and sketch a proof of the second.

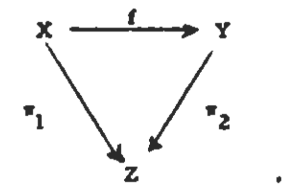
Throughout this section, we let M and P be smooth manifolds, $f : M \to P$ a smooth mapping, and S a closed subset of M which admits a Whitney pre-stratification.

Proposition 11.1. Thom's first isotopy lemma. Suppose $f|S : S \to P$ is proper and $f|X : X \to P$ is a submersion for each stratum X of . Then the bundle (S, f, P) is locally trivial.

Proof: By Proposition 7.1, we can find a system of control data for S which is compatible with f. This provides S with a structure of an abstract stratified set in such a way that f is a controlled submersion. Then the conclusion of the theorem is an immediate consequence of Corollary 10.2. Q. E. D.

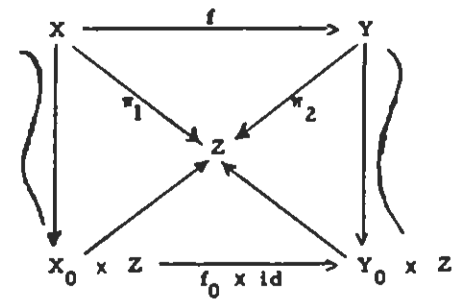
Remark: Thom considered the case $P = \mathbb{R}$. If $a, b \in \mathbb{R}$, then the proof of Proposition 10.1 constructs an isotopy from the fiber S_a to the fiber S_b, whence the name "isotopy lemma".

The second isotopy lemma is an analogous result for mappings instead of spaces. Consider a diagram of spaces and mappings:



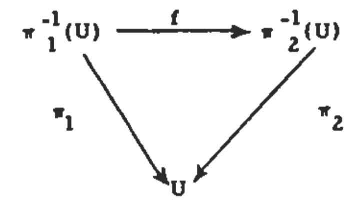
.

We say that f is trivial over Z if there exists spaces X_0 and Y_0, a mapping $f_0 : X_0 \to Y_0$ and homeomorphisms $X \approx X_0 \times Z$, $Y \approx Y_0 \times Z$ such that the following diagram of spaces and mappings is commutative:



.

We say f is locally trivial over Z if for any $z \in Z$, there is a neighborhood U of z in Z such that in the diagram



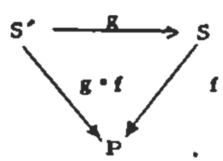
we have that f is trivial over U.

Local triviality of a mapping f over a space Z has a consequence which will be very important in what follows. We think of f as a family $\{f_a : a \in Z\}$ of mappings, where $f_a : X_a \to Y_a$ is the mapping obtained by restricting f to the fiber X_a of X over a. If Z is connected and f is locally trivial over Z, then for any a and b in Z, the mappings f_a and f_b are <u>equivalent</u> in the sense that there exist homeomorphisms $h : X_a \to X_b$ and $h' : Y_a \to Y_b$ such that $h' f_a = f_b h$.

This is the relation of equivalence that is used in the definition of topologically stable mapping, and a step in the proof that the topologically stable mappings form an open dense set will be to show that certain families of mappings are locally trivial in the sense defined above, by an application of Thom's second isotopy lemma.

Now suppose M' is a smooth manifold and S' is a closed subset of M', which admits a Whitney pre-stratification $\mathcal{S}'$. Let $g : M' \to M$ be a smooth mapping and suppose $g(S') \subseteq S$. Thom's second isotopy lemma gives sufficient conditions for the following diagram to be locally trivial:

diagram 11.1



To state Thom's second isotopy lemma, we must introduce Thom's condition a_g. Let X and Y be submanifolds of M' and let y be a point in Y. Suppose $g|X$ and $g|Y$ are of constant rank. We say the pair (X, Y) <u>satisfies condition</u> a_g <u>at</u> y if the following holds:

Let x_i be any sequence of points in X converging to y. Suppose that the sequence of planes $\ker(d(g|X')_{x_i}) \subseteq TM'_{x_i}$ converges to a plane $\tau \subseteq TM'_{x_i}$ in the appropriate Grassmannian bundle. Then $\ker(d(g|Y')_y \in \tau$.

We say that the pair (X, Y) satisfies condition a_g if it satisfies condition a_g at every point y of Y.

Now, we return to the situation of Diagram 11.1. We will say that g is a <u>Thom mapping</u> (over P) if the following conditions are satisfied.

(a) $g|S'$ and $f|S$ are proper.

(b) For each stratum X of $\mathcal{S}$, $f|X$ is a submersion.

(c) For each stratum X' of $\mathcal{S}'$, $g(X')$ lies in a stratum X of $\mathcal{S}$, and $g : X' \to X$ is a submersion (whence $g|X'$ is of constant rank).

(d) Any pair (X', Y') of strata of $\mathcal{S}'$ satisfies condition a_g (which makes sense in view of (c)).

In the case P is a point, we will drop over P.

PROPOSITION II. 2 (Thom's second isotopy lemma). If g is a Thom mapping over P, then g is locally trivial over P.

The proof of this requires new machinery. Let $\{T\}$ be a system of control data for the stratification $\mathcal{S}$ of S. We need the notion of a system $\{T'\}$ of control data over $\{T\}$ for the stratification $\mathcal{S}'$ of S'.

CAUTION: A system of control data over $\{T\}$ is not a system of control data as previously defined. If we were to require that a system of control data over $\{T\}$ also be a system of control data tout court then the fundamental existence theorem for control data over $\{T\}$ (Proposition II. 3, below) would not be true.

DEFINITION: Suppose g is a Thom mapping. A system $\{T'\}$ of control data for $\mathcal{S}'$ over $\{T\}$ is a family of tubular neighborhoods, indexed by $\mathcal{S}'$, where T'_X is a tubular neighborhood of X in M' with the following properties:

(a) If X' and Y' are strata of $\mathcal{S}'$ and $X' < Y'$, then the commutation relation

$$\pi_{X'}\pi_{Y'}(v) = \pi_{X'}(v)$$

holds for all v for which both sides are defined, i.e., all $v \in |T_{X'}| \cap |T_{Y'}|$ such that $\pi_{Y'}(v) \in |T_{X'}|$.

Furthermore, if $g(X')$ and $g(Y')$ lie in the same stratum of $\mathcal{S}$, then the commutation relation

$$\rho_{X'}\pi_{Y'}(v) = \rho_{X'}(v)$$

holds for all v for which both sides of this equation are defined.

(b) If X' is a stratum of $\mathcal{S}'$ and X is a stratum of $\mathcal{S}$ which contains $g(X')$, then

$$g\pi_{X'}(v) = \pi_X g(v)$$

for all v for which both sides of this equation are defined, i.e., for all $v \in |T_{X'}| \cap g^{-1}|T_X|$.

Note that a is weaker than the commutation relation for control data in the case $g(X')$ and $g(Y')$ are not in the same stratum of $\mathcal{S}$.

PROPOSITION II. 3. If g is a Thom mapping then for any system $\{T\}$ of control data for $\mathcal{S}$ there exists a system $\{T'\}$ of control data for $\mathcal{S}'$ over $\{T\}$.

The proof of this is similar to the proof of the existence theorem for control data (Proposition 7.1). We will only outline it.

Proof (Outline): Let $\mathcal{S}'_k$ be the family of all strata of $\mathcal{S}'$ of dimension $\leq k$, and let S'_k denote the union of all strata in $\mathcal{S}'_k$. We will show by induction on k that the proposition is true for $\mathcal{S}'_k$ and S'_k in place of $\mathcal{S}'$ and S'. This will suffice to prove the proposition.

The case $k = 0$ is trivial. For the inductive step, we suppose that for each stratum X' of $\mathcal{S}'$ of dimension $< k$, we are given a tubular neighborhood $T_{X'}$ of X' and that this family of tubular neighborhoods satisfies conditions (a) and (b) above.

By shrinking the $T_{X'}$ if necessary, we may suppose that if X' and Y' are strata of dimension $< k$ which are not comparable, then $|T_{X'}| \cap |T_{Y'}| = \emptyset$. To construct the $T_{X'}$ on the strata of dimension k, we may do it one stratum at a time, since the relations (a) and (b) impose no conditions on pairs of strata of the same dimension. Let X' be a stratum of $\mathcal{S}'$ of dimension k.

We construct the tubular neighborhood $T_{X'}$ in two steps as follows. For each $\ell \leq k$, we let U'_ℓ denote the union of all $|T_{Y'}|$

for $Y' < X'$ and $\dim Y' \geq \ell$. We let $X'_\ell = U'_\ell \cap X'$. In the first step, we construct a tubular neighborhood T'_ℓ of X'_ℓ by decreasing induction on ℓ, shrinking various T'_Y where necessary.

This step is carried out in essentially the same way as the first step in the proof of Proposition 7.1. We start the induction at $\ell = k$, where there is nothing to prove. For the inductive step, we suppose $T'_{\ell+1}$ has been constructed. We observe that to construct T'_ℓ it is enough to construct T'_ℓ on $|T_{Y'}| \cap X'$ for each stratum $Y' < X'$ of dimension ℓ separately. Then there are two cases.

Case 1. If $g(Y')$ and $g(X')$ are in the same stratum of ,then the construction is carried out in the same way as the corresponding construction in the proof of Proposition 7.1. In this way we define T'_ℓ on $|T_{Y'}| \cap X'$ so that the commutation relations (a) hold. (Note that condition (b) follows from (a) in this case.)

Case 2. In the case $g(Y')$ and $g(X')$ are not in the same stratum of $\mathcal{S}$, the proof must be modified. Let X be the stratum which contains $g(X')$ and let Y be the stratum which contains $g(Y')$. Then $Y < X$. By shrinking $|T_{Y'}|$ if necessary, we may suppose that $g(|T_{Y'}|) \subseteq |T_Y|$. Let

$$V = (|T_Y| \cap X) \times_Y Y'$$

where the fiber product is taken with respect to the mappings

$$\pi_Y : |T_Y| \cap X \longrightarrow Y$$

$$g : Y' \longrightarrow Y \quad .$$

Then the mapping

$$G = (g, \pi_{Y'}) : |T_{Y'}| \cap X' \longrightarrow V$$

is defined because the following diagram commutes:

$$\begin{array}{ccc} |T_{Y'}| \cap X' & \xrightarrow{\pi_{Y'}} & Y' \\ \downarrow g & & \downarrow g \\ |T_Y| \cap X & \xrightarrow{\pi_Y} & Y \end{array} \quad ,$$

by the inductive hypothesis that (b) is satisfied for those tubular neighborhoods which are already defined.

LEMMA 11.4. <u>There exists a neighborhood</u> N <u>of</u> Y' <u>in</u> $|T_{Y'}|$ <u>such that</u>

$$G|N \cap X' : N \cap X' \longrightarrow V$$

<u>is a submersion.</u>

<u>Proof</u>: Let Σ be the set of points in $|T_{Y'}| \cap X'$ where the differential of G is not onto. It suffices to show that $Y' \cap \overline{\Sigma} = \emptyset$.

Let $x' \in |T_{Y'}| \cap X'$, $x = g(x')$, $y' = \pi_{Y'}(x')$, and $y = g(y') = \pi_Y(x)$. Then

$$dG_{x'} = \left(d(\pi_{Y'X'})_{x'}, d(g|X')_{x'}\right) : TX'_{x'} \longrightarrow TV_{G(x')} = TX_x \times_{TY_y} TY'_{y'}$$

By definition, $x' \in \Sigma$ if and only if this mapping is not onto. Since

$$d(g|X')_{x'} : TX'_{x'} \longrightarrow TX_x$$

is onto (by hypothesis), it follows that this mapping is onto if and only if

$$d(\pi_{Y'X'})_{x'} : \ker\left(d(g|X')_{x'}\right) \longrightarrow \ker\left(d(g|Y')_{y'}\right)$$

is onto. From condition a_g, it follows that Y' does not meet the closure $\overline{\Sigma}$ of the set of points where this mapping is not onto. Q. E. D.

Now we extend T'_ℓ over $|T_{Y'}| \cap X'$ in such a way that (a) holds (the weak (a)!) and (b) holds. We may do this by the generalized existence theorem for tubular neighborhoods and Lemma 11.4.

This completes the inductive step.

Now the second step (extension of T'_ℓ from U'_0 over all of X') is carried out in exactly the same way as in the proof of Proposition 7.1. Q. E

The rest of the proof of Proposition 11.2 will be carried out in three steps. First, we define the notion of a controlled vector field over another controlled vector field. (WARNING: this is not a special case of the notion of a controlled vector field.) Then we prove a lifting theorem for controlled vector fields. Finally, we show that every controlled vector field over another controlled vector field generates a local one parameter group.

Now we suppose g is a Thom mapping. We suppose that we are given a system $\{T\}$ of control data for S and a system $\{T'\}$ of control data for S' over $\{T\}$. Let $\eta = \{\eta_X\}_{X \in S}$ be a controlled vector field on S.

DEFINITION: By a controlled vector field on S' over η, we will mean a collection $\{\eta_{X'}\}_{X' \in S'}$, where $\eta_{X'}$ is a vector field on X', such that the following conditions are satisfied.

(a) For any $X' \in S'$ and $x' \in X'$, we have

$$(g|X')_* \eta_{X'}(x') = \eta_X(g(x')) \quad .$$

(b) For any $X', Y' \in S'$ with $Y' < X'$, there is a neighborhood $N_{Y'}$ of Y' in $|T_{Y'}|$ such that for $y' \in |T_{Y'}| \cap X'$, we have

$$(\pi_{Y'X'})_* \eta_{X'}(x') = \eta_{Y'}(\pi_{Y'X'}(x'))$$

and if $g(X')$ and $g(Y')$ are in the same stratum of S then we have

$$\eta_{X'} \rho_{Y'X'}(x') = 0 \quad .$$

(Note that condition b is weaker than the condition that we imposed on a controlled vector field in Section 9 in the case $g(Y')$ and $g(X')$ are not in the same stratum of S.)

PROPOSITION 11.5. <u>There exists a controlled vector field on S' over η.</u>

The proof is completely analogous to the proof of Proposition 9.1, and we omit it.

PROPOSITION 11.6. <u>If η' is a controlled vector field on S' over η, then η' generates a local one parameter group, which commutes with the one-parameter group on S generated by η.</u>

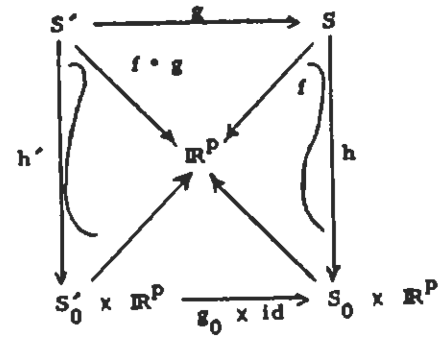
The proof of this is essentially the same as the proof of Proposition 10.1. The only additional remark to be made is that if X' and Y' are strata of S with $Y' < X'$, and $g(Y')$ lies in Y and $g(X')$ lies in X, then, in the case $Y < X$, a trajectory y' of η' starting at a point of X' cannot approach Y' because the image of y' is a trajectory of η and therefore cannot approach a point of Y.

We omit the proof.

Proof of Proposition 11.2. To prove that g is locally trivial over P, it suffices to consider the case $P = \mathbb{R}^p$ and prove that g is trivial over P in this case. By Proposition 7.1 we can find a system $\{T\}$ of control data for S compatible with f, and by Proposition 11.3 there exists a system $\{T'\}$ of control data for S' over $\{T\}$. Let $\partial_1, \cdots, \partial_p$ be the coordinate vector fields on $\mathbb{R}^p$. By Proposition 9.1, we can lift ∂_i to a controlled vector field $\tilde{\partial}_i$ on S, and by Proposition 11.5 we can lift $\tilde{\partial}_i$ to a controlled vector field $\tilde{\tilde{\partial}}_i$ on S' over $\tilde{\partial}_i$.

By Propositions 10.1 and 11.6 the vector fields $\tilde{\partial}_i$ and $\tilde{\tilde{\partial}}_i$ generate local one parameter groups $\tilde{\varphi}_i$ and $\tilde{\tilde{\varphi}}_i$. Since the mappings f and g are proper and ∂_i generates a (global) one parameter group φ_i, it follows that $\tilde{\varphi}_i$ and $\tilde{\tilde{\varphi}}_i$ are (global) one parameter groups.

Let S_0 (resp. S_0') denote the fiber of S (resp. S') over 0. To complete the proof, it is enough to construct local homeomorphisms h and h' such that the following diagram commutes.



.

We define h and h' as follows.

$$h'(x) = \left(\varphi'_{p,-t_p} \cdots \varphi'_{1,-t_1}(x), t\right) \quad \text{where} \quad t = (t_1, \cdots, t_p) = f \circ g(x) \quad ,$$

$$h(x) = \left(\varphi_{p,-t_p} \cdots \varphi_{1,-t_1}(x), t\right) \quad \text{where} \quad t = (t_1, \cdots, t_p) = f(x) \quad .$$

It is easily verified that the above diagram commutes and that h and h' are homeomorphisms. Q. E. D.

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