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A new approach to the study of the $3\mathcal{D}$-Navier-Stokes System

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Dedicated to M. Feigenbaum on the occasion of his 60^{th} birthday

ABSTRACT. In this paper we study the Fourier transform of the $3\mathcal{D}$-Navier-Stokes-System without external forcing on the whole space $\mathbb{R}^3$. The properties of solutions depend very much on the space in which the system is considered. In this paper we deal with the space $\Phi(\alpha,\alpha)$ of functions $v(k) = c(k)/|k|^\alpha$ where $\alpha = 2+\epsilon$, $\epsilon > 0$ and $c(k)$ is bounded, $\sup_{k\in\mathbb{R}^3\setminus 0} |c(k)| < \infty$. We construct the power series which converges for small t and gives solutions of the system for bounded intervals of time. These solutions can be estimated at infinity (in k-space) by $\exp\{-\text{const}\sqrt{t}\,|k|\}$.

1. The spaces $\Phi(\alpha,\alpha)$ and the ruling parameter for the Navier-Stokes system in $\Phi(\alpha,\alpha)$

Consider $3\mathcal{D}$-Navier-Stokes System (NSS) on $\mathbb{R}^3$ for incompressible free fluids. After Fourier transform and an elementary transformation it becomes a non-linear non-local equation for an unknown function $v(k,t)$ with values in $\mathbb{R}^3$ having the form

$$(1)\quad v(k,t) = e^{-t|k|^2} v(k,0) + i\int_0^t e^{-|k|^2(t-s)}\,ds \cdot \int_{\mathbb{R}^3} \langle k, v(k-k',s)\rangle P_k\, v(k',s)\,dk'.$$

The function $v(k,t)$ must satisfy the condition $v(k,t) \perp k$ for any $k \in \mathbb{R}^3$, $k \neq 0$ and $t \geq 0$; P_k is the orthogonal projection to the subspace orthogonal to k; the viscosity is taken to be one, i.e., $\nu = 1$. Classical solutions of (1) on $[0,t_0]$ are functions $v(k,t)$, $0 \leq t \leq t_0$, for which all integrals in (1) converge absolutely and (1) becomes the identity.

There are several reasons by which it is natural to consider (1) in the spaces of functions having singularities near $k=0$ or $k=\infty$. In this paper, we restrict ourselves to the spaces of functions $v(k) = c(k)/|k|^\alpha$ where $\alpha = 2+\epsilon$, $\epsilon > 0$ and sufficiently small, $c(k)$ is continuous everywhere outside $k=0$ and uniformly bounded, i.e., $\sup_{k\in\mathbb{R}^3\setminus 0}|c(k)| = \| c \| < \infty$ (see [**7**], [**1**]). If the solution of (1) belongs to $\Phi(\alpha,\alpha)$ then $v(k,t) = c(k,t)/|k|^\alpha$, $0 \leq t \leq t_0$, $c(k,t) \perp k$ for any $k \in \mathbb{R}^3 \setminus 0$, $t \geq 0$, and $c(k,t)$ satisfies the equation which is equivalent to (1):

$$(2)\quad c(k,t) = e^{-|k|^2 t} c(k,0) + i|k|^\alpha \int_0^t e^{-|k|^2(t-s)}\,ds \cdot \int_{\mathbb{R}^3} \frac{\langle k, c(k-k',s)\rangle\, P_k\, c(k',s)\,dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}.$$

It is easy to check that for typical $c \in \Phi(\alpha,\alpha)$ the initial condition has infinite energy and enstrophy.

Assume that $\|c(k,0)\| = 1$ and take a one-parameter family of initial conditions $c_A(k,0) = A\,c(k,0)$, where A is a parameter taking positive values. In [**1**] the local existence theorem for solution of (2) was proven. Below we outline this proof and show that if $\lambda = A \cdot t^{\epsilon/2} \le \lambda_0$, where λ_0 is an absolute constant which may depend on α, then there exists the unique solution of (2) in the corresponding time interval.

Usual arguments are based on the classical iteration scheme. Put $c_A^{(0)}(k,t) = Ae^{-|k|^2 t} c(k,0)$ and

$$(3)\quad c_A^{(n)}(k,t) = c_A^{(0)}(k,t) + i|k|^\alpha \int_0^t e^{-|k|^2(t-s)}\,ds \cdot \int_{\mathbb{R}^3} \frac{\left\langle k, c_A^{(n-1)}(k-k',s)\right\rangle P_k\, c_A^{(n-1)}(k',s)\,dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}\,.$$

The first step in the proof of the convergence of the iterations $c_A^{(n)}(k,t)$ as $n \to \infty$ is to show that all $c_A^{(n)}(k,t)$ remain close to $c_A^{(0)}(k,t)$ in the sense of the norm in $\Phi(\alpha,\alpha)$. If $c_A^{(n)} = \sup_{0\le s\le t,\, k\in\mathbb{R}^3\setminus 0} |c_A^{(n)}(k,s)|$ then we would like to show that $c_A^{(n)} \le 2c_A^{(0)} = 2A$ for all n. By induction and with the use of (3) we can write

$$(4)\qquad c_A^{(n)} \le c_A^{(0)} + \sup_{\substack{k\in\mathbb{R}^3\setminus 0\\ 0\le s\le t}} |k|^{\alpha-1} \cdot (1 - e^{-|k|^2 t}) \cdot (c_A^{(n-1)})^2 \cdot \int \frac{dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}\,.$$

The last integral satisfies the inequality

$$(5)\qquad \int_{\mathbb{R}^3} \frac{dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \le \frac{B_1}{|k|^{2\alpha-3}}\,.$$

Here and below the letter B with indices is used for various absolute constants which appear during the proofs. These constants may depend on α.

Now, we have to show that

$$\sup_k |k|^{2-\alpha}(1-e^{-|k|^2 t}) \cdot (c_A^{(n-1)})^2 \cdot B_1 \le c_A^{(0)}\,.$$

By induction $c_A^{(n-1)} \le 2c_A^{(0)}$. Therefore we have to show that

$$\sup_k |k|^{2-\alpha}(1-e^{-|k|^2 t}) \cdot 4 \cdot c_A^{(0)} \le 1\,.$$

Consider two cases.

(1) $|k|^2 \le 1/t$. Then

$$|k|^{2-\alpha}(1-e^{-|k|^2 t}) \le |k|^{4-\alpha} \cdot t \le t^{-\frac{4-\alpha}{2}+1} = t^{\epsilon/2}\,.$$

(2) $|k|^2 \ge 1/t$. Then

$$|k|^{2-\alpha} \cdot (1-e^{-|k|^2 t}) \le |k|^{2-\alpha} \le t^{\frac{\alpha-2}{2}} = t^{\epsilon/2}\,.$$

Thus we can write

$$c_A^{(n)} \le A \cdot + 4A^2 \cdot (c^{(0)})^2 \cdot t^{\epsilon/2} B_1 = A(1 + 4A \cdot t^{\epsilon/2} B_1) = A(1 + 4B_1\lambda)\,.$$

We used the fact that $c^{(0)} = c_1^{(0)} \le 1$. If $\lambda < 1/(4B_1)$ then $c_A^{(n)} \le 2c_A^{(0)}$. This argument shows how the parameter λ arises.

The next step in the proof of the existence of solutions is to show that the iterations $c_A^{(n)}$ converge to a limit. We have from (3)

$$c_A^{(n)}(k,t) - c_A^{(n-1)}(k,t) = i\,|k|^\alpha \cdot \int_0^t e^{-|k|^2(t-s)}\,ds$$
$$\cdot \Bigg[\int_{\mathbb{R}^3} \frac{\left\langle k, c_A^{(n-1)}(k-k',s) - c_A^{(n-2)}(k-k',s)\right\rangle P_k\, c_A^{(n-1)}(k',s)\,dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}$$

$$+\int_{\mathbb{R}^3} \frac{\left\langle k, c_A^{(n-2)}(k-k',s)\right\rangle P_k\left(c_A^{(n-1)}(k',s)-c_A^{(n-2)}(k',s)\right) dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}\Bigg]$$

and

$$\left|c_A^{(n)}(k,t)-c_A^{(n-1)}(k,t)\right| \le 2A\cdot \left\|c_A^{(n-1)}-c_A^{(n-2)}\right\|\cdot |k|^{\alpha-1}\left(1-e^{-|k|^2 t}\right)\int_{\mathbb{R}^3}\frac{dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\,. \tag{6}$$

¿From (5) and (6),

$$\left\|c_A^{(n)}-c_A^{(n-1)}\right\| \le 2A\,B_1\cdot\left\|c_A^{(n-1)}-c_A^{(n-2)}\right\|\cdot \sup_{k\in\mathbb{R}^3\setminus 0}|k|^{\alpha-1}\left(1-e^{-|k|^2t}\right)\cdot\frac{1}{|k|^{2\alpha-3}}\,.$$

The same arguments as before give that

$$\sup_{k\in\mathbb{R}^3\setminus 0}|k|^{\alpha-1}\left(1-e^{-|k|^2t}\right)\cdot\frac{1}{|k|^{2\alpha-3}}\le B_2\cdot t^{\epsilon/2}\,.$$

Therefore, for some constant B_3,

$$\left\|c_A^{(n)}-c_A^{(n-1)}\right\|\le B_3\cdot\lambda\cdot\left\|c_A^{(n-1)}-c_A^{(n-2)}\right\|\,.$$

¿From the last inequality it follows that if λ is less than some absolute constant then the iteration scheme converges and gives the desired solution. Thus λ is really a ruling parameter in the current situation. The main purpose of this paper is to construct a general power series in λ which provides the solution of (2) with the given initial condition.

Write down the solution of (2) with the initial condition $A\cdot c(k,0)$, $\|c(k,0)\|\le 1$, in the form

$$\begin{aligned} c_A(k,t) &= A\Bigg(c(k,0)\,e^{-t|k|^2}+\int_0^t e^{-(t-s)|k|^2}\sum_{p\ge 1}\lambda^p h_p(k,s)\,ds\Bigg)\\ &= A\Bigg(c(k,0)\,e^{-t|k|^2}+\sum_{p\ge1}A^p\int_0^t e^{-(t-s)|k|^2}s^{p\epsilon/2}\,h_p(k,s)\,ds\Bigg)\end{aligned} \tag{7}$$

where now $\lambda = A\cdot s^{\epsilon/2}$. Substituting this expression into (2) we get the system of recurrent relations for h_p. Below we give the explicit formulas for h_1, h_2 and then the general formula for h_p, $p\ge 3$. We have

$$A^2 s^{\epsilon/2}h_1(k,s) = iA^2|k|^\alpha\int_{\mathbb{R}^3}\frac{\langle k, c(k-k',0)\rangle\, P_k\, c(k',0)\, e^{-s|k-k'|^2-s|k'|^2}\,dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\,, \tag{8}$$

$$\begin{aligned} &A^3 s^\epsilon h_2(k,s) = \\ &iA^3\cdot|k|^\alpha\Bigg[\int_0^s s_1^{\epsilon/2}ds_1\int_{\mathbb{R}^3}\frac{\langle k, h_1(k-k',s_1)\rangle\, P_k\, c(k',0)\cdot e^{-(s-s_1)|k-k'|^2-s|k'|^2}\,dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\\ &\qquad+\int_0^s s_2^{\epsilon/2}ds_2\int_{\mathbb{R}^3}\frac{\langle k, c(k-k',0)\rangle\, P_k\, h_1(k',s_2)\, e^{-s|k-k'|^2-(s-s_2)|k'|^2}\,dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\Bigg]\end{aligned} \tag{9}$$

and

$$\begin{aligned} &A^{p+1}s^{p\epsilon/2}h_p(k,s) = iA^{p+1}\cdot|k|^\alpha\cdot\\ &\cdot\Bigg[\int_0^s s_1^{(p-1)/2}ds_1\int_{\mathbb{R}^3}\frac{\langle k, h_{p-1}(k-k',s_1)\rangle\, P_k\, c(k',0)e^{-(s-s_1)|k-k'|^2-s|k'|^2}\,dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\\ &\quad+\int_0^s s_2^{(p-1)\epsilon/2}ds_2\int_{\mathbb{R}^3}\frac{\langle k, c(k-k',0)\rangle\, P_k\, h_{p-1}(k',0)e^{-s|k-k'|^2-(s-s_2)|k'|^2}\,dk'}{|k-k'|^\alpha\cdot|k'|^\alpha}\end{aligned} \tag{10}$$

$$+\sum_{\substack{p_1,p_2\geq 1\\ p_1+p_2=p-1}} \int_0^s s_1^{p_1\epsilon/2} ds_1 \int_0^s s_2^{p_2\epsilon/2} ds_2$$

$$\left.\int_{\mathbb{R}^3} \frac{\langle k, h_{p_1}(k-k',s_1)\rangle\, P_k\, h_{p_2}(k',s_2)\, e^{-(s-s_1)|k-k'|^2-(s-s_2)|k'|^2} dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}\right].$$

Use the following Ansatz: $h_p(k,s) = s^{\epsilon/2}|k|^\alpha g_p(k\sqrt{s}, s)$ and in all integrals above make the change of variables: $s_1 = s\cdot \tilde{s}_1$, $s_2 = s\cdot\tilde{s}_2$, $k\sqrt{s}=\tilde{k}$, $k'\sqrt{s}=\tilde{k}'$. Thus $h_p(k,s) = s^{\epsilon/2}|k|^\alpha\, g_p(\tilde{k},s)$. Instead of (8), (9), (10) we shall get the system of recurrent relations for the functions $g_p(\tilde{k},s)$:

$$A^2 s^\epsilon |k|^\alpha \cdot g_1(\tilde{k},s) = iA^2\cdot |k|^\alpha \cdot s^\epsilon \int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\right\rangle P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right) e^{-|\tilde{k}-\tilde{k}'|^2-|k'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha \cdot |\tilde{k}'|^\alpha}$$

or

$$g_1(\tilde{k},s) = i\int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\right\rangle P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right) e^{-|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha \cdot |\tilde{k}'|^\alpha}, \tag{11}$$

$$A^3 s^{3\epsilon/2}\cdot |k|^\alpha g_2(\tilde{k},s) = iA^3\cdot |k|^\alpha s^{3\epsilon/2}$$

$$\left[\int_0^1 \tilde{s}_1^\epsilon\, d\tilde{s}_1 \int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, g_1\left((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\tilde{s}_1\right)\right\rangle P_{\tilde{k}} c\left(\frac{\tilde{k}}{\sqrt{s}},0\right) e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}'|^\alpha}\right.$$

$$\left. + \int_0^1 \tilde{s}_2^\epsilon\, d\tilde{s}_2 \int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\right\rangle g_1\left(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2\right) e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2} d\,\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}\right]$$

or

$$g_2(\tilde{k},s) = \tag{12}$$

$$\int_0^1 \tilde{s}_1^\epsilon\, d\tilde{s}_1 \int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, g_1\left((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\cdot\tilde{s}_1\right)\right\rangle P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)\ e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2}\ d\tilde{k}'}{|\tilde{k}'|^\alpha}$$

$$+ \int_0^1 \tilde{s}_2^\epsilon\, d\tilde{s}_2 \int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\right\rangle g_1\left(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2\right) e^{-|\tilde{k}-\tilde{k}'|^2-(|-\tilde{s}_2)|\tilde{k}'|^2} d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}.$$

For general $g_p(\tilde{k},s)$, $p\geq 3$,

$$A^{p+1}\cdot s^{\frac{p+1}{2}\cdot\epsilon}\, |k|^\alpha \cdot g_p(\tilde{k},s) = iA^{p+1}\cdot s^{\frac{p+1}{2}\cdot\epsilon}\, |k|^\alpha$$

$$\left[\int_0^1 \tilde{s}_1^{p\epsilon/2}\, d\tilde{s}_1 \int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, g_{p-1}\left((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\cdot\tilde{s}_1\right)\right\rangle P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)\ e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2}\ d\tilde{k}'}{|\tilde{k}'|^\alpha}\right.$$

$$+ \int_0^1 \tilde{s}_2^{p\epsilon/2}\, d\tilde{s}_2 \int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\right\rangle P_{\tilde{k}} g_{p-1}\left(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2\right) e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}\ d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}$$

$$+ \sum_{\substack{p_1\geq 1,\, p_2\geq 1\\ p_1+p_2=p-1}} \int_0^1 \tilde{s}_1^{(p_1+1)\epsilon/2}\, d\tilde{s}_1 \int_0^1 \tilde{s}_2^{(p_2+1)\epsilon/2}\, d\tilde{s}_2 \int_{\mathbb{R}^3} \left\langle \tilde{k}, g_{p_1}\left((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\tilde{s}_1\right)\right\rangle$$

$$\left. P_{\tilde{k}} g_{p_2}\left(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2\right)\cdot e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}\ d\tilde{k}'\right]$$

or

$$
\begin{aligned}
(13)\quad & g_p(\tilde{k},s) = \\
& i\Bigg[\int_0^1 \tilde{s}_1^{p\epsilon/2}\, d\tilde{s}_1 \int_{\mathbb{R}^3} \frac{\big\langle \tilde{k}, g_{p-1}\big((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\tilde{s}_1\big)\big\rangle\, P_{\tilde{k}} c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)\, e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2}\, d\tilde{k}'}{|\tilde{k}'|^{\alpha}} \\
& + \int_0^1 \tilde{s}_2^{p\epsilon/2}\, d\tilde{s}_2 \int_{\mathbb{R}^3} \frac{\left\langle \tilde{k}, c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\right\rangle P_{\tilde{k}}\, g_{p-1}\big(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2\big)\, e^{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}\, d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^{\alpha}} \\
& + \sum_{\substack{p_1,p_2\ge 1\\ p_1+p_2=p-1}} \int_0^1 \tilde{s}_1^{(p_1+1)\epsilon/2}\, d\tilde{s}_1 \int_0^1 \tilde{s}_2^{(p_2+1)\epsilon/2}\, d\tilde{s}_2 \int_{\mathbb{R}^3} \left\langle \tilde{k}, g_{p_1}\big((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1}, s\cdot\tilde{s}_1\big)\right\rangle \\
& \qquad\qquad P_{\tilde{k}} g_{p_2}\big(\tilde{k}'\sqrt{\tilde{s}_2}, s\tilde{s}_2\big)\, e^{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2}\, d\tilde{k}'\Bigg].
\end{aligned}
$$

These recurrent relations allow to express each $g_p(k\sqrt{s},s)$ through the initial conditions $c(k,0)$. It is easy to see that this expression will be the sum of not more than b^p $4p$-dimensional integrals containing products of $c(\cdot,0)$ with different values of the arguments where b is some constant. We shall discuss the related questions in another paper.

Write down the inequality

$$|g_p(\tilde{k},s)| \le C_p\, f\big(|\tilde{k}|\big)\, e^{-|k|^2 s/(p+1)} = C_p\, f\big(|\tilde{k}|\big)\, e^{-|\tilde{k}|^2/(p+1)}$$

where

$$f(x) = \begin{cases} x & 0 \le x \le 1, \\ x^{-1} & x \ge 1. \end{cases}$$

The main result of this paper is the following theorem.

Main Theorem: *The numbers C_p can be chosen in such a way that*

$$C_p = B \sum C_{p_1}\cdot C_{p_2}\cdot \frac{(p_1+1)(p_2+1)}{(p+1)}. \tag{14}$$

We prove the main theorem in the next section. First we analyze $p = 1,2,3$ and then the general case $p > 3$. We use the identity

$$a_1|k-k'|^2 + a_2|k'|^2 = \frac{a_1a_2}{a_1+a_2}\,|k|^2 + (a_1+a_2)\left|k' - \frac{a_1}{a_1+a_2}\,k\right|^2 \tag{15}$$

valid for arbitrary k, k'.

It follows easily from (14) that the C_p grow no faster than exponentially (see §2), $C_p \le b_1 b_2^p$ for some constants $b_1, b_2 < \infty$ depending on α.

Corollary: *If $At^{\epsilon/2} < b_2^{-1}$ then the series* (7) *converges for every $k \in \mathbb{R}^3\setminus 0$.*

It is interesting to remark that all but one of the terms of the series (7) have finite energy and enstrophy.

Other expansions for the Navier-Stokes system which are formal can be found in the monographs [**4**], [**5**]. General approach to the existence problem for the Navier-Stokes System is discussed in [**3**].

2. The discussion of the results

First we show that the constants C_p grow not faster than exponentially. Denote $C'_p = C_p(p+1)$. The numbers C'_p satisfy the relation

$$C'_p = B \sum_{\substack{p_1,p_2\ge 0\\ p_1+p_2=p-1}} C'_{p_1}\, C'_{p_2}\,.$$

Using the induction, let us prove that $C'_p \le (\widetilde{B})^p \cdot (p+1)^{-3/2}$ for any $p \ge 0$ and some constant $\widetilde{B}$. For $p = 0$ we can choose $\widetilde{B}$. Assuming that the last inequality is valid for all $q < p$ we can write

$$C'_p \;\le\; B \cdot (\widetilde{B})^{p-1} \sum_{\substack{p_1, p_2 \ge 0 \\ p_1 + p_2 = p-1}} \frac{1}{(p_1+1)^{3/2}\,(p_2+1)^{3/2}} \;\le\; B \cdot B_1 \cdot (\widetilde{B})^{p-1} \cdot \frac{1}{(p+1)^{3/2}}\,.$$

Take $\widetilde{B} = B \cdot B_1$. This gives the result.

Now we see that the series (7) converges if $\lambda = A \cdot t^{\epsilon/2} < (\widetilde{B})^{-1}$.

In many estimates done for the NSS system, people assumed that solutions $v(k,t)$ at infinity in k satisfy the inequality $|v(k,t)| \le e^{-f(t)|k|}$ which is different from the diffusion-like asymptotics. If this asymptotics represents the true decay of solutions, it is an interesting question how does it appear. The series (7) sheds some light on this question. We can write

$$|v(k,t)| \le \mathrm{Const} \sum_{p \ge 0} \widetilde{B}^p \cdot \lambda^p \cdot \frac{1}{(p+1)^{3/2}}\, e^{-t|k|^2/(p+1)}\,.$$

The usual asymptotical method shows that the largest term in this last sum is when $t|k|^2/(p_{\max}+1)^2 = -\ln(b_2\lambda)$, i.e., $p_{\max} = \sqrt{t} \cdot |k| / \sqrt{-\ln(b_2\lambda)}$ and the whole sum behaves as $e^{2p_{\max}\ln(b_2\lambda)} = e^{-2\sqrt{t}\,\sqrt{-\ln(b_2\lambda)}\cdot|k|}$. This is the asymptotics which was mentioned above. It also shows that in the domain of convergence of the series the enstrophy of the solution is finite for $t > 0$.

This type of decay of solutions in various situations was obtained earlier in the works [**2**], [**6**].

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