Some Problems in the Theory of Dynamical Systems & Mathematical Physics*

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Problems which are considered below are not new. They were discussed earlier in many publications and important results were obtained. However it seems that an adequate understanding is still lacking and a serious progress is expected and can be even more important than initial goals.

A brief look at the history of the theory of dynamical systems during the last fifty years shows that it developed by bursts around great discoveries when new ideas, methods, phenomena appeared. Let me mention these discoveries explicitly (in a more or less chronological order):

1. KAM - theory.
2. Works of Smale, Anosov and others on the structure stability and hyperbolicty of dynamical systems, strange attractors and Lorenz model, hyperbolic billiards.
3. Entropy theory which started with the classical work by Kolmogorov and ended with the famous works of Ornstein and his coauthors.
4. Feigenbaum universality in period-doubling bifurcations.

Remarkable results were obtained in other directions like non-local bifurcations, holomorphic dynamics, flows on surfaces, ergodic theory on homogeneous spaces, billiards but it seems that 1-4 were dominating events.

Fifty years ago it was absolutely impossible to anticipate any of the breakthroughs 1-4. This unpredictability is one of the attracting features of the theory of dynamical systems. By this reason on the occasion of this Conference one can only hope to indicate some directions where a new serious progress looks possible. The text below presents some suggestions.

§I. Generically quasi-periodic and hyperbolic dynamics are the only structurally stable types of dynamics.

The problem goes back to Kolmogorov. In the case of Hamiltonian dynamics a positive answer would mean that the phase space can be decomposed onto subsets of positive measure consisting of invariant tori with quasi-periodic motions and stochastic layers where at least some Lyapunov exponents are non-zero. The statement of this type is important for applications in differential geometry (the structure of geodesic flows), astronomy, statistical mechanics and other fields using Hamiltonian dynamics.

For dissipative systems some version of this problem was formulated recently as the whole program by J.Palis (see [P])

There are some results which can be considered as related to this problem.

First of all I mean the one-dimensional dynamics where due to the important results of Jakobson [J], Benedicks and Carleson [BC], Lyubich [L], Graczyk and Swiatek [GS] we know that in analytic families of one-dimensional maps the space of parameters can be decomposed onto a subset where the dynamics has a stable periodic orbit, a subset where the dynamic has an absolutely continuous invariant measure and is chaotic and a negligible subset of measure zero.

KAM-theory gives the conditions under which the invariant tori with quasi-periodic dynamics persist in Hamiltonian systems. In spite of a recent activity around the construction of these tori it is hard to expect new discoveries based on this technique even in its modernized form. From the other side many problems which grew up from KAM-theory remain completely open. The most notable one is the problem of classification of KAM-islands. It is highly non-trivial even within the class of two-dimensional twist maps, i.e. symplectic maps of the two-dimensional cylinder $T(z,\varphi) = (z^{'},\varphi^{'})$, $z^{'} = z + kV^{'}(\varphi)$, $\varphi^{'} = \varphi + z^{'}$. Here V is a smooth periodic function of φ, k is a parameter. It is widely believed that the families of twist maps are the next candidates for the deep studies after the families of one-dimensional maps. One of the non-direct consequences of KAM-theory is the existence for generic twist maps of KAM-islands, i.e. invariant or periodic domains bounded by invariant or periodic curves. Near the islands one can find similar islands of a smaller scale and so on. We clearly see a hierarchical structure of the islands with some scale invariance. The basic problems here are the following:

1) to establish the existence of this hierarchical structure;

2) to describe rotation numbers for the induced maps on the boundaries of the islands;

3) to estimate the global measure of KAM-islands as $k \to \infty$.

Apparently, 1) can be studied with the help of re-normalization group technique because in typical situations asymptotically the structure of the corresponding power of the map near any small island depends only very weakly on the structure of the initial map and belongs to some class of universality. These ideas were proposed in the works of Lichtenberg [Li], Zaslavsky [Z], Melnikov [M] and others. The renormalization group approach to the construction of KAM-tori can be found in [AK], [KS] (in [AK] one can find more related references).

The problem 2) is connected with another important question. Consider an invariant curve Γ given by the equation $z = f_\Gamma(\varphi)$. If Γ is a KAM-curve then it is not isolated and is in a natural sense a density point of other KAM-curves. Therefore boundary curves which separate stochastic layers from KAM-curves should be non-smooth. From the other side they are "last" curves with the given rotation numbers and this is their characteristics feature. (See Figure 2.) With the change of parameter they bifurcate into Aubry-Mather sets. Thus another problem along these lines is

4) to construct or to prove the existence of non-smooth invariant curves of twist maps.

At the moment the only hope to attack this problem is again the renormalization group theory.

Chirikov [Ch] stresses that rotation numbers of induced maps on these curves should be special. In this connection one can formulate the next problem:

5)why the rotation number of the last invariant curve is golden mean.

It is easy to show that for large enough k twist maps have no invariant curves Γ described by the equations $z = f_\Gamma(\varphi)$ (see for example, [Si]). Therefore there exists a critical value k_{cr} such that for $k > k_{cr}$ there are no such curves. It was a numerical discovery by J.Greene [G] that for some twist maps the rotation number on the last curve is golden mean. R.Mac Kay [Mac] proposed a renormalization group technique to study these curves.

A satisfactory answer to the last problem can explain why golden mean appears so often in mathematics and not only in mathematics (architecture, construction, art,...).

The problem 3) is connected with the following well-known hypothesis:

Hypothesis 1. For any $k \neq 0$ the twist map has positive metric entropy.

Hypothesis 2. For generic V the set of k for which T has no islands at all and is ergodic has density 1.

This hypothesis is motivated by the analogy with the Jakobson theorem in one-dimensional dynamics and Benedicks-Carleson results for Henon map. It means that for k in the Hypothesis 2 the map T is hyperbolic but the hyperbolicity should be very weak. In particular, stable and unstable manifolds are not transversal everywhere and can have tangency up to some order. It would be interesting to show that typically the tangency is of finite order. However, even the precise formulation of an exact statement can be non-trivial. One should mention the result by Duarte [D] who showed that the set of k for which T has KAM-islands is open and everywhere dense.

The extension of these problems to the multidimensional setting is certainly very important but apparently very difficult.

§II. **Qualitative Theory of PDE and Infinite-Dimensional Systems.**

It seems that this theory will be different from its finite-dimensional counterpart started by Poincare, Denjoy, Lyapunov and others. So far the concept of PDE as a vector field on a topological manifold was not so fruitful, probably, because the topology of the phase space did not play an essential role. From the other side except general questions of existence, uniqueness and smoothness of solutions the existing theory is much more concrete. There are relatively few PDE which are important for applications: Navier-Stokes and other hydrodynamical systems, Korteweg-de-Vries equation, KP-equation, Ginzburg-Landau equation. Kuramoto-Sivashinski equation, reaction-diffusion equations, equation of general relativity, mean curvature flows and few others. Basic qualitative and numerical results are concentrated around the construction of various types of space patterns with different geometry (rolls, dislocations and others). The transition from one type of pattern to another seems to be a new type of bifurcations. The basic problem can be formulated as follows:

1) to classify spatial patterns which are displayed by PDE and to describe possible types of their bifurcations. Some results in this direction can be found in the book by Golubitsky and Schaeffer [GoS]..

In previous years there were many examples of computer-assisted proofs in the theory of dynamical systems. The most notable one was the proof by Lanford of the existence of Feigenbaum-Collet-Tresser fixed point in the theory of period-doubling bifurcations. Recently Tucker gave a computer-assisted proof of the existence of Lorenz attractor (see[T]). However in general this direction did not attract much of attention. People usually do not want to provide detailed mathematical arguments if numerical evidence is enough convincing. It is quite possible that in the case of PDE rigorous results will go together with serious numerical studies. It is already clear that complete proofs require in many cases the most powerful computers.

Navier-Stokes equations deserve a special attention. The existence problem of strong solutions for 3D-Navier-Stokes system remains still an open outstanding problem in spite of efforts of many researchers having different opinions about the answer. If we adopt a point of view that blow-ups in finite time are possible even in the presence of viscosity, then we can make the next step and ask which flows with finite energy and infinite enstrophy can be limits of local solutions of 3D-Navier-Stokes system. Probably this formulation of the basis problem is more amenable to direct attacks since it is local.

The problem of turbulence will be in the center of interest of many mathematicians, physicists, engineers. For mathematicians one of the main problem is to formalize notions which are widely used by physicists, engineers, etc. Here is one example.

2) To propose definitions of solutions displaying the flow of energy over the spectrum and to prove the existence of such solutions.

It is believed that one of the mathematical models of turbulence is Navier-Stokes system with random forcing. Some results concerning the existence of invariant measure for this model were obtained by Flandoli, Maslowski (see [FM]) and others. However, basic questions remain unanswered. The main problem can be formulated as follows.

3) To construct Markov random fields for Navier-Stokes systems where forcing is white in time random field and to study the properties of their correlation functions in the limit of zeroth viscosity.

A progress in this direction can lead to the mathematical justification of Kolmogorov's theory of turbulence.

§III. **Quantum chaos and Anderson localization.**

Quantum chaos as a part of physics appeared about twenty years ago. For mathematicians its main purpose was to describe properties of quantum systems which depend on their classical limits. Problems of quantum chaos are discussed also by P. Sarnak (at this conference). The first mathematical result in quantum chaos was proven by Schnirelman (see [S]) who showed that if a geodesic flow on a smooth compact Riemannian manifold is ergodic then the squares of corresponding eigen-functions of Laplacian are in some sense uniformly distributed.

Actually problems of quantum chaos in mathematics started even earlier. Assume that Q is a compact smooth d-dimensional Riemannian manifold and $N(\lambda) = \sharp\{\lambda_i \leq \lambda\}$ where λ_i are eigen-values of the Laplacian. Weyl's famous asymptotics tells us that

$$N(\lambda) = c\lambda^{d/2} + n(\lambda)$$

where c a dimension-dependent constant and $n(\lambda)$ is a remainder.

It is widely believed that the behavior of $n(\lambda)$ is non-universal and is determined by ergodic properties of the geodesic flow on the unit tangent bundle of Q. It follows basically from the works of Colin de Verdiere (see [C], two-dimensional case in [KMS]) that in the integrable case the analysis of $n(\lambda)$ can be reduced to number - theoretic problems and the result essentially has the form $n(\lambda) = \lambda^{\frac{d-1}{4}} Q(\lambda) + n_1(\lambda)$ where $Q(\lambda)$ is a quasi-periodic function and $n_1(\lambda)$ has a smaller order of magnitude.

In the case of manifolds of negative curvature where the geodesic flow is strongly mixing numerical and qualitative results indicate that $n(\lambda)$ grows only as $(\log \lambda)^\gamma$ for some γ. This resembles the asymptotics of the number of zeros of Riemann ζ-function. Other results suggest that the limiting probability distribution of spacings between the nearest eigen-values behaves similar to the distribution of spacings in some ensembles of random matrices of growing dimension. Let me mention in this connection results by Hejhal and Rackner (see [HR]) who convincingly showed that the probability distributions of eigen-functions of Laplacians on such manifolds are Gaussian in the limit $\lambda \to \infty$. All these results lead to the following problem.

1) Why Laplacians on compact manifolds of negative curvature behave as elements of Wigner ensembles of random matrices.

   So far asymptotic expressions for eigen-functions of Laplacians are known for above mentioned cases of Q with integrable geodesic flows. Schnirelman's theorem shows that in the case of mixing flows they are much more random.

Random eigen-functions appear in a quite different part of mathematical physics, the so - called Anderson localization (AL). AL is a property of Schroedinger operators with random potentials and means that with probability 1 all eigen - functions decay exponentially at infinity AL is based on quantum resonances, i.e. of the appearance or disappearance of approximate eigen-functions with close eigen-values. One of the main open problems here is

2) to construct for $d \geq 3$ non-localized eigen-function belonging to the continuous spectrum.

It would be interesting to connect the theory of AL and quantum chaos. It looks plausible that the asymptotic of eigen-functions of Laplacians in mixing cases has something common with AL.

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