Absence of the Local Existence Theorem in the Critical Space for the 3D-Navier-Stokes System

Yakov Sinai*

Dedicated to: L. P. Shilnikov on the occasion of his 70th birthday

Abstract

We consider the 3D-Navier-Stokes system (NSS) on $\mathbb{R}^3$ without external forcing. After Fourier transform it becomes the system of non-linear integral equations. For one-parameter families of initial conditions $A \cdot c(0)(k) |k|^2$ it is known that if $|A|$ is sufficiently small then NSS has global solution. We show that if $c(0)$ satisfies some natural conditions at infinity then for sufficiently large $A$ NSS has no local solutions with this initial condition.

Keywords: Navier-Stokes system, Fourier transform, critical space.

*Mathematics Department of Princeton University & Landau Institute of Theoretical Physics
Consider the 3D-Navier-Stokes System on $R^3$ without external forcing and viscosity $\nu = 1$. After Fourier transform it can be written as

$$v(k, t) = e^{-|k|^2 t} v(0)(k) + i \int_0^t e^{-|k|^2 (t-s)} ds .$$

$$\int_{R^3} < k, v(k - k', s) > P_k v(k', s) dk'. \quad (1)$$

Here $k \in R^3$, $v(k, t) \perp k$ for any $k$ and $P_k$ is the orthogonal projection to the subspace orthogonal to $k$.

In [1], see also [2], the subspaces $\Phi(\alpha, \omega)$ were introduced. By definition, $v(k) \in \Phi(\alpha, \omega)$ iff

1) $v(k) = \frac{c(k)}{|k|}, \; c(k) \in R^3, \ c(k) \perp k$ for $|k| \leq 1$ where $c(k)$ is continuous outside $k = 0$,  
\[ \sup_{|k| \leq 1, \; k \neq 0} |c(k)| = c < \infty; \]

2) $v(k) = \frac{d(k)}{|k|}, \; d(k) \in R^3, \ d(k) \perp k$ for $|k| \geq 1$ and $d(k)$ is continuous,  
\[ \sup_{|k| \geq 1} |d(k)| = d < \infty. \]

We put $\| v \| = c + d$. With this norm the curve $e^{-t|k|^2} v(0)(k), v(0)(k) \in \Phi(\alpha, \omega)$ may not be continuous at $t = 0$ in the sense that $\| v(0)(k)e^{-t|k|^2} - v(0)(k) \|$ may not tend to zero as $t \to 0$. In the spaces $\Phi(\alpha, \omega), \; 0 \leq \alpha < 3, \; \omega > 2$ the local existence theorem is valid (see [1]). More precisely (see [1]),

**Theorem 1** Let $v(0) \in \Phi(\alpha, \omega)$. Then for some $t_0 > 0$ depending on $\alpha, \omega$ and $v(0)$ there exists $v(k, t) = e^{-t|k|^2} v(0)(k) + v(1)(k, t), \; 0 \leq t \leq t_0$ where $v(1)(k, t) \in \Phi(\alpha, \omega)$ and the family $\{v(1)(k, t), \; 0 \leq t \leq t_0\}$ is continuous in $\Phi(\alpha, \omega)$ including $t = 0$.

This means that $\| v(1)(k, t) \| \to 0$ as $t \to 0$. In the so-called critical case $\alpha = \omega = 2$ where $v(k, t) = \frac{c(k,t)}{|k|^2}, \; c(k,t)$, Le Jan and Sznitman (see [3]) and later Cannone and Planchon (see [4]) proved that if $\| v(0) \|$ is sufficiently small then there exists global solution of (1) defined for all $t > 0$.

In this paper we consider $v(0)(k)$ satisfying some regularity condition at $\infty$. Namely, for each $r > 0$ consider the sphere $S_r = \{k : |k| = r\}$. The condition $v(k) \perp k$ implies that for each $r$ we have the vector field on $S_r$ consisting of vectors $v(k), \; |k| = r$.

**Basic Assumption:** There exist a continuous vector field $w = \{w(k), \; |k| = 1\}$ on the unit sphere such that

$$\max_{k \in S_r} |c(0)(k) - w \left( \frac{k}{|k|} \right) | \to 0 \text{ as } |k| \to \infty.$$
This assumption implies the existence of the limits of $c^0(k)$ when $k \to \infty$ along any direction.

Having $c^0(k)$ satisfying the basic assumption take a one-parameter family of initial conditions $v_A^{(0)}(k) = \frac{A}{|k|^2} \cdot c^0(k)$, $A > 0$. For sufficiently small $A$ the result by Le Jan and Sznitman and Cannone and Planchon can be applied and it gives the existence of global solution $v_A(k, t)$. The purpose of this paper is to prove the following theorem.

**Main Theorem:** Let $c^0(k)$ satisfy the basic assumption and some non-degeneracy condition (see below). For all sufficiently large $A, A \geq A_1$ there does not exist a solution of (1) $v(k, t) = \frac{c(k,t)}{|k|^2} = \frac{A e^{-|k|^2} c^0(k)}{|k|^2} + \frac{c^{(1)}(k,t)}{|k|^2}$ where $\sup_{k \in \mathbb{R}^3} |c^{(1)}(k,t)| \to 0$ as $t \to 0$.

Certainly, $c^{(1)}(k,t)$ may depend on $A$. To prove the theorem we show that for any $t > 0$ and $k \sim O\left(\frac{1}{\sqrt{t}}\right)$ the solution $c(k,t)$ takes values of order $A^2$ and hence cannot be small, i.e. $\|v(k, t) - v^{(0)}(k)\| \sim O(A^2)$ for all sufficiently small $t$, where $O$ does not depend on $t$.

**Proof:** We write $v(k, t) = A e^{-|k|^2} c^0(k) + c^{(1)}(k,t)$.

From (1)
\[
c^{(1)}(k,t) = i \int_0^t e^{-(t-s)|k|^2} ds \cdot |k|^2.
\]

\[
\begin{align*}
&\left[ A^2 \int_{\mathbb{R}^3} \frac{< k, c^{(0)}(k-k') > P_k \cdot c^{(0)}(k') e^{-|k-k'|^2-s|k'|^2} dk'}{|k-k'|^2 \cdot |k'|^2} \right. \\
&+ A \int_{\mathbb{R}^3} \frac{< k, c^{(0)}(k-k') > e^{-s|k-k'|^2} P_k c^{(1)}(k', s) dk'}{|k-k'|^2 \cdot |k'|^2} \\
&+ A \int_{\mathbb{R}^3} \frac{< k, c^{(1)}(k-k', s) > P_k c^{(0)}(k') e^{-|k'|^2} dk'}{|k-k'|^2 \cdot |k'|^2} \\
&+ A \int_{\mathbb{R}^3} \frac{< k, c^{(1)}(k-k', s) > P_k c^{(1)}(k', s) dk'}{|k-k'|^2 \cdot |k'|^2} = \\
&\left. \text{def } \int_0^t e^{-(t-s)|k|^2} ds \cdot |k|^2 [I_1(k, s) + I_2(k, s) + I_3(k, s) + I_4(k, s)] \right].
\end{align*}
\]

Take $t > 0$ and make the rescaling: $s = \xi \cdot t$, $k = x t^{-\frac{1}{2}}$. 

Using the formula
\[ a_1|k - k'|^2 + a_2|k'|^2 = \frac{1}{a_1^{-1} + a_2^{-1}} |k|^2 + a_2|k| - \frac{a_1}{a_1 + a_2} |k|^2 \]
we can write
\[
\int_0^t e^{-(t-s)|k|^2} ds \cdot |k|^2 \cdot A^2 \int_{R^3} \frac{< k, c^{(0)}(k - k') >}{|k - k'|^2} \cdot \frac{P_k c^{(0)}(k') e^{-s|k-k'|^2} dk'}{|k'|^2}
\]
\[ = \int_0^1 e^{-(1-\xi)|x|^2} d\xi \cdot x^2 \cdot A^2 \cdot e^{-\frac{2}{\xi}|x|^2} \int_{R^3} \frac{< x, c^{(0)}(\frac{x'}{\sqrt{t}} - \frac{x'}{\sqrt{t}}) >}{|x - x'|^2} \cdot \frac{P_x \cdot c^{(0)}(\frac{x'}{\sqrt{t}}) e^{-\xi|x-x'|^2} dx'}{|x'|^2} \cdot . \quad (3)
\]
It follows from the Basic Assumption that
\[ c^{(0)}(\frac{x}{\sqrt{t}} - \frac{x'}{\sqrt{t}}) - w(\frac{x - x'}{|x|}) \rightarrow 0, c^{(0)}(\frac{x'}{\sqrt{t}}) - w(\frac{x'}{|x'|}) \rightarrow 0 \]
as \( t \rightarrow 0 \). Therefore (3) converges to the limit
\[
A^2 \int_0^t e^{-(1-\xi)|x|^2} d\xi \cdot |x|^2 \cdot e^{-\frac{2}{\xi}|x|^2} \int_{R^3} \frac{< x, w(\frac{x-x'}{|x-x'|}) >}{|x - x'|^2} \cdot \frac{P_x w(\frac{x'}{|x'|}) e^{-\xi|x-x'|^2}}{|x'|^2} \cdot \quad (4)
\]
Non-degeneracy condition which we meant in the formulation of the theorem just says that the integral in (1) is non-zero. Thus we have that the first term in (2) for \( x \sim \frac{1}{\sqrt{t}} \) is proportional to \( A^2 \) and in the main order of magnitude the coefficient near \( A^2 \) does not depend on \( t \). In the Appendix 1 we estimate the other terms in (2). The estimates show that they tend to zero as \( t \rightarrow 0 \) and this gives the statement of the theorem.

Comments.

1. Critical case \( \alpha = \omega = 2 \) is remarkable because after rescaling main terms do not depend on \( t \) explicitly.

2. For the main theorem only the behavior of \( c^{(0)}(k) \) at infinity is important. Therefore \( c^{(0)}(k) \) can tend to zero as \( k \rightarrow 0 \) so that \( v^{(0)}(k) \) has the finite energy. But the enstrophy \( \Omega = \int |v^{(0)}(k)||k|^2 dk = \infty \).

3. The main theorem gives the precise meaning to the intuitive feeling that for sufficiently large \( A \) the iteration scheme corresponding to (1) diverges.
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Appendix 1. Estimates of $I_2, I_3, I_4$

We shall estimate

$$E_2 = \int_0^t e^{-(t-s)|k|^2} ds \cdot |k|^2 \cdot \int_{R^3} \frac{< k, c^{(0)}(k-k') > e^{-s|k-k'|^2} P_k c^{(1)}(k', s) dk'}{|k-k'|^2 \cdot |k'|^2}$$

The functions $c^{(0)}, c^{(1)}$ satisfy the inequalities:

1. $|c^{(0)}(k)| \leq c^{(0)}$
2. $|c^{(1)}(k', s)| \leq \epsilon(s)$

where $\epsilon(s)$ is continuous function on $[0, t_0]$ and $\epsilon(s) \to 0$ as $s \to 0$; by this reason we may assume that $\epsilon(s) \leq \epsilon, 0 \leq s \leq t_0$ for any given $\epsilon$ and appropriate $t_0$.

We have

$$|E_2| \leq \int_0^t e^{-(t-s)|k|^2} ds \cdot |k|^3 \cdot c^{(0)} \cdot \epsilon \cdot \int_{R^3} \frac{dk'}{|k-k'|^2 \cdot |k'|^2}. \quad (5)$$

It is easy to check that for some constant $B_1$

$$\int_{R^3} \frac{dk'}{|k-k'|^2 \cdot |k'|^2} \leq \frac{B_1}{|k|}.$$

Therefore, the right-hand side of (5) is not more than

$$\int_0^t e^{-(t-s)|k|^2} ds \cdot |k|^3 \cdot c^{(0)} \cdot \epsilon \cdot B_1 \leq (1 - e^{-t|k|^2}) \cdot c^{(0)} \cdot \epsilon \cdot B_1 \leq c^{(0)} \cdot \epsilon B_1.$$

This is the estimate which we need. The estimation of $E_3$,

$$E_3 = \int_0^t e^{-(t-s)|k|^2} ds \cdot |k|^2 \cdot \int_{R^3} \frac{< k, c^{(1)}(k-k', s) > e^{-s|k'|^2} P_k c^{(0)}(k') dk'}{|k-k'|^2 \cdot |k'|^2}$$

is done in a similar way. The estimate of $E_4$,

$$E_4 = \int_0^t e^{-(t-s)|k|^2} ds \cdot |k|^2 \int_{R^3} \frac{< k, c^{(1)}(k-k', s) > P_k c^{(1)}(k', s) dk'}{|k-k'|^2 \cdot |k'|^2}$$

is also simple and $|E_4| \leq \epsilon^2 \cdot B_1$. This completes the proof of the theorem.