Absence of the Local Existence Theorem in the Critical Space for the $3\mathcal{D}$-Navier-Stokes System

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Dedicated to: L. P. Shilnikov on the occasion of his 70th birthday

Abstract

We consider the $3\mathcal{D}$-Navier-Stokes system (NSS) on R^3 without external forcing. After Fourier transform it becomes the system of non-linear integral equations. For one-parameter families of initial conditions $\frac{A\cdot c^{(0)}(k)}{|k|^2}$ it is known that if $|A|$ is sufficiently small then NSS has global solution. We show that if $c^{(0)}$ satisfies some natural conditions at infinity then for sufficiently large A NSS has no local solutions with this initial condition.

Keywords: Navier-Stokes system, Fourier transform, critical space.

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Consider the $3\mathcal{D}$-Navier-Stokes System on R^3 without external forcing and viscosity $\nu = 1$. After Fourier transform it can be written as

$$v(k,t) = e^{-|k|^2 t} v^{(0)}(k) + i \int_0^t e^{-|k|^2(t-s)}\, ds \,.$$

$$\int_{R^3} < k,\, v(k-k',s) > P_k\, v(k',s)\, dk' \,. \tag{1}$$

Here $k \in R^3$, $v(k,t) \perp k$ for any k and P_k is the orthogonal projection to the subspace orthogonal to k.

In [1], see also [2], the subspaces $\Phi(\alpha,\omega)$ were introduced. By definition, $v(k) \in \Phi(\alpha,\omega)$ *iff*

1) $v(k) = \frac{c(k)}{|k|^\alpha}$, $c(k) \in R^3$, $c(k) \perp k$ for $|k| \le 1$ where $c(k)$ is continuous outside $k=0$, $\sup_{|k|\le 1,\ k\neq 0} |\, c\,(k)| = c < \infty$;

2) $v(k) = \frac{d(k)}{|k|^\omega}$, $d(k) \in R^3$, $d(k) \perp k$ for $|k| \ge 1$ and $d(k)$ is continuous, $\sup_{|k|\ge 1} |d(k)| = d < \infty$.

We put $\| v \| = c + d$. With this norm the curve $e^{-t|k|^2} v^{(0)}(k), v^{(0)}(k) \in \Phi(\alpha,\omega)$ may not be continuous at $t = 0$ in the sense that $\| v^{(0)}(k) e^{-t|k|^2} - v^{(0)}(k) \|$ may not tend to zero as $t \to 0$. In the spaces $\Phi(\alpha,\omega)$, $0 \le \alpha < 3$, $\omega > 2$ the local existence theorem is valid (see [1]). More precisely (see [1]),

Theorem 1 *Let $v^{(0)} \in \Phi(\alpha,\omega)$. Then for some $t_0 > 0$ depending on α,ω and $v^{(0)}$ there exists $v(k,t) = e^{-t|k|^2} v^{(0)}(k) + v^{(1)}(k,t)$, $0 \le t \le t_0$ where $v^{(1)}(k,t) \in \Phi(\alpha,\omega)$ and the family $\{v^{(1)}(k,t),\, 0 \le t \le t_0\}$ is continuous in $\Phi(\alpha,\omega)$ including $t = 0$.*

This means that $\| v^{(1)}(k,t) \| \to 0$ as $t \to 0$. In the so-called critical case $\alpha = \omega = 2$ where $v(k,t) = \frac{c(k,t)}{|k|^2}$, Le Jan and Sznitman (see [3]) and later Cannone and Planchon (see [4]) proved that if $\| v^{(0)} \|$ is sufficiently small then there exists global solution of (1) defined for all $t > 0$.

In this paper we consider $v^{(0)}(k)$ satisfying some regularity condition at ∞. Namely, for each $r > 0$ consider the sphere $S_r = \{k : |k| = r\}$. The condition $v(k) \perp k$ implies that for each r we have the vector field on S_r consisting of vectors $v(k)$, $|k| = r$.

Basic Assumption: There exist a continuous vector field $w = \{w(k)\,, |k| = 1\}$ on the unit sphere such that

$$\max_{k \in S_r} |\, c^{(0)}(k) - w\left(\frac{k}{|k|}\right) | \to 0 \text{ as } |k| \to \infty \,.$$

This assumption implies the existence of the limits of $c^{0)}(k)$ when $k \to \infty$ along any direction.

Having $c^{(0)}(k)$ satisfying the basic assumption take a one-parameter family of initial conditions $v_A^{(0)}(k) = \frac{A}{|k|^2} \cdot c^{(0)}(k)$, $A > 0$. For sufficiently small A the result by Le Jan and Sznitman and Cannone and Planchon can be applied and it gives the existence of global solution $v_A(k,t)$. The purpose of this paper is to prove the following theorem.

Main Theorem: *Let $c^{(0)}(k)$ satisfy the basic assumption and some non-degeneracy condition (see below). For all sufficiently large $A, A \geq A_1$ there does not exist a solution of (1) $v(k,t) = \frac{c(k,t)}{|k|^2} = \frac{Ae^{-t|k|^2}c^{(0)}(k)}{|k|^2} + \frac{c^{(1)}(k,t)}{|k|^2}$ where $\sup\limits_{k \in R^3 \setminus 0} |c^{(1)}(k,t)| \to 0$ as $t \to 0$.*

Certainly, $c^{(1)}(k,t)$ may depend on A. To prove the theorem we show that for any $t > 0$ and $k \sim O\left(\frac{1}{\sqrt{t}}\right)$ the solution $c(k,t)$ takes values of order A^2 and hence cannot be small, i.e. $\| v(k,t) - v^{(0)}(k) \| \sim O(A^2)$ for all sufficiently small t, where O does not depend on t.

<u>Proof</u>: We write $v(k,t) = \frac{Ae^{-t|k|^2}c^{(0)}(k)}{|k|^2} + \frac{c^{(1)}(k,t)}{|k|^2}$.

From (1)

$$
\begin{aligned}
c^{(1)}(k,t) \;=\; & i\int_0^t e^{-(t-s)|k|^2}\, ds \cdot |k|^2 \cdot \\
& \left[A^2 \int_{R^3} \frac{< k, c^{(0)}(k-k') > \; P_k \cdot c^{(0)}(k') e^{-s|k-k'|^2 - s|k'|^2}\, dk'}{|k-k'|^2 \cdot |k'|^2}\right. \\
& + A \int_{R^3} \frac{< k, c^{(0)}(k-k') > \; e^{-s|k-k'|^2}\, P_k c^{(1)}(k',s)\, dk'}{|k-k'|^2 \cdot |k'|^2} + \\
& + A \int_{R^3} \frac{< k, c^{(1)}(k-k',s) > \; P_k c^{(0)}(k')\, e^{-s|k'|^2} dk'}{|k-k'|^2 \cdot |k'|^2} \\
& + A \int_{R^3} \frac{< k, c^{(1)}(k-k',s) > \; P_k c^{(1)}(k',s)\, dk'}{|k-k'|^2 \cdot |k'|^2} = \\
\underline{\underline{\text{def}}}\; & i \int_0^t e^{-(t-s)|k|^2}\, ds \cdot |k|^2 \left[I_1(k,s) + I_2(k,s) + I_3(k,s) + I_4(k,s)\right]. \qquad (2)
\end{aligned}
$$

Take $t > 0$ and make the rescaling: $s = \xi \cdot t$, $k = xt^{-\frac{1}{2}}$.

Using the formula

$$a_1|k-k'|^2 + a_2|k'|^2 = \frac{1}{a_1^{-1}+a_2^{-1}}|k|^2 + a_2|k' - \frac{a_1}{a_1+a_2}k|^2$$

we can write

$$\int_0^t e^{-(t-s)|k|^2}\,ds \cdot |k|^2 \cdot A^2 \int_{R^3} \frac{< k, c^{(0)}(k-k') > P_k\, c^{(0)}(k')\, e^{-s|k-k'|^2 - s|k'|^2}\, dk'}{|k-k'|^2 \cdot |k'|^2}$$

$$= \int_0^1 e^{-(1-\xi)|x|^2}\, d\xi \cdot x^2 \cdot A^2 \cdot e^{-\frac{\xi}{2}\cdot|x|^2} \cdot$$

$$\int_{R^3} \frac{< x, c^{(0)}\left(\frac{x}{\sqrt{t}} - \frac{x'}{\sqrt{t}}\right) > P_x \cdot c^{(0)}\left(\frac{x'}{\sqrt{t}}\right)\, e^{-\xi|x' - \frac{x}{2}|^2}\, dx'}{|x-x'|^2 \cdot |x'|^2}\,. \tag{3}$$

It follows from the Basic Assumption that

$$c^{(0)}\left(\frac{x}{\sqrt{t}} - \frac{x'}{\sqrt{t}}\right) - w\left(\frac{x-x'}{|x-x'|}\right) \to 0, c^{(0)}\left(\frac{x'}{\sqrt{t}}\right) - w\left(\frac{x'}{|x'|}\right) \to 0$$

as $t \to 0$. Therefore (3) converges to the limit

$$A^2 \int_{0^1} e^{-(1-\xi)|x|^2}\, d\xi \cdot |x|^2 \cdot e^{-\frac{\xi}{2}|x|^2} \int_{R^3} \frac{< x, w\left(\frac{x-x'}{|x-x'|}\right) > P_x\, w\left(\frac{x'}{|x'|}\right)\, e^{-\xi|x'-\frac{x}{2}|^2}}{|x-x'|^2 \cdot |x'|^2} \tag{4}$$

Non-degeneracy condition which we meant in the formulation of the theorem just says that the integral in (1) is non-zero. Thus we have that the first term in (2) for $x \sim \frac{1}{\sqrt{t}}$ is proportional to A^2 and in the main order of magnitude the coefficient near A^2 does not depend on t. In the Appendix 1 we estimate the other terms in (2). The estimates show that they tend to zero as $t \to 0$ and this gives the statement of the theorem.

Comments.

1. Critical case $\alpha = \omega = 2$ is remarkable because after rescaling main terms do not depend on t explicitly.

2. For the main theorem only the behavior of $c^{(0)}(k)$ at infinity is important. Therefore $c^{(0)}(k)$ can tend to zero as $k \to 0$ so that $v^{(0)}(k)$ has the finite energy. But the enstrophy $\Omega = \int |v^{(0)}(k)||k|^2\, dk = \infty$.

3. The main theorem gives the precise meaning to the intuitive feeling that for sufficiently large A the iteration scheme corresponding to (1) diverges.

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Appendix 1. Estimates of I_2, I_3, I_4

We shall estimate

$$\mathcal{E}_2 = \int_0^t e^{-(t-s)|k|^2}\, ds \cdot |k|^2 \cdot \int_{R^3} \frac{< k, c^{(0)}(k-k') > e^{-s|k-k'|^2}\, P_k\, c^{(1)}(k',s)\, dk'}{|k-k'|^2 \cdot |k'|^2}$$

The functions $c^{(0)}$, $c^{(1)}$ satisfy the inequalities:

1. $|c^{(0)}(k)| \le c^{(0)}$
2. $|c^{(1)}(k',s)| \le \epsilon(s)$

where $\epsilon(s)$ is continuous function on $[0,t_0]$ and $\epsilon(s) \to 0$ as $s \to 0$; by this reason we may assume that $\epsilon(s) \le \epsilon$, $0 \le s \le t_0$ for any given ϵ and appropriate t_0.

We have

$$|\mathcal{E}_2| \le \int_0^t e^{-(t-s)|k|^2}\, ds \cdot |k|^3 \cdot c^{(0)} \cdot \epsilon \cdot \int_{R^3} \frac{dk'}{|k-k'|^2 \cdot |k'|^2}\,. \tag{5}$$

It is easy to check that for some constant B_1

$$\int_{R^3} \frac{dk'}{|k-k'|^2 \cdot |k'|^2} \le \frac{B_1}{|k|}\,.$$

Therefore, the right-hand side of (5) is not more than

$$\int_0^t e^{-(t-s)|k|^2}\, ds \cdot |k|^2 \cdot c^{(0)} \cdot \epsilon \cdot B_1 \le (1-e^{-t|k|^2}) \cdot c^{(0)} \cdot \epsilon \cdot B_1 \le c^{(0)} \cdot \epsilon\, B_1\,.$$

This is the estimate which we need. The estimation of $\mathcal{E}_3$,

$$\mathcal{E}_3 = \int_0^t e^{-(t-s)|k|^2}\, ds \cdot |k|^2 \cdot \int_{R^3} \frac{< k, c^{(1)}(k-k',s) > e^{-s|k'|^2}\, P_k\, c^{(0)}(k')\, dk'}{|k-k'|^2 \cdot |k'|^2}$$

is done in a similar way. The estimate of $\mathcal{E}_4$,

$$\mathcal{E}_4 = \int_0^t e^{-(t-s)|k|^2}\, ds \cdot |k|^2 \int_{R^3} \frac{< k, c^{(1)}(k-k',s) >\ P_k\, c^{(1)}(k',s)\, dk'}{|k-k'|^2 \cdot |k'|^2}$$

is also simple and $|\mathcal{E}_4| \le \cdot \epsilon^2 \cdot B_1$. This completes the proof of the theorem.