Kolmogorov Lecture

Renormalization Group Method in Probability Theory and Theory of Dynamical Systems

Ya. G. Sinai*

01.29.2007

Royal Holloway, University of London

*Mathematics Department, Princeton University, Princeton, New Jersey, U.S.A.

§1. Introduction

I was a graduate student in the Mathematics department of Moscow State University during the years 1957-1960. A.H. Kolmogorov was my thesis advisor and he was my advisor for one more year when I was an undergraduate student. It is widely-known that Kolmogorov had many students among which there were such great people as I.M. Gelfand, S.M. Nikolski, A.M. Obukhov, Yu. V. Prokhorov and others. However, literally speaking, Kolmogorov didn't teach many people around him. He supplied his students with interesting challenging problems, was always open for discussion, could suggest some general direction of thought but not much more. I don't know cases when Kolmogorov formulated a problem and provided its solution.

Due to his enormous intellectual power and great energy, Kolmogorov worked in many fields of mathematics and in each field he was one of the leading experts, in many cases, the best. Somehow this explains why Kolmorov's students worked in so many different directions.

It is a common point of view that the main field for Kolmogorov were probability theory and mathematical statistics. He had ground-breaking results in each part of these subjects. Even during his last active years he published papers where he generalized results of other mathematicians, improved the inequalities, asymptotics, etc. I am quite sure that if Kolmogorov were active now, he would work in such new topics as stochastic Loewner equations, percolation, etc.

The main content of this lecture is the renormalization group method (RGM). This method appeared in statistical mechanics and quantum field theory. It is connected with scaling ideas and limit theorems in probability theory. In early works by physicists on RGM there were the references to Kolmogorov works on turbulence. Two new ideas which were not known before were brought from physics. Namely, the idea that for a stationary field (on a lattice) the transition to the distribution of normalized sums over cells of sub-lattice is a non-linear transformation in the functional space of all possible probability distributions. From this point of view, limiting distributions of probability theory are fixed points of the corresponding semi-group of these transformations. The second idea directly related to the first one is natural from the point of view of non-linear analysis and is related to stability properties of these fixed points. We shall discuss below. Actually, RGM mostly works in cases when we have families of probability distributions depending on one or several parameters. The basic problem is to show that for some special values of parameters there appear non-trivial limiting distributions which are fixed points of the mentioned above transformations and are separating points of different regimes like in the theory of phase transitions. Usually, the situation is very far from the well-studied case of sequences of independent or weakly dependent random variables. In this sense, one can say that RGM provides a natural approach to limiting theorems for strongly dependent random variables.

From the point of view of the theory of dynamical systems the situation looks very familiar. We have a fixed point of some non-linear transformation and consider the spectrum of the linearized map near this point. If the number of unstable ($|\lambda| > 1$) and neutral ($|\lambda| = 1$) eigen-values is finite then the fixed point has a stable manifold $\Gamma^{(s)}$ which consist of points approaching to the fixed point under the iterations of our transformation. If d is the sum of the number of unstable and neutral eigen-values then there is an open set in the space of d-parameter families such that each family from this set intersects $\Gamma^{(s)}$ in a single point. In probability theory this gives a limiting theorem of the type mentioned above.

In §2 we discuss in more detail some applications of RGM to probability theory, in §3 we consider several problems from the theory of dynamical systems which can be attacked with the help of RGM. In §4 we describe the application of RGM to the problem of blow-ups of complex solutions of 3-dimensional Navier-Stokes system.

§2. RGM in Probability Theory

Consider the simplest case of a sequence of *iidrv* $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ having a density $p(x)$. For simplicity we assume that $p(x)$ is even and $\int_{-\infty}^{\infty} x^2 p(x)\,dx = 1$. Let $\zeta_m = \frac{1}{2^{m/2}}\ (\xi_1 + \xi_2 + \cdots + \xi_{2^m})$. Then $\zeta_{m+1} = \frac{\zeta'_m + \zeta''_m}{\sqrt{2}}$ where ζ'_m, ζ''_m are independent random variables having the same distribution as ζ_m. If $p_m(x)$ is the density of distribution of ζ_m then

$$p_{m+1}(x) = \sqrt{2} \int_{-\infty}^{\infty} p_m\left(x\sqrt{2} - y\right)\, p_m(y)\,dy\,. \tag{1}$$

The formula (1) shows that $p_{m+1} = f(p_m)$ where f is the quadratic map given by the *rhs* of (1). The purpose of probability theory is to study the behavior of p_m as $m \longrightarrow \infty$. As the first step we find fixed points of f, i.e., the densities q for which

$$q(x) = \sqrt{2} \int_{-\infty}^{\infty} q\left(x\sqrt{2} - y\right)\, q(y)\,dy\,. \tag{2}$$

It is easy to check that the family of Gaussian densities $q_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{x^2}{2\sigma}\right\}$, $\sigma > 0$ is a one-parameter family of such fixed points. The next step of RGM is the linearization of (2) near q_σ and the analysis of the spectrum of the corresponding linearization. The linear map L is given by the linear integral operator L_σ:

$$L_\sigma\, h(x) \;=\; 2\sqrt{2} \int\limits_{-\infty}^{\infty} h\left(x\sqrt{2} \,-\, y\right)\, q_\sigma(y)\, dy\,.$$

Its eigen-functions are given by Hermite functions $He_m(x)\, g_\sigma(x)$. The corresponding eigen-values are $\lambda_m = \frac{1}{2^{\frac{m}{2}-1}}$. Since we consider the space of even probability densities m must be even. Since we consider the space of probability densities the projection to the zeroth eigen-vector is zero. Then $\lambda_2 = 1$ is a neutral eigen-value. It is connected with the fact that we have a family of fixed points depending on σ and the second moment of the distribution is the first integral of f. The remaining part of the spectrum is stable. The methods of non-linear analysis allow to prove the following theorem.

Theorem 1. *Let* $p_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} (1 + h(x))$ *where* h *is even, small in*

$L^2(R^1\,,\, \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\})$

and

$$\int\limits_{-\infty}^{\infty} h(x)\, \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \;=\; \int\limits_{-\infty}^{\infty} h(x)\, \frac{x^2}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx = 0\,.$$

Then $f^m(p_1) \longrightarrow \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$ *in the sense of* L^2.

The statement of the theorem is a local version of the central limit theorem. Probability theory has special tools to prove the global version of Theorem 1 and global stability of the Gaussian fixed point. However, non-linear methods allow to prove Theorem 1 without the assumption that $p_1(x) \geq 0$.

There is a modification of (1) which is connected with the so-called Dyson hierarchical model in statistical mechanics. Namely, consider a family of transformations $f_\beta^{(1)}$, $p_1 = f_\beta^{(1)}(p)$ where $1 < c < 2$, $0 < \beta < \infty$ and

$$p_1(x;\beta) \;=\; L(\beta)\,\exp\left\{\beta x^2\right\} \int\limits_{-\infty}^{\infty} p\left(\frac{x}{\sqrt{c}} \,+\, u;\,\beta\right)\, p\left(\frac{x}{\sqrt{c}} \,-\, u;\beta\right)\, du\,, \tag{3}$$

$L(\beta)$ is the normalizing factor so that $f_\beta^{(1)}$ acts in the space of probability distributions.

The equation (3) generates the equation for the fixed point

$$q(x;\beta) \;=\; \exp\left\{\beta x^2\right\} \int\limits_{-\infty}^{\infty} q\left(\frac{x}{\sqrt{c}} \,+\, u;\,\beta\right)\, q\left(\frac{x}{\sqrt{c}} \,-\, u;\,\beta\right)\, du\,. \tag{4}$$

As before (4) has a curve of Gaussian solutions $q(x;\,\beta) \;=\; \sqrt{\frac{\beta c}{\pi(2-c)}}\,\exp\left\{-\,\frac{\beta c\,x^2}{2-c}\right\}$. However, these solutions are stable only if $\sqrt{2} < c < 2$. For $c < \sqrt{2}$ there appear new fixed points which decay approximately as $\exp\{-O(1)x^4\}$. The analysis of the Gaussian case and the construction of new points with the help of bifurcation theory were done in our joint papers with Bleher (see [BS1], [BS2]). A detailed exposition of the whole theory can be found in the book [CE] by P. Collet and J-P Eckmann.

Consider now a more general case. Take a stationary random field $\{\xi(n), n \in \mathbb{Z}^d\}$. Denote by P its probability distribution. Then $\xi'(n) = \frac{1}{2^{d\gamma}} \sum\limits_{e} \xi(2n{+}e)$ is again a stationary random field. Here $2n = (2n_1, 2n_2, \ldots, 2n_d)$ and e is any d-dimensional vector whose components are 0 or 1. The transition from the probability distribution P to the probability distribution P' of all random variables $\xi'(n)$ is sometimes called Kadanoff block-spin transformation (see [Ka]).

Probability distribution P is called scale-invariant (for given γ) if $P' = P$. The condition of scale-invariance is an infinite-dimensional analog of the equation for the fixed point of RGM (see above). The next step is to find scale-invariant Gaussian fields.

Theorem 2. *Let $f(\lambda_1, \ldots, \lambda_d)$ be an homogeneous positive function of degree $d(\gamma{+}1)$. Then the function*

$$\rho(\lambda_1, \ldots, \lambda_d) \;=\; \prod_{j=1}^{d} |e^{2\pi i\,\lambda_1} \;-1|^2 \sum_{m\in\mathbb{Z}^d} \frac{1}{f(\lambda) + m}$$

is the spectral density of the scale-invariant Gaussian random field with this γ.

The continuous version of the scale-invariant random field appeared actually in the Kolmogorov's paper [Ko], see also the paper by R. Dobrushin [D].

The case $f(\lambda_1, \lambda_2, \ldots, \lambda_d) = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_d^2$ and $\gamma = \frac{2}{d} - 1$ is of special importance. The Gibbs distribution corresponding to this field can be written as $\sim \exp\left\{-\frac{1}{2}\langle \nabla\xi, \nabla\xi\rangle\right\}$. Sometimes it is called Gaussian free field. The interaction corresponding to the Lagrangian of this field is short-ranged. Therefore it is natural to expect that this field can appear as the limiting distribution in the Ising-type models at the critical temperature. However, this Gaussian scale-invariant field is stable only if its dimension $d \geq 5$. The case $d = 4$ is marginal and requires a non-standard normalization. There were many papers and books by physicists dedicated to all these questions and we mention only the book by L. Kadanoff [Ka] and the book by Patashinski and Pokrovski [PP]. Mathematical results can be found in the papers by Aizenman [A], Frohlich [Fr], Gawedski and Kupiainen [GK]. The modern development of this topic is connected with conformal quantum field theory and Loewner stochastic equations.

§3. RGM in the Theory of Dynamical Systems

Fractal theory provides many examples of scale invariance and in this sense can be considered as related to RGM. The first case where RGM was used in dynamics, was connected with the universality in sequences of period-doubling bifurcations discovered by M. Feigenbaum.

Consider the quadratic family of one-dimensional maps $x \longrightarrow ax(1-x) = f_a(x)$, $0 \leq x \leq 1$ and $0 \leq a \leq 4$. It has a sequence of parameters a_n such that in a small neighborhood (a'_n, a_n) the maps f_a have stable periodic orbits of period 2^n. As $a \uparrow a_n$ this orbit loses its stability and for $a > a_n$ becomes unstable. At the same time a new stable periodic orbit of period 2^{n+1} appears. This bifurcation is called period-doubling bifurcation and the sequence of such bifurcations is typical in one-parameter families of one-dimensional maps. Usually $a_n \longrightarrow a_\infty$ as $n \longrightarrow \infty$. Feigenbaum discovered (see [F1], [F2]) that a_n converges to a_∞ as a geometric progression whose exponent is a universal constant which is called Feigenbaum constant. There were many mathematical papers related to this problem. An interested reader can find a lot of information in the book by W. de Melo and S. Van Strien (see [MS]).

The theory starts with the equation for the fixed point which now takes the form

$$\psi(x) = -\theta\,\psi(\psi(\theta^{-1}x))\,,\ \theta = -\frac{1}{\psi(1)} \tag{5}$$

where $-1 \le x \le 1$, $\psi(0) = 1$ and ψ is convex. The first proofs of existence of its solutions were computer-assisted (Coulet, Tresser, Lanford, Epstein, Eckmann, Collett and others). Later Sullivan gave a complete proof without any use of computer (see [Su1]).

The next step is the analysis of the spectrum of linearization near the solution of (5). The corresponding operator is not self-adjoint but nevertheless its spectrum can be analyzed. It is important that it has the eigen-value $\lambda_1 > 1$ which determines the universality in period-doubling bifurcations.

Khanin and I used RGM in the theory of rectifying circle maps. Let f be such that it is strictly monotone, continuous and $f(x+1) = f(x) + 1$. Any such function f generates a homeomorphism of the unit circle $x \longrightarrow \{f(x)\}$. It follows from Denjoy theory that if $f \in C^1$ and the rotation number is irrational then there exists the change of variables $y = \chi(x)$ such that in the variable y the homeomorphism φ is a rotation, $\varphi(y) = y + \rho$. The basic problem which sometimes is called Arnold problem is to study the dependence of smoothness of χ on the smoothness of f and arithmetic properties of ρ. The basic results here were proven by M. Herman and J-C Yoccoz (see [H1], [Y1]).

Denote by q_n the denominator of the n-th approximant of ρ in the expansion of ρ into continued fraction. We consider φ^{q_n} on intervals of the length $O\left(\frac{1}{q^n}\right)$. It turns out that under very mild assumptions on the smoothness of f the map φ^{q_n} is asymptotically linear in the rescaled coordinates. This statement is of a RGM-character and it shows that linear map is a fixed point of RGM. The corresponding spectrum can be analyzed completely and this gives a detailed information about the smoothness of χ. Recently, Khanin and Teplitsky improved the previous technique and made basic proofs much shorter (see [KT]). There are many other problems of this type but we shall not discuss it here.

§4. RGM in the Problem of Blow-Ups of Solutions of the 3D-Navier-Stokes System

In this section I shall describe briefly recent results of Dong Li and myself (see [LS]) related to the famous sixth Clay problem. These results were obtained with the help of RGM and led to fixed points with unusual properties. This is one of the reasons why it is natural to discuss related technique and results in this lecture.

The Navier-Stokes System on R^3 without external forcing takes the form:

$$\mathrm{div}\ u = 0$$

$$\frac{\partial u}{\partial t} + (u, \nabla) u = \triangle u - \nabla p \tag{6}$$

Here $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, the viscosity is taken to be one and p is the pressure. If we write the Fourier transform of u as $-iv(k,t)$ then for $v(k,t)$ we have the equation

$$v(k,t = \exp\left\{-t|k|^2\right\} v(k,0) + \int_0^t \exp\{-(t-s)|k|^2\} \, ds$$
$$\int_{R^3} < v(k-k', s), k > P_k v(k', s) \, dk' . \tag{7}$$

In the last expression $v(k,t) \perp k$ and P_k is the orthogonal projection to the plane orthogonal to k.

We study real solutions of (7). They do not satisfy the energy inequality which means that the energy can grow with time.

The main result of [LS] says that there exists an open set in the space of 10-parameter families of initial conditions such that for each family from this set one can find an interval on the time axis $s = [s^-, s^+]$ such that for each $t \in S$ there exist values of parameters for which the solution blows up in finite time t or so that $E(s) \to \infty$, $\Omega(s) \to \infty$ as $s \to t$ where $E(t)$, $\Omega(t)$ are the energy and the enstrophy.

The proof uses special power series which were proposed earlier in [S2] for solutions of the Navier-Stokes system. RGM is needed for the analysis of asymptotics of elements of the series. The number 10 is the sum of the number of unstable and neutral eigen-values of the corresponding fixed point. Our initial conditions are concentrated in neighborhoods of the type $\{(k_1, k_2, k_3) : |k_3 - k| \le C_1\sqrt{k}, \ |k_2| \le C_1\sqrt{k}, \ |k_1| \le C_1\sqrt{k}\}$. where C_1, k are large parameters. The equations for the coefficients of the series resemble convolutions in probability theory. Therefore their analysis has a clear probabilistic flavor. Details can be found in [LS].

References

[A] M. Aizenman, "Geometric Analysis of φ^4-fields and Ising models," *Comm. in Math. Physics*, (1982), v. 86, p. 1-48.

[BS1] P.M. Bleher and Ya. G. Sinai, "Investigation of the Critical Point in Models of the Type of Dyson's Hierarchical Model," *Comm. in Math. Physics*, (1973), v. 33, N1, p. 23-42.

[BS2] P.M. Bleher and Ya. G. Sinai, "Critical Indices for Dyson's Asymptotically-hierarchical Models," *Comm. in Math. Physics*, (1975), v. 45, p. 247.

[CE] P. Collet, J-P. Eckmann, "A Renormalization Group Analysis of Hierarchical Model in Statistical Mechanics," Lecture Notes in Physics, (1978), v. 74, Springer-Verlag.

[D] R. L. Dobrushin, "Automodel Randon Fields," *Annals of Probability*, (1979), v. 7, N1, p. 1.

[F1] M. Feigenbaum, "Quantative Universality for a class of non-linear transformations, *Journal of Stat. Physics*, (1978), v. 19, p. 25-52.

[F2] M. Feigenbaum, "The universal metric properties of non-linear transformations," *Journal of Stat. Physics*, (1979), **21**, p. 669-706.

[Fr] J. Fröhlich, "On the triviality of $\lambda\phi^d$-theories and the approach to the critical point in d-dimensions," *Nuclear Physics* B, (1982), **200**, p. 281-296.

[GK] K. Gawedzki, A. Kupiainen, "Massless lattice φ^4_4-theory: Rigorous Control of a renormalizable asymptotically free models," *Comm. in Math. Physics*, (1985), **99**, p. 195.

[H] M. Herman, "Sur la conjugasion differentiable des difféomorphisms du cercle á des rotátions," *Publ. Math. IHES*, (1979), **49**, p. 5-233.

[Ka] L. Kadanoff, "Statistical Physics," World Scientific Publishing Co., River Edge, NJ, (2000), p. 483.

[KS] K. M. Khanin and Ya. G. Sinai, "Smoothness of conjugacies of diffeomorphisms of the circle to rotations," *Russian Math. Surveys*, (1989), v. 49, p. 57-82.

[KT] K. Khanin, A. Teplitsky, "Robust Rigidity for Circle Diffeomorphisms with Singularities," *Math. Inv.*, (submitted).

[Ko] A. N. Kolmogorov, "Wiener's Spiral and Some Interesting Curves in Hilbert Space," Doklady of Russian Academy of Sci., (1940), v. 26, p. 115-118.

[LS] Dong Li and Ya.G. Sinai, "Blow-Ups of Complex Solutions of the 3D-Navier-Stokes System and Renormalization Group Method," *Journal of American Math. Society*, (submitted).

[MS] W. de Melo and S. van Strien, "One-dimensional Dyanamics," Spring-Verlag, (1993).

[PP] A. Patashinski, V. Pokrovski, "Fluctuation Theory of Phase Transitions," *International Series in Natural Philosophy*, Pergamon Press, (1979), **98**, p. 321.

[S1] Ya. G. Sinai, "Theory of Phase Transitions, Rigorous results," North Holland, (1981).

[S2] Ya. G. Sinai, "Power Series for Solutions of the Navier-Stokes System on R^3," *Journal of Stat. Physics*, (2005), **121**, N 5/6, p. 779-804.

[Su] D. Sullivan, "Bounds, Quadratic Differentials and Renormalization Conjecture," *AMS Centennial Publications*, (1992), v. 2.

[Y] J-C. Yoccoz, "Conjugasion différentiables des difféomorphisms du cercle dont le nombre de rotation vérifie une conditions diophantienne," *Ann. Sci. Ec. Norm. Sup.*, (1984), v. 4, N17, p. 333-359.