The Two Eyes Lemma: a linking problem for horoball necklaces

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In the course of our work on low-volume hyperbolic 3-manifolds, we came upon a linking problem for horoball necklaces in $\mathbb{H}^3$. A horoball necklace is a collection of sequentially tangent beards (i.e. spheres) with disjoint interiors lying on a flat table (i.e. a plane) such that each bead is of diameter at most one and is tangent to the table. In this note, we analyze the possible configurations of an 8-bead necklace linking around two other diameter-one spheres on the table. We show that all the beads are forced to have diameter one, the two linked spheres are tangent, and that each bead must kiss (i.e. be tangent to) at least one of the two linked spheres. In fact, there is a 1-parameter family of distinct configurations.

1 Introduction

Start with a disc D of radius r in the Euclidean plane. What is the maximal number of discs of radius r with disjoint interiors that each *kiss* D? We say two discs *kiss* if they intersect on their boundaries but not in their interiors. The answer is 6, as can be seen by noting that the visual angle (as measured from the center of D) of a kissing disc is 60 degrees. Further, all such configurations are the same up to rotation about D, and the centers of the 6 discs are the vertices of a regular hexagon.

This leads to the classical kissing problem: what is the maximal number of equal radius spheres that simultaneously kiss a base sphere of the same radius? This question was the subject of a correspondence between Isaac Newton and James Gregory in the 17th century. Newton thought the answer was 12 but Gregory wondered whether 13 might work. Newton was correct, as was first proven in the nineteenth century. One could also ask about how many essentially distinct 12-kissings there are. It turns out that there are infinitely many that are fundamentally different and then one could ask for a description of this parameter space. Similarly, this question is of interest in higher dimensions. Good references for this material are the classic text "Sphere Packings, Lattices and Groups" by Conway and Sloane (Chapter 2) [CS99] and the semi-expository paper "The Twelve Spheres Problem" by Kusner, Kusner, Lagarias, and Shlosman [KKLS16].

In the course of our work on low-volume hyperbolic 3-manifolds [GHM$^+$], we faced a different generalization of the kissing problem. Here we came upon a cycle (or necklace) of ≤ 8 kissing

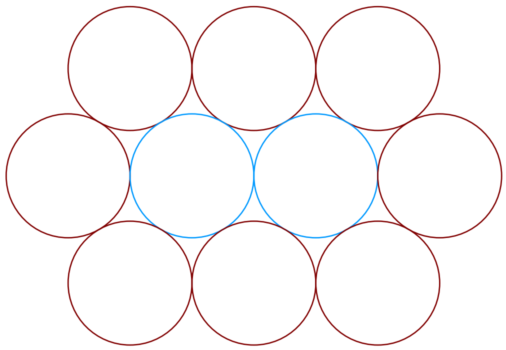

Figure 1: Hexagonal configuration.

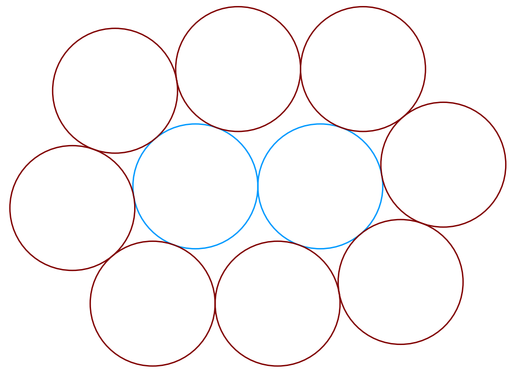

Figure 2: Non-hexagonal configuration.

spheres (or beads) of diameter at most one lying on a flat table. Suppose they link around spheres D_1 and D_2, also on the table, with disjoint interiors and of height (i.e. diameter) exactly one. As a consequence of the *Two-Eyes Lemma* (see below), we are able to prove that D_1 and D_2 must kiss, that each bead must kiss D_1 or D_2, and that each bead must be of height one. An example of this is obtained by taking a hexagonal packing of height-one spheres, labeling two abutters as D_1 and D_2, and then observing the cycle of 8 spheres encircling them. In fact, there is a 1-parameter family of essentially different solutions that is gotten by sliding one sphere along D_1 (or D_2) and then all other sphere positions are forced. Further, these are the only possible solutions. See Figures 1 and 2. We note that when all the beads are assumed to be of height one, our result reduces to a planar problem that is quite easy to address.

We are naturally lead to the following question, which we simply pose, but do not address. Given two abutting spheres of radius r in $\mathbb{R}^3$, what is the kissing number for these two spheres? That is, what is the maximal number of (non-overlapping) radius r spheres that each kiss either of the two abutting spheres?

2 Set-Up and Statement of Main Proposition

Definition 2.1 Consider the upper-half-space model of hyperbolic 3-space $\mathbb{H}^3$ with the standard projection $\pi : \mathbb{H}^3 \to \mathbb{R}^2$. We say that a horoball B is *full-sized* if radius$(\pi(B)) = 1/2$ and *less than full-sized* if radius$(\pi(B)) < 1/2$. Denote by *center(B)* the point at infinity of B.

Definition 2.2 A *k-necklace* $\eta = N_1 \cup \cdots \cup N_k$ is a cyclicly ordered set of k horoballs with disjoint interiors such that one is tangent to the next. In what follows indices for a k-necklace are always modulo k. The N_i's are called the *beads* and k is called the *necklace* or *bead number* of η. The hyperbolic geodesics connecting the centers of successive horoballs are called *ties*.

In this note, we will fix a horoball $H_\infty \subset \mathbb{H}^3$ centered at ∞ with ∂H_∞ a plane of Euclidean height 1 in the upper half-space model. We see that a horoball is full-sized (or *full*) if it is tangent to H_∞. We would like to understand how necklaces can wind around full-sized horoballs. The main result fo this note is

Proposition 2.3 *If C_1 and C_2 are full-sized horoballs with disjoint interiors then the minimum bead number of a necklace η with less-than-or-equal-to full-sized horoballs encircling C_1 and C_2 is 8. If the bead number is 8, then all horoballs in η must be full-sized; one example of this arises from the hexagonal packing of full-sized horoballs in the upper-half-space model. Further, all examples with bead number 8 are obtained by sliding N_1 in η along C_i and then placing the remaining N_i cyclically in turn making sure that each N_i abuts C_1 and/or C_2.*

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3 The Two-Eyes Lemma

Since we will be working with projections of horoballs to the plane, we will need the following useful formula.

Lemma 3.1 *(Horoball distance) Let B_1, B_2 be two horoballs with disjoint interiors in the upper half space model with $b_i = \partial_\infty B_i \in \mathbb{R}^2$ and of Euclidean height h_i. Then the hyperbolic distance $d_{\mathbb{H}}(B_1, B_2)$ between B_1, B_2 is given by*

$$d(B_1, B_2) = \log\left(\frac{d_{\mathbb{E}}(b_1, b_2)^2}{h_1 h_2}\right).$$

Proof Consider the point $b_2' = b_1 - (b_2 - b_1)$ and let γ be the geodesic between b_2, b_2'. Note that γ has Euclidean radius $d_{\mathbb{E}}(b_1, b_2)$. The highest point of γ lies directly above (or below) the highest point h_1 of B_1 in the upper-half-space model. In particular, we have the distance $d_{\mathbb{H}}(\gamma, B_1) = \log\big(d_{\mathbb{E}}(b_1, b_2)/h_1\big)$. Note that $d_{\mathbb{H}}(\gamma, B_1) > 0$ if and only if $\gamma \cap B_1 = \emptyset$. Rotating 180° around γ by an elliptic isometry, we see that B_1 has to map to a horoball at infinity of Euclidean height $d_{\mathbb{E}}(b_1, b_2)^2/h_1$. Since B_1, B_2 had disjoint interiors, it follows that $h_2 \le d_{\mathbb{E}}(b_1, b_2)^2/h_1$ and

$$d(B_1, B_2) = \log\left(\frac{d_{\mathbb{E}}(b_1, b_2)^2}{h_1 h_2}\right).$$

□

A direct corollary of this computation is a statement about visual angles.

Corollary 3.2 *(Visual Angle) Let C be a full-sized horoball and let B be an at most full-sized horoball tangent to C, then the visual angle of $\pi(B)$ from center(C) is $\leq \pi/3$ with equality if and only if B is full-sized.*

We now turn to the Two-Eyes Lemma, which is depicted in Figures 3, 4, 5 and 6.

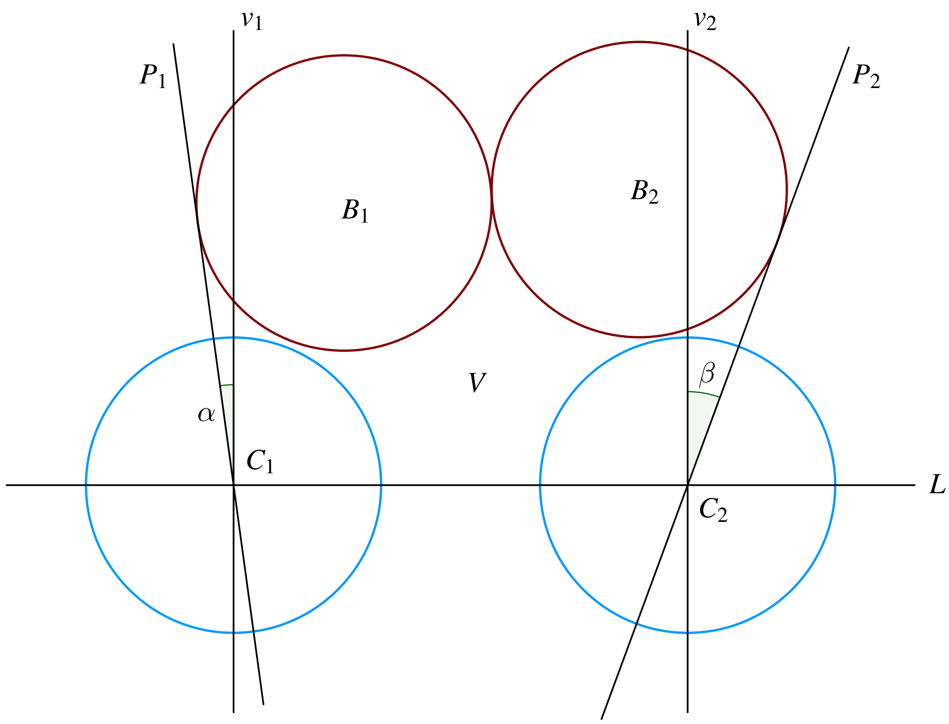

Figure 3: $\alpha + \beta \leq \pi/3$.

Lemma 3.3 *(Two-Eyes Lemma) Let C_1 and C_2 be full-sized horoballs with disjoint interiors. Let B_1 and B_2 be tangent horoballs with heights h_1 and h_2, respectively, with interiors disjoint from $C_1 \cup C_2$. Assume $h_i \leq 1$ for $i = 1, 2$. Let L be the line through center(C_1) and center(C_2) and let v_1 and v_2 be lines orthogonal to L passing through center(C_1) and center(C_2), respectively. Let V_1 and V_2 be the geodesic planes with boundaries containing v_1 and v_2 and let V be the closure of the region bounded by $V_1 \cup V_2$. Suppose that for each $i, B_i \cap V_i \neq \emptyset$. Let P_i denote the line tangent to $\pi(B_i)$ through center(C_i) such that $\pi(B_1 \cup B_2)$ lies to one side. Finally let α (resp. β) be the acute angle between P_i and v_i. Then,*

1. $\alpha + \beta \leq \pi/3$
2. *If $\alpha + \beta = \pi/3$, then*

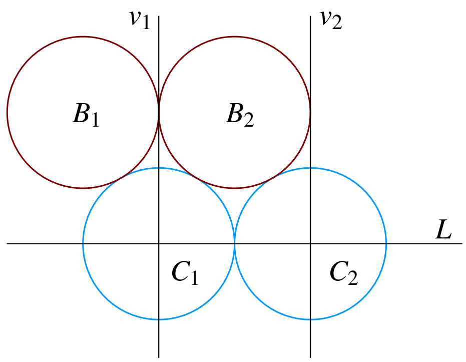

Figure 4: $\alpha = \pi/3, \beta = 0$

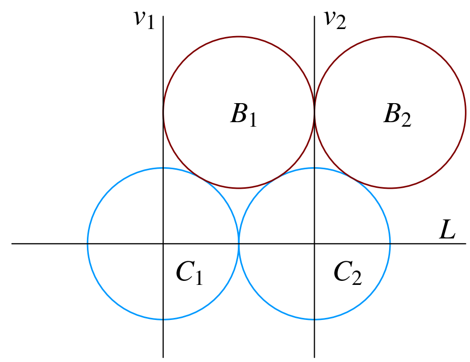

Figure 5: $\alpha = 0, \beta = \pi/3$

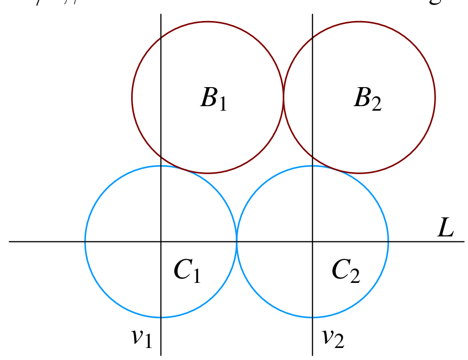

Figure 6: $0 < \alpha, \beta < \pi/3, \alpha + \beta = \pi/3$

(a) *C_1 is tangent to C_2*

(b) *B_1 and B_2 are full-sized*

(c) *for $i = 1, 2$ wehavethatB_i is tangent to C_i and the line J through center(B_1) and center(B_2) is parallel to L.*

Proof To start with, we can assume that L is parallel to the x-axis. The proof involves a series of steps whereby the positions of B_1, B_2, C_1, C_2 are repeatedly *improved*. The reader should note that any *improvement* strictly increases $\alpha + \beta$. In the end $\alpha + \beta = \pi/3$ and the various horoballs satisfy the equality conclusions. We repeatedly use the fact that an operation that moves center(B_2) infinitesimally closer to P_2 is β increasing with the analogous fact holding for α.

Let $b = d_{\mathbb{E}}(\text{center}(B_1), \text{center}(B_2)), c = d_{\mathbb{E}}(\text{center}(C_1), \text{center}(C_2))$ and $d_{ij} = d_{\mathbb{E}}(\text{center}(B_i), \text{center}(C_j))$ for $i, j \in \{1, 2\}$. We can assume that center$(C_1) = (0, 0)$, center$(C_2) = (c, 0)$, center$(B_1) = (x_1, y_1)$ and center$(B_2) = (x_2, y_2)$. Note that $c \geq 1$, $-h_1/2 \leq x_1 \leq h_1/2$ and $-h_2/2 \leq x_2 - c \leq h_2/2$. By Lemma 3.1, we also have that $b = \sqrt{h_1 h_2}$ and $d_{ij} \geq \sqrt{h_i}$, with equality if and only if B_i is tangent to C_j.

Step 1. At the cost of possibly increasing $\alpha+\beta$ we can assume that either B_1 is tangent to C_1 or B_2 is tangent to C_2.

Proof. If both $B_1 \cap C_1 = \emptyset$ and $B_2 \cap C_2 = \emptyset$, then we can translate $B_1 \cup B_2$ in the $(0,-1)$ direction until a first tangency occurs. Note that both α and β increase. If $B_1 \cap C_2 \neq \emptyset$ but $B_1 \cap C_1 = \emptyset$, then we can obtain a contradiction as follows: we have $(x_1-c)^2 + y_1^2 = d_{12}^2 = h_1$ and $x_1^2 + y_1^2 = d_{11}^2 > h_1$. However, since $x_1 \leq h_1/2$, we obtain $1 \leq c < h_1 \leq 1$, a contradiction. A similar fact holds for B_2, thus the tangency is of the type claimed. □

Step 2. At the cost of possibly increasing $\alpha+\beta$ we can additionally assume that either $C_1 \cap C_2 \neq \emptyset$ or each of B_1 and B_2 are respectively tangent to C_1 and C_2.

Proof. It suffices to consider the case where B_1 is tangent to C_1. If C_2 is disjoint from B_2, then translate C_2 in the $(-1,0)$ direction until a first tangency occurs. Note that β increases. If C_2 becomes tangent to B_1 first, then by the computation in Step 1, $c = 1$ and C_2 is also tangent to C_1. Lastly, we observe that $B_2 \cap V_2 \neq \emptyset$ remains true as we translate by computation: if $-h_2/2 \leq x_2 - c \leq h_2/2$ fails as we decrease c, we have that $x_2 > c + h_2/2$. But $x_2 \leq x_1 + b = x_1 + \sqrt{h_1 h2} \leq h_1/2 + (h_1+h_2)/2$, so we obtain $1 \leq c < h_1 \leq 1$, a contradiction. □

Step 3. At the cost of possibly increasing $\alpha+\beta$ we can further assume that for each $i, B_i \cap C_i \neq \emptyset$.

Proof. It suffices to consider the case that $B_1 \cap C_1 \neq \emptyset$ and $B_2 \cap C_2 = \emptyset$. Let J denote the ray from center(B_1) through center(B_2). First assume that $J \cap P_2 \neq \emptyset$. For each $t \geq 0$ we translate B_2 away from B_1 by moving its center Euclidean distance t along J away from center(B_2) to obtain $B_2'(t)$. We then expand $B_2'(t)$ keeping its center fixed until it first hits B_1 to obtain $B_2(t)$. Let B_2(new) be the first $B_2(t)$ that is either full-sized or satisfies $B_2(t) \cap C_2 \neq \emptyset$. Note that if B_2(new) $\neq B_2$, then β increases. We now abuse notation by denoting B_2(new) by B_2. Thus, if $B_2 \cap C_2 = \emptyset$, then B_2 is full-sized and by Step 2, $C_1 \cap C_2 \neq \emptyset$.

If $J \cap P_2 = \emptyset$, then apply a clockwise rotation about the geodesic γ through center(B_1) and ∞ until either $B_2 \cap C_2 \neq \emptyset$ or $J \cap P_2 \neq \emptyset$. This operation is strictly β increasing. If now $J \cap P_2 \neq \emptyset$, then argue as in the first paragraph to conclude that either Step 3 holds or B_2 is full sized and $C_1 \cap C_2 \neq \emptyset$.

We have now reduced to the case that B_2 is full-sized, $C_1 \cap C_2 \neq \emptyset$ and $B_2 \cap C_2 = \emptyset$. Observe that $y_2 \geq y_1$. This is immediate if B_1 is full-sized. In general, center(B_1) lies on the line perpendicular to the midpoint of the segment between center(C_1) and center(B_2) since B_1 is tangent to the full-sized horoballs C_1 and B_2. Since $x_1 \leq 1/2 \leq x_2$, the maximal y_1 is obtained when B_1 is full-sized and hence $y_2 \geq y_1$. Since P_2 has non-negative slope, a clockwise rotation about γ both transforms B_2 to a horoball tangent to C_2 and increases β. □

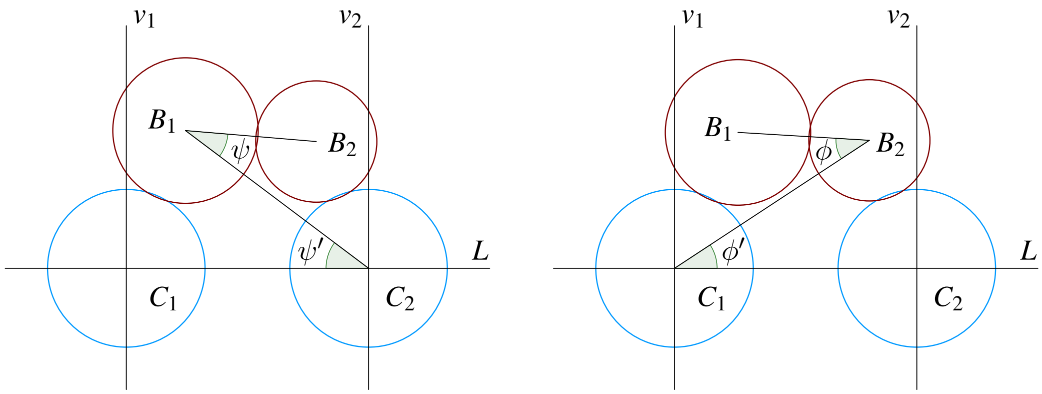

Figure 7: Transforming B_2 by increasing ψ up to $\pi/2$ increases both radius(B_2) and β. Similarly for B_1.

Step 4. At the cost of possibly increasing $\alpha+\beta$ we can further assume that both B_1 and B_2 are full-sized.

Proof. Consider the hyperbolic geodesic γ_1 from center(B_1) to center(C_2) and define the angles ϕ, ϕ' and ψ, ψ' as in Figure 7. An elliptic rotation of angle θ about γ_1 transforms B_2 to the horoball $B_2(\theta)$. Being a hyperbolic isometry setwise fixing B_1 and C_2, it follows that $B_2(\theta)$ is tangent to both B_1 and C_2. Oriented appropriately, as θ increases so does $\psi(\theta)$, where $\psi(\theta)$ is defined as in Figure 7, where B_2 is replaced by $B_2(\theta)$. If $\psi=\psi(0)<\pi/2$, the next two lemmas show that increasing ψ up to $\pi/2$ strictly increases β as well as the radius of $\pi(B_2(\theta))$. Note that $\psi', \phi' \leq \pi/3$, since say ψ' is the angle at the base of a right triangle whose height is at most 1 and whose base is at least 1/2. Since $\phi+\psi=\phi'+\psi'\leq 2\pi/3$, it follows that one of ϕ or ψ has angle at most $\pi/3$. Without loss of generality we can assume that the latter holds.

To prove Step 4, we first rotate B_2 as above so that it either becomes full-sized or $\psi=\pi/2$. Next we rotate B_1 until either it becomes full-sized or $\phi=\pi/2$. Here the rotation has axis γ_2, the geodesic from center(B_2) to center(C_1). Next rotate B_2 so that either it becomes full sized or $\psi=\pi/2$ and so on. After finitely many such rotations one of B_2 or B_1 becomes full sized. Since each step involves a $\geq \pi/6$ rotation, the process stops after a finite and computable time. After some B_i becomes full-sized, the third lemma below shows that one more rotation suffices to bring the other to full size. Thus Step 4 follows from the next three lemmas. □

Lemma 3.4 *If* $\psi(0)<\psi(\theta)\leq\pi$, *then* $\text{radius}(\pi(B_2(\theta))) > \text{radius}(\pi(B_2(0))) = \text{radius}(\pi(B_2))$. *The analogous result holds for transformations of* B_1.

Proof Let H_∞ denote the horoball $z\geq 1$. Apply a hyperbolic isometry that takes center(C_2) to ∞. See Figure 8 which shows the projections of the transformed B_1, B_2, C_1 and H_∞ to the new xy plane. We abuse notation by continuing to call the transformed horoballs by their original names. Notice that H_∞ and B_2 are full-sized. Since γ_1 is now a vertical geodesic an

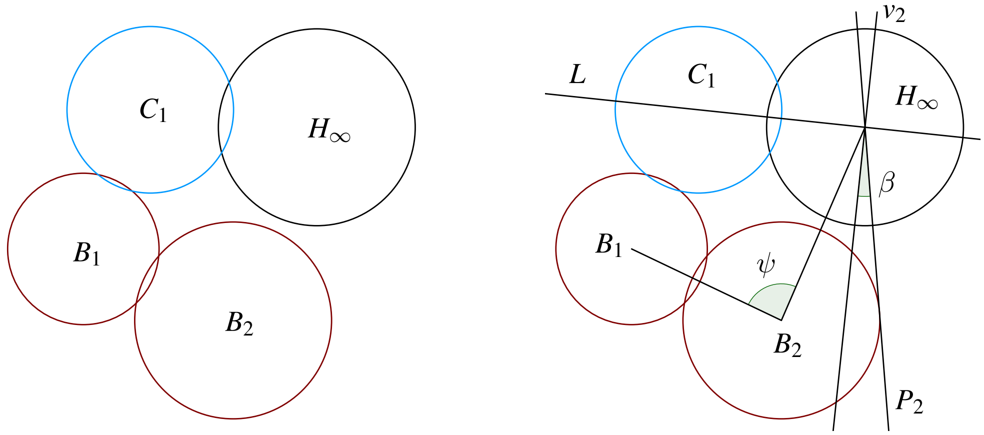

Figure 8: Sending C_2 to infinity and computing β.

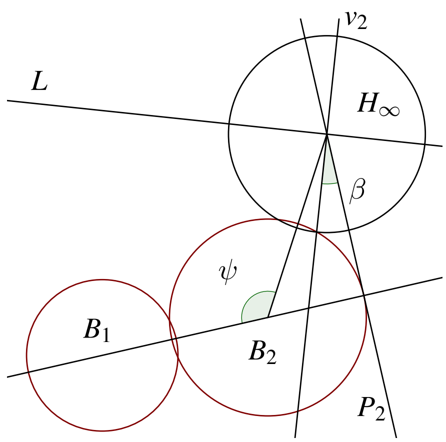

Figure 9: Maximizing β

elliptic transformation fixing γ_1 is a Euclidean rotation in these coordinates. A counterclockwise rotation by angle θ takes $B_2 = B_2(0)$ to $B_2(\theta)$. Consider the ideal tetrahedron T_θ with vertices center(B_1), center$(B_2(\theta))$, center(H_∞), center(C_2). Since opposite dihedral angles of T_θ are equal, the angle ψ in Figure 7 is equal to the angle of the same name in Figure 8.

In the original coordinates radius$(\pi(B_2(\theta)))$ monotonically increases as the γ_1 dihedral angle decreases and is maximized when this angle equals 0. As this angle decreases to 0 the angle $\psi(\theta)$ increases to π. $\square$

Lemma 3.5 *If $\psi(0) < \psi(\theta) \leq \pi/2$, then $\beta(\theta) > \beta(0) := \beta$. The analogous result holds for transformations of B_1.*

Proof Figure 8 shows how to compute β. Note that β is maximized when the line from center(B_1) through center(B_2) is orthogonal to P_2 at which point $\psi > \pi/2$. See Figure 9 $\square$

Lemma 3.6 *If B_1 is full sized, then for some θ with $\psi(0) < \psi(\theta) < \pi/2, B_2(\theta)$ is full-sized. The analogous result holds for transformations of B_1.*

Proof Since B_1 is full sized it is tangent to H_∞ in addition to B_2. Since a horoball in the original coordinates is full sized exactly when it is tangent to H_∞, we observe that $B_2(\theta)$ becomes full sized for some $\psi < \pi/2$. □

Step 5. For $i = 1, 2$ let L_i denote the line through center(B_i) and center(C_i). At the cost of possibly increasing $\alpha + \beta$ we can further assume that both L_1 and L_2 are parallel and hence B_1 (resp C_1) is tangent to B_2 (resp. C_2) and the line J through the centers of B_1 and B_2 is parallel to L.

Proof. A clockwise rotation of $\mathbb{H}^3$ applied to B_2 using the vertical geodesic through C_2 as axis takes L_2 to a line parallel to L_1. Let B_2' denote the rotated B_2. This operation increases β and makes J parallel to L but loses the B_1, B_2 tangency. Next translate B_2' and C_2 in the $(-1, 0)$ direction until the translated C_2 becomes tangent to C_1, in which case the translated B_2' also becomes tangent to B_1. □

Step 6. $\alpha + \beta = \pi/3$ and the conclusions (a)-(c) also hold.

Proof. We have already shown that conclusions (a)-(c) hold. Since B_2 is full-sized and tangent to C_2, the visual angle of $\pi(B_2)$ from center(C_2) is equal to $\pi/3$. Using the fact that $B_1 \cup C_1$ is a translate of $B_2 \cup C_2$ it follows that this visual angle decomposes into $\alpha + \beta$. □

This completes the proof of the Two-Eyes Lemma. □

4 Proof of Proposition 2.3

The proof of the main proposition is now just a counting argument.

Proof of Proposition 2.3 As in the proof of the Two-Eyes Lemma, consider the hyperplanes V_1, and V_2. Since the necklace η winds around C_1 and C_2, it follows that V_1 and V_2 each intersect at least two horoballs of η. For $i = 1, 2$, let B_i^U, B_i^L be these horoballs intersecting V_i with centers in the upper and lower half-planes, respectively. These four horoballs are distinct. Further, we can assume that B_2^U, B_2^L have the largest x-coordinates and B_1^U, B_1^L are the smallest x-coordinates amongst all choices in η satisfying the non-empty intersection conditions. Since all horoballs are at most full-sized, visual angle around center(C_i) tells us that, away from the critical case where *both* B_i^U and B_i^L are tangent to V_i, we need at least two more horoballs to connect B_1^L to B_1^U and at least two more to connect B_2^U to B_2^L in the clockwise direction along η. Away from this critical case, the necklace must have at least 8 horoballs.

Assume we are in the critical case where B_i^U and B_i^L are tangent to V_i for some i. Without loss of generality, we can take $i = 1$. By the minimality of the x-coordinates and the fact that necklace horoballs are sequently tangent, we can assume that B_1^U and B_1^L lie entirely to the left of V_1, aside from the points of tangency. The region V, between V_1 and V_2, will then contain at least two more horoballs, but these cannot be B_2^U, B_2^L by the maximality of the x-coordinate and because all the horoballs are at most full-sized. Therefore, we need at least 1 horoball to join B_1^L to B_1^U, 2 more horoballs in V, and at least 1 more horoball to join B_2^U to B_2^L, giving us a total of 8.

We turn to the case where η has exactly 8 horoballs. It remains to show that all are full sized and the configuration is obtained by sliding the hexagonal example. For this, we will use the Two-Eyes Lemma and visual angle arguments. Assume that in each of the pairs $\{B_1^U, B_2^U\}$ and $\{B_2^L, B_1^L\}$ at least one of the horoballs is not tangent to the associated V_i. In this setting, our counting argument in the first paragraph gives that the horoballs B_1^U and B_2^U are tangent. Similarly for B_1^L and B_2^L. Let α, β be the angles from the Two-Eyes Lemma applied to the pair $\{B_1^U, B_2^U\}$ and α', β' be the angles for the pair $\{B_2^L, B_1^L\}$. It follows that $\alpha + \beta \leq \pi/3$ and $\alpha' + \beta' \leq \pi/3$. For each i, we have exactly two horoballs in η connecting B_i^L to B_i^U with centers in the complement of V. Let δ_i, φ_i be the visual angles from center(C_i) of these horoballs. Then, cutting out V, we have that the sum of the angles satisfies

$$2\pi \leq (\beta + \alpha) + \delta_1 + \varphi_1 + (\beta' + \alpha') + \delta_2 + \varphi_2 \leq \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} = 2\pi.$$

It follows that $\delta_i = \varphi_i = \pi/3$ and $\alpha + \beta = \alpha' + \beta' = \pi/3$. Thus, all horoballs in η are full-sized and tangent to C_1 or C_2. Hence, all the horoballs in η are tangent to C_i are part of the hexagonal packing. This allows us to compute $\alpha = \pi - \delta_1 - \varphi_1 - \beta' = \pi/3 - \beta'$ and, similarly, $\beta = \pi/3 - \alpha'$. Since $\alpha + \beta = \pi/3$ and $\alpha' + \beta' = \pi/3$, we obtain a one-parameter family of horoballs parametrized by, say, α.

Without loss of generality, the remaining case is where B_1^U is tangent to V_1 and B_2^U is tangent to V_2 (with the x coordinate max/min condition above). There is then at least one horoball from B_1^U to B_2^U in the clockwise direction along η. Let $D_{1,1}, D_{1,2}$ be the next two horoballs in the counter-clockwise direction from B_1^U and $D_{2,1}, D_{2,2}$ the next two horoballs in the clockwise direction from B_2^U along η. If the visual angle of at least one of $D_{i,j}$ from center(C_i) is $< \pi/3$, then $D_{i,2} \neq B_i^L$. Counting the horoballs tells us that at least one of B_i^L has to be tangent to V_i. In fact, both must. Indeed, if B_2^L is tangent to V_2 then it cannot be tangent to B_1^L and one more horoball is required in the clockwise direction. Similarly, if B_1^L is tangent to V_1. Thus, $D_{i,2} = B_i^L$ for $i = 1, 2$ and $D_{i,j}$ have visual angle $\pi/3$, which means they are full-sized and tangent to C_i. The horoballs that connect B_1^U to B_2^U and B_2^U to B_1^U in the clockwise direction must also be full-sized and tangent to both C_1 and C_2 to bridge the “width” of V. Thus, we are in the configuration above where $\alpha = \pi/3$.

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