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Hyperbolic Geometry and 3-Manifold Topology

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David Gabai

Introduction

This paper is based on a course the author gave at the July 2006 Park City Mathematics Institute. The audience primarily consisted of graduate students, although there were a number of researchers in attendance. Starting at the very beginning we gave a detailed exposition of a proof of the Tameness theorem, that was independently proven by Agol [**Ag**] and Calegari - Gabai [**CG**].

Theorem 0.1. *(Tameness Theorem) If N is a complete hyperbolic 3-manifold with finitely generated fundamental group, then N is topologically and geometrically tame.*

In 1974 Marden conjectured that N is topologically tame and proved it for geometrically finite manifold. The assertion of topological tameness became known as the Marden Tameness Conjecture. In 1993 Canary [**Ca2**] proved that topological tameness implies geometric tameness. The argument we give simultaneously proves both.

For applications of Theorem 0.1 see the introduction to [**CG**].

The argument presented here follows the broad outline of [**CG**]; however, we invoke two ideas of Soma and a lemma of Bowditch to greatly simplify the details. In particular we will work with simplicial hyperbolic surfaces instead of the more general Cat(-1) ones and replace Shrinkwrapping in the smooth category by PL Shrinkwrapping. We will also give an elementary proof of the Tameness Criterion that eliminates the use of results about the Thurston and Gromov norm on H_2.

In order to focus on the central ideas all manifolds in this paper are orientable and all hyperbolic 3-manifolds are parabolic free. I.e. if $N = \mathbb{H}^3/\Gamma$, then $\Gamma \subset \text{Isom}(\mathbb{H}^3)$ has no nontrivial parabolic elements. Section §7 [**CG**] gives a detailed discussion of the parabolic case. While this paper is a bit different than [**CG**], it should be clear as to how to adapt our arguments to the parabolic case.

The paper is organized as follows. In §1 we discuss examples of non tame 3-manifolds and state foundational results in the subject. In particular we describe the classical Whitehead and Fox - Artin manifolds and mention an example of Freedman - Gabai. We state Tucker's characterization of tameness and Scott's Core Theorem. In §2 we give background material in the theory of hyperbolic 3-manifolds. In particular we state the Thick - Thin decomposition, define simplicial hyperbolic surfaces and prove the bounded diameter lemma. We define the notion of geometrically infinite end and state Bonahon's characterization of such ends as

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well as the upgraded version of [**CG**]. We then state the *Tameness Criterion* which gives three conditions that are sufficient to show that an end is both geometrically and topologically tame. In §3 we discuss Shrinkwrapping and using Soma's argument prove the PL shrinkwrapping theorem. In §4 we demonstrate how to use shrinkwrapping to prove Canary's theorem which asserts that a topologically tame end is geometrcially tame. That proof, which motivates the proof of the general case, involves showing that the end is either geometrically finite or satisfies the conditions of the Tameness Criterion. In §5 we prove that an end which satisfies the conditions of the Tameness Criterion is geomtrically and topologically tame. In §6 we prove the Tameness Theorem, by showing that a geometrically infinite end satisfies the conditions of the Tameness Criterion.

There are many exercises scattered throughout this paper which establish both peripheral and required results. In order to focus on the main ideas, many results in this paper are not stated in their most general form.

These 2006 PCMI lectures are based on Spring 2005 lectures given at Princeton University. I thank all the attendees at Princeton and PCMI for their participation, interest and comments as well as Sucharit Sarkar who served as teaching assistant for the PCMI lectures. I thank John Polking for his patient assistance in the preparation of the manuscript.

For other expositions of the Tameness theorem see Agol [**Ag**], Soma [**Som**], Choi [**Ch**] and Bowditch [**Bo**]. Much of the needed basic results about hyperbolic geometry or 3-manifold topology can be found in either Thurston's [**T3**] or Jaco's [**Ja**] books.

Notation 0.2. If Y is a subspace of the metric space X, then $N(Y,t)$ denotes the metric t-neighborhood of Y in X. The notation $\langle S, \alpha\rangle$ denotes algebraic intersection number and $X \approx Y$ denotes X is homeomorphic to Y. A notation such as $\text{length}_S(\alpha)$ suggests that length of α is computed with the S-metric. Also $|Y|$ denotes the number of components of Y and $\text{int}(C)$ denotes the interior of C.

1. Topological Tameness; Examples and Foundations

In his seminal paper [**Wh**], Whitehead gave the first example of an open contractible 3-manifold $\mathcal{W}$ not homeomorphic to $\mathbb{R}^3$. Let $V_1 \subset V_2 \subset \cdots$ be a nested union of closed solid tori, one embedded into the interior of the next. Embed $V_1 \to V_2$ as in Figure 1.1a and for $i \in \mathbb{N}$ embed $V_i \to V_{i+1}$ similarly. Define $\mathcal{W} = \cup V_i$ and topologize $\mathcal{W}$ so that U is open in $\mathcal{W}$ if and only if for all i, $U \cap V_i$ is open in V_i. Figure 1.1b shows the inclusion of V_1 in V_3.

Theorem 1.1. *$\mathcal{W} \neq \mathbb{R}^3$.*

Proof. By Moise [**Ms**], if $\mathcal{W}$ is homeomorphic to $\mathbb{R}^3$, then it is PL-homeomorphic. It is an exercise, using the Seifert - Van Kampen theorem, to show that $\pi_1(\mathcal{W} \setminus V_1)$ has infinitely generated fundamental group. Now apply Lemma 1.2. (See [**Ro**] for a slightly more direct argument.) □

Lemma 1.2. *If C is a compact codimension-0 PL-submanifold of $\mathbb{R}^3$, then $\pi_1(\mathbb{R}^3 \setminus int(C))$ is finitely generated.*

Proof. Let $B \subset \mathbb{R}^3$ be a large 3-ball with $C \subset \text{int}(B)$. Then $B \setminus \text{int}(C)$ is a compact PL-manifold homotopy equivalent to $\mathbb{R}^3 - \text{int}(C)$, hence $\mathbb{R}^3 - \text{int}(C)$ has finitely generated fundamental group. □

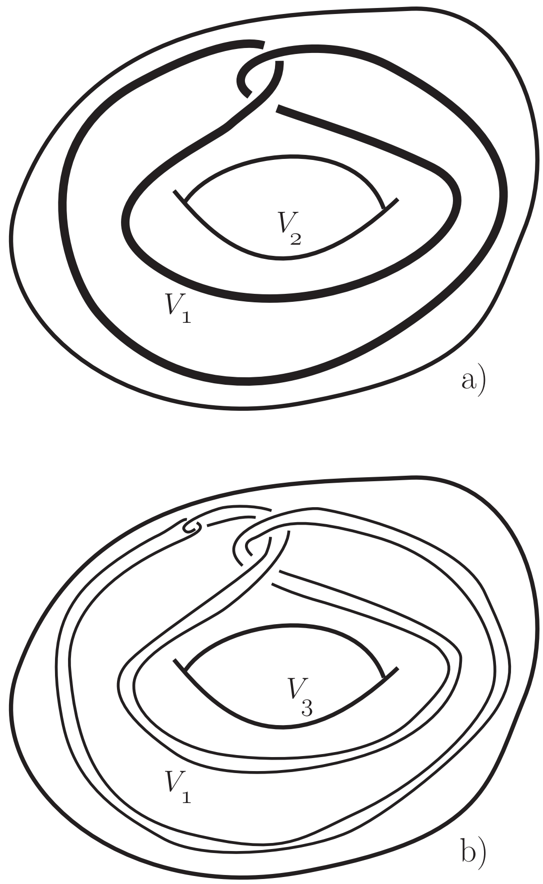

FIGURE 1.1

Definition 1.3. Let M be a 3-manifold, possibly with boundary. We say M is *tame* if it naturally compacitifies to a compact manifold. More precisely, there exists a compact 3-manifold X and a proper embedding $i : M \to X$ with $X = \text{cl}(i(M))$. Such an M is also called a *missing boundary manifold*.

Remark 1.4. Note that if M is tame, non compact and $\partial M = \emptyset$, then M is homeomorphic to $M_1 \cup S \times [0, \infty)$ where M_1 is a compact 3-manifold, S is a compact closed surface and $S \times [0, \infty) \cap M_1 = \partial M_1 = S \times 0$.

The following characterization of tame manifolds is due to Tom Tucker. Recall that an *irreducible* 3-manifold is one such that every smooth 2-sphere bounds a 3-ball.

Theorem 1.5. *(Tucker* [**Tu**]*) Let M be an irreducible 3-manifold. M is tame if and only if for every smoothly embeddded compact submanifold Y transverse to ∂M, $\pi_1(M \setminus Y)$ is finitely generated. (Y is allowed to have* corners *along $\partial(Y \cap \partial M)$.)*

If $\partial M = \emptyset$, then ignore the transversality condition.

Example 1.6. (Fox - Artin manifold) Let $N(k)$ denote a smooth closed regular neighborhood of the smooth properly embedded ray $k \subset \mathbb{R}^3$ shown in Figure 1.2. We call $\mathcal{F} = \mathbb{R}^3 \setminus \text{int}(N(k))$ the Fox - Artin manifold.

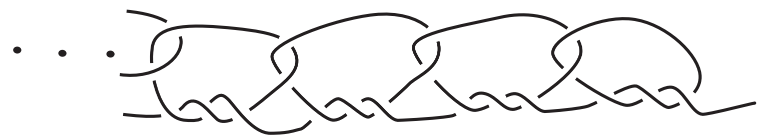

FIGURE 1.2. The Fox Artin Manifold.

Exercise 1.7. Show that $\text{int}(\mathcal{F}) \approx \mathbb{R}^3$ and $\partial\mathcal{F} \approx \mathbb{R}^2$. Is $\mathcal{F}$ tame? Is $(\mathcal{F}, \partial\mathcal{F}) \approx (\mathbb{R}^3, \mathbb{R}^2)$ as a manifold pair? Compare with Remark 6.7.

The Whitehead manifold is a non tame contractible 3-manifold. Variations of the theme lead to many different types of non tame manifolds, e.g. see [**ST**]. Indeed, there are uncountably many different homeomorphism types of non tame contractible 3-manifolds [**McM**], non tame manifolds homotopy equivalent to $\text{int}(D^2 \times S^1)$ and non tame manifolds homotopy equivalent to open handlebodies, i.e. manifolds that have a free, finitely generated fundamental group.

Example 1.8. Here is a non tame homotopy handlebody $\mathcal{H}$ of genus-2 discovered by Mike Freedman and the author [**FG**]. Let $\mathcal{H} = \cup V_i$, where V_i is a standard compact handlebody of genus-2 and $V_1 \subset V_2 \subset \cdots$ with one embedded in the interior of the next. Figure 1.3 shows a simple closed curve $\gamma \subset V_1$ and the embedding of $V_1 \to V_2$. For $i \in \mathbb{N}$, the embedding $V_i \to V_{i+1}$ is defined similarly. Let $\tilde{\mathcal{H}}$ denote the universal covering of $\mathcal{H}$. We have

i) $\tilde{\mathcal{H}} = \mathbb{R}^3$,
ii) The preimage Γ of γ in $\tilde{\mathcal{H}}$ is the infinite unlink. I.e.

$$(\tilde{\mathcal{H}}, \Gamma) \approx (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{Z} \times 0 \times \mathbb{R}) \text{ and}$$

iii) $\pi_1(\mathcal{H} - \gamma)$ is infinitely generated

Figure 1.3

Remark 1.9. The manifold $\mathcal{H}$ is not tame by applying Tucker's theorem to iii). The curve γ is called a *nontaming* knot, i.e. it is complicated enough to expose the non tameness of $\mathcal{H}$, yet simple enough to lift to the unlink in the universal covering. This example dispatched a provocative conjecture of Mike Freedman. Much contemplation of this example ultimately led to the proof of Theorem 0.1. The author is grateful to Mike for introducing him to this problem.

The following result is central to the theory of open 3-manifolds.

Theorem 1.10. *(Scott Core Theorem* [**Sc**]*). If M is a connected irreducible 3-manifold with finitely generated fundamental group, then there exists a compact submanifold $C \subset M$ such that the inclusion $C \to M$ is a homotopy equivalence.*

Remark 1.11. See [**Mc**] for a version for manifolds with boundary. It asserts that if ∂M has finitely many boundary components, then the core C can be chosen so that for each component T of ∂M, the inclusion $C \cap T \to T$ is a homotopy equivalence. Such a C is called a *relative core*.

Relative cores are not needed for this paper. They play an important role in the parabolic case.

Definition 1.12. An *end* $\mathcal{E}$ of a manifold M is an equivalence class of nested sequences of connected open sets $U_1 \supset U_2 \supset \cdots$ with the property that $\cap U_i = \emptyset$ and each U_i is a component of $M \setminus C_i$ where C_i is a compact submanifold. Two such sequences $\{U_i\}$, $\{V_i\}$ are equivalent if for each i there exists j, k such that $V_j \subset U_i$ and $U_k \subset V_i$. Any open set U_i as above is called a *neighborhood* of $\mathcal{E}$.

Exercise 1.13. Let M be a connected, irreducible, open (i.e. $\partial M = \emptyset$) 3-manifold with finitely generated fundamental group. If C is a core of M, then there is a 1-1 correspondence between ends of M and components of $M \setminus C$ which in turn is in 1-1 correspondence with components of ∂C. In particular, M has finitely many ends.

If instead, $|\partial M| < \infty$, prove the above result where C is a relative core.

Remark 1.14. A consequence of the above exercise is that if $\mathcal{E}$ is an end of M, then $\mathcal{E}$ is homeomorphic to the unique end of a 1-ended submanifold $M_{\mathcal{E}}$ of M.

We will often abuse notation by referring to an end of a 3-manifold with finitely generated fundamental group as a complementary component of a core.

Definition 1.15. Let $\mathcal{E}$ be an end of the 3-manifold M. We say that $\mathcal{E}$ is *tame* if $\mathcal{E}$ has a closed neighborhood homeomorphic to $S \times [0, \infty)$. This means that there exists a proper embedding $i : S \times [0, \infty) \to M$, where S is a connected compact surface, such that $\mathcal{E}$ is identified with $i(S \times (0, \infty))$.

Exercise 1.16. Let M be an open irreducible 3-manifold with finitely generated fundamental group. The end $\mathcal{E}$ of M is tame if and only if there exists a core C of M such that the closure of the component Z of $M \setminus C$ containing $\mathcal{E}$ is homeomorphic to $S \times [0, \infty)$ for some closed surface S.

2. Background Material for Hyperbolic 3-Manifolds

Recall that in this paper all manifolds are orientable and unless said otherwise all hyperbolic manifolds are parabolic free.

See [**Mo**] for a quick introduction to 3-dimensional hyperbolic geometry, in particular the concepts of limit set, domain of discontinuity, convex hull and geometrically finite manifold.

An end $\mathcal{E}$ of a complete hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ is *geometrically finite* if it has a neighborhood disjoint from the convex core $\mathcal{C}(N)$ of N. It follows from [**EM**] that if $\partial_{\mathcal{E}}\mathcal{C}(N)$ is the component of $\partial\mathcal{C}(N)$ that faces $\mathcal{E}$, then the component E of $N \setminus \partial_{\mathcal{E}}\mathcal{C}(N)$ that contains $\mathcal{E}$ is homeomorphic to $\partial_{\mathcal{E}}\mathcal{C}(N) \times (0, \infty)$. Orthogonal projection of $E \to \partial_{\mathcal{E}}\mathcal{C}(N)$ infinitesimally reduces length at least by the factor $\cosh(d)$, where d is the distance to $\partial_{\mathcal{E}}\mathcal{C}(N)$. Consequently if $\partial_{\mathcal{E}}\mathcal{C}(N) \times t$ are the points of E at distance t from $\partial_{\mathcal{E}}\mathcal{C}(N)$, then $\operatorname{area}(\partial_{\mathcal{E}}\mathcal{C}(N) \times t) \geq \cosh(d)^2 \operatorname{area}(\partial_{\mathcal{E}}\mathcal{C}(N))$ which, for t large, is approximately $(e^{2t}/4) \operatorname{area}(\partial_{\mathcal{E}}\mathcal{C}(N))$, i.e. cross sectional area grows exponentially in distance from $\mathcal{C}(N)$.

Example 2.1. If $S = \mathbb{H}^2/\Gamma$ is a closed surface, then the natural extension of Γ to a subgroup of $\operatorname{Isom}(\mathbb{H}^3)$ gives rise to the geometrically finite manifold $N = \mathbb{H}^3/\Gamma$ which is homeomorphic to $S \times \mathbb{R}$.

We say that the end $\mathcal{E}$ of N is *geometrically infinite* if it is not geometrically finite.

Example 2.2. (Thurston [**T1**]) Let X be a closed hyperbolic 3-manifold that fibers over the circle with fiber S. Let N be the infinite cyclic covering space corresponding to S. Topologically, $N \approx S \times \mathbb{R}$ but area$(S \times t)$ is uniformly bounded. Further, since $\pi_1(N)$ is normal in $\pi_1(X)$, they both have the same limit set which is the 2-sphere and hence $\mathcal{C}(N) = N$.

Definition 2.3. Let $\mathcal{E}$ be an end of the 3-manifold M with finitely generated fundamental group. Let $A_1, A_2, \cdots$ be a sequence of subsets of M. We say that the sequence $\{A_i\}$ *exits* $\mathcal{E}$ if for any neighborhood sequence $\{U_k\}$ of $\mathcal{E}$; for any j, $A_i \subset U_j$ for all but finitely many i.

Proposition 2.4. *(Bonahon, Proposition 2.3* [**Bo**]*) Let $\mathcal{E}$ be an end of the complete hyperbolic 3-manifold N with finitely generated fundamental group. Then $\mathcal{E}$ is geometrically infinite if and only if there exists a sequence $\delta_1, \delta_2, \cdots$ of closed geodesics that exit $\mathcal{E}$.*

Remark 2.5. This result can be upgraded as follows. An elementary argument in [**CG**] shows that the δ_i's can be take to be simple closed curves, each the core of an embedded tube of radius 0.025. An argument [**CG**] using the rigorous computer assisted proof in [**GMT**] further improves that number to $\log(3)/2 = .5493...$.

The following central result is due to Margulis. See [**T3**].

Theorem 2.6. *(Thick - Thin Decomposition) There exists $\mu > 0$, such that if $\epsilon \leq \mu$ and if N is a complete, parabolic free hyperbolic 3-manifold, then $N = N_{(0,\epsilon]} \cup N_{[\epsilon,\infty)}$ where $N_{[\epsilon,\infty)} = \{x \in N |\, \mathrm{injrad}(x) \geq \epsilon\}$ and $N_{(0,\epsilon]} = \mathrm{cl}\{x \in N |\, \mathrm{injrad}(x) < \epsilon\}$. Furthermore $N_{(0,\epsilon]}$ is a disjoint union of solid tubes about simple closed geodesics.*

Definition 2.7. These tubes are called *Margulis tubes*.

Definition 2.8. A *simplicial hyperbolic surface* in the hyperbolic 3-manifold N is a map $f : T \to N$ where T is a triangulated surface and the restriction of f to each simplex is a totally geodesic immersion. Furthermore, the cone angle at each vertex is $\geq 2\pi$. We abuse notation by calling the corresponding image surface S a simplicial hyperbolic surface. We say that S is *useful* if the triangulation has a unique vertex v and if there exists an edge e, called the *preferred edge* such that $e \cup v$ is a closed geodesic in N. (The ends of the other edges of the triangulation may meet at v in an angle $\neq \pi$.). A *pre-simplicial* hyperbolic surface is a surface satisfying all the conditions of a simplicial hyperbolic surface except possibly the cone angle condition.

We require the following crucial yet elementary Bounded Diameter Lemma. Versions of this result are needed for many of the results in hyperbolic 3-manifold theory discovered over the last 30 years.

Theorem 2.9. *(Bounded Diameter Lemma) Let $\epsilon > \delta > 0$, where ϵ is less than the Margulis constant. Let S be a simplicial hyperbolic surface in the complete hyperbolic 3-manifold N such that S is π_1-injective on δ-short curves, i.e. if γ is an essential simple closed curve in S and length$_S(\gamma) \leq \delta$, then γ is homotopically nontrivial in N. Then there exists $C_0 > 0$, depending on $\chi(S)$, ϵ and δ such that diam$_N(S) \leq C_0$ modulo Margulis tubes. I.e. if $x, y \in S \setminus N_{(0,\epsilon]}$, then there exists a path α from x to y in S such that length$_N(\alpha \setminus N_{(0,\epsilon]}) \leq C_0$.*

PROOF. Let $\text{injrad}_N(x)$ (resp. $\text{injrad}_S(x)$) denote the injectivity radius of x in N (resp. the injectivity radius of x in S with the induced metric). Since S is π_1-injective on δ-short loops, if $\text{injrad}_S(x) \leq \delta/2$, then $\text{injrad}_N(x) \leq \delta/2 \leq \epsilon/2$ and hence $x \in N_{(0,\epsilon]}$.

Since S has intrinsic curvature ≤ -1, if D is an embedded disc in S of radius $\delta/2$, then $\text{area}_S(D) > \pi(\delta/2)^2$.

On the other hand the Gauss - Bonnet theorem implies that $\text{area}(S) \leq 2\pi|\chi(S)|$.

The three previous paragraphs imply that $S \backslash N_{(0,\epsilon]}$ can be covered by $8|\chi(S)|/\delta^2$ metric δ-balls, where measurement is taken with respect to the induced metric on S. □

Remarks 2.10. 1) By reducing the value of ϵ, we can assume that distinct Margulis tubes are separated by some fixed amount ϵ_1. Therefore, for g fixed, a genus-g simplicial hyperbolic surface can only intersect a uniformly bounded number of Margulis tubes. Hence given g and the hyperbolic 3-manifold N with core C, there exists a function $f : \mathbb{R} \to \mathbb{R}$ so that if S is a genus-g simplicial hyperbolic surface in $N \setminus C, x \in S$ and $d_N(x, C) > t$, then $d_N(S, C) > f(t)$. Furthermore, $\lim_{t\to\infty} f(t) = \infty$.

2) By Gauss - Bonnet, there exists a constant C_1 depending only on $\chi(S)$ so that if S is a closed simplicial hyperbolic surface, then there exists a simple closed geodesic $\gamma \subset S$ such that $\text{length}_S(\gamma) \leq C_1$.

3) Fix $C_2 > 0$. Let γ be a simple closed curve in N with $\text{length}(\gamma) \leq C_2$. If γ is homotopically trivial, then a trivializing homotopy has diameter $\leq C_2$ since it lifts to one lying in a ball in $\mathbb{H}^3$ of radius C_2.

If γ as above is homotopically nontrivial, then it can be homotoped either to a geodesic or into $N_{(0,\epsilon]}$ via a homotopy which moves points a uniformly bounded amount. Indeed, if γ is homotopic to the geodesic η, then any homotopy lifts to one in N_η, the cover of N with fundamental group generated by η. If $N_t(\eta)$ is a radius-t, tubular neighborhood of η in N_η, then orthogonal projection of $N_\eta \setminus N_t(\eta)$ to $N_t(\eta)$ infinitesimally reduces arc length by at least $\cosh(d)$ where d is distance to $N_t(\eta)$. Since γ is a uniformly bounded distance from one of η or $N_{(0,\epsilon]}$, it follows that a homotopy into either η or $N_{(0,\epsilon]}$ can be chosen to move points a uniformly bounded distance.

Definition 2.11. An end $\mathcal{E}$ of the complete hyperbolic 3-manifold N is *simply degenerate* if lies in a submanifold homeomorphic to $S \times [0, \infty)$, where S is a closed surface, and there exists a sequence $S_1, S_2, \cdots$ of simplicial hyperbolic-surfaces exiting $\mathcal{E}$ such that for each i, S_i is homotopic within $\mathcal{E}$ to a homeomorphism onto S.

Remark 2.12. This concept is due to Thurston [**T1**] and was clarified and generalized by Bonahon [**Bo**] and Canary [**Ca1**]. Actually, the S_i need only be Cat(-1) or have intrinsic curvature ≤ -1, e.g. the S_i's could be pleated surfaces or minimal surfaces .

Definition 2.13. An end $\mathcal{E}$ of the complete hyperbolic 3-manifold N is *geometrically tame* if it is either geometrically finite or simply degenerate.

To prove the Tameness Theorem we will prove, in §5, the following *Tameness Criterion* and then using *shrinkwrapping* developed in §3, show in §6 that each end of a complete hyperbolic 3-manifold with finitely generated fundamental group satisfies the three conditions of this criterion.

Theorem 2.14. *(Tameness Criterion) Let N be a complete hyperbolic 3-manifold with finitely generated fundamental group. Let C be a codimension-0 core of N and $\mathcal{E}$ an end of N. Then $\mathcal{E}$ is geometrically and topologically tame if the following hold.*

1) There exists a sequence $S_1, S_2, \cdots$ of closed mapped surfaces exiting $\mathcal{E}$ such that for each i, genus(S_i) $\leq$ genus($\partial_{\mathcal{E}} C$), where $\partial_{\mathcal{E}} C$ is the component of ∂C facing $\mathcal{E}$.

2) Each S_i is a simplicial hyperbolic surface.

3) Each S_i homologically separates $\mathcal{E}$ from C, i.e. $[S_i] = [\partial_{\mathcal{E}} C] \in H_2(N \setminus int(C))$.

On the road to proving Theorem 2.14 we require the following

Proposition 2.15. *The end $\mathcal{E}$ of the complete hyperbolic 3-manifold N with finitely generated fundamental group is simply degenerate if and only if $\mathcal{E}$ is topologically tame and satisfies conditions 1)-3) of the tameness criterion.*

Corollary 2.16. *The end $\mathcal{E}$ of the complete hyperbolic 3-manifold N with finitely generated fundamental group is geometrically tame if $\mathcal{E}$ is topologically tame and satisfies conditions 1)-3) of the tameness criterion.* □

Proof of Proposition. It is immediate that for any core C of N, a simply degenerate end satisfies conditions 1)-3). We now prove the converse. We will assume that N is 1-ended, for the general case is similar. It is exercise in algebraic topology (e.g. see Exercise 5.2) that after passing to subsequence, a sequence $\{S_i\}$ satisfies 1)-3) with respect to one core of N if and only if it satisfies 1)-3) for any core of N. Thus we can assume that $Z = N \setminus \text{int}(C)$ is homeomorphic to the product $\partial C \times [0, \infty]$. We need to show that each S_i can be homotoped within Z to a homeomorphism onto ∂C. Fix i. Since S_i homologically separates, the projection of S_i to ∂C induced from the product structure on Z, is a degree-1 map. Now apply the equality case of Exercise 2.17. □

Exercise 2.17. If $f : S \to T$ is a degree-1 map between closed surfaces, then genus(S) $\geq$ genus(T). If equality holds, then f is homotopic to a homeomorphism and if inequality holds, then there exists an essential simple closed curve $\gamma \subset S$ such that $f|\gamma$ is homotopically trivial.

Remark 2.18. Actually when genus(S) $>$ genus(T) the map is homotopic to a composition of a *pinch* and a homeomorphism [**Ed**]. A pinch is a quotient map which identifies a compact subsurface with connected boundary to a point. See also [**Kn**].

Remark 2.19. The very simplest cases of the Tame Ends Theorem are the cases that $\pi_1(N) = 1$ and $\pi_1(N) = \mathbb{Z}$. In the former case $N = \mathbb{H}^3$ and the latter case $N \approx \text{int}(D^2 \times S^1)$ and is geometrically finite. (Recall, that N is parabolic free.) Here $N = \mathbb{H}^3 / < g >$, where g is a loxadromic isometry of $\mathbb{H}^3$. It is interesting that the case of $\pi_1(N) = \mathbb{Z} * \mathbb{Z}$ was the most problematic.

3. Shrinkwrapping

Shrinkwrapping was introduced in [**CG**] as a new method to construct Cat(-1) surfaces in hyperbolic 3-manifolds. We developed shrinkwrapping in the smooth category and suggested that a PL theory exists. Subsequently, Teruhiko Soma developed a *ruled wrapped* version which as discussed below is routinely transformed into a PL one. In this chapter we prove Theorem 3.4 using Soma's argument.

Definition 3.1. Let Δ be a collection of simple closed curves in the 3-manifold N. The embedded surface $S \subset N$ is *2-incompressible* [**CG**] relative to Δ if for each essential compressing disc D, we have $|D \cap \Delta| \geq n$.

Remark 3.2. It is an immediate consequence of Waldhausen's generalized loop theorem [**Wa1**] (see I.13 [**Ja**]) that it is equivalent to consider either embedded or mapped discs in the definition of 2-incompressible. More precisely, if S is an embedded surface in N, D is a disc and $f : D \to N$ is such that $f^{-1}(S) = \partial D$, $f|\partial D$ is homotopically non trivial in S and $|f^{-1}(\Delta)| \leq 1$, then there exists an embedded disc $E \subset N$ with these same properties.

Definition 3.3. A Δ-*homotopy* is a homotopy $f : X \times [0,1] \to N$ such that if $f(x,0) \notin \Delta$, then $f(x,t) \notin \Delta$ for $t < 1$ and if $f(x,0) \in \Delta$, then $f(x,t) = f(x,0)$ all $t \in [0,1]$. Typically $f|X \times 0$ is a surface disjoint from Δ or a loop disjoint from Δ or a path transverse to Δ that intersects Δ only within its endpoints. We say that X_1 is Δ-*homotopic* to X_0.

Theorem 3.4. *Let N be a complete hyperbolic 3-manifold and $\Delta \subset N$ a locally finite collection of simple geodesics. If $S \subset N$ is an embedded surface disjoint from Δ that separates components of Δ and is 2-incompressible rel Δ, then S is Δ-homotopic to a mapped simplicial hyperbolic surface S_1.*

Remark 3.5. The shrinkwrapping theorem of [**CG**] is the same as Theorem 3.4 except that S_1 is a mapped Cat(-1) surface and away from Δ the surface is smooth and of mean curvature zero, i.e. is locally a minimal surface. Additionally each $F|S \times t, t < 1$, is an embedding.

One can define the idea of 2-incompressibility for mapped (resp. immersed) surfaces and using nearly identical arguments, do PL (resp. smooth) shrinkwrapping for such surfaces.

The following application, true in both the PL and smooth categories, is extremely useful for controlling the location of simplicial hyperbolic surfaces. Here S, N and Δ are as in Theorem 3.4. To make the statement cleaner we assume that Δ is η *separated*, a condition true in applications thanks to Lemma 2.5. η-separated means that if $\beta \subset N$ is a path with endpoints in Δ and $\text{length}_N(\beta) \leq \eta$, then β is path homotopic to a path lying in Δ.

Exercise 3.6. Show that the 2-incompressibility of S implies that S_1 is π_1-injective on $\eta/2$-short loops.

Corollary 3.7. *(Geodesics Trap Surfaces* [**CG**]*) Let α be a path from δ_1 to δ_2 where $\delta_1 \cup \delta_2 \subset \Delta$ and S separates δ_1 from δ_2. Or, if $\mathcal{E}$ is an end of N and S separates δ_1 from $\mathcal{E}$, let α be a proper ray from δ_1 to $\mathcal{E}$. Assume that Δ is η separated. Then any surface S_1 obtained by shrinkwrapping S satisfies $S \cap \alpha \neq \emptyset$ and there exists $K_0 > 0$ that depends only on genus$(S), \epsilon$ and η such that*

$$Max\{d_N(x,\alpha)|x \in S_1\} \leq K_0 \, modulo \, N_{(0,\epsilon]}.$$

Proof. Note that $S_1 \cap \alpha \neq \emptyset$, since S is homotopic to S_1 via a homotopy disjoint from $\delta_1 \cup \delta_2$ except possibly at the last instant and that each intermediary surface intersects α. The result now follows from the Bounded Diameter Lemma 2.9. $\square$

Remark 3.8. To prove the Tameness theorem in the PL category we require the following more general version of Theorem 3.4 which is the actual result analogous to Soma's ruled wrapped theorem [**So**].

Consider a branched covering $p : (\hat{N}, \hat{\Delta}) \to (N, \Delta)$ where N and Δ are as in Theorem 3.4, the image of the branched locus is Δ and if $x \in \hat{\Delta}$, then the local branching near x is either 1-1 (i.e. a local homeomorphism) or is infinite to one. In the general version the manifold N is replaced by $\hat{N}$ and Δ is replaced by $\hat{\Delta}$. In this chapter we prove Theorem 3.4. Very similar ideas prove the more general result.

Lemma 3.9. *(Existence, Uniqueness and continuity of Δ-geodesics* [**So**]*) Let $x, y \in N$. If $\alpha_0 \subset N - \Delta$ is a path from x to y or a loop disjoint from Δ, then there exists a unique piecewise geodesic α_1 both path and Δ-homotopic to α_0 such α_1 is length minimizing among all paths both path and Δ-homotopic to α_0. The function that takes α_0 to α_1 is continuous.*

A similar result holds if α_0 is transverse to Δ and $\alpha_0 \cap \Delta \subset \partial\alpha_0$.

Proof. We give the proof in the case where α_0 is a path disjoint from Δ, the general case being similar. If g_0 denotes the original metric on N, then there exists a continuous family of complete metrics g_t on $N \setminus \Delta$ such that

a) $g_t|N \setminus N(\Delta, t) = g_0|N \setminus N(\Delta, t)$ and

b) each g_t has negative sectional curvature.

Using the Cartan Hadamard theorem, the lemma is true for the g_t metrics. A limiting argument shows that it is true for the g_0 metric. See [**So**] for an explicit construction of the g_t metrics and more details of this limiting argument. □

Definition 3.10. A piecewise geodesic α_1 constructed as above is called a Δ-*geodesic*. A Δ-geodesic implicitly includes information as to how to locally perturb it off of Δ. If α_1 is not a component of Δ, then distinct local push offs differ by local winding about Δ.

Lemma 3.11. *(Local view of Δ-geodesics* [**So**]*) If α is a Δ-geodesic, $x \in int(\alpha) \cap \Delta$, then either*

i) α is a geodesic segment of Δ or

ii) near x, α lies in the union U of two totally geodesic half discs glued along Δ.

With respect to the induced hyperbolic metric on U, $\alpha_x \cap U$ is a geodesic where α_x is a small neighborhood of x in α. □

Lemma 3.12. *(How to prove cone angle $\geq 2\pi$* [**So**]*) Let N be a complete hyperbolic 3-manifold with the locally finite collection of simple geodesics Δ. Let $S_1 \subset N$ be a closed pre-simplicial hyperbolic surface Δ-homotopic to a surface S_0 disjoint from Δ. If through each vertex of S_1 passes a Δ-geodesic α lying in S_1, then S_1 is a simplicial hyperbolic surface.*

Remark 3.13. We assume that the implicit push off of the Δ-geodesic is compatible with the Δ-homotopy from S_0 to S_1, i.e. if the homotopy is defined by the surfaces S_t and $\alpha_t \subset S_t$ is the path corresponding to $\alpha \subset S_1$, then the implicit push off of α is given by α_t for t close to 1.

Proof. Let $D \subset S_1$ a small disc about v and E the closure of a component of $D \setminus \alpha$. The cone angle contribution of E is the length of the piecewise geodesic path

β lying in the unit tangent space $\mathbb{U}$ to $v \in N$, where β is the path of unit vectors which point from v to ∂E. Using Lemma 3.11 it follows that $\text{length}_{\mathbb{U}}(\beta) \geq \pi$. Since this is true for each component of $D \setminus \alpha$ the result follows. Compare with Lemma 5.6 □

Lemma 3.14. *(Local Shrinkwrapping) Let $B \subset N$ be a round 3-ball which intersects Δ in a single component passing through its center. Let σ be a 2-cell with edges α, β, γ and δ cyclically ordered about $\partial\sigma$. We allow for the possibility that δ is a point and hence σ is a 2-simplex.*

Let $f : \sigma \to B \subset N$ be such that $f|\alpha$ and $f|\gamma$ are $\Delta-$geodesics and $f|\beta$ and $f|\delta$ are N-geodesics and $f^{-1}(\Delta) \subset int(\alpha) \cup int(\gamma)$. Then, via a homotopy supported in B, f is Δ-homotopic to g rel $\partial\sigma$ such that g is a simplicial hyperbolic surface. □

Remark 3.15. Saying that $g : \sigma \to N$ is a simplicial hyperbolic surface means that it is a pre-simplicial hyperbolic surface such that the cone angle of each interior vertices is at least 2π.

We abuse notation by letting α, β, γ and δ denote their images in N. The implicit local push off of α and γ are given by $f|\sigma$. For applications, we only require that β and δ are extremely short compared with the radius of B and that α and γ are nearly parallel (as Δ-geodesics) with endpoints very close to ∂B.

Proof of the PL Shrinkwrapping Theorem. Let γ be a non separating simple closed curve on S. Δ-homotop S so that γ becomes a Δ-geodesic and so the resulting mapped surface S_1 satisfies $(S_1 \setminus \gamma) \cap \Delta = \emptyset$. Note that this Δ-geodesic is non trivial since S is 2-incompressible.

Let v be a point of $\gamma \subset S_1$. Let $\alpha_1, \cdots, \alpha_n$ be a minimal collection of embedded arcs in S_1 based at v which cut $S_1 \setminus \gamma$ into a disc D_1. Now Δ-homotop S_1 to S_2 rel γ via a homotopy that takes each α_i to a Δ-geodesic, which is non trivial by 2-incompressibility. Again assume that D_2, the homotoped D_1 has interior disjoint from Δ. Our D_2 is defined by a map $h_2 : E \to N$, where E is a convex $4g$-gon $E \subset \mathbb{R}^2$. Let w be a vertex of E. Foliate E by line segments $\{\sigma_x\}$ with one endpoint on v and the other on points $x \in \partial E$.

It follows from the existence property of Lemma 3.9 the restriction of h_2 to each σ_x is path homotopic and Δ-homotopic to a Δ-geodesic, which conceivably is just a point. In what follows it is routine to deal with this degenerate case, thus from now on we will assume that it does not occur. The continuity property implies that these restricted homotopies extend to a Δ-homotopy $h_3 : E \to N$. The resulting mapped surface S_3 is not in general a pre-simplicial hyperbolic surface.

Let V be the union of vertices of E together with the isolated points of $\Delta \cap \partial E$. Let $\sigma_0, \sigma_1, \cdots, \sigma_m$ be a subset of the σ_x's such that σ_0 and σ_n are edges of E, $V \subset \cup\sigma_i$ and the various σ_i's are naturally linearly ordered in E. By appropriately choosing these σ_i's we can assume that they chop E into triangles which are mapped under h_3 into *very narrow* triangles. I.e. if $\sigma \subset E$ is one triangle bounded by σ_{i-1}, σ_i and a *short* edge in ∂E, then $h_3(\sigma_x)$ is *nearly parallel* to $h_3(\sigma_{i-1})$ for any σ_x lying in σ. Let $h_4 : E \to N$ be Δ-homotopic to $h_2 : E \to N$ via a homotopy fixing ∂E pointwise such that if $i \in \{0, \cdots, m\}$, then $h_4|\sigma_i = h_3|\sigma_i$ and if $y \in E$, then $h_4(y)$ is *very close* to $h_3(y)$. Finally, $h_4(E \setminus (\partial E \cup \sigma_0 \cup \cdots \cup \sigma_m)) \cap \Delta = \emptyset$. View $\partial E \cup \sigma_0 \cup \cdots \cup \sigma_m$ as the 1-skeleton of a triangulation Σ_4 of E.

The expressions *very narrow*, *nearly parallel*, *short*, and *very close* in the previous paragraph mean that Σ_4 can be subdivided to a cellulation Σ_5 so that after a

Δ-homotopy of h_4 to h_5 that fixes pointwise the 1-skeleton of Σ_4, each 2-cell σ of Σ_5 is either a 2-simplex with geodesic boundary whose convex hull has interior disjoint from Δ or $h_5|\sigma$ satisfies the hypothesis of Lemma 3.14. Furthermore, $h_5^{-1}(\Delta)$ is contained in the 1-skeleton of Σ_4. See Figure 3.1.

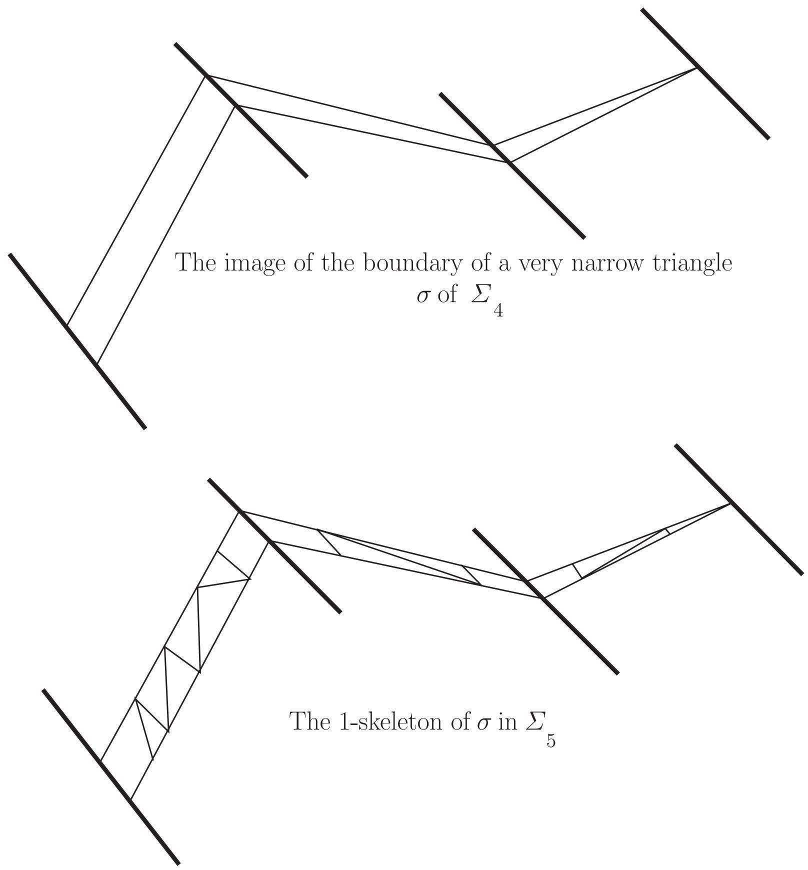

Figure 3.1

If σ is a 2-simplex of Σ_5 whose convex hull has interior disjoint from Δ, then Δ-homotope it rel boundary to a totally N-geodesic simplex. Otherwise, apply Lemma 3.14 to $h_5|\sigma$. Thus we obtain a map $h_6 : E \to N$. By Lemma 3.12 the corresponding pre-simplicial hyperbolic surface S_6 satisfies the cone angle condition hence, is simplicial hyperbolic.

The homotopy from S to S_6 is not a Δ-homotopy, for intermediate surfaces may encounter Δ. However, this homotopy is the concatenation of Δ-homotopies which evidently can be perturbed to be a single Δ-homotopy. □

Remark 3.16. In [**So**] Soma argued as above to produce the surface, that we denoted S_3. He called it a *ruled wrapping* and noted that it is Cat(-1).

4. Proof of Canary's Theorem

In this section we present the proof of Dick Canary's theorem, for parabolic free hyperbolic 3-manifolds which motivated our proof of the Tameness Theorem. While stated in the PL category, the argument here is identical to the one given in [**CG**] with many passages quoted verbatim. Our proof that N satisfies the tameness criterion uses several of the arguments presented in this chapter.

At the end of this section we present the example which suggested how to promote our proof of Canary's theorem to the proof of the tameness theorem.

Theorem 4.1 (Canary [**Ca1**]). *If $\mathcal{E}$ is a topologically tame end of the complete, hyperbolic 3-manifold N, then $\mathcal{E}$ is geometrically tame.*

Proof of the parabolic free case. It suffices to consider the case that $\mathcal{E}$ is geometrically infinite. By Corollary 2.16 it suffices to show that $\mathcal{E}$ satisfies conditions 1)-3) of the Tameness Criterion.

By Remark 2.5 there exists a sequence of pairwise disjoint η-separated simple closed geodesics $\Delta = \{\delta_i\}$ exiting $\mathcal{E}$. Assume that Δ and the parametrization of $\mathcal{E}$ are chosen so that for all $i \in \mathbb{N}$, $\delta_i \subset S\times(i-1, i)$. Let g=genus(S), $\Delta_i = \{\delta_1, \cdots, \delta_i\}$ and $\{\alpha_i\}$ a locally finite collection of embedded proper rays exiting $\mathcal{E}$ such that $\partial\alpha_i \in \delta_i$.

Warm-up Case 4.2. $N = S \times \mathbb{R}$

Proof. Apply Theorem 3.4 to shrinkwrap $S \times i$ rel. Δ_{i+1} to a simplicial hyperbolic surface S_i. By Corollary 3.7, max $\{d_N(x, \alpha_i)|x \in S_i\} < K_0$ modulo $N_{(0,\epsilon]}$. Since $\{\alpha_i\}$ is locally finite, the S_i's exit $\mathcal{E}$. Thus for i sufficiently large, the homotopy from $S \times i$ to S_i, never intersects δ_1 and so $\langle S_i, \alpha_1\rangle = \langle S \times i, \alpha_1\rangle = 1$. Thus $\{S_i\}$ satisfy 1)-3) of the Tameness Criterion. See Figure 4 [**CG**] for a schematic view.

General Case 4.3. (e.g. N is an open handlebody)

Proof. Without loss of generality we can assume that every closed orientable surface in N separates. Indeed, we can pass to a covering space $\hat{N}$ of N with finitely generated fundamental group such that every orientable surface in $\hat{N}$ separates and $\mathcal{E}$ lifts isometrically to an end of $\hat{N}$. See Lemmas 5.1 and 5.6 of [**CG**].

We use a purely combinatorial/topological argument to find a particular sequence of embedded surfaces exiting $\mathcal{E}$. We then shrinkwrap these surfaces and show that they have the desired escaping and homological properties.

Fix i. If possible, compress $S \times i$, via a compression which either misses Δ or crosses Δ once say at $\delta_{i_1} \subset \Delta_i$. If possible, compress again via a compression meeting $\Delta\backslash\delta_{i_1}$ at most once say at $\delta_{i_2} \subset \Delta_i$. After at most $n \le 2g-2$ such operations and deleting 2-spheres we obtain embedded connected surfaces $S^i_1, \cdots, S^i_{i_r}$, none of which is a 2-sphere and each is 2-incompressible rel $\Delta_{i+1}\backslash\{\delta_{i_1} \cup \cdots \cup \delta_{i_n}\}$. With at most $2g-2$ exceptions, each $\delta_j, j \le i$, is separated from $\mathcal{E}$ by exactly one surface S^i_k. Call Bag^i_k the region separated from $\mathcal{E}$ by S^i_k. Note that all compressions in the passage of S_i to $\{S^i_1, \cdots, S^i_{i_r}\}$ are on the non $\mathcal{E}$-side.

Since the i_r's are uniformly bounded, a pigeon hole type argument shows that we can find a $p \in \mathbb{N}$; and for each i, a reordering of the S^i_j's (and their bags) so that for infinitely many $i \ge p$, $\delta_p \in \text{Bag}^i_1$. Furthermore, for those i such that $\delta_p \in \text{Bag}^i_1$, if $p(i)$ denotes the maximal index such that $\delta_{p(i)} \in \text{Bag}^i_1$ then the set $\{p(i)\}$ is unbounded. By Theorem 3.4, S^i_1 is $(\Delta_{i+1}\backslash\{\delta_{i_1}, \cdots, \delta_{i_n}\})$-homotopic to a simplicial hyperbolic surface surface R_i. Since the collection $\{\alpha_{p(i)}\}$ is infinite and locally finite, the Bounded Diameter Lemma implies that a subsequence of these R_i's with $\delta_p \in \text{Bag}^i_1$ must exit $\mathcal{E}$. Call this subsequence $T_1, T_2, \cdots$, where T_i denotes the shrinkwrapped $S_1^{n_i}$. Therefore, for i sufficiently large, T_i must lie in $S \times (p, \infty)$ and $\langle T_i, \alpha_p \rangle = \langle S_1^{n_i}, \alpha_p, \rangle = 1$. Therefore, $\{T_i\}$ satisfy 1)-3) of the Tameness Criterion. See Figure 5 [**CG**] for a schematic view. □

Remark 4.4. This argument together with the proof of Proposition 2.15 shows that for i sufficiently large, T_i is homeomorphic to $S \times 0$. Thus, for i sufficiently large, $S \times i$ is 2-incompressible in $N \backslash \Delta_i$, these surfaces never compressed and for k sufficiently large T_k is simply a $S \times j$ shrinkwrapped with respect to Δ_{j+1}.

Remark 4.5. A key advantage of trying to verify the Tameness Criterion for a topologically tame hyperbolic manifold is that we already have an exiting sequence of embedded homologically separating surfaces of the right genus. Hence we have surfaces which are candidates to shrinkwrap to produce the desired simplicial hyperbolic surfaces.

On the other hand the general genus-2 homotopy handlebody M does not have an exiting sequence of homologically separating genus-2 surfaces. Nevertheless, in the presence of an exiting sequence of homotopically nontrivial simple closed curves, there are interesting *immersed* genus-2 surfaces. (For a hint, see Example 4.6.) If M was also hyperbolic, then after *appropriately* shrinkwrapping these surfaces we obtain surfaces that have points both far out in the manifold and points that hit the core. This contradicts the Bounded Diameter Lemma. See §6 for the details.

Expressed positively, as explained in §6, if M is hyperbolic, then the Bounded Diameter Lemma miraculously implies that these simplicial hyperbolic surfaces exit M. A homological argument modeled on the proof of Canary's theorem shows that they homologically separate. Thus M is geometrically and topologically tame.

Example 4.6. Construct a homotopy genus-2 handlebody $M = \cup V_i$, such that $V_1 \subset V_2 \subset \cdots$, where the embedding $V_1 \to V_2$ is shown in Figure 4.1a) and the other embeddings $V_i \to V_{i+1}$ are defined similarly. Here each V_i is a genus-2 handlebody. Figure 4.1b) shows an immersed genus-2 surface $T' \subset V_2$. Note that it lifts to an embedded genus-2 surface $T \subset X$, where X is the $\pi_1(V_1)$ cover to V_2. See Figure 4.1c). (This figure appeared as Figure 3 in [**CG**].)

5. The Tameness Criterion

In this section we provide a sufficient condition for showing that an end $\mathcal{E}$ of a complete hyperbolic 3-manifold N is topologically and geometrically tame, namely that $\mathcal{E}$ satisfies the three conditions of the following Theorem 5.1. That result, inspired by earlier work of Juan Souto [**So**], is a slight reformulation of the one given on p. 431, [**CG**]

Recall, that unless said otherwise, in this paper, all manifolds are orientable and all hyperbolic manifolds are parabolic free. By a *mapped surface* in a 3-manifold N, we mean a function $f : T \to N$ where T is homeomorphic to a surface. We often abuse notation by identifying a mapped surface with either its domain or its range.

Theorem 5.1. *(Tameness Criterion) Let N be a complete hyperbolic 3-manifold with finitely generated fundamental group. Let C be a codimension-0 core of N and $\mathcal{E}$ an end of N. Then $\mathcal{E}$ is geometrically and topologically tame if the following hold.*

1) There exists a sequence $S_1, S_2, \cdots$ of closed mapped surfaces exiting $\mathcal{E}$ such that for each i, genus(S_i) $\leq$ genus($\partial_{\mathcal{E}} C$), where $\partial_{\mathcal{E}} C$ is the component of $\partial C facing \mathcal{E}$.

FIGURE 4.1

2) Each S_i is a simplicial hyperbolic surface.
3) Each S_i homologically separates $\mathcal{E}$ from C, i.e. $[S_i] = [\partial_{\mathcal{E}} C] \in H_2(N \setminus int(C))$.

Here is the idea of the proof of the Tameness Criterion. By Corollary 2.16 it suffices to show that $\mathcal{E}$ is topologically tame. To do this we first observe that each S_i can be homotoped through simplicial hyperbolic surfaces to one which lies near C, via a process called *surface interpolation*. By patching together pieces of these interpolations we obtain a proper map of $S \times [0, \infty)$ into $\mathcal{E}$. In the course of straightening this map we show $\mathcal{E}$ is topologically tame.

We leave to the reader the following algebraic topology exercises which are essential for what follows.

Exercise 5.2. Let $\mathcal{E}$ be an end of the open 3-manifold N with codimension-0 core C. Let Z be the component of $N \setminus \text{int}(C)$ which contains $\mathcal{E}$. Let T be a closed surface where genus$(T) \leq$ genus$(\partial_{\mathcal{E}} C)$ and let S be a mapped surface in Z which homologically separates $\mathcal{E}$ from C and is defined by the function $f : T \to Z$.

1) Show that the inclusion $\partial_{\mathcal{E}} C \to Z$ induces an isomorphism on H_1.

2) Show that genus(T) =genus$(\partial_{\mathcal{E}} C)$ and that $f : T \to Z$ is an H_1-isomorphism.

3) Let A be the (possibly disconnected) surface obtained by maximally compressing $\partial_{\mathcal{E}} C$ in C and isotoping the result into int(C). Let N_1 be the closure of the component of $N \setminus A$ which contains Z and let $C_1 = N_1 \cap C$. Show that the inclusion $C_1 \to N_1$ is a homotopy equivalence, i.e. C_1 is a core of N_1. In particular if $\partial_{\mathcal{E}} C$ is incompressible, then the inclusion $\partial_{\mathcal{E}} C \to Z$ is a homotopy equivalence. (The manifold N_1 is called the *end-manifold* associated to $\mathcal{E}$.)

4) Show that S is incompressible in Z. See Definition 5.3. A more challenging exercise, not needed for this paper, is to show that S is π_1-injective.

5) Show that if $\partial_{\mathcal{E}} C$ is compressible in C, then S is compressible in N.

6) Show that if $\partial_{\mathcal{E}} C$ is incompressible, then S is incompressible in N and via a homotopy supported in Z, can be homotoped to a homeomorphism onto $\partial_{\mathcal{E}} C$.

Definition 5.3. We say that the mapped surface S defined by the mapping $f : T \to M$ is *compressible* if some essential simple closed curve in S is homotopically trivial in M, i.e. there exists an essential simple closed curve $\gamma \subset T$ such the $f|\gamma$ is homotopically trivial. We say S is *incompressible* if it is not compressible.

Remark 5.4. Many authors use compressible synonymously with π_1-injective. However, given the unresolved *simple loop conjecture* for 3-manifolds, these may represent distinct concepts.

We state a simple criterion [**Ca2**] for showing that a pre-simplicial hyperbolic surface satisfies the cone angle condition.

Definition 5.5. Let N be a hyperbolic 3-manifold. A mapped surface $S \subset N$ is *convex busting* if for each $x \in S$ and every neighborhood $U \subset S$ of x, $P \cap (U \setminus x) \neq \emptyset$, where P is any totally geodesic disc in N with $x \cap \text{int}(P) \neq \emptyset$. (I.e. S does not locally lie to any one side of a geodesic disc.)

Lemma 5.6. [**Ca2**] *A convex busting, pre-simplicial hyperbolic surface is a simplicial hyperbolic surface.*

Proof. It suffices to check the cone angle condition at each vertex v of the simplicial hyperbolic surface S. The link of v in S gives rise to a piecewise geodesic α in the unit tangent space $\mathbb{U}$ of N at v where length$_{\mathbb{U}}(\alpha)$ = cone angle(v). Note that $\mathbb{U}$ is isometric to the round unit 2-sphere and S is not convex busting at v if and only if there exists a geodesic $\beta \subset \mathbb{U}$ disjoint from α. The proof now follows from the next exercise. □

Exercise 5.7. If α is a piecewise closed geodesic in the round unit 2-sphere and length$(\alpha) < 2\pi$, then there exists a geodesic disjoint from α.

Exercise 5.8. Show that we can assume that the surfaces in the Tameness Criterion are *useful* simplicial hyperbolic surfaces.

Hint: Generalize the ideas in Remarks 2.10 to show that for i sufficiently large a uniformly bounded homotopy, modulo Margulis tubes, transforms each S_i to a useful pre-simplicial hyperbolic surface. Now apply Lemma 5.6 to show that useful pre-simplicial hyperbolic surfaces are actually simplicial hyperbolic surface.

Our proof of the Tameness Criterion requires understanding surface interpolation of simplicial hyperbolic surfaces. As part of his revolutionary work on hyperbolic geometry, William Thurston developed surface interpolation for pleated surfaces. The simplicial version of this theory, which we now discuss, was later introduced by Francis Bonahon [**Bo**] and further developed by Dick Canary [**Ca2**] and Canary - Minsky [**CaM**].

If Δ is a 1-vertex triangulation, with preferred edge e, on the mapped surface $S' \subset N$ and each closed edge of Δ is a null homotopic curve in N, then Δ gives rise to a simplicial hyperbolic surface S in N homotopic to S'. To see this first homotop the preferred edge to a closed geodesic e in N. Next, homotop the various edges to be geodesic via homotopies fixing the vertex v. Finally homotop, rel boundary, the 2-simplices to geodesic ones; however, note that if two adjacent edges of a 2-simplex meet at angle π, then the resulting 2-simplex is degenerate. Note that the resulting simplicial hyperbolic surface is unique up to sliding v along e.

Remark 5.9. [**Ca2**] If S_0 and S_1 are two useful simplicial hyperbolic surfaces which differ by sliding the vertex along the preferred edge, then S_0 is homotopic to S_1 via homotopy through simplicial hyperbolic surfaces.

In what follows we will suppress discussion of choice of vertex, up to sliding, in the preferred edge.

Definition 5.10. Let S be a closed surface with a 1-vertex triangulation Σ. A *quadrilateral move* is an operation which transforms Σ to a triangulation Σ', where Σ' is obtained from Σ by deleting an edge to create a quadrilateral 2-cell and then inserting the opposite edge.

If Δ is a 1-vertex triangulation on S with preferred edge e, then an *elementary move* is either the replacement of e by another edge of Δ or a quadrilateral move on Δ not involving e.

The following is a well known result in surface topology, e.g. see [**Ha**].

Theorem 5.11. *If Δ_0 and Δ_1 are two 1-vertex triangulations on the closed surface S, then up to isotopy, Δ_0 can be transformed to Δ_1 via a finite number of quadrilateral moves.*

The following result and proof of Canary shows that simplicial hyperbolic surfaces that differ by an elementary move are homotopic through simplicial hyperbolic surfaces.

Proposition 5.12. *(Chapter 5,* [**Ca2**]*) Let Σ_0 and Σ_1 be 1-vertex triangulations with preferred edge e on the closed oriented surface $S \subset N$, where N is a complete hyperbolic 3-manifold. If both Σ_0 and Σ_1 have no null homotopic edges and Σ_1 is obtained from Σ_0 by either a quadrilateral move not involving e or by changing the choice of preferred edge, then the corresponding simplicial hyperbolic surfaces S_0 and S_1 are homotopic through simplicial hyperbolic surfaces.*

Proof. If Σ_0 and Σ_1 differ by a quadrilateral move, then let Σ be the 2-vertex triangulation of S obtained by including both edges. In a natural way construct a map $F : S \times [0,1] \to N$ so that $F|S \times i = S_i$ for $i \in \{0,1\}$ and for $t \in (0,1)$, $F|S \times t$ is a simplicial hyperbolic surface S_t which realizes Σ. Each S_t is convex busting at each vertex, hence by Lemma 5.6 satisfies the cone angle condition.

If Σ_0 and Σ_1 differ by the choice of preferred edges e_0 and e_1, then consider the associated closed geodesics e_0^* and e_1^* in N and the orthogonal geodesic arc α connecting them. Assuming that the vertex of each Σ_i lies on $\partial\alpha$, construct a homotopy $F : S \times [0,1] \to N$ where $F|S \times t$ is a pre-simplicial hyperbolic surface with the unique vertex v_t lying on α. Check that S_t is convex busting at v_t to show that S_t is a simplicial hyperbolic surface. □

Theorem 5.13. [**CaM**] *Let N be a complete hyperbolic 3-manifold with finitely generated fundamental group. Let C be a codimension-0 core for N. Let Z be a component of $N \setminus int(C)$. If $S \subset Z$ is a mapped useful simplicial hyperbolic surface which is compressible in N but incompressible in Z, then there exists a homotopy $F : S \times [0,1] \to N$ such that for each $t, F|S \times t$ is a simplicial hyperbolic surface, $F|S \times 0 = S$, $F(S \times 1) \cap C \neq \emptyset$ and $F(S \times [0,1]) \subset Z$.*

If S and S' are homotopic mapped useful simplicial hyperbolic surfaces in Z that are incompressible in N, then they are either homotopic through simplicial hyperbolic surfaces in Z or, as above, each is homotopic through simplicial hyperbolic surfaces in Z to ones which meet C.

Proof. Let Σ_0 denote the 1-vertex triangulation with preferred edge given by the simplicial hyperbolic surface S. Let $\Sigma_0, \Sigma_1, \cdots, \Sigma_n$ denote 1-vertex triangulations on S where one differs from the next by an elementary move and some edge of Σ_n is null homotopic in N. Furthermore no edge in Σ_k is null homotopic in N if $k < n$. Let S_{n-1} denote the simplicial hyperbolic surface with triangulation Σ_{n-1}. By Theorem 5.12 S_0 is homotopic through simplicial hyperbolic surfaces to S_{n-1}. Let $G : S \times [0, n-1] \to N$ denote the homotopy. We will show that for some $t \leq n-1, G|S \times t \cap C \neq \emptyset$. If t is the first such event, then the desired F is obtained by restricting G to $[0,t]$ and reparametrizing.

Triangulation Σ_n is obtained from Σ_{n-1} by deleting some edge f and inserting edge g. Let σ and σ' denote the 2-simplices of Σ_{n-1} which meet along f. Since g is null homotopic in N, the only possibility is that the corresponding geodesic simplices in N coincide as do the edges of $\partial\sigma$ and $\partial\sigma'$. Indeed, σ and σ' are folded along f. It follows that a null homotopy of g lies within the point set S_{n-1}. Therefore, if $G(S \times [0, n-1]) \cap C = \emptyset$, then S is compressible in Z, a contradiction.

If S and S' are homotopic useful simplicial hyperbolic surfaces incompressible in N, then there exist triangulations $\Sigma_0, \cdots, \Sigma_n$ where Σ_0, Σ_n are the triangulations associated with S and S' and for each i, Σ_{i+1} is obtained from Σ_i by an elementary move. Use Theorem 5.12 to homotop S to S' through simplicial hyperbolic surfaces. If the homotopy is supported in Z, then the first conclusion holds; if not, then the second one holds. □

We need the following finiteness theorem for simplicial hyperbolic surfaces. The first theorem of this type was proven by Thurston [**T1**] for pleated surfaces. A version for simplicial hyperbolic surfaces is given in [**So**]. Our version basically says that any simplicial hyperbolic surface which is π_1-injective on short simple loops and intersects a fixed compact set is homotopic, via an ϵ-homotopy, to one of finitely many simplicial hyperbolic surfaces.

Definition 5.14. Two maps $f, g : S \to N$ are *unparametrized ϵ-homotopic* if there exists a homeomorphism $h : S \to S$ and a homotopy $F : S \times I \to N$ from f to $g \circ h$ such that for each $x \in S, \operatorname{diam}(F(x \times I)) \leq \epsilon$, i.e. after reparametrization there exists a homotopy such that tracks of points have diameter $\leq \epsilon$.

Lemma 5.15. *(Finiteness of Simplicial Hyperbolic Surfaces) Let N be a complete hyperbolic 3-manifold, $K \subset N$ compact, $\epsilon > 0$ and $\delta > 0$. Then there exists $S_1, \cdots, S_n$ genus-g simplicial hyperbolic surfaces in N such that if S is a genus-g simplicial hyperbolic surface which intersects K and is π_1-injective on essential simple closed curves of length $\leq \delta$, then S is unparametrized ϵ-homotopic to some S_i.*

Proof. Let S be a genus-g simplicial hyperbolic surface which intersects K and is π_1-injective on essential simple closed curves of length $\leq \delta$. By the Bounded Diameter Lemma 2.9 and Remark 2.10 1), there exists a compact codimension-0 submanifold $K_1 \subset N$ which contains any such surface S. There exists $\eta > 0$ such that the injectivity radius of each point in the 1-neighborhood of K_1 is $\geq \eta$. We can assume that $\eta < \min(\delta/2, \epsilon)$ so if S is as above, then $\operatorname{injrad}_S(x) \geq \eta$ for $x \in S$. Thus, there exists $C_0 \in \mathbb{N}$ such that given S as above, there exists a triangulation Δ on S with at most C_0 vertices; furthermore, after homotopy which move points at most $\eta/10$, each simplex of Δ is mapped to a totally geodesic simplex of diameter $\eta/10$. Since there are only finitely many combinatorial types of triangulations with at most C_0 vertices, it follows by a compactness argument that any such S is $\eta/2$-homotopic to one of finitely many surfaces and hence ϵ-homotopic to one of finitely many simplicial hyperbolic surfaces. ◻

Proof of the Tameness Criterion: To simplify notation we will assume that N has a unique end $\mathcal{E}$. Let C be a core of N and let $Z = N \setminus \operatorname{int}(C)$. Note that ∂C is connected and equals $\partial_{\mathcal{E}} C$. By invoking Exercise 5.2, we will assume that for all i, $\operatorname{genus}(S_i) = \operatorname{genus}(\partial C) = g$. Therefore S_i corresponds to a map $f_i : S \to N \setminus \operatorname{int}(C)$, where S is a closed surface of genus-g. By Exercise 5.8 we can further assume that each S_i is a useful simplicial hyperbolic surface.

Step 1. There exists a compact submanifold $K \subset N$ such that $C \subset K$ and every S_i can be homotoped into K via a homotopy of simplicial hyperbolic surfaces supported in Z.

Proof of Step 1. If ∂C is compressible in N, then by Exercise 5.2 each S_i is compressible in N but incompressible in Z. By Theorem 5.13 each S_i can be homotoped through simplicial hyperbolic within Z to one which hits C. There exists $\eta > 0$ so that the η-neighborhoodof ∂C has the natural product structure and so by Exercise 5.2 4) each S_i is π_1-injective in N on η-short simple closed curves. Now apply Lemma 2.9 and Remark 2.10 1).

If ∂C is incompressible in C, then by Exercise 5.2 each S_i is homotopic within Z to S_1, hence by Theorem 5.13 they are homotopic through simplicial hyperbolic surfaces. By passing to subsequence all such homotopies either hit C or are disjoint from C. In the former case each S_i is homotopic within Z to a simplicial hyperbolic surface which hits C. Now apply Lemma 2.9. In the latter case choose K to be a compact submanifold which contains $C \cup S_1$. ◻

Step 2. There exists a proper map $F : S \times [0, \infty) \to Z$ such that for each t, $F|S \times t$ is π_1-injective on simple loops and homologically separates $\mathcal{E}$ from C.

Proof of Step 2. Let $Y_1, Y_2, \cdots$ be a sequence of embedded surfaces in N which exit $\mathcal{E}$ and homologically separate $\mathcal{E}$ from K. (There is no assumption on the genus of the Y_i's.) Assume that if $i > j$, then $d(Y_j, Y_i) \geq 10$ and Y_j separates K from Y_i. Also $d_N(Y_1, K) \geq 10$. The Y_i's will serve as coordinates for the end $\mathcal{E}$. Let Z_m denote the closure of the component of $N \setminus Y_m$ disjoint from C.

For each i, fix a homotopy of S_i through simplicial hyperbolic surfaces to one which lies in K. If such exists, let S_i^j denote the first surface in this homotopy which intersects Y_j. By passing to a subsequence $S_{1_1}, S_{1_2}, \cdots$ of $\{S_i\}$ we can assume that for $k \in \mathbb{N}$, $S_{1_k} \subset Z_1$ and by Theorem 5.15 any two $S^1_{1_j}$'s are 1-homotopic. By passing to a subsequence $S_{2_1}, S_{2_2}, \cdots$ of the $\{S_{1_k}\}$ sequence we can assume that each $S_{2_k} \subset Z_2$ and any two $S^2_{2_j}$'s are 1-homotopic. In an inductive manner construct the sequence $\{S_{n_k}\}$ with the property that each $S_{n_k} \subset Z_n$ and any two $S^n_{n_j}$'s are 1-homotopic.

Each $S^m_{m_1}$ is homotopic to $S^{m+1}_{(m+1)_1}$ via a homotopy supported in the 1-neighborhood of Z_m. To see this observe that $S_{(m+1)_1} = S_{m_k}$ for some k, so $S^{m+1}_{(m+1)_1}$ is homotopic to $S^m_{(m+1)_1} = S^m_{m_k}$ via a homotopy supported in Z_m and $S^m_{m_k}$ is homotopic to $S^m_{m_1}$ via a 1-homotopy.

The desired homotopy is obtained by concatenating these homotopies from $S^m_{m_1}$ to $S^{m+1}_{(m+1)_1}$ for $m = 1, 2, \cdots$. □

Remark 5.16. The homotopy of Step 2 can be viewed as the concatenated reverse of surface interpolations.

Step 3. $\mathcal{E}$ is topologically tame.

The proof of Step 3 is purely topological and relies on the following basic result in 3-manifold topology, due to Stallings (more or less in sections 3 and 4 of [**St**]) whose proof is left as an exercise.

Lemma 5.17. [**St**] *Let X and Y be irreducible 3-manifolds, $T \subset Y$ an embedded incompressible surface, $f : X \to Y$ a map transverse to T with $R = f^{-1}(T)$. If D is a compressing disc for R, then f is homotopic to g via a homotopy supported near D such that $g^{-1}(T)$ equals R compressed along D.*

If R_1 is an innermost 2-sphere component of R, then f is homotopic to g via a homotopy supported near the 3-ball B bounded by R_1 such that $g^{-1}(T) = R \setminus R_1$. (Innermost means that $B \cap R = R_1$.) □

Step 3 will be proved on route to proving the following result.

Proposition 5.18. *Let N be an open irreducible 3-manifold with finitely generated fundamental group, C a codimension-0 core of N, $\mathcal{E}$ an end of N, Z the component of $N \setminus int(C)$ containing $\mathcal{E}$, S a closed genus-g surface and $F : S \times [0, \infty) \to Z$ a map such that*

i) F is proper,

ii) $F|S \times 0$ is π_1-injective on simple loops of S and

iii) $F(S \times 0)$ homologically separates $\mathcal{E}$ from C, (i.e. represents $\partial_{\mathcal{E}} C \in H_2(Z)$).

Then F is properly homotopic to a homeomorphism onto its image.

Proof. If N is simply connected, then N is properly homotopy equivalent to $\mathbb{R}^3$ and hence, being irreducible, is well known to be homeomorphic to $\mathbb{R}^3$. To see this, first note that C is a closed 3-ball and hence S is a 2-sphere and so $F|[S \times i]$, $i \in \mathbb{N}$ is an exiting sequence of mapped 2-spheres essential in Z. Use the Sphere Theorem to find an escaping sequence of embedded 2-spheres which separate C from $\mathcal{E}$. Each 2-sphere bounds a 3-ball, hence the region between any two is a $S^2 \times I$.

We now assume that $\pi_1(N)$ is non trivial. Hence, if $U \subset N$ is a compact submanifold containing C, then $N \setminus U$ is irreducible.

To simplify the notation we will assume that N is 1-ended.

Step A. Given a compact connected codimension-0 submanifold $K \subset N$ with $C \subset K$ and $F^{-1}(K) \subset S \times [0, m-1]$, then F is homotopic to G rel $S \times [0, m-1]$ such that $G(S \times [m, \infty)) \cap K = \emptyset$ and $G(S \times i)$ is embedded for some $i > m$.

Proof of Step A. Let $Y \subset N$ be an embedded surface which separates $F(S \times m)$ from $\mathcal{E}$. We can assume Y is incompressible in $N \setminus K$. Indeed, if Y is compressible in $N \setminus K$, then let Y_1 be the result of compressing Y along the essential compressing disc D. We rectify the possible problem $Y_1 \cap F(S \times m) \neq \emptyset$ as follows. Assume F is transverse to $Y \cup D$. Since $F|S \times m$ is injective on simple loops, it follows that $F^{-1}(D) \cap S \times m$ is a union of embedded circles bounding discs in $S \times m$. $N \setminus K$ is irreducible, hence via a homotopy supported near $S \times m$, we can homotop F to F_1 so that $F_1(S \times m) \cap Y_1 = \emptyset$ and $F_1^{-1}(K) \subset S \times [0, m-1]$. Note that the resulting $F_1(S \times m)$ lies to the C side of Y_1. The case that Y can be compressed along finitely many discs is similar.

By Lemma 5.17 we can further assume that $F^{-1}(Y) \cap S \times [m, \infty)$ is incompressible in $S \times [m, \infty)$. Up to isotopy, the only embedded incompressible surfaces in $S \times [m, \infty)$ are of the form $S \times t$, thus we can assume each component of $F^{-1}(Y) \cap S \times [m, \infty)$ is of this type. Note that $F^{-1}(Y) \cap S \times [m, \infty) \neq \emptyset$, else Y does not separate $F(S \times m)$ from $\mathcal{E}$. Indeed, consider any proper ray α in $S \times [m, \infty)$ starting at $S \times m$. $F(\alpha)$ is a proper ray starting at $F(S \times m)$, so must intersect Y and hence $\alpha \cap F^{-1}(Y) \neq \emptyset$.

If $F(S \times t) \subset Y$, then $F|S \times t$ is homotopic to a homeomorphism onto Y. Indeed, since $F|S \times t$ is injective on simple loops, $F|S \times t$ is not degree-0. Since $[F(S \times t)]$ is a generator of $H_2(Z)$, $F|S \times t$ is degree-1. By Exercise 2.17 a degree-1 map between closed surfaces which is π_1-injective on simple loops is homotopic to a homeomorphism. □

It follows from Step A that we can assume that for each $i \in \mathbb{N}, F(S \times i)$ is embedded and homologically separates $\mathcal{E}$ from C. Furthermore, if $i > j$, then $F(S \times j)$ separates $F(S \times i)$ from C.

Step B. $\mathcal{E}$ is tame.

Proof of Step B. Let T_i denote $F(S \times i)$. It suffices to show that the region P between T_i and T_{i+1} is a product. By Lemma 5.17 we can assume that $F^{-1}(T_i \cup T_{i+1})$ is incompressible in $S \times [0, \infty)$ and hence, after homotopy of F, are components of the form $S \times u$ and the restriction of F to such a component is a homeomorphism onto its image. Thus, there exists a component Q of $F^{-1}(P)$ such that $F|Q$ is a degree-1 map onto P and $F|\partial Q$ is a homeomorphism onto ∂P. By hypothesis, as a map into $Z, F|\partial Q$ is injective on simple loops, hence by the loop theorem $F|\partial Q : \partial Q \to Z$ is

π_1-injective. It follows that $F|Q : Q \to P$ is π_1-injective. By Waldhausen [**Wa2**] $F|Q$ is homotopic rel ∂Q to a homeomorphism onto P. □

Exercise 5.19. Complete the proof of Proposition 5.18.

Remark 5.20. The above proof of the Tameness Criterion follows the same outline as that of [**CG**]. However, [**CG**] used the result that the Thurston norm for embedded surfaces equals the norm for singular surfaces [**G1**]. Here, we got around this by reversing the surface interpolation process (Step 2), using the Finiteness Lemma 5.15 and using standard 1960's 3-manifold topology.

6. Proof of the Tameness Theorem

An elementary covering space argument §5 [**CG**] shows that if $\mathcal{E}$ is an end of a complete hyperbolic 3-manifold N with finitely generated fundamental group, then there exists a covering space $\hat{N}$ of N such that $\mathcal{E}$ lifts isometrically to an end of $\hat{N}$ and $\pi_1(\hat{N})$ is a *free/surface group* i.e. is the free product of a (possibly trivial) free group and finitely many (possibly zero) closed surface groups. [Recall that here all manifolds are orientable and all hyperbolic manifolds are parabolic free.] So to prove the theorem in general it suffices to consider the case that $\pi_1(N)$ is a free/surface group.

We restrict our attention here to homotopy handlebodies, i.e. the case that $\pi_1(N)$ is a free group. In the non homotopy handlebody case we need to focus on the *end manifold* associated to $\mathcal{E}$ which is a 1-ended submanifold containing $\mathcal{E}$ homotopy equivalent to N. By restricting to the homotopy handlebody case we avoid this excess verbiage of end-manifold (which is the whole of N in the homotopy handlebody case) and the minor technicalities in dealing with the boundary components of this end-manifold (which are nonexistant in the homotopy handlebody case). Otherwise the proof is identical.

The following purely topological result is a special case of Theorem 5.21 [**CG**].

Theorem 6.1. *Let M be an irreducible homotopy handlebody, $\gamma_1, \gamma_2, \cdots$ an exiting sequence of pairwise disjoint, locally finite, homotopically nontrivial simple closed curves. After passing to subsequence and allowing γ_1 to have multiple components, there exists an open irreducible submanifold $\mathcal{W} \subset M$ which is both H_1 and π_1-injective. Furthermore, $\mathcal{W}$ is exhausted by compact submanifolds $W_1 \subset W_2 \subset \cdots$ such that each ∂W_i is connected and*

1) If $\Gamma_i = \gamma_1 \cup \cdots \cup \gamma_i$, then $\Gamma_i \subset W_i$ and ∂W_i is 2-incompressible with respect to Γ_i and ∂W_i is incompressible in $M \setminus int(W_i)$.

2) There exists a codimension-0 core D for $\mathcal{W}$ such that $D \subset W_1$ and for each i, Γ_i can be homotoped into D via a homotopy supported in W_i. □

The conclusion of this theorem is schematically shown in Figure 6.1.

Addendum to Theorem 6.1 If M is complete and hyperbolic and the γ_i's are geodesic, then each W_i is atoroidal. □

Remark 6.2. An elementary, self contained proof of the general form of Theorem 6.1 can be found on pages 417–423 of [**CG**]. It is based on the end-reduction theory Brin and Thickston [**BT1**],[**BT2**] developed in the 1980's. See p. 426 [**CG**] for a proof of the Addendum.

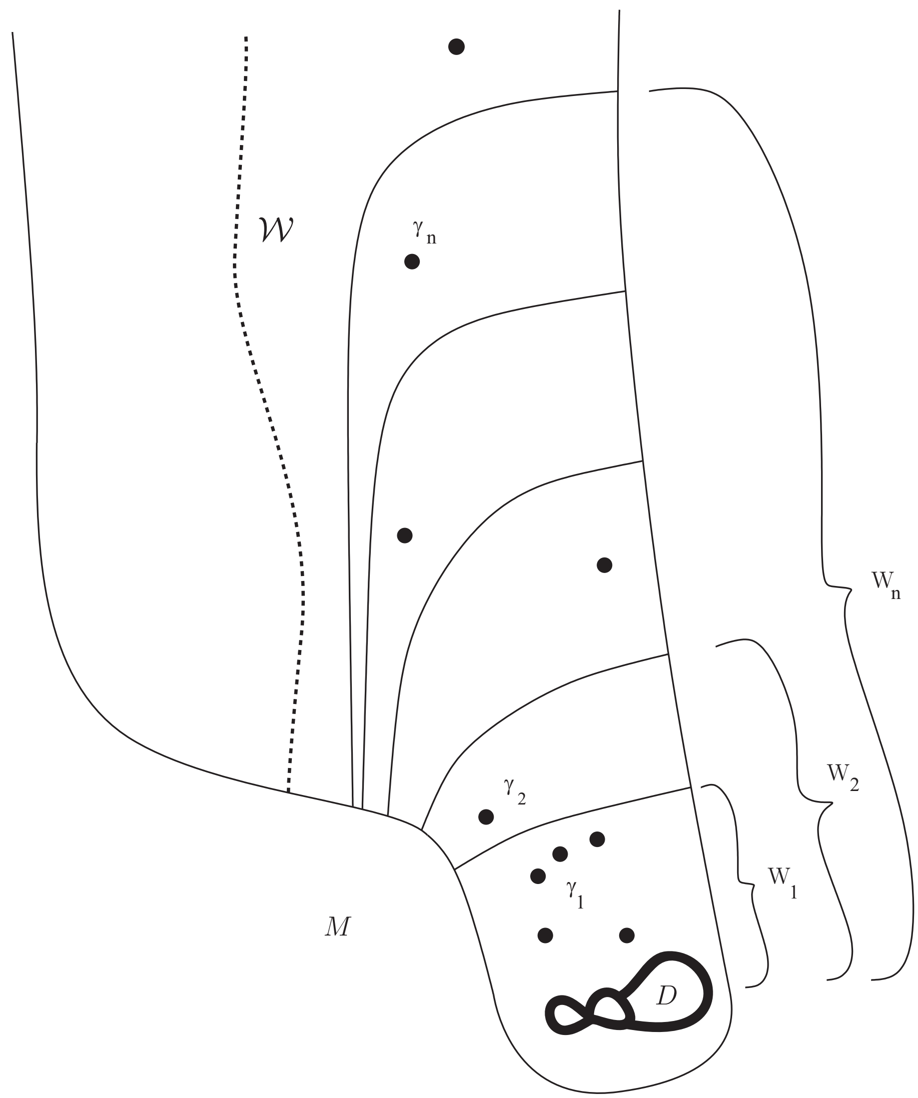

Figure 6.1

Actually the last assertion of conclusion 1) is not explicitly stated in Theorem 5.21. It follows because W_i is a term in an end-reduction of Γ_i.

Theorem 6.3. *(Thurston's Tameness Theorem) Let W be a compact, irreducible, atoroidal 3-manifold with $\chi(\partial W) < 0$. If $\hat{W} \to W$ is a covering map such that $\pi_1(\hat{W})$ is finitely generated, then int$(\hat{W})$ is topologically tame.*

Proof. Thurston's hyperbolization theorem implies that int(W) is homeomorphic to $\mathbb{H}^3/\Gamma$ where Γ is geometrically finite. Since a finitely generated subgroup of a geometrically finite group is geometrically finite, $\pi_1(\hat{W}) \subset \Gamma$ is geometrically finite and hence $\text{int}(\hat{W})$ is tame. See page 71 [**Mo**] or [**Ca2**] for more details. □

Problem 6.4. *Find a topological proof of Thurston's Tameness Theorem for the relevant case that $\pi_1(\hat{W}_i)$ is a free/surface group.*

Corollary 6.5. *Given the hypotheses of Theorem 6.1 and its Addendum, if D and W_i are as in the conclusion of Theorem 6.1 and if $\hat{W}_i$ denotes the $\pi_1(D)$ cover of W_i, then int$(\hat{W}_i)$ is topologically tame.*

Problem 6.6. *Does the boundary of $\hat{W}_i$ extend to a manifold compactification of int$(\hat{W}_i)$?*

Remark 6.7. This question is a special case of Simon's Missing Boundary Problem [**Si**]. Indeed, it is the essential unresolved one. By Tucker [**Tu2**], if X is a 3-manifold with boundary, such that $\text{int}(X)$ is topologically tame, then the union of the essential non simply connected components of ∂X extends to a manifold compactification of $\text{int}(X)$. Poenaru has observed that if $\mathcal{D}$ is the union of the disc components of ∂X, then $\mathcal{D}$ corresponds to a locally finite collection of embedded proper rays in $X \setminus \mathcal{D}$. I.e. by deleting an open neighborhood of these rays from $X \setminus \mathcal{D}$ we obtain a manifold homeomorphic to X. These rays are called the *wick* of $\mathcal{D}$. Thus $\mathcal{D}$ extends if and only if the components of the wick are individually unknotted and globally unlinked in $\text{int}(X)$.

Recall that $\mathcal{E}$ denotes the end of our hyperbolic homotopy handlebody N.

Here is the idea to find surfaces $\{T_i\}$ which satisfy the three conditions of the Tameness Criterion. First apply Theorem 6.1 and its Addendum to an exiting sequence of η-separated simple closed geodesics given by Remark 2.5. Let $\delta_1, \delta_2, \cdots$ denote the resulting subsequence, where δ_1 is allowed to have multiple components. Let Δ denote $\cup\delta_i$ and Δ_i denote $\cup_{k=1}^{i}\delta_k$. The Theorem and Addendum also produce the open $\mathcal{W} \subset N$ exhausted by $W_1 \subset W_2 \subset \cdots$ with $D \subset W_1$ a core for $\mathcal{W}$.

Note that $d = \text{rank}(\pi_1(D)) \leq \text{rank}(\pi_1(N))$, since D is a core of $\mathcal{W}$ which H_1 and π_1-injects in N. (This crucial fact is also true in the general free/surface group case.) Let $\hat{W}_i$ denote the $\pi_1(D)$ cover of W_i and let $\hat{D}_i$ denote the canonical lift of D to $\hat{W}_i$. If $j \leq i$, then $\delta_j \subset W_i$ can be homotoped into D via a homotopy supported in W_i. Therefore, δ_j has the canonical lift $\hat{\delta}_j \subset \hat{W}_i$. Let $\hat{\Delta}_i$ denote $\hat{\delta}_1 \cup \cdots \cup \hat{\delta}_i$. By Corollary 6.5, $\text{int}(\hat{W}_i)$ has a manifold compactification, which is necessarily a compact handlebody of genus d since its fundamental group is free of rank d. Let P_i denote the manifold boundary of $\text{int}(\hat{W}_i)$ pushed *slightly* into $\text{int}(\hat{W}_i)$. Slightly means that $\hat{\Delta}_i$ remains to the inside. The surface P_i bounds a genus-d handlebody. As in the proof of Canary's theorem, maximally compress P_i along a sequence of finitely many discs which are either disjoint from $\hat{\Delta}_i$ or intersect $\hat{\Delta}_i$ (or more precisely an appropriate subset, as detailed in the proof of 4.3) in a single point to obtain the surface Q_i. Again as in that proof, there exists $p > 0$ and $i_1, i_2, \cdots$ such that for each j, there exists a component R_{i_j} of Q_{i_j} which bounds the region $\text{Bag}_{i_j} \subset \hat{W}_{i_j}$ such that $\hat{\delta}_p \subset \text{int}(\text{Bag}_{i_j})$ and $\lim_{j\to\infty} p(i_j) \to \infty$. Here $p(i_j)$ is the largest subscript of a $\hat{\delta}_k$ that lies interior to Bag_{i_j}. By construction R_{i_j} is 2-incompressible with respect to those $\hat{\delta}_k$'s which lie interior to Bag_{i_j}. See Figure 6.2.

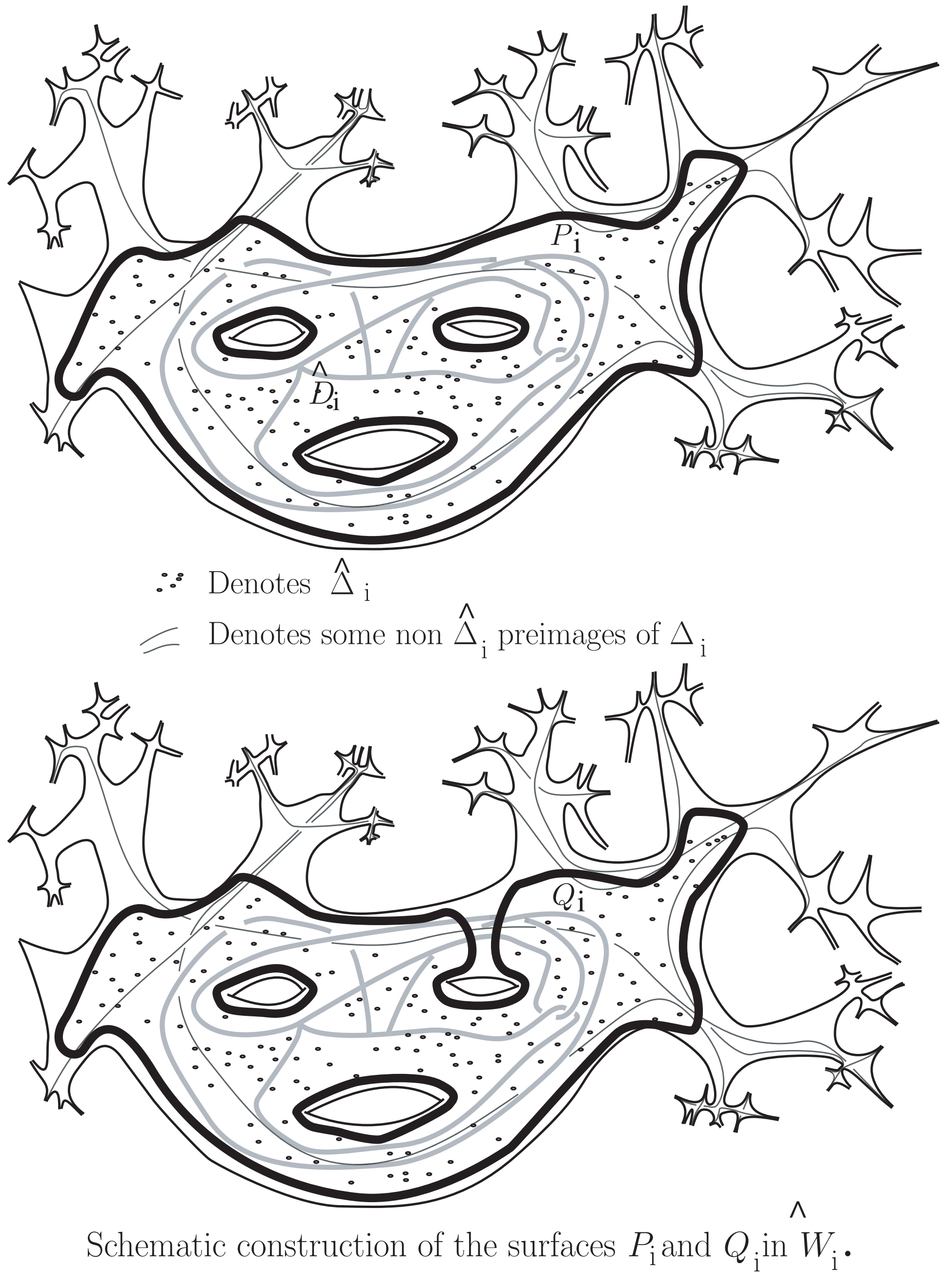

Figure 6.2

We will abuse notation by henceforth letting R_j, Bag_j, etc. denote R_{i_j}, Bag_{i_j} etc.

The heuristic idea for obtaining the simplicial hyperbolic surface $T_j \subset N$ is as follows. Construct a surface $S_j \subset \hat{W}_j$ by shrinkwrapping R_j with respect to the

$\hat{\delta}_i$'s which lie inside Bag_j, then project S_j into N via the covering projection to get T_j.

Remark 6.8. (Technical Problem) The shrinkwrapping of R_j occurs in $\hat{W}_j$ which is an incomplete hyperbolic manifold. The incompleteness occurs along $\partial\hat{W}_j$. The surface might want to jump out of $\hat{W}_j$ in the act of shrinkwrapping.

Remark 6.9. (Original Solution [**CG**]) In the smooth category shrinkwrap $\partial W_j \subset N$ with respect to Δ_{j+1} to obtain the surface U_j. If $U_j \cap \Delta_{j+1} = \emptyset$, then U_j is embedded, smooth and has mean curvature zero. Also U_j is isotopic to ∂W_j via an isotopy disjoint from Δ_j, so we can abuse notation and view W_j as the submanifold bounded by U_j that contains Δ_j. Thus $\partial\hat{W}_j$ has mean curvature 0, so acts as a barrier for the smooth shrinkwrapping process. In this case, we really can do shrinkwrapping in $\hat{W}_j$. (Since there is a barrier to the outside, we can shrinkwrap R_j with respect to geodesics lying completely to the inside.)

If $U_j \cap \Delta_{j+1} \neq \emptyset$, then we do a limit argument. We deform the hyperbolic metric of N near Δ_{j+1} to obtain a metric g_t such that the boundary of a small regular neighborhood of Δ_{j+1} is a union of totally geodesic tori. Shrinkwrapping ∂W_j with this metric yields a surface U_j^t disjoint from Δ_{j+1}. As above, we shrinkwrap R_j in $\hat{W}_j$ with respect to the induced metric $\hat{g}_t$ to obtain S_j^t, then project S_j^t to N to obtain T_j^t. With an appropriate family of metrics g_t, $t \in [0,1)$, our desired T_j is a limit of a subsequence of these T_j^t's as $t \to 1$.

Remark 6.10. (Soma's Solution [**So**]) Let $p : \hat{W}_j \setminus p^{-1}(\Delta_j) \to W_j \setminus \Delta_j$ be the restriction of the covering projection $\hat{W}_j \to W_j$. Note that $W_j \setminus \Delta_j$ π_1-injects into $N \setminus \Delta_j$ since by Theorem 6.1, ∂W_j π_1-injects into $N \setminus \mathrm{int}(W_j)$. Let $Y_j \to N \setminus \Delta_j$ denote the cover corresponding to $p_*(\pi_1(\hat{W}_j \setminus p^{-1}(\Delta_j)))$. Let $\bar{Y}_j$ denote the metric completion of Y_j. The induced map $q : \bar{Y}_j \to N$ is a branched cover over Δ_j, $\hat{W}_j \setminus p^{-1}(\Delta_j)$ embeds in Y_j and $\hat{W}_j$ embeds in $\bar{Y}_j$. Let δ^*_{j+1} be a component of $q^{-1}(\delta_{j+1})$. Apply Theorem 3.4 as generalized in Remark 3.8 to shrinkwrap R_j with respect to $\delta^*_{j+1} \cup q^{-1}(\Delta^j) \setminus \hat{W}^i$ plus the components of $\hat{\Delta}_j$ lying inside of Bag_j, to obtain the simplicial hyperbolic surface $S_j \subset \bar{Y}_j$. Let T_j be the projection to N.

We will show that the T_j's arising from Soma's solution satisfy 1)-3) of the Tameness Criterion. By constructon T_j is a simplicial hyperbolic surface and $\mathrm{genus}(T_j) = \mathrm{genus}(S_j) = \mathrm{genus}(R_j) \leq \mathrm{genus}(Q_j) \leq \mathrm{genus}(P_j) = \mathrm{rank}(\pi_1(D)) \leq \mathrm{rank}(\pi_1(N)) = \mathrm{genus}(\partial C)$. Therefore, it suffices to show that the T_j's exit and the T_j's homologically separate. We argue, as in [**CG**], that the T_j's have these properties.

Proof that the T_j's exit. As in the proof of Canary's theorem, let $\{\alpha_k\}$ denote a locally finite collection of proper rays in N such that for each k, α_k starts at δ_k. If $k \leq j$, let $\hat{\alpha}_k^j$ denote the lift of α_k to $\bar{Y}_j$ starting at $\hat{\delta}_k$. Since $\hat{\delta}_{p(j)} \subset \mathrm{Bag}_j$, $\langle R_j, \hat{\alpha}^j_{p(j)} \rangle = 1$ which implies $S_j \cap \hat{\alpha}^i_{p(j)} \neq \emptyset$ which implies that $T_j \cap \alpha_{p(j)} \neq \emptyset$. Since $p(j) \to \infty$ as $j \to \infty$, the result follows from the Bounded Diameter Lemma 2.9.

Remark 6.11. Note how the Bounded Diameter Lemma miraculously forces the T_j's to exit N.

Since the S_i's that are not obviously approximated by embedded surfaces isotopic to R_i (which is the case for smooth shrinkwrapping) we need the following result.

Lemma 6.12. *(Bowditch* [**Bo**]*) For each $i \leq j$, $\hat{\delta}_i$ is the only closed curve preimage of δ_i in $\hat{W}_j$.*

Proof. Let $\delta \subset \hat{W}_j$ be a closed preimage of δ_i. We first find a mapped annulus $\hat{X} \subset \hat{W}_j$ connecting δ and $(\hat{\delta}_i)^n$ where n is the degree of the map from δ to δ_i.

Here is how to build $\hat{X}$. Since the inclusion of $\hat{D}$ to $\hat{W}_j$ is a homotopy equivalence, δ (resp. $(\hat{\delta}_i)^n$) can be homotoped into $\hat{D}$ via a homotopy supported in $\hat{W}_j$. This gives rise to a mapped annulus $\hat{A}$ (resp. $\hat{B}$) connecting δ (resp. $(\hat{\delta}_i)^n$) to a curve $\hat{\beta}$ (resp. $\hat{\gamma}$) in $\hat{D}$. Let A (resp. β, B, γ) be the projection of $\hat{A}$ (resp. $\hat{\beta}$, $\hat{B}$, $\hat{\gamma}$) into W_j. Concatenating B with A produces a mapped annulus $F \subset W_j \subset \mathcal{W}$ connecting β to γ. Since the inclusion $D \to \mathcal{W}$ is a homotopy equivalence, F can be homotoped rel ∂F into D to produce a mapped annulus G connecting β and γ. Thus concatenating A with G and B along β and γ we obtain a mapped annulus X which lifts to the desired annulus $\hat{X}$ connecting $(\delta_i)^n$ and δ.

After projecting this annulus to W_j and gluing the boundary components we obtain a mapped torus T in W_j such that a simple closed curve in T maps onto δ_i and represents the class $(\delta_i)^n$. Since W_j is atoroidal, this torus can be homotoped into δ_i via a homotopy supported in W_j. By lifting this homotopy to $\hat{W}_j$ we see that $\delta = \hat{\delta}_i$. □

Proof that T_j is homologically separating for j sufficiently large. Let C be a codimension-0 core of N, $Z = N \setminus \text{int}(C)$ and α any embedded proper ray in Z starting at ∂C, oriented away from C. By Exercise 5.2, $H_2(Z) = \mathbb{Z}$ and is generated by $[\partial C]$. Therefore, if T is a mapped closed oriented surface in Z, then $[T] = n[\partial C]$ where $n = \langle T, \alpha \rangle$. Let α be the concatenation of α_p and an arc $\beta \subset Z$ from ∂C to $\partial \alpha_p$. If j is sufficiently large, then $T_j \subset Z$ and $T_j \cap (\beta \cup \delta_p) = \emptyset$. Therefore, $[T_j] = n[\partial C]$ where $n = \langle T_j, \alpha_p \rangle$ and also $q^{-1}(\delta_p) \cap S_j = \emptyset$. Note that n is the sum of the local intersection numbers corresponding to $\{S_j \cap q^{-1}(\alpha_p)\}$.

If $\hat{\alpha}_p$ denotes the component of $q^{-1}(\alpha_p)$ with endpoint on $\hat{\delta}_p$, then $\langle \hat{\alpha}_p, S_j \rangle = 1$. Indeed, $\hat{\delta}_p \subset \text{Bag}_j$ implies that $\langle \hat{\alpha}_p, R_j \rangle = 1$ and R_j is homotopic to S_j via a homotopy disjoint from $\hat{\delta}_p$.

To complete the proof that $< T_j, \alpha_p >= 1$ it suffices to show that if γ is a component of $q^{-1}(\alpha_p) \setminus \hat{\alpha}_p$, then $< \gamma, S_j >= 0$. Recall that j is sufficiently large so that $T_j \cap (\delta_p \cup \beta) = \emptyset$. This implies that $S_j \cap q^{-1}(\delta_p) = \emptyset$. Since $R_j = \partial \,\text{Bag}_j$, $[S_j] = [R_j] = 0 \in H_2(\bar{Y}_j)$, so $S_j = \partial L_j$ for some 3-cycle L_j. Since the homotopy from R_j to S_j is disjoint from $q^{-1}(\Delta_j) \setminus \hat{W}_j$ we can assume that $L_j \cap (q^{-1}(\Delta_j) \setminus \hat{W}_j) = \emptyset$. Let δ be the component of $q^{-1}(\delta_p) \setminus \hat{\delta}_p$ that contains $\partial\gamma$. If $\delta \subset \bar{Y} \setminus \hat{W}_i$, then $\partial\gamma \cap L_j = \emptyset$ and so $\langle \gamma, S_j \rangle = 0$. If $\delta \subset \hat{W}_i$, then by Lemma 6.12, δ is a line. Therefore, γ is properly homotopic to γ_1, where $\partial\gamma_1 \cap L_j = \emptyset$, via a homotopy disjoint from S_j that slides $\partial\gamma_1$ along δ. It follows that $\langle \gamma, S_j \rangle = \langle \gamma_1, S_j \rangle = 0$. □

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