MOM TECHNOLOGY AND VOLUMES OF HYPERBOLIC
3-MANIFOLDS

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0. Introduction

This paper is the first in a series whose goal is to understand the structure of low-volume complete orientable hyperbolic 3-manifolds. Here we introduce Mom technology and enumerate the hyperbolic Mom-n manifolds for \( n \leq 4 \). Our long-term goal is to show that all low-volume closed and cusped hyperbolic 3-manifolds are obtained by filling a hyperbolic Mom-n manifold, \( n \leq 4 \) and to enumerate the low-volume manifolds obtained by filling such a Mom-n.

William Thurston has long promoted the idea that volume is a good measure of the complexity of a hyperbolic 3-manifold (see, for example, [Th] page 6.48). Among known low-volume manifolds, Jeff Weeks ([We]) and independently Sergei Matveev and Anatoly Fomenko ([MF]) have observed that there is a close connection between the volume of closed hyperbolic 3-manifolds and combinatorial complexity. One goal of this project is to explain this phenomenon, which is summarized by the following:

**Hyperbolic Complexity Conjecture 0.1.** (Thurston, Weeks, Matveev-Fomenko)
The complete low-volume hyperbolic 3-manifolds can be obtained by filling cusped hyperbolic 3-manifolds of small topological complexity.

**Remark 0.2.** Part of the challenge of this conjecture is to clarify the undefined adjectives *low* and *small*. In the late 1970’s, Troels Jorgensen proved that for any positive constant \( C \) there is a finite collection of cusped hyperbolic 3-manifolds from which all complete hyperbolic 3-manifolds of volume less than or equal to \( C \) can be obtained by Dehn filling. Our long-term goal stated above would constitute a concrete and satisfying realization of Jorgensen’s Theorem for “low” values of \( C \).

A special case of the Hyperbolic Complexity Conjecture is the long-standing

**Smallest Hyperbolic Manifold Conjecture 0.3.** The Weeks Manifold \( M_W \), obtained by \((5, 1), (5, 2)\) surgery on the two components of the Whitehead Link, is the unique oriented hyperbolic 3-manifold of minimum volume.

Note that the volume of \( M_W \) is 0.942...

All manifolds in this paper will be orientable and all hyperbolic structures are complete. We call a compact manifold *hyperbolic* if its interior supports a complete hyperbolic structure of finite volume.

**Definition 0.4.** A *Mom-n structure* \((M, T, \Delta)\) consists of a compact 3-manifold \( M \) whose boundary is a union of tori, a preferred boundary component \( T \), and a handle decomposition \( \Delta \) of the following type. Starting from \( T \times I \), \( n \) 1-handles

\[\text{...}\]
and \( n \) 2-handles are attached to \( T \times 1 \) such that each 2-handle goes over exactly three 1-handles, counted with multiplicity. Furthermore, each 1-handle encounters at least two 2-handles, counted with multiplicity. We say that \( M \) is a Mom-\( n \) if it possesses a Mom-\( n \) structure \((M, T, \Delta)\).

**Remarks 0.5.** On a Mom-\( n \), the handle decomposition \( \Delta \) deformation retracts to an almost simple 2-complex which has \( 2n \) true vertices, in the sense of Matveev [Mv2]. Therefore Mom-\( n \) manifolds are a subset of those with Matveev complexity at most \( 2n \).

Here is the fundamental idea at the foundation of our project. Given a complete finite-volume hyperbolic 3-manifold \( N \), start with either a slightly shrunken maximal horotorus neighborhood \( V \) of a cusp or slightly shrunken maximal tube \( V \) about a geodesic. After expanding \( V \) in the normal direction, it eventually encounters itself, thereby creating a 1-handle. Subsequent expansions give rise to the creation of 1, 2, and 3-handles. In the presence of low volume we expect that \( V \) will rapidly encounter 1 and 2-handles and \( \partial V \) together with a subset of these handles (perhaps somewhat perturbed to allow for the “valence-3 2-handle condition”) will create a Mom-\( n \) manifold \( M \), for some \( n \leq 4 \). Furthermore, the complement of \( M \) will consist of cusp neighborhoods and tubular neighborhoods of geodesics. In practice, the handle structure may arise in a somewhat different manner; e.g., as a sub-complex of the dual triangulation of the Ford Domain (see [GMM3]).

The papers [GM] and [GMM] can be viewed as steps in this direction when \( V \) is a tubular neighborhood about a geodesic \( \gamma \). Indeed, [GM] gives a lower bound on \( \text{Vol}(N) \) in terms of the tube radius of \( \gamma \) and [GMM] gives a lower bound in terms of the first two ortholengths, or equivalently the radii of the expanding \( V \) as it encounters its first and second 1-handles.

**Definition 0.6.** If \( i : M \to N \) is an embedding, then we say that the embedding is elementary if \( i_*\pi_1(M) \) is abelian, and non-elementary otherwise.

In §1 we give the basic definitions regarding Mom-\( n \) manifolds embedded in hyperbolic 3-manifolds and state for later use some standard results about hyperbolic 3-manifolds and embedded tori in such. The end result of §2 - §4 is that if \( n \leq 4 \), then given a non-elementary Mom-\( n \) in a hyperbolic manifold one can find a non-elementary hyperbolic Mom-\( k \), where \( k \leq n \). Weaker results are given for general values of \( n \). In §5 we enumerate the hyperbolic Mom-\( n \) manifolds for \( n \leq 4 \); Theorem 5.1 and Conjecture 5.2 together imply the following:

**Theorem 0.7.** There are 3 hyperbolic Mom-2 manifolds, 21 hyperbolic Mom-3 manifolds (including the 3 hyperbolic Mom-2’s, which are also Mom-3’s).

**Conjecture 0.8.** There are 138 hyperbolic Mom-4 manifolds (including the hyperbolic Mom-2’s and Mom-3’s, which are also Mom-4’s).

In §6 we show that any non-elementary embedding of a hyperbolic Mom-\( n \) manifold \( M \), \( n \leq 4 \), into a compact hyperbolic manifold \( N \) gives rise to an internal Mom-\( n \) structure on \( N \), i.e. every component of \( \partial M \) either splits off a cusp of \( N \) or bounds a solid torus in \( N \).

In §7 we give examples of internal Mom-2 structures on cusped hyperbolic 3-manifolds, including in particular a detailed exposition of one of our key motivating examples.
In future papers we will use the Mom Technology to directly address the Hyperbolic Complexity Conjecture, and the Smallest Hyperbolic Manifold Conjecture. Indeed, in [GMM3] we will identify all 1-cusped hyperbolic 3-manifolds with volume less than 2.7 by showing that all such manifolds possess internal Mom-n structures with \( n \leq 3 \). This result, in combination with work of Agol-Dunfield (see [AST]), gives a lower bound of 0.86 for the volume of an orientable hyperbolic 3-manifold (but see the note below). The Agol-Dunfield result is an improvement of an earlier result of Agol which provides a tool for controlling the volume of a hyperbolic 3-manifold in terms of the volume of an appropriate cusped hyperbolic 3-manifold from which it is obtained via Dehn filling; the improved version utilizes recent work of Perelman.

This leads to three very promising directions towards the Smallest Hyperbolic Manifold Conjecture. Either, improve the technology of [GMM3] to identify the 1-cusped manifolds of volume less than 2.852, and then apply Agol-Dunfield. Or, extend the tube radius results of [GMI] from \( \log(3)/2 \) to 0.566 (the Agol-Dunfield volume bound involves the radius of a solid tube around a short geodesic in the original closed hyperbolic 3-manifold). Or, extend the method of [GMM3] to the closed case, thereby providing an essentially self-contained proof of the Smallest Hyperbolic Manifold Conjecture.

Note that all three approaches require an analysis of volumes of hyperbolic 3-manifolds obtained by Dehn filling a Mom-2 or Mom-3 manifold. These Dehn filling spaces have been extensively studied by J. Weeks and others, and it is highly likely that all low-volume manifolds in these Dehn filling spaces have been identified. However, some work will need to be done to bring these studies up to a suitable level of rigor.

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1. Basic Definitions and Lemmas

**Definition 1.1.** Let \( M \) be a compact connected 3-manifold with \( B \subset \partial M \) a compact surface which may be either disconnected or empty. A handle structure \( \Delta \) on \((M, B)\) is the structure obtained by starting with \( B \times I \) adding a finite union of 0-handles, then attaching finitely many 1 and 2-handles to \( B \times 1 \) and the 0-handles. We call \( B \times I \) (resp. \( B \times I \cup 0\)-handles) the base (resp. extended base) and say that the handle structure is based on \( B \). The valence of a 1-handle is the number of times, counted with multiplicity, the various 2-handles run over it and the valence of a 2-handle is the number of 1-handles, counted with multiplicity, it runs over.

Following the terminology of Schnbert [Sch] and Matveev [Mv1] we call the 0-handles, 1-handles and 2-handles balls, beams and plates respectively. We call islands (resp. bridges) the intersection of the extended base with beams (resp. plates) and the closed complement of the islands and bridges in \( B \times 1 \cup \partial(0\text{-handles}) \) are the lakes. We say that \( \Delta \) is full if each lake is a disc. If \( B = \emptyset \), then we say that \( \Delta \) is a classical handle structure.

Let \( M \) be a compact 3-manifold with \( \partial M \) a union of tori and let \( T \) be a component of \( \partial M \). We say that \((M, T, \Delta)\) is a weak Mom-n if \( \Delta \) is a handle structure based on \( T \) without 0-handles or 3-handles and has an equal number of 1 and 2-handles, such that each 1-handle is of valence \( \geq 2 \) and each 2-handle is of valence...
2 or 3. Furthermore, there are exactly $n$ 2-handles of valence 3. A weak $\text{Mom-n}$ with no valence-2 2-handles is a $\text{Mom-n}$. A weak $\text{Mom-n}$ is strictly weak if there exists a valence-2 2-handle.

The following is a well-known existence result stated in our language.

**Proposition 1.2.** A compact 3-manifold $M$ has a weak $\text{Mom}$-structure if and only if $\partial M$ is a union of at least two tori.

**Proof.** If $M$ has a weak $\text{Mom-n}$ structure, then by definition all of its boundary components are tori and there is at least one such boundary component. Further, because there are no 3-handles in $\Delta$, there must be another (torus) boundary component.

The converse is not much more difficult. In fact, if $M$ has at least two boundary components, then it is standard to create it by starting with a thickened boundary component and adding 1 and 2-handles where the 1 and 2-handles are of valence $\geq 2$. By subdividing the 2-handles with 1-handles we satisfy the condition that the 2-handles have valence $\leq 3$. Since $\chi(M) = 0$, there are an equal number of 1 and 2-handles. □

**Definition 1.3.** Call a torus that bounds a solid torus a tube and call a torus bounding a tube with knotted hole a convolutube. Recall that a tube with knotted hole is a $B^3 - \overline{N}(\gamma)$, where $\gamma$ is a knotted proper arc.

The following standard result follows from the loop theorem (see for example [Jac]).

**Lemma 1.4.** If $S$ is a torus in an irreducible 3-manifold $N$, then either $S$ is incompressible or $S$ is a tube or a convolutube. If $S \subset \hat{N}$, $\partial N$ is incompressible and $S$ is compressible, and there exists an embedded essential annulus connecting $S$ to a component of $\partial N$, then $S$ is a tube. □

**Proposition 1.5.** If $M$ is a non-elementary compact, connected 3-manifold embedded in the compact hyperbolic 3-manifold $N$ and $\partial M$ is a union of tori, then, up to isotopy, $N$ is obtained from $M$ by first filling a subset of the components of $\partial M$ by solid tori to obtain the manifold $M_1$, then compressing a subset of the components of $\partial M_1$ to obtain the manifold $M_2$, then attaching 3-balls to the 2-sphere components of $\partial M_2$ to obtain $M_3$. Furthermore all of these operations can be performed within $N$.

**Proof.** The components of $\partial M$ that bound solid tori in $N$ are exactly those boundary components which compress to the non-M side. Fill in all such tori to obtain the manifold $M_1$. If $P$ is a component of $\partial M_1$ which is not boundary parallel in $N$, then $P$ is compressible in $N$ and hence is a convolutube. These convolutubes can be isotoped to lie in pairwise disjoint 3-balls in $N$. Therefore we can compress all the compressible components of $\partial M_1$ (to obtain $M_2$) and cap the resulting 2-spheres with 3-cells to obtain $M_3$ which is isotopic to $N$.

Since $M_3$ must have all boundary components boundary parallel in $N$ and $M_3$ is non-elementary, the result follows. □

**Corollary 1.6.** Let $M \subset \hat{N}$ be a connected compact non-elementary submanifold in the compact hyperbolic 3-manifold $N$. If $\partial M$ is a union of tori, then each component of $\partial N$ is parallel to a component of $\partial M$ via a parallelism disjoint from $M$. □
The following result is due to Kerckhoff (see [Koj]).

Lemma 1.7. If $\gamma$ is a simple closed geodesic in the complete, finite-volume hyperbolic 3-manifold $N$, then $N - \gamma$ has a complete finite-volume hyperbolic structure. □

Lemma 1.8. Let $M$ be a compact submanifold of the compact hyperbolic 3-manifold $N$.

i) If $M_1 = M - \tilde{X}$, where $X \subset N$ is either a solid torus or 3-ball with $\partial X \subset M$, or $M_1$ is obtained by deleting a 2-handle or more generally deleting an open regular neighborhood of a properly embedded arc from $M$, then the inclusion $M \rightarrow N$ is a non-elementary embedding if and only if the inclusion $M_1 \rightarrow N$ is a non-elementary embedding.

ii) Suppose $M$ is non-elementary, $\partial M$ is a union of tori and $A$ is an essential annulus in $M$. Split $M$ along $A$; then some component $M_1$ of the resulting manifold is non-elementary.

iii) Suppose $F \subset M$ is an embedded torus essential in $M$, and $M \subset N$ is non-elementary. Split $M$ along $F$; then exactly one component of the resulting manifold is non-elementary.

Proof. The conclusion is immediate in case (i) because both $M_1$ and $M$ have the same $\pi_1$-image.

Under the hypotheses of (ii) the boundary of each component of $M$ split along $A$ is also a union of tori. We consider the case where the split manifold connected, for the general case is similar. Since all tori in $N$ separate, $M$ is obtained from $M_1$ by attaching a thickened annulus $A$ to a boundary parallel torus, a tube or a convolutube. In the first case $M$ and $M_1$ have the same $\pi_1$-image since $N$ is an annular. If $A$ is attached to the outside of a convolutube, then $\pi_1(M) \subset \mathbb{Z}$. If $A$ is attached to the outside of a tube, then $M$ can be enlarged to a Seifert fibered space in $N$ and hence is elementary.

To prove (iii) note that $F$ is either boundary parallel or is a tube or convolutube. In each case the result follows immediately. □

Definition 1.9. Let $N$ be a compact hyperbolic 3-manifold. An internal Mom-$n$ structure on $N$ consists of a non-elementary embedding $f : M \rightarrow N$ where $(M,T,\Delta)$ is a Mom-$n$ and each component of $\partial M$ is either boundary parallel in $N$ or bounds a solid torus in $N$. We will sometimes suppress mention of the embedding and simply say that $(M,T,\Delta)$ is an internal Mom-$n$ structure on $N$. In the natural way we define the notion of weak internal Mom-$n$ structure on $N$.

Lemma 1.10. A non-elementary embedding of the Mom-$n$ manifold $M$ into the compact hyperbolic 3-manifold $N$ will fail to give an internal Mom-$n$ structure on $N$ if and only if some component of $\partial M$ maps to a convolutube. In that case, a reimbedding of $M$, supported in a neighborhood of the convolutubes gives rise to an internal Mom-$n$ structure on $N$. □

Definition 1.11. A general based Mom-$n$ $(M,B,\Delta)$ consists of a compact manifold $M$ with $\partial M$ a union of tori, $B \subset \partial M$ a compact codimension-0 submanifold of $\partial M$ that is $\pi_1$-injective in $\partial M$, and $\Delta$ a handle structure based on $B$ without 0-handles such that every 1-handle is of valence $\geq 2$, every 2-handle is of valence $\geq 3$ and there are exactly $n$ of each of them. A weak general based Mom-$n$ is as above with $\Delta$
perhaps having $k \geq 0$ extra valence-2 2-handles in addition to the $n$ valence-3 2-handles and hence has $k + n$ 1-handles.

A general based internal Mom-n structure on $N$ consists of a non-elementary embedding $f : M \to N$, where $N$ is a compact hyperbolic 3-manifold and $(M, B, \Delta)$ is a general based Mom-n structure. Along similar lines we have the notion of weak general based internal Mom-n structure on $N$.

**Notation 1.12.** If $X$ is a space, then let $|X|$ denote the number of components of $X$, $\hat{X}$ denote the interior of $X$ and $N(X)$ denote a regular neighborhood of $X$. If $\sigma$ is a 2-handle, then let $\delta \sigma$ denote the lateral boundary, i.e. the closure of that part of $\partial \sigma$ which does not lie in lower index handles. If $b$ is a bridge which lies in the 2-handle $\sigma$, then define $\delta b = b \cap \delta \sigma$.

2. Handle structures and Normal surfaces

We slightly modify Haken’s [Ha] theory of surfaces in handlebodies to our setting. We closely parallel the excellent exposition given by Matveev in [Mv1].

**Definition 2.1.** Let $\Delta$ be a handle structure on $M$ based on $B \subset \partial M$. A compact surface $F \subset M$ is called normal if

1. $F$ intersects each plate $D^2 \times I$ in parallel copies of the form $D^2 \times \text{pt}$.
2. Each component of the intersection of $F$ with a beam $D^2 \times I$ is of the form $\alpha \times I$, where $\alpha$ is a proper arc whose endpoints are disjoint from $\delta$ (bridges). Furthermore, each component of $D^2 \times 0 - \alpha$ intersects $\delta$ (bridges) in at least two points.
3. Each component $U$ of $F \cap B \times [0, 1] \cup 0$-handles is $\pi_1$-injective in $B \times [0, 1] \cup 0$-handles. If $U \cap B \times 0 \neq \emptyset$, then $U$ is a product disc or annulus, i.e. The inclusion $(U, U \cap (B \times 0), U \cap (B \times I), U \cap (\partial B \times I)) \to (B \times I, B \times 1, \partial B \times I)$, can be relatively isotoped to a vertical embedding.
4. If $U$ is a component of $F \cap B \times I$ with $F \cap B \times 0 \neq \emptyset$, then $U$ is an essential vertical disc or annulus, i.e. $U = \alpha \times I$ where $\alpha \times 0$ is either an essential simple closed curve or essential proper arc in $B \times 0$.

**Remark 2.2.** i) For $F$ closed, the second condition can be restated by requiring that $\alpha$ intersect distinct components of $D^2 \times 0 \cap (\text{bridges})$. When $\partial F \neq \emptyset$, the second condition implies that $F$ is locally efficient in that it neither can be locally boundary compressed nor can its weight be reduced via an isotopy supported in the union of a 2-handle and its neighboring 1-handles.

ii) Note that $\partial F$ lies in the union of the beams, lakes and $B \times 0$.

**Lemma 2.3.** (Haken) If $F$ is a compact, incompressible, boundary-incompressible surface in a compact irreducible 3-manifold, then $F$ is isotopic to a normal surface.

**Definition 2.4.** Let $\Delta$ be a handle structure on $M$ based on $B \subset \partial M$. The valence $v(b)$ of a beam (resp. plate) is the number of plates (resp. beams) that attach to it, counted with multiplicity. Define the complexity $C(\Delta)$ to be $(\rho_1(\Delta), |\Delta|)$ lexicographically ordered, where $\rho_1(\Delta) = \Sigma_{\text{beams}} \max(v(b) - 2, 0)$ and $|\Delta|$ is the number of 1-handles. In particular we have

$\rho_1$-formula: $\rho_1(\Delta) = \sum_{\text{2-handles } \sigma} \text{valence}(\sigma) - 2|\Delta| + |\text{valence-1 1-handles}| + 2|\text{valence-0 1-handles}|$
Lemma 2.5. (Matveev) Let $\Delta$ be a handle structure on $M$ based on $B$, $F \subset M$ a closed normal surface and let $M'$ be $M$ split along $F$. If each component of $M' \cap B \times [0,1]$ disjoint from $B \times 0$ is a 3-ball, then $M'$ has a handle structure $\Delta'$ based on $B$ with $\rho_1(\Delta') = \rho_1(\Delta)$.

Proof. This follows almost exactly as in §3 and §4 of [Mv1]: $M'$ naturally inherits a handle structure $\Delta_1$ from $\Delta$ as follows. The surface $F$ splits $B \times I$ into various submanifolds one of which is homeomorphic to $B \times [0,1]$ with $B \times 0 = B$. All of the other submanifolds which lie in $M'$ are 3-balls. This new $B \times [0,1]$ becomes the base and the 3-balls become 0-handles. The various 1 and 2-handles are split by $F$ into 1 and 2-handles and as in [Mv1], $\rho_1(\Delta_1) = \rho_1(\Delta)$. □

Lemma 2.6. Given the handle structure $\Delta$ on $(M,B)$, if some 1-handle is valence-1, then there exists another structure $\Delta_1$ on $(M,B)$ with $C(\Delta_1) < C(\Delta)$. □

Lemma 2.7. If $(M,T,\Delta)$ is a Mom-$n$, then $C(\Delta) = (n,n)$. □

Lemma 2.8. Let $\Delta$ be a handle structure on $(M,B)$, $F \subset M$ a connected separating normal surface and $M_1$ be the component of $M - \overline{N(F)}$ which does not contain $B$. If each component of $F \cap B \times [0,1]$ is a disc, then $M_1$ has a classical handle structure $\Delta_1$ with $\rho_1(\Delta_1) \leq \rho_1(\Delta)$.

Proof. This follows as in the proof of Lemma 2.5 after noting that each component of $M_1 \cap B \times [0,1]$ is a 3-ball and these 3-balls correspond to the 0-handles of the induced handle structure on $M_1$. □

Lemma 2.9. If $\partial M$ is a union of tori, and $\Delta$ is a handle structure on $(M,T)$ with $T$ a component of $\partial M$, then there exists a weak Mom-$n$ $(M,T,\Delta_1)$ with $n \leq \rho_1(\Delta_1)$.

Proof. First apply Lemma 2.6 then add 1-handles to subdivide the valence-$k$, $k \geq 4$, 2-handles into valence-3 2-handles. □

Definition 2.10. If $B \neq \emptyset$ is a compact submanifold of $\partial M$, then define $\text{rank}_{\rho_1}(M,B)$ to be the least $n$ such that there exists a handle decomposition $\Delta$ on $(M,B)$ with $\rho_1(\Delta) = n$.

Problem 2.11. Is there an example of a compact hyperbolic 3-manifold $N$ with $T$ a component of $\partial N$ and $A$ an essential annulus in $T$ such that $\text{rank}_{\rho_1}(N,A) < \text{rank}_{\rho_1}(N,T)$?

3. Estimates for the reduction of $\rho_1$ under splitting

The main result of these next two sections is Theorem 4.1 which shows that if a hyperbolic 3-manifold $N$ has an internal Mom-$n$ structure $(M,T,\Delta)$ with $\Delta$ full and $n \leq 4$, then it has an internal Mom-$k$ structure $(M_1,T_1,\Delta_1)$ where $k \leq n$, $\Delta_1$ is full, and $M_1$ is hyperbolic. If $n > 4$, we obtain the similar conclusion except that “full” is replaced by “general based” and hence $T_1$ can be a union of tori and annuli.

As far as we know, transforming a structure based on an annulus lying in $T_1$ to one based on the whole torus $T_1$ may require an increase in $\rho_1$. This issue is responsible for many of the technicalities of this section and the next. See Problem 2.11.
Lemma 3.1. Let $f : M \rightarrow N$ be a non-elementary embedding of a compact connected manifold into a compact irreducible 3-manifold. Suppose $\partial M$ is a union of tori, $T$ is a component of $\partial M$ and that $\Delta$ is a handlebody structure on $(M,T)$ without 0-handles such that each 2-handle is of valence $\geq 3$. Then there exists a non-elementary embedding $g : M' \rightarrow N$ with $\partial M'$ a union of tori and a handle structure $\Delta'$ on $(M',T')$ with $T'=T$ such that

$$\rho_1(\Delta') + 2|\text{valence-1 1-handles of } \Delta| + 3|\text{valence-0 1-handles of } \Delta| \leq \rho_1(\Delta).$$

If instead each 2-handle of $\Delta$ is of valence $\geq 2$ then we have

$$\rho_1(\Delta') + |\text{valence-1 1-handles of } \Delta| + 2|\text{valence-0 1-handles of } \Delta| \leq \rho_1(\Delta).$$

Proof. Both assertions follow similarly by induction on the number of 1-handles of $\Delta$. If $\eta$ is a valence-1 1-handle, then cancelling $\eta$ with its corresponding 2-handle creates a handle structure $\Delta_1$. The Lemma follows by applying the $\rho_1$-formula to $\Delta_1$ and induction. If $\eta$ is a valence-0 1-handle, then the manifold $M_1$ obtained by deleting $\eta$ is connected and $\chi(M_1) = 1$, hence has a 2-sphere boundary component $S$ which bounds a 3-ball disjoint from $M_1$. Let $(M_2,T,\Delta_2)$ be obtained from $(M,T,\Delta)$ by deleting $\eta$ as well as a 2-handle which faces $S$. $M_2$ is a non-elementary embedding by Lemma 3.2. Now apply the $\rho_1$-formula and induction. \qed

Lemma 3.2. Let $f : M \rightarrow N$ be a non-elementary embedding of a manifold into a compact hyperbolic 3-manifold $N$, where $\partial M$ is a union of tori. Suppose that $M$ has a full handle structure $\Delta$ without 0-handles based on the component $T$ of $\partial M$ such that every 2-handle is of valence $\geq 3$. If either of the following are true then there exists a non-elementary embedding $f : M' \rightarrow N$ with handle structure $\Delta'$ on $(M',T')$ such that $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$, $T' = T$ and $\partial M'$ is a union of tori:

i) There exists a 2-handle $\sigma \subset N - \hat{M}$ that can be added to $\Delta$.

ii) There exists a disc $D \subset \partial M$ such that $\partial D$ is the union of two arcs $\alpha \cup \beta$, where $\beta$ lies in a lake and $\alpha$ lies in a 2-handle $\lambda$. Furthermore, within $\lambda \cap \partial M$, $\alpha$ separates components of $\lambda \cap (1-\lambda)$.

Proof. By Lemma 3.1 we can assume that every 1-handle of $\Delta$ is of valence $\geq 2$. To prove (i) let $\Delta_1$ be the handle structure on the manifold $M_1$ obtained by attaching $\sigma$ to $\Delta$ along $\partial M$. Let $\eta$ denote the 1-handle which $\sigma$ meets. Let $\Delta_2$ and $M_2$ be obtained by deleting a 2-handle $\lambda \neq \sigma$, which faces the resulting 2-sphere boundary component. Let $\Delta_3$ be obtained by cancelling $\sigma$ and $\eta$. Finally, in the usual way, reduce to a non-elementary $M_4$ with structure $\Delta_4$ on $(M_4,T)$ whose 1 and 2-handles are of valence $\geq 2$. Applying the $\rho_1$-formula shows that $\rho_1(\Delta_4) + 2 \leq \rho_1(\Delta)$ unless, measured in $\Delta$, $\text{valence}(\eta) = \text{valence}(\lambda) = 3$ and $\lambda$ attaches to $\eta$ at least twice. If $\lambda$ attaches to $\eta$ twice, then in the passage from $\Delta_3$ to $\Delta_2$ delete a 2-handle $\lambda_1 \neq \lambda$ which faces the 2-sphere. If $\lambda$ attaches to $\eta$ thrice, then either $\Delta$ is not full or $\eta$ is the unique 1-handle of $\Delta$. In the latter case $M_2 = T \times I$ which is elementary, a contradiction.

ii) Under these hypotheses we can attach a 2-handle $\sigma \subset N - \hat{M}$ to $\Delta$ such that either $\text{valence}(\sigma) = 1$ or $\text{valence}(\sigma) \leq \text{valence}(\lambda) + 2$ and $\lambda$ faces the resulting 2-sphere boundary component. If $\text{valence}(\sigma) = 1$, then apply (i). Otherwise let $\Delta_1$ be obtained by deleting $\lambda$ and apply the $\rho_1$-formula and if necessary Lemma 3.1. \qed
Lemma 3.3. Let $M$ be a non-elementary embedding of a compact 3-manifold into the compact hyperbolic 3-manifold $N$ with $\partial M$ a union of tori, and let $\Delta$ be a handle structure of $M$ based on $R \subset \partial M$. If there exists a valence $\geq 3$ 1-handle $\eta$ of $\Delta$ which attaches to a 0-handle $\zeta$, then there exists a non-elementary embedding $M' \to N$, and a handle structure $\Delta'$ based on $R' \subset \partial M'$ such that $\rho_1(\Delta') < \rho_1(\Delta)$.

Here either $(M', R') = (M, R)$ or $M' = M - \bar{\nu}$ and $R' = R \cup \partial V$ where $V$ is an embedded solid torus in $\bar{M}$.

Proof. If $\eta$ also attaches to either the base or a 0-handle distinct from $\zeta$, then cancelling $\eta$ with $\zeta$ gives rise to $\Delta'$ on $(M, R)$ with $\rho_1(\Delta') < \rho_1(\Delta)$. If $\eta$ attaches only to $\zeta$, then let $M'$ be obtained by deleting $\bar{\nu}$, the open solid torus gotten by hollowing out $\zeta$ and $\eta$. Let $R' = R \cup \partial V$ and let $\Delta'$ be the induced structure on $(M', R')$.

Lemma 3.4. Let $N$ be a compact hyperbolic 3-manifold. If $(M, T, \Delta)$ is a full internal weak Mom-n structure on $N$ and $M$ is reducible then there exists $(M_p, T_p, \Delta_p)$, a weak internal Mom-k structure on $N$ such that $k + 2 = \rho_1(\Delta_p) + 2 \leq \rho_1(\Delta) = n$. If $T$ does not lie in a 3-cell, then $T_p = T$.

Proof. We first consider the case that no reducing sphere in $M$ bounds a ball in $N$ containing $T$. Let $F$ be a least-weight normal reducing 2-sphere and note that $F \cap T \times I$ is a union of discs. Let $M_0$ and $M'_0$ be the components of $M$ split along $F$.

By Lemma 1.8 and the irreducibility of $N$ exactly one of $M_0$, $M'_0$ is non-elementary. We let $M_0$ (resp. $M'_0$) denote the non-elementary (resp. elementary) component with $\Delta_0$ (resp. $\Delta'_0$) its induced structure. By hypothesis $T \subset M_0$.

We show that $\rho_1(\Delta'_0) > 0$. Let $X$ denote the union of the islands and bridges of $\Delta$ and $Y' = X \cap M'_0$. If $\rho_1(\Delta'_0) = 0$, then each component $A$ of $Y'$ is an annulus. If some component $A$ of $Y'$ is disjoint from the lakes of $\Delta$, then $Y'$ would be disjoint from the lakes and so $F$ would 2-fold cover a projective plane, which contradicts the fact that $N$ is irreducible. Therefore each component of $Y'$ has one boundary component in a lake and one component in $\bar{X}$. Since $\Delta$ is full, this implies that $F$ is a boundary parallel 2-sphere and contradicts the fact that $\partial M$ is a union of tori.

If $Y = X \cap M_0$ then a similar argument shows that some component of $Y$ is not an annulus and furthermore some 2-handle $\sigma$ of $\Delta_0$ faces $F$ and attaches to a valence $\geq 3$ 1-handle. Delete $\sigma$ from $\Delta_0$ to obtain $(M_1, T, \Delta_1)$ with $\rho_1(\Delta_1) + 1 \leq \rho_1(\Delta_0)$. The standard simplifying moves as in Lemma 3.7 transform $(M_1, T, \Delta_1)$ to $(M_2, T, \Delta_2)$, a weak Mom-k with $k = \rho_1(\Delta_2) \leq \rho_1(\Delta_1)$. Since $\rho_1(\Delta_0) + \rho_1(\Delta'_0) = \rho_1(\Delta) = n$ we have $k + 2 = \rho_1(\Delta_p) + 2 \leq \rho_1(\Delta) = n$. If $M_2$ is reducible, then split along an essential least-weight sphere $F_2$ which is normal with respect to $\Delta_2$. Retain the component which contains $T$ and do the usual operations to obtain the weak Mom-k $(M_3, T, \Delta_3)$ with $k_3 \leq k$ and $M_3$ non-elementary. By Haken finiteness, this procedure terminates in a finite number of steps, completing the proof in this case.

We now consider the case that $T$ is compressible in $M$. Let $F$ be a least-weight compressing disc for $T$. Note that $F \cap T \times I$ consists of discs and a single annulus. If $M_1$ is $M$ split along $F$ with induced handle structure $\Delta_1$ and $A$ is $T$ split along $F \cap T$, then $\Delta_1$ is based on $A$. Since $M$ is obtained by attaching a 1-handle to a spherical component of $\partial M_1$, it follows that $M_1$ is non-elementarily embedded.
in $N$. Let $M_2$ be obtained by attaching 2-handles to $\partial M_1$ along the components of $\partial A$, then capping off the resulting 2-sphere which faces $A$ with a 3-cell. The resulting manifold $M_2$ has two 2-sphere boundary components and the induced handle structure $\Delta_2$ is classical. Since $\Delta$ is full some 2-handle of $\Delta_2$ facing a 2-sphere of $\partial M_2$ attaches to a 1-handle of valence $\geq 3$. Delete this 2-handle to obtain the non-elementary $M_3$ with handle structure $\Delta_3$ which satisfies $\rho_1(\Delta_3) + 1 \leq \rho_1(\Delta)$. Delete another 2-handle to create $M_4$ and $\Delta_4$ such that $\partial M_4$ is a union of tori. Now apply Lemma 3.4 to $\Delta_4$ to create $(M_5, T_5, \Delta_5)$ so that $\rho_1(\Delta_5) + 1 \leq \rho_1(\Delta_4)$. Finally cancel the extraneous 0-handles and valence-1 1-handles to create the desired weak internal Mom-k structure $(M_6, T_6, \Delta_6)$.

Let $F$ be a least-weight essential normal 2-sphere for $\Delta$. Since we can assume that $T$ is incompressible in $M$, $F \cap T \times I$ is a union of discs. If $T$ does not lie in the ball bounded by $F \subset N$, then proceed as in the first part of the proof. Otherwise let $M_0$ and $M_0'$ be the components of $M$ split along $F$, with $M_0'$ the component containing $T$. Since $\Delta$ is full and some component of $T \times 1 \cap M_0'$ is nonplanar, it follows that $\rho_1(\Delta_0') \geq 1$ and hence $\rho_1(\Delta_0) + 1 \leq \rho_1(\Delta)$. To complete the proof apply Lemma 3.4 to $\Delta_0$, delete a 2-handle to create a manifold with torus boundary components and cancel low valence handles as in the previous paragraph. □

**Remark 3.5.** If $(M, T, \Delta)$ is a weak internal Mom-n structure on the compact hyperbolic manifold $N$, and $\partial M$ is compressible in $M$, then $M$ is reducible. Therefore, if $M$ is irreducible and $\Delta$ has an annular lake $A$ that is homotopically inessential in $T \times 1$, then the core of $A$ bounds a disc in $T \times I$ which separates off a 3-cell in $M$. Absorbing this 3-cell into $T \times I$ simplifies $\Delta$ and transforms $A$ into a disc lake.

From now on we will assume that if a homotopically inessential lake appears it is immediately removed via the above operation.

**Definition 3.6.** If $\Delta$ is a handle structure on $M$, then the sheets of $\Delta$ are the connected components of the space $S$ which is the union of the 2-handles and the valence-2 1-handles. So sheets are naturally thickened surfaces which are attached to a 3-manifold along their thickened boundaries. The valence of a sheet is the number of times the boundary runs over 1-handles counted with multiplicity.

**Lemma 3.7.** Let $N$ be a compact hyperbolic 3-manifold and $f : M \to N$ a non-elementary embedding where $\partial M$ is a union of tori. Let $\Delta$ be a handle structure on $M$ with no 0-handles based on a component $T$ of $\partial M$ such that the valence of each 2-handle is at least 3 and $T$ does not lie in a 3-cell of $N$. If some sheet $S$ of $\Delta$ is not a thickened disc then there exists a non-elementary $M' \subset N$ with handle structure $\Delta'$ based on a component $T'$ of $\partial M'$ such that $\rho_1(\Delta') + 1 \leq \rho_1(\Delta)$. If equality must hold then $S$ is a thickened annulus or Mobius band and if $\Delta$ is full then $S$ is a thickened Mobius band and $\Delta'$ is full.

**Proof.** By Lemma 3.1 and the proof of Lemma 3.4 we can assume that $M$ is irreducible and each 1-handle of $\Delta$ has valence $\geq 2$. If $M_1$ denotes the manifold obtained by deleting the sheet $S$, then $\chi(M) = \chi(M_1) + \chi(S)$. Since $\chi(M) = 0$, if $\chi(S) < 0$, then $\partial M_1$ contains a 2-sphere. This implies that $M$ is reducible. Note that if $\chi(S) = 0$, then $M_1$ is non-elementary.

Now assume that $\chi(S) = 0$. In this case $S$ is either an annulus$\times I$ or a non-trivial $I$-bundle over a Mobius band. Since $\chi(S) = 0$, if $S$ contains more than one 2-handle, then valence$(S) \geq 2$. If valence$(S) > 1$, then $\rho_1(\Delta_1) + 2 \leq \rho_1(\Delta)$ holds
where $\Delta_1$ is the induced structure on $M_1$. If $S$ is a valence-1 annulus $\times I$, then $\Delta$ has an annular lake and $\rho_1(\Delta_1) + 1 = \rho_1(\Delta)$.

If $S$ is a thickened Mobius band of valence-1, then $\rho_1(\Delta_1) + 1 = \rho_1(\Delta)$. If $S$ attached to the component $R$ of $\partial M_1$ and $R$ had an annular lake, then $R$ must be a tube with a compressing disc $D$ whose boundary goes over a 1-handle $\eta$ of $\Delta_1$ exactly once. Let $M_2$ be obtained by attaching the 2-handle $\sigma$ with core $D$ to $M_1$ and $\Delta_2$ the induced handle structure. Proceed as in the proof of Lemma 3.3 to show that exists a non-elementary embedding $(M_3, T)$ with handle structure $\Delta_3$ with $\rho_1(\Delta_3) + 2 \leq \rho_1(\Delta_1) + 1 \leq \rho_1(\Delta)$. □

**Lemma 3.8.** Let $(M, T, \Delta)$ be an internal Mom-n structure on the compact hyperbolic manifold $N$. Assume that every sheet of $\Delta$ is a disc and $\Delta$ is full. If there exists an embedded annulus $A$ connecting the component $S$ of $\partial M - T$ to $T$, then there exists a non-elementary embedding of a manifold $M'$ into $N$ such that $\partial M'$ is a union of tori and $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$. Here $\Delta'$ is a handle structure on $M'$ based on $T'$. Either $T' = T$ or $T'$ is the union of an essential annulus in $T$ and possibly a component of $\partial M'$.

**Proof.** By Lemma 3.4 we can assume that $M$ is irreducible. Since all sheets are discs and letting $M_2 = M$, we obtain from $\Delta$ a full handle structure $\Delta_2$ with no 0-handles based on the component $T_2 = T$ of $\partial M_2$ where each 1 and 2-handle of $\Delta_2$ is of valence $\geq 3$. Finally $\rho_1(\Delta_2) \leq \rho_1(\Delta)$. We can assume that $A \subset M_2$ is a least-weight normal annulus connecting $T_2$ to the boundary component $S \neq T_2$. Since $A$ is least weight, $A \cap T_2 \times I$ is a disc and a single annulus.

Let $\partial A_0$ (resp. $\partial A_1$) denote $A \cap T_2$ (resp. $A \cap S$). Our $A$ has an induced handle structure $\Phi$ based on $\partial A_0$ as follows. The base consists of the annular component of $A \cap T_2 \times [0,1]$, the 0-handles consist of the disc components of $A \cap T_2 \times [0,1]$, the 1-handles (resp. 2-handles) consist of the intersections of $A$ with the 1-handles (resp. 2-handles).

Let $M_3$ denote the manifold obtained by splitting $M_2$ along $A$ and let $\Delta_3$ denote the induced handle structure. As in §2, the ball components of $T_2 \times I$ split along $A$ to become 0-handles of $\Delta_3$. The remaining component is $B' \times I$, where $B'$ is $T_2$ split along $\partial A_0$ and hence $\Delta_3$ is based on $B' \subset T_2$. By Lemma 1.8, $M_3$ is non-elementarily embedded in $N$.

If $\eta$ is a 1-handle of $\Delta_2$ and if $\{\eta_i\}$ denotes the 1-handles of $\Delta_3$ which descended from $\eta$, then $\sum_i (\text{valence}(\eta_i) - 2) \leq \text{valence}(\eta) - 2$ with equality if and only if $A_1$ does not run over $\eta$. In fact, counting with multiplicity, if $A_1$ runs over the 1-handles of $\Delta_2$ more than once then by operations as in Lemma 3.4 we can pass to $M_4$ and $\Delta_4$ with $\rho_1(\Delta_4) + 2 \leq \rho_1(\Delta)$. We will now assume that $A_1$ ran over a unique 1-handle $\eta$ and it did so with multiplicity one. This implies that $\rho_1(\Delta_3) + 1 = \rho_1(\Delta_2)$. Also $A_1$ is the union of two arcs $\alpha$ and $\beta$, where $\alpha$ lies in a 1-handle of $\Delta_2$ and $\beta$ lies in a lake. $\beta$ lies either in the base of $\Phi$ or in a 0-handle $v^* \Phi$.

Give $A$ a transverse orientation. Call a 0-handle $v$ of $\Phi$ plus (resp. minus) if the transverse orientation of the disc $v \subset T_2 \times I$ points away from (resp. towards) $T_2$. Each such disc $v$ separates off, in $T_2 \times I$, a 3-ball $v_B$. Let $v_D$ denote $v_B \cap T_2 \times 1$.

If $\beta$ lies in a 0-handle $v^* \Phi$, then Lemma 3.3 applies to $\Delta_4$ and the Lemma is proved. Indeed, since $A$ is least weight, the disc $v_D^\ast$ contains a bridge $b$ in its interior and this bridge is not parallel to $\beta$. In other words, there is no embedded disc in a lake whose boundary is a concatenation of four arcs $\beta_1, \beta_2, \beta_3$ and $\beta_4$ such that $\beta_1 = \beta, \beta_3$ lies in $\partial b$, and $\beta_2$ and $\beta_4$ are arcs lying in islands. Otherwise, since $A_1$
runs over a unique 1-handle, this implies that $\Delta_2$ had a valence-1 2-handle. If $v_B^i$ is disjoint from the other 0-handles of $\Phi$, then Lemma 3.3 applies to the 0-handle $v_B^i$ of $\Delta_3$. Otherwise, $v_B^i$ is split into balls by the various 0-handles of $\Phi$ and Lemma 3.3 applies to one of these balls.

From now on we assume that $\beta$ lies in the base of $\Phi$. Let $X \subset T_2 \times 1$ denote the union of the islands and bridges of $\Delta_2$. A similar but easier argument to the one given in the previous paragraph shows that for each 0-handle $v$ of $\Phi$ either $\partial v$ is boundary parallel in $X$ or Lemma 3.3 applies. This implies that if $v \neq w$ are 0-handles of $\Phi$, and $v_B \subset w_B$, then $\partial v$ and $\partial w$ are normally parallel in $X$. Furthermore, no 1-handle of $\Phi$ connects a plus 0-handle to a minus 0-handle of $\Phi$. Also, if $v^0$ and $v^1$ are two 0-handles of $\Phi$ that are connected by a 1-handle, then $v_B^0 \cap v_B^1 = \emptyset$. Finally, there do not exist 0-handles $w^0, w^1, \ldots, w^n$ of $\Phi$ such that for $i = 1, 2, \ldots, n - 1$, $w^i$ is connected to $w^{i+1}$ by a 1-handle and $w_B^n \subset w_B^0$.

It follows that there exists a disc $E \subset A$ whose boundary is the union of two arcs $\phi$ and $\psi$ where $\phi$ is a proper arc in a $T$-handle of $\Phi$ and $\psi$ lies in the $T \times 1 \cap (\text{base}(A))$. Furthermore $E \cap (0, 1)$-handles of $\Phi \neq \emptyset$ and is connected. By the previous paragraph if $v, w$ are 0-handles of $\Phi$ lying in $E$, then they are of the same parity and $v_B \cap w_B = \emptyset$.

Therefore, $E$ can be normally isotoped, with respect to $\Delta_2$ to a disc $G \subset \partial M$ such that $\partial G$ is a union of two arcs, one lying in a lake and the other in a 2-handle. Now apply (ii) of Lemma 3.2 to $\Delta_2$. 

Lemma 3.9. Let $(M, T, \Delta)$ be a full internal Mom-$n$ structure on the compact hyperbolic 3-manifold $N$ such that every sheet is a disc. Suppose that there exists an essential embedded annulus $A$ with $\partial A \cap T = \emptyset$. Then either there exists a full internal Mom-$k$ structure $(M', T', \Delta')$ on $N$ with $k < n$ or there exists a non-elementary embedding $M' \to N$ with handle structure $(M', T', \Delta')$ where $\partial M'$ is a union of tori, $T'$ is a component of $\partial M'$, and $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$.

Proof. Assume that $M$ is irreducible. By Lemma 3.3 we can assume that no essential annulus connects $T$ to a component of $\partial M - T$. Let $A$ be a least-weight essential normal annulus with $\partial A \cap T = \emptyset$. By the second sentence, $A \cap T \times I$ is a union of discs. Let $M_1$ be the non-elementary component of $M$ split along $A$ and let $(M_1, T_1, \Delta_1)$ be the induced handle structure. As in the previous proof, $\rho_1(\Delta_1) + 1 \leq \rho_1(\Delta)$. Indeed, since $\partial A$ traverses at least two 1-handles of $\Delta$, counted with multiplicity, the inequality will be strict unless $A$ normally double covers a Mobius band. If $M_1$ is disjoint from $T$, then the result follows from Lemma 3.3. Assume now that $T \subset \partial M_1$ and $\Delta_1$ is full.

Cancel the 0-handles with 1-handles to obtain $(M_2, T_2, \Delta_2)$. If $\Delta_2$ is not full, then some 1-handle of valence $\geq 3$ was cancelled and hence $\rho_1$ is reduced. Next cancel valence-1 2-handles to obtain $(M_3, T_3, \Delta_3)$. Again fullness is preserved or $\rho_1$ is reduced. If $\Delta_3$ has a valence-2 2-handle $\sigma$ connecting distinct 1-handles, then cancel one of these 1-handles with $\sigma$. If the 2-handle $\sigma$ attaches to the same 1-handle $\alpha$, then the core of $\sigma \cap \alpha$ can be viewed as an embedded annulus or Mobius band $C$ in $M_4$. Splitting $M_4$ along $C$ we obtain $(M_4, T_4, \Delta_4)$ where $T_4$ is the newly created boundary component of $M_4$ and $\Delta_4$ is the induced structure. Note that $\rho_1(\Delta_4) < \rho_1(\Delta_3)$ since $\Delta_3$ is full. After finitely many such operations the lemma is proved. 

Lemma 3.10. Let $(M, T, \Delta)$ be a full internal Mom-$n$ structure on the compact hyperbolic 3-manifold $N$ such that every sheet is a disc. Suppose that there exists an
essential embedded annulus $A$ with $\partial A \subset T$. Then either there exists a full internal Mom-k structure $(M', T', \Delta')$ on $N$ with $k < n$ or there exists a non-elementary embedding $M' \to N$ with handle structure $(M', T', \Delta')$ where $\partial M'$ is a union of tori, $T'$ is a component of $\partial M'$, and $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$.

Proof. If $M$ is reducible or there exists an essential annulus with some boundary component disjoint from $T$, then apply Lemma 3.4, Lemma 3.8, or Lemma 3.9. Let $(M, \partial M - T, \Sigma)$ be the dual handle structure. I.e. the 1-handles (resp. 2-handles) of $\Sigma$ are in 1-1 correspondence with the 2-handles (resp. 1-handles) of $\Delta$ and for each handle $\sigma$, the core of $\sigma$ is the cocore of the dual handle and vice versa. Note that $\rho_1(\Sigma) = \rho_1(\Delta)$ and each 1-handle of $\Sigma$ is of valence 3, but $\Sigma$ may have valence-2 2-handles.

Let $A$ be an essential annulus, least weight with respect to $\Sigma$. Since each essential annulus has its entire boundary in $T$, $A \cap (\partial M - T) \times I$ is a union of discs. Let $M_1$ be the non-elementary component of $M$ split along $A$ with $(M_1, R_1, \Sigma_1)$ the induced structure. As before $\rho_1(\Sigma_1) < \rho_1(\Sigma)$. Eliminate the 0-handles and low valence-1 and 2-handles of $\Sigma_1$ to obtain $(M_2, R_2, \Sigma_2)$ and note that either $\Sigma_2$ is full or $\rho_1(\Sigma_2) + 2 \leq \rho_1(\Sigma) = \rho_1(\Delta)$. If $\Sigma_2$ is full, then $(M_3, T_3, \Delta_3)$ the handle structure dual to $(M_2, R_2, \Sigma_2)$ is full. Let $(M_4, T_4, \Delta_4)$ be obtained by eliminating the valence-2 2-handles as in the previous proof. As before $\rho_1(\Delta_4) + 1 \leq \rho_1(\Delta)$ with equality holding only if $\Delta_4$ is full.

Lemma 3.11. Let $(M, T, \Delta)$ be a full internal Mom-n structure on the compact hyperbolic 3-manifold $N$. If $M$ is not hyperbolic, then either there exists a full internal Mom-k structure $(M', T', \Delta')$ on $N$ with $k < n$ and $M'$ hyperbolic or there exists a general based internal Mom-l, $l + 2 \leq n$, structure $(M', B', \Delta')$ on $N$.

Proof. By Lemma 3.4, 3.8, or 3.9 it suffices to consider the case that $M$ is irreducible and anannular. By Thurston [Th2], if $M$ is not hyperbolic, then it contains an essential torus. Let $F$ be a least-weight essential torus. Since there are no essential annuli, $F \cap T \times I$ is a union of discs.

Let $M_1$ denote the component of $M$ split along $F$ which contains $T$ and let $M'_1$ denote the other component. Let $\Delta_1$ and $\Delta'_1$ denote the induced handle structures. Note that $\Delta_1$ is based on $T$, $\Delta'_1$ is a classical structure and $\rho_1(\Delta_1) + \rho_1(\Delta'_1) = \rho_1(\Delta)$. By Lemma 1.2 one of $M_1$ or $M'_1$ is non-elementary. Let $X \subset T \times I$ be the union of the islands and bridges of $\Delta$, $Y = M'_1 \cap X$ and $Z = M_1 \cap X$.

If $M_1$ is elementary, $\rho_1(\Delta_1) \geq 1$ since $Z$ is nonplanar. In the usual way obtain a handle structure $\Sigma$ on $M'_1$ such that $\rho_1(\Sigma) \leq \rho_1(\Delta')$, $\Sigma$ has exactly one 0-handle and each 1-handle is of valence $\geq 2$. Since $M'_1$ is non-elementary, some 1-handle is of valence $\geq 3$. To complete the proof apply Lemma 3.3.

From now on assume that $M_1$ is non-elementary. We first show that $\rho_1(\Delta'_1) \geq 1$. If $\rho_1(\Delta'_1) = 0$, then $Y$ is a union of annuli, each component of which intersects $\partial X$ in $\leq 1$ circle, since $\Delta$ is full. If $Y \cap \partial X \neq \emptyset$, then each component must have this property and hence $F$ is boundary parallel in $M$. If $Y \cap \partial X = \emptyset$, then $M'_1$ is an I-bundle over a Klein bottle, which is impossible in an orientable hyperbolic 3-manifold.

In the usual way pass from $(M_1, T_1, \Delta_1)$ to a Mom structure $(M_2, T_2, \Delta_2)$ with $\rho_1(\Delta_2) \leq \rho_1(\Delta_1)$. Since $M$ is anannular, $\Delta_2$ has no annular lakes. If some component of $\partial M_2$ is a convolutube, then reimbed $M_2$ in $N$ to get a full internal Mom-k structure where $k < n$. \qed
4. From Mom-n to Hyperbolic Mom-n

The following is the main result of this section:

**Theorem 4.1.** If \((M,T,\Delta)\) is a full internal Mom-n structure on the compact hyperbolic 3-manifold \(N\) and \(n \leq 4\), then there exists a full internal Mom-k structure \((M',T',\Delta')\) on \(N\) where \(M'\) is hyperbolic and \(k \leq n\).

For general \(n\), either there exists an internal Mom-k structure \((M',T',\Delta')\) in \(N\) with \(k \leq n\) and \(M'\) hyperbolic or there exists a general based internal Mom-k structure \((M',B',\Delta')\) on \(N\) such that \(M'\) is hyperbolic and each component of \(B'\) is either a component of \(\partial M'\) or an annulus which is essential in a component of \(\partial M\). If \(M \neq M'\), then \(k + 2 \leq n\).

We first prove some preliminary lemmas about complexity (see Definition 2.4).

**Lemma 4.2.** (Clean-Up Lemma) Let \(M\) be a compact 3-manifold with \(\partial M\) a union of tori and \(N\) a compact hyperbolic 3-manifold. If \(M \to N\) is a non-elementary embedding and \(\Delta\) is a handle structure on \((M,B)\), then there exists \((M',B',\Delta')\) a weak general based internal Mom-n structure on \(N\) with \(C(\Delta') \leq C(\Delta)\).

If \(M\) is hyperbolic, then either \(\rho_1(\Delta') < \rho_1(\Delta)\) or \(M' = M\).

*Proof.* Proof by induction on \(C(\Delta)\). By the usual cancellation operations pass to a handle structure \(\Delta_2\) on \((M,B)\) without 0-handles or 1 and 2-handles of valence 1. If \(\Delta\) has a valence-0 1-handle, then delete it and proceed as in Lemma 3.1 to obtain \(\Delta_3\) on \((M_3,B_3)\) with \(M_3\) non-elementary and \(\partial M_3\) a union of tori. All of these operations reduce \(C(\Delta)\).

If \(\Delta_2\) has a valence-0 2-handle \(\sigma\), then compress \(M_3\) along a disc \(D\) which passes once through \(\sigma\) and intersects \(B \times I\) in an annulus. Let \(\Delta_4\) be the resulting handle structure on \((M_4,B_4)\) where \(M_4\) is the non-elementary component of \(M_3\) split along \(D\) and \(B_4\) is \(B_3 \cap M_4\). This splits \(\sigma\) into two 2-handle components, at least one of which lies in \(\Delta_4\). Absorb such 2-handles into \(B_4 \times I\). This creates \((M_5,B_5,\Delta_5)\) where \(M_5 = M_4\) and \(B_5\) is obtained by attaching one or two discs to \(\partial B_4\). If some component of \(B_5\) is a 2-sphere \(S\), then create \((M_6,B_6,\Delta_6)\) by filling in \(S\) with a 3-cell \(E\) and identifying \(S \times I \cup E\) as a 0-handle of \(\Delta_6\). Now cancel with a 1-handle to obtain \((M_7,B_7,\Delta_7)\) with \(C(\Delta_7) < C(\Delta_3)\). If one or two components of \(B_5\) are discs and there exists a non-simply connected component of \(B_5\), then the ball components of \(B_5 \times I\) are now re-identified as 0-handles and cancelled with 1-handles to obtain \((M_7,B_7,\Delta_7)\). If all the components of \(B_5\) are discs, then transforming the components of \(B_5 \times I\) to 0-handles produces a classical handle structure \(\Delta_{5,5}\) on \(M_5\). Now apply the proof of Lemma 2.8 to create \((M_6,B_6,\Delta_6)\) with \(C(\Delta_6)\) reduced. If some component of \(\partial M_6\) is a 2-sphere, then delete a 2-handle facing it to obtain \((M_7,B_7,\Delta_7)\).

By repeatedly applying the above operations we can assume that our \(\Delta_7\) has no 0-handles, and each 1-handle and 2-handle is of valence \(\geq 2\). Furthermore, no component of \(B_7\) is a disc. If some component \(G\) of \(B_7\) is not \(\pi_1\)-injective in \(\partial M_7\) then attach a 2-handle to a component of \(\partial G \times I\) whose restriction to \(\partial M_7\) bounds a disc, to create \((M_8,B_8,\Delta_8)\). Delete a 2-handle which faces the resulting 2-sphere boundary component and simplify as in the previous paragraphs to create \((M_9,B_9,\Delta_9)\) with \(C(\Delta_9) < C(\Delta_7)\). We can now assume that each component of \(B_9\) is essential in \(\partial M_9\), \(\Delta_9\) is 0-handle free and all 1 and 2-handles are valence \(\geq 2\).
If $\partial M_9$ contains convolutubes, then reimbed $M_9$ in $N$ to eliminate them. If $M$ is not hyperbolic, then the resulting $(M_{10}, B_{10}, \Delta_{10})$ satisfies the conclusions of our clean up lemma.

If $M$ is hyperbolic, then the proof also follows as above. However, note that $\partial M$ is incompressible so in the above process valence-0 1-handles or lakes compressible in $B \times I$ can be eliminated without changing $M$ and with reducing complexity. The topology of $M$ may change if we delete a 2-handle or apply Lemma 3.3; however, if such operations must be done, they can be done in such a manner that reduces $\rho_1$.

**Lemma 4.3.** If $(M, B, \Delta)$ is a strictly weak general based Mom-$n$ structure with $M$ hyperbolic, then there exists a general based Mom-$k$ structure $(M, B', \Delta')$ with $C(\Delta') < C(\Delta)$.

**Proof.** Proof by induction on $C(\Delta)$. If $\sigma$ is a valence-2 2-handle which goes over distinct 1-handles, then cancelling a 1-handle with $\sigma$ creates $(M, B, \Delta_1)$ with $C(\Delta_1) < C(\Delta)$.

We now assume that no valence-2 2-handle $\sigma$ goes over distinct 1-handles. Suppose that $\sigma$ goes over the same 1-handle $\alpha$ twice. Then $\sigma \cup \alpha$ can be viewed as an embedded annulus or Möbius band $A$ with boundary on $B$. Since $M$ is hyperbolic and orientable, $A$ is either a boundary parallel annulus or a compressible annulus.

If $A$ is boundary parallel, then it together with an annulus on $T$ cobound a solid torus $V \subset B$ such that $A$ wraps longitudinally around $V$ exactly once. Melting $V$ into $B \times [0, 1]$ eliminates both $\alpha$ and $\sigma$, together with all the 2-handles and 1-handles inside of $V$. The resulting manifold $(M, B_1, \Delta_1)$ is a weak provisional Mom-$k$ with $k \leq n$ whose induced handle structure $\Delta_1$ satisfies $C(\Delta_1) < C(\Delta)$.

If $A$ is not boundary parallel, then $M$ is reducible, a contradiction. □

**Lemma 4.4.** If $(M, B, \Delta)$ is a general based Mom-$n$ and $M$ is hyperbolic, then $n \geq 2$.

**Proof.** This follows by direct calculation. □

**Lemma 4.5.** If the compact hyperbolic 3-manifold $N$ has a general based internal Mom-2 structure $(M, B, \Delta)$, then it has a full internal Mom-2 structure.

**Proof.** By splitting along annuli in $B \times I$ we can assume that every non-peripheral lake of $B$ is a disc. If $B$ is the union of two tori $T_1$ and $T_2$, then each 1-handle must connect $T_1$ to $T_2$. This implies that each 2-handle is of even valence which is a contradiction. If $B$ consists of a single torus, then $(M, B, \Delta)$ is a full Mom-2 structure. We finally assume that $B$ contains an annulus. We only discuss the case that $B$ is connected; the case where $B$ is either the union of an annulus and torus or two annuli is similar and easier.

First suppose that $\Delta$ has a single sheet of valence four. Let $\eta$ denote the valence-4 1-handle of $\Delta$. There exists an essential compressing disc $D$ for $B \times I$ which cuts across the bridges in at most three components. View $N(D)$ as a 1-handle and $B \times I - \tilde{N}(D)$ as a 0-handle to obtain a classical handle structure with two 1-handles respectively of valence 4 and $\leq 3$. Now as in Lemma 3.3 hollow out the 0-handle and $\eta$ to get a non-elementary $M_1$ with handle structure $\Delta_1$ based on a torus with a single 1-handle of valence $\leq 3$. Therefore we obtain a Mom-1 structure on a hyperbolic 3-manifold, which is a contradiction.
Now consider the case that $\Delta$ has two valence-3 1-handles. If some essential disc $D$ in $B \times I$ cuts the bridges in $\leq 2$ components, then as above, we obtain a handle structure with one 0-handle and three 1-handles. Hollowing out a valence-3 1-handle and the 0-handle produces a handle structure $\Delta_1$ on a non-elementary manifold $M_1$ with $\rho_1(\Delta_1) = 1$, which is a contradiction. If no essential disc $D$ exists as above, then we can find one which cuts the bridges in exactly three components. One readily enumerates the possible handle structures that satisfy our assumptions of valence-3 1-handles and non-peripheral disc lakes. After applying the hollowing out procedure, a Mom-2 handle structure is created. Of the two choices of which 1-handle is to be hollowed out, one will produce a full Mom-2 structure.

\[ \blacklozenge \]

Remark 4.6. As an example of the type of non-full Mom-2 discussed in the previous lemma, note that the figure-8 knot complement contains a non-full Mom-2 structure with two 1-handles of valence 3, in addition to having a full Mom-2 structure.

Proof of Theorem 4.1 The proof is by induction on $\rho_1(\Delta)$. If $(M, T, \Delta)$ is not hyperbolic, then apply Lemma 3.11 to obtain $(M_1, T_1, \Delta_1)$ where $M_1 \to N$ is a non-elementary embedding, $\partial M_1$ is a union of tori and either $(M_1, T_1, \Delta_1)$ is a full internal Mom-$k$ structure, $k < n$, or $\rho_1(\Delta_1) + 2 \leq \rho_1(\Delta)$. In the former case the proof follows by induction. In the latter case as seen in the next paragraph we will produce a general based internal Mom-$k$ structure $(M_4, B_4, \Delta_4)$ with $k \leq n - 2$ and $M_4$ hyperbolic. Lemma 4.4 then implies that $k \geq 2$ and hence $n \geq 4$. If $k = 2$, then Lemma 4.5 implies that $N$ has a full internal Mom-2 structure $(M_5, B_5, \Delta_5)$. $M_5$ must be hyperbolic, or else the above argument for $n = 2$ will give a contradiction to Lemma 4.4, completing the proof.

So assume that $(M_1, T_1, \Delta_1)$ is not a full internal Mom structure. If $M_1$ is reducible, then split along a normal reducing 2-sphere, retain the non-elementarily embedded component, cap off the resulting 2-sphere boundary component with a 3-handle and cancel that 3-handle with a 2-handle. After a sequence of such operations we obtain $(M_2, T_2, \Delta_2)$ with $\rho_1(\Delta_2) \leq \rho_1(\Delta_1)$ and $M_2$ irreducible. If $M_2$ contains an embedded essential torus $R$, then split along $R$ and retain the non-elementarily embedded component. After a sequence of such operations we obtain $(M_3, T_3, \Delta_3)$ with $\rho_1(\Delta_3) \leq \rho_1(\Delta_2)$ and $M_3$ is irreducible and geometrically toroidal. Since $M_3$ is non-elementarily embedded in $N$, it is not a Seifert fibered space and hence is an annular and so by Thurston [Th2] it is hyperbolic. After repeatedly applying Lemmas 4.2 and 4.3 we obtain a general based internal Mom-$k$ structure $(M_4, T_4, \Delta_4)$ on $N$ where $M_4$ is hyperbolic and $\rho_1(\Delta_4) \leq \rho_1(\Delta_3)$. \[ \blacklozenge \]

5. Enumeration of hyperbolic Mom-$n$’s for $2 \leq n \leq 4$

Let $(M, T, \Delta)$ be a full hyperbolic Mom-$n$, with $2 \leq n \leq 4$. The handle structure $\Delta$ collapses to a cellular complex $K$ in the following fashion. Each 1-handle collapses to the arc at its core, and each 2-handle collapses to the disc at its core (expanded as necessary so that it is still attached to the cores of the appropriate 1-handles). Also, $T \times I$ collapses to $T \times 1$, subdivided into 0-cells, 1-cells, and 2-cells corresponding to the islands, bridges, and lakes of $(M, T, \Delta)$. (Note that if $(M, T, \Delta)$ were not full, we might have a non-simply connected lake and $K$ would not be a proper cellular complex.)
The resulting complex $K$ is a spine for $M$ in the sense of [MF]. If all of the 1-handles of $\Delta$ are of valence 3, then it is also a special spine in the sense of [MF]; however $K$ is not a special spine in general. In particular, in a special spine the link of each point is either a circle or a circle with two or three radii, but if $\Delta$ has a 1-handle of valence $n$ then the endpoints of the corresponding arc in $K$ will have links which are a circle with $n$ radii. This, however, is the only way in which $K$ fails to be a special spine.

In section 2 of [MF] Matveev and Fomenko describe how a manifold with a special spine can be reconstructed by gluing together truncated or ideal simplices dual to the vertices of the spine. This construction is easily generalized to our situation, and shows that $M$ can be reconstructed from $K$ by gluing together ideal polyhedra dual to the vertices of $K$. The result is an ideal cellulation of $M$ which is dual to the cellular complex $K$.

The 3-cells of this cellulation will be dual to the elements of $K^0$, which consist of the endpoints of the cores of the 1-handles of $\Delta$. In addition, since we’ve assumed each 1-handle of $\Delta$ meets at least two 2-handles, each point $v \in K^0$ will be the endpoint of at least two curves in $T \times 1 \cap K^1$. Hence if $n_v$ is the valence of $v$ in the 1-skeleton of $K$ then $n_v \geq 3$. If $n_v \geq 4$ then $v$ is dual to an $(n_v - 1)$-sided pyramid: the base of the pyramid is dual to a core of a 1-handle while the sides are dual to curves in $T \times 1 \cap K^1$. If $n_v = 3$ then $v$ is dual to a “digonal pyramid”, which we eliminate from the cellulation by collapsing it to a face in the obvious fashion. Thus $K$ is dual to a cellulation of $M$ by ideal pyramids. Since the bases of these pyramids correspond to the ends of the 1-handles of $\Delta$, we can pair them up into a collection of ideal dipyramids.

We can say more concerning the possible types and combinations of dipyramids. On one hand, each vertex $v$ is adjacent to $n_v - 1$ edges in $T \times 1 \cap K^1$, and each such edge has two endpoints; on the other hand, the core of each 2-cell of $\Delta$ contributes three edges to $T \times 1 \cap K^1$, and there are $n$ such cores in a Mom-$n$. Therefore $\Sigma_v(n_v - 1) = 6n$ in a Mom-$n$. Furthermore, $n_v - 1$ must be at least 2 and (if it’s greater than 2) equals the number of sides of the pyramid dual to $v$. Finally the vertices $v$ occur in pairs since each one corresponds to an end of a 1-handle, and the vertices in each pair have the same valence. Therefore for a Mom-2, there are only two possibilities: four three-sided pyramids, which glue together to form two three-sided dipyramids, or two four-sided pyramids and two “digonal pyramids”, which (after eliminating the “digonal pyramids”) glue together to form a single ideal octahedron. Similarly, there are only three possibilities for a Mom-3: three three-sided dipyramids, a three-sided dipyramid together with an octahedron, or a five-sided dipyramid by itself. The five possibilities for a Mom-4 are: four three-sided dipyramids, two three-sided dipyramids and an octahedron, one three-sided dipyramid and one five-sided dipyramid, two octahedra, or one six-sided dipyramid.

Thus, if $(M, T, \Delta)$ is a hyperbolic Mom-2, Mom-3, or Mom-4 then $M$ can be obtained by gluing together the faces of one of these ten sets of ideal polyhedra. Enumerating the possibilities for $M$ then becomes a matter of enumerating the ways in which the faces of these polyhedra can be glued together to form a hyperbolic 3-manifold.

This task is simplified somewhat by the following observation: the faces of each dipyramid always have exactly one vertex which is dual to the cusp neighborhood $T \times [0, 1)$. When gluing the polyhedra together to form $N$, all such vertices must be
identified with one another and with no other vertices. Thus given any two faces, there is only one possible orientation-preserving way that those two faces could be glued together.

Hence it is sufficient to enumerate the number of ways in which the faces of one of the ten sets of polyhedra can be identified in pairs. Although it is almost trivial to program a computer to do this, care must be taken as the number of possibilities is a factorial function of the number of faces, and a naive approach can rapidly exhaust a computer’s memory. To reduce the demands on the computer, a refinement to the naive approach was employed. First, for each possible set of polyhedra a symmetry group was computed. Each dipyramid has dihedral symmetry, while if a given set of polyhedra contains two dipyramids with the same number of sides then they can be exchanged to provide an additional symmetry. Secondly, an ordering was chosen for the set of all possible pairings of faces, namely the lexicographic ordering of the pairings when represented as permutations. Our computer program considered the set of pairings in order, and any pairing was immediately rejected if it was conjugate to a previous pairing via an element of the symmetry group. This considerably reduced the running time of the program.

The next step in the process is to eliminate pairings which result in obviously non-hyperbolic manifolds. While the program SnapPea can in principle handle this, for reasons of speed our program checked one necessary criterion itself: whether the link of every ideal vertex was Euclidean. Computing the Euler characteristic of the link of each ideal vertex in the cellular complex resulting from a pairing was easy to do and eliminated many cases from consideration. Our program also eliminated any pairing in which the vertices supposedly dual to the original cusp neighborhood or solid torus in fact glued together to form two or more ideal vertices.

The above considerations resulted in a list of gluing descriptions of 4,231 manifolds which might be hyperbolic Mom-2’s or Mom-3’s. At this point, SnapPea was employed to try and compute hyperbolic metrics for each of these manifolds, and to find further hyperbolic symmetries among the manifolds which admitted such metrics. SnapPea claimed to find hyperbolic metrics in 164 cases. In three of those cases SnapPea had experienced an obvious floating-point error and “found” a hyperbolic metric with an absurdly low volume. In those three cases the programs Regina and GAP (which, unlike SnapPea, do not rely on floating-point arithmetic) were used to calculate the fundamental groups of the corresponding manifolds. In all three cases the fundamental group was isomorphic to the group \( \langle a, b | [a, b^n] \rangle \) where \( n = 3 \) or 5, which has a non-trivial center. Therefore, these three manifolds could not possibly be hyperbolic and were rejected. That left 161 cases, which SnapPea identified as belonging to a total of 21 isometry classes of hyperbolic manifolds.

Some comments about rigor are in order here. Since SnapPea relies on floating-point arithmetic, some of its results are unavoidably inexact. In particular, there is no guarantee that SnapPea will find a hyperbolic metric on a manifold even if one exists, or that SnapPea will correctly discern the absence of a hyperbolic metric in cases where it doesn’t exist. In practice it is our experience that if one is careful to allow SnapPea to simplify a triangulation before attempting to find a metric, then if a metric exists SnapPea will either find it or fail to make a determination, while if a metric doesn’t exist SnapPea will either correctly say so or on rare occasions “find” a metric with absurdly low volume due to floating-point error. Still, from a standpoint of rigor this is problematic. Fortunately there is at least one task
which SnapPea does perform exactly, and that is finding isometries between two different cusped manifolds: SnapPea will only report that an isometry exists if it finds identical triangulations of the two manifolds. (See \cite{W}, in particular see the comments in the source code file \texttt{isometry.c}.) This is a combinatorial operation, not a floating-point one, and hence we are confident that SnapPea performs this operation rigorously.

Those familiar with SnapPea’s source code may object that SnapPea re-triangulates each manifold before determining if an isometry exists, and that SnapPea uses floating-point information to choose the re-triangulation. To this objection we would reply that while floating-point information is used to choose the re-triangulation, the actual re-triangulating is still a combinatorial operation, i.e. it uses integer arithmetic. The new triangulation is guaranteed to have the same topological type (see the comments in \texttt{canonize_part_1.c} from \cite{W}), and hence the possibility of floating-point error does not invalidate the result when SnapPea reports that it has found an isometry.

Thus while we are trusting SnapPea when it says that the 161 manifolds mentioned above are all in the isometry class of one of 21 manifolds from the SnapPea census, we are confident that we are not sacrificing rigor in so doing. Furthermore, the census manifolds were recently confirmed to be hyperbolic by Harriet Moser in \cite{M}, establishing that we have found 21 different hyperbolic Mom-2’s and Mom-3’s.

Unfortunately, we still can’t trust SnapPea when it fails to find an hyperbolic metric for a given manifold, as that result is not guaranteed to be rigorous. This means that there are 4,067 manifolds from the above list of 4,231 which may still be hyperbolic despite SnapPea evidence to the contrary. These manifolds were analyzed separately in the same way as the three manifolds for which SnapPea claimed to have found an absurdly low hyperbolic volume. Namely, we used Regina and GAP as before to compute the fundamental groups of the manifolds in question, and then examined the list of groups to see if any of them might be the fundamental group of a hyperbolic manifold. The vast majority of the groups on the list either had a non-trivial center, or else had two rank-2 Abelian subgroups which intersected in a rank-1 Abelian subgroup (also impossible in the fundamental group of a hyperbolic 3-manifold). Some of the groups required further analysis but were still eventually rejected; for example, many groups had an index-two subgroup with one of the above properties even when it was not clear that the whole group had such properties.

In the end the hand analysis did not reveal any new hyperbolic 3-manifolds in the list of gluing descriptions. This completes the proof of the following:

\textbf{Theorem 5.1.} There are 3 hyperbolic manifolds $M$ such that $(M,T,\Delta)$ is a Mom-2 for some $T$ and $\Delta$: the manifolds known in SnapPea’s notation as m125, m129, and m203. There are 18 additional hyperbolic manifolds $M$ such that $(M,T,\Delta)$ is a Mom-3 structure for some $T$ and $\Delta$: the manifolds known in SnapPea’s notation as m202, m292, m295, m328, m329, m359, m366, m367, m391, m412, s596, s647, s774, s776, s780, s785, s898, and s959.

Some comments about this list are in order. The manifold m129, better known as the complement of the Whitehead link, is the only manifold on this list which is obtained by gluing together the faces of an ideal octahedron. Also, all but one of these manifolds have two cusps. The exception is the three-cusped s776, which
Figure 1. Conjectured list of SnapPea manifolds which are strict Mom-4’s.

| m357 | s579 | s883 | v2124 | v2943 | v3292 | v3450 |
| m388 | s602 | s887 | v2208 | v2945 | v3294 | v3456 |
| s441 | s621 | s895 | v2531 | v3039 | v3376 | v3468 |
| s443 | s622 | s906 | v2533 | v3108 | v3379 | v3497 |
| s503 | s638 | s910 | v2644 | v3127 | v3380 | v3501 |
| s506 | s661 | s913 | v2648 | v3140 | v3383 | v3506 |
| s549 | s782 | s914 | v2652 | v3211 | v3384 | v3507 |
| s568 | s831 | s930 | v2731 | v3222 | v3385 | v3518 |
| s569 | s843 | s937 | v2732 | v3223 | v3393 | v3527 |
| s576 | s859 | s940 | v2788 | v3224 | v3396 | v3544 |
| s577 | s864 | s941 | v2892 | v3225 | v3426 | v3546 |
| s578 | s880 | s948 | v2942 | v3227 | v3429 |        |

is the complement in $S^3$ of a three-element chain of circles (the link $6_1^3$ in Rolfsen’s notation).

Enumerating hyperbolic Mom-4’s was more difficult: merely enumerating the possible gluing descriptions resulted in a list of 1,033,610 possibilities (compared to 4,231 possibilities in the previous case). From this list, SnapPea identified 138 different hyperbolic manifolds. In another 493 cases, SnapPea was either unable to make a determination or else experienced an obvious floating-point error. In each such case, the fundamental group of the corresponding manifold was again computed by Regina and GAP, and in each case the fundamental group was isomorphic to $\langle a, b | [a^n, b] \rangle$ for some $n$, or else had two rank-2 Abelian subgroups which intersected in a rank-1 Abelian subgroup. Therefore these exceptional cases do not correspond to hyperbolic manifolds. Note that all of the Mom-2’s and Mom-3’s appear in the Mom-4 list; the same manifold can admit multiple handle structures.

Based on the above result, we propose the following:

**Conjecture 5.2.** There are 138 hyperbolic manifolds $M$ such that $(M, T, \Delta)$ is a Mom-2, Mom-3, or Mom-4 for some $T$ and $\Delta$. Of these, 117 are strict Mom-4’s, i.e. Mom-4’s which are not Mom-2’s or Mom-3’s.

Of the 117 strict Mom-4’s, SnapPea was successfully used to identify 83 of them as manifolds from the SnapPea census. Those manifolds appear in Figure 1. SnapPea was not able to identify the remaining 34 manifolds, and in fact 33 of those manifolds have volumes which do not appear anywhere in the SnapPea census, presumably because the Matveev complexity of the corresponding manifolds is greater than 7 (see [MF]). The remaining manifold has the same volume and homology as the census manifold v3527; it is conceivable that SnapPea was simply unable to find a corresponding isometry.

The unidentified manifolds are listed in Figure 2. The notation used can be interpreted as follows: the numbers before the semi-colon describe the type of ideal polyhedra used to construct the manifold. For example, the first entry in the figure has the numbers “3, 3, 4” to the left of the semi-colon; each “3” indicates an ideal triangular dipyrramid, while each “4” indicates an ideal square dipyrramid (i.e. an ideal octahedron). Each ideal dipyrramid has two “polar” vertices and either three
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\begin{center}
\begin{tabular}{c}
(3, 3, 4 : 3, 6, 8, 0, 1, 15, 2, 17, 14, 18, 16, 4, 10, 7, 12, 9, 11, 5) \\
(3, 3, 4 : 3, 6, 11, 0, 10, 9, 1, 15, 4, 5, 2, 16, 18, 7, 12, 19, 13, 17) \\
(3, 3, 4 : 3, 6, 11, 0, 9, 18, 1, 19, 13, 4, 15, 2, 16, 8, 17, 10, 12, 14, 5, 7) \\
(3, 3, 3, 3 : 3, 6, 9, 0, 13, 19, 1, 22, 14, 2, 17, 12, 11, 4, 8, 23, 18, 10, 16, 5, 21, 20, 7, 15) \\
(3, 3, 3, 3 : 3, 6, 0, 1, 19, 2, 9, 13, 7, 14, 15, 23, 8, 10, 11, 20, 22, 5, 16, 17, 18, 12) \\
(3, 3, 3, 3 : 3, 6, 12, 0, 17, 9, 1, 11, 18, 5, 23, 7, 2, 20, 15, 14, 21, 4, 8, 22, 13, 16, 19, 10) \\
(3, 3, 4 : 3, 4, 6, 0, 1, 18, 2, 11, 16, 10, 9, 7, 15, 19, 17, 12, 8, 14, 5, 13) \\
(3, 3, 4 : 3, 4, 6, 0, 1, 18, 2, 14, 12, 13, 19, 17, 8, 9, 7, 16, 15, 11, 5, 10) \\
(3, 3, 4 : 3, 4, 6, 0, 1, 18, 2, 14, 16, 13, 19, 17, 5, 9, 7, 12, 8, 11, 5, 10) \\
(3, 3, 4 : 3, 6, 10, 0, 13, 9, 1, 16, 19, 5, 2, 17, 14, 4, 12, 18, 7, 11, 15, 8) \\
(3, 3, 4 : 3, 6, 10, 0, 8, 13, 1, 16, 4, 18, 2, 17, 15, 5, 19, 12, 7, 11, 9, 14) \\
(3, 3, 4 : 3, 6, 10, 0, 8, 13, 1, 18, 4, 16, 2, 19, 15, 5, 17, 12, 9, 14, 7, 11) \\
(3, 3, 4 : 3, 6, 10, 0, 15, 17, 1, 18, 14, 16, 2, 19, 13, 12, 8, 4, 9, 5, 7, 11) \\
(3, 3, 4 : 3, 6, 11, 0, 8, 19, 1, 15, 4, 17, 14, 2, 16, 18, 10, 7, 12, 9, 13, 5) \\
(3, 3, 4 : 3, 6, 11, 0, 8, 13, 17, 0, 1, 3, 15, 2, 19, 16, 4, 18, 9, 12, 5, 14, 11) \\
(3, 3, 3, 3 : 3, 4, 6, 0, 1, 9, 2, 15, 17, 5, 13, 18, 19, 10, 23, 7, 22, 8, 11, 12, 21, 20, 16, 14) \\
(3, 3, 4 : 3, 6, 10, 0, 8, 14, 1, 16, 4, 18, 2, 17, 13, 12, 5, 19, 7, 11, 9, 15) \\
(3, 3, 4 : 3, 6, 10, 0, 8, 14, 1, 18, 4, 16, 2, 19, 13, 12, 5, 17, 9, 15, 7, 11) \\
(3, 3, 4 : 3, 6, 7, 0, 16, 19, 1, 2, 10, 12, 8, 14, 9, 18, 11, 17, 4, 15, 13, 5) \\
(3, 3, 4 : 3, 6, 10, 0, 13, 17, 0, 12, 3, 15, 1, 16, 7, 4, 18, 9, 11, 5, 14, 2) \\
(3, 3, 4 : 3, 6, 7, 10, 8, 9, 13, 0, 1, 3, 4, 2, 19, 16, 5, 17, 18, 12, 14, 15, 11) \\
(3, 3, 4 : 3, 6, 10, 0, 8, 18, 1, 14, 4, 16, 2, 19, 13, 12, 7, 17, 9, 15, 5, 11) \\
(3, 3, 3, 3 : 3, 6, 12, 0, 9, 16, 1, 18, 23, 4, 20, 22, 2, 19, 15, 4, 14, 5, 21, 7, 13, 10, 17, 11, 8) \\
(3, 3, 3, 3 : 3, 4, 6, 0, 1, 9, 2, 15, 17, 5, 14, 13, 19, 11, 10, 7, 23, 8, 22, 12, 21, 20, 18, 16) \\
(3, 3, 3, 3 : 3, 6, 12, 0, 9, 16, 1, 10, 18, 4, 7, 22, 2, 20, 15, 14, 5, 21, 8, 23, 13, 17, 11, 19) \\
(3, 3, 3, 3 : 3, 6, 12, 0, 9, 16, 1, 18, 11, 4, 23, 8, 2, 19, 15, 14, 5, 21, 7, 13, 22, 17, 20, 10) \\
\end{tabular}
\end{center}

\textbf{Figure 2.} Conjectured Mom-4's not identified by SnapPea.

or four “equatorial vertices”. Number the faces of all the polyhedra sequentially in such a way that the faces “north” of each equator are numbered before the faces “south” of each equator. For example, in the first entry the first triangular dipyramid has faces 0, 1, and 2 next to one polar vertex, and faces 10, 11, and 12 next to the other polar vertex. The next triangular dipyramid has faces 3, 4, and 5 as well as faces 13, 14, 15, and the square dipyramid has faces 6 through 9 and 16 through 19. (This somewhat unintuitive numbering scheme was chosen for convenience when writing the computer software for this part of the paper.) Then the numbers to the right of the semi-colon form a permutation which describes how to glue together the faces of the ideal polyhedra. For example, in the first entry the string of numbers which begins with “3, 6, 8, 0, . . .” imply that face 0 is glued to

(3, 3, 4 : 3, 6, 8, 0, 1, 15, 2, 17, 14, 18, 16, 4, 10, 7, 12, 9, 11, 5)
face 3, face 1 is glued to face 6, and so on. Since we are requiring “polar” vertices to be identified solely with other “polar” vertices, no other information is needed to reconstruct the polyhedral gluing.

One additional point of information: all but eight of the manifolds in the list satisfy $|\partial M| = 2$; seven satisfy $|\partial M| = 3$ and one satisfies $|\partial M| = 4$. Thanks to the timely assistance of Morwen Thistlethwaite, the authors were able to positively identify all eight of these manifolds:

**Conjecture 5.3.** There are 8 hyperbolic manifolds $M$ such that $\Delta$ is a Mom-$n$ for some $2 \leq n \leq 4$ and $|\partial M| > 2$. All eight manifolds are complements of links in $S^3$: the links $6_3^1$, $6_3^2$ (the Borromean rings), $8_3^1$, $8_3^2$, $8_3^3$, and the links with Gauss codes $jcceddEGHjJBFCa$, $jcbecceaHbIJDGF$, and $mccdfiEhAjKLcmdhFG$.

At the time of writing we are still searching for an efficient way to verify SnapPea’s computations in the Mom-4 case: clearly, examining over a million fundamental groups by hand is not a practical solution. Until a better way is found, our enumeration results in the Mom-4 case should properly be considered speculative.

6. **Hyperbolic Mom-$n$’s in hyperbolic 3-manifolds are internal Mom-$n$ structures for $n \leq 4$**

Let $R$ be a convolotube in the interior of a compact hyperbolic 3-manifold $N$ and let $V$ be the cube with knotted hole bounded by $R$. By drilling out solid tori from $N - \bar{V}$, we can create a manifold $M$ which is non-elementarily embedded in $N$ and whose boundary contains a convolutube. We call such an embedding knotted. The goal of this chapter is to show that if $n \leq 4$, any embedding of a Mom-$n$ manifold $(M,T)$ into a compact hyperbolic manifold $(N,T)$ is unknotted.

**Definition 6.1.** Let $M$ be a compact 3-manifold and $T$ a possibly empty union of components of $\partial M$. We say that $(M,T)$ is hereditarily unknotted, if every non-elementary embedding into a compact hyperbolic 3-manifold $N$, taking $T$ to components of $\partial N$, has the property that each component of $\partial M$ is either boundary parallel or bounds a solid torus.

**Remark 6.2.** If $(M,T)$ is hereditarily unknotted and $M_1$ is obtained by filling a component of $\partial M - T$, then $(M_1,T)$ is hereditarily unknotted.

**Lemma 6.3.** If $(M,T)$ is a hereditarily unknotted Mom-$n$ manifold non-elementarily embedded in the hyperbolic 3-manifold $N$ such that $T$ bounds a tubular neighborhood of a geodesic, then $(M,T)$ is an internal Mom-$n$ structure.

**Proof.** Let $V$ be the solid torus bounded by $T$. By Lemma 1.7 if $N_1 = N - V$ with cusp neighborhoods deleted, then $N_1$ is compact hyperbolic. Therefore $(M,T) \subset (N_1,T)$ is a non-elementary embedding and hence any component of $\partial M - T$ either bounds a solid torus or is boundary parallel in $N_1$. Therefore similar properties hold in $N$ and hence $(M,T)$ is an internal Mom-$n$ structure on $N$. \qed

**Remark 6.4.** The condition that $T$ bounds a neighborhood of a geodesic is essential.

**Lemma 6.5.** Let $M$ be a compact hyperbolic 3-manifold with $T$ a union of components of $\partial M$. If $\partial M - T$ is connected, $(M,T)$ is hereditarily unknotted.
Figure 3. Eight links whose complements are Mom-4's with 3 or more cusps.
Proof. If under a non-elementary embedding \((M,T) \rightarrow (N,T)\), \(\partial M - T\) was a convolutube, then \(M\) would be reducible.

The following result establishes criteria for showing that \((M,T)\) is hereditarily unknotted.

**Lemma 6.6.** Let \(M\) be a compact hyperbolic 3-manifold with \(V_1, \ldots, V_n\) components of \(\partial M\) and \(T\) a nonempty union of some other components. If any of the following hold, there exists no non-elementary embedding \((M,T) \rightarrow (N,T)\) such that \(N\) is compact hyperbolic and \(\{V_1, \ldots, V_n\}\) is exactly the set of convolutubes of \(\partial M \subset N\).

i) The manifold obtained by some filling of \(M\) along \(V_1, \ldots, V_n\) is a 3-manifold without any hyperbolic part. (That is, after applying sphere and torus decompositions there are no hyperbolic components.)

ii) After some filling of \(M\) along \(V_1, \ldots, V_n\), the surface \(T\) is compressible.

iii) For every filling on a non-empty set of components of \(\partial M - T \cup V_1 \cup \cdots \cup V_n\), either \(V_1 \cup \cdots \cup V_n\) is incompressible or the filled manifold has no hyperbolic part.

Proof. Suppose that \((M,T)\) embeds in \((N,T)\), where among the components of \(\partial M\), \(V_1, \ldots, V_n\) are the set of convolutubes and \(W_1, \ldots, W_m\) are the tubes. Let \(W_i^*\) denote the solid torus bounded by \(W_i\) and \(V_i^*\) denote the cube with knotted hole bounded by \(V_i\). Let \(B_1, \ldots, B_n\) be pairwise disjoint 3-balls in \(N\) such that for each \(i\), \(V_i \subset B_i\).

i) Let \(\tilde{M}\) be a manifold obtained by filling the \(V_i^*\)'s. Let \(\tilde{N}\) be obtained by deleting the \(V_i^*\)'s and doing the corresponding fillings along the \(V_i^*\)'s. Therefore \(\tilde{N}\) is obtained from \(\tilde{M}\) by Dehn filling and \(\tilde{N}\) is a connected sum of \(N\) with \(S^2 \times S^1\)'s and/or lens spaces and/or \(S^3\)'s. This implies that \(\tilde{M}\) has a hyperbolic part.

ii) If \(T\) is compressible in \(\tilde{M}\) it is compressible in \(\tilde{N}\) and hence in \(N\), which is a contradiction.

iii) First observe that \(V_i\) compresses in the manifold \(M'\) obtained by filling \(M\) where each \(W_i\) is filled with \(W_i^*\). Topologically, \(M'\) is homeomorphic to \(N\) with \(n\) open unknotted and unlinked solid tori removed and so has a hyperbolic part.

**Theorem 6.7.** If the Mom-\(n\) manifolds for \(n \leq 4\) with three or more boundary components are exactly those listed in Figure 3 (i.e., if Conjecture 5.1 is true), then any hyperbolic Mom-\(n\) manifold \((M,T)\) with \(n \leq 4\) is hereditarily unknotted.

Proof. By Lemma 6.6 it suffices to consider the case where \(M\) is one of the eight Mom-4 manifolds with at least three boundary components listed in Figure 3. If \(M\) is any of the first six manifolds and \(T\) is any component of \(\partial M\), then \((M,T)\) is hereditarily unknotted by criterion (i) of Lemma 6.6. For manifolds 7 and 8, depending on which boundary component is used for \(T\), applications of (i) and (iii) imply that they are hereditarily unknotted.

7. Examples of Mom-\(n\) Structures

In this section we give some representative examples of hyperbolic manifolds \(N\) which contain an internal Mom-2 or Mom-3 structure \((M,T,\Delta)\). Our goal in this section is to give the reader an intuitive feel for how these particular cell complexes arise inside hyperbolic manifolds. All of the manifolds in this section involve manifolds \(N\) with torus boundary, with the base torus of the Mom-structure \((M,T,\Delta)\) being \(\partial N\). To obtain Mom-\(n\) structures on closed manifolds, note that if
Figure 4. The figure-8 knot complement equipped with a Mom-2.

$T = \partial N$ then a Mom-$n$ structure $(M, T, \Delta)$ on $N$ passes to a Mom-$n$ structure on any manifold obtained by filling $\partial N$.

**Example 7.1.** The first example is the figure-8 knot complement. We construct a Mom-2 $(M, T, \Delta)$ inside this manifold as follows. The torus $T$ is just the boundary of the manifold. The 1-handles $\lambda_1$ and $\lambda_2$ span the two tangles which make up the standard diagram of this knot, as seen in figure 4. Finally the 2-handles $\sigma_1$ and $\sigma_2$ are symmetrically placed as shown in the diagram. Note that, as required, each 2-handle meets three 1 handles counting multiplicity. Specifically, each 2-handle meets $\lambda_1$ twice and $\lambda_2$ once. Also, one can see from the diagram that the complement of $T \cup \{\lambda_i\} \cup \{\sigma_j\}$ consists of a solid torus, and that the solid torus retracts onto a homotopically non-trivial simple closed curve (which is a geodesic in $N$). Thus this is a valid hyperbolic Mom-2 structure on $N$.

Moreover, we can quickly determine the nature of the ideal triangulation of $M$ described in the Section 5. The ends of $\lambda_1$ are each dual to a four-sided pyramid in this triangulation, and the two endpoints of $\lambda_2$ are each dual to a “digonal pyramid”, so that each get eliminated. Thus the figure-8 knot complement possesses a Mom-2 structure $(M, T, \Delta)$ where $M$ is a two-cusped hyperbolic manifold which is in turn obtained by gluing together the faces of an ideal octahedron. By the comments after Theorem 5.1, $M$ must be the complement of the Whitehead link. And indeed, it is easy to verify that if one drills out the core of the solid torus in the complement of $M$ one obtains a manifold homeomorphic to the complement of the Whitehead link.

**Example 7.2.** Next we will let $N$ be the manifold known as m003 in the SnapPea census. This manifold has first homology group $\mathbb{Z} \oplus \mathbb{Z}/5$, and hence is not a knot complement; instead, we will present this manifold as the union of two regular ideal hyperbolic tetrahedra; see figure 5. Note that in the diagram each face is glued to the corresponding face on the other tetrahedron, in such a way that the edges match...
up into two equivalence classes as shown. To make $N$ a compact manifold with
torus boundary, assume the ideal tetrahedra are truncated. Now suppose that we
construct $(M, T, \Delta)$ in this case as follows. For the 1-handles, we use neighborhoods
of the two edges shown in the diagram, truncated by the torus $T = \partial N$. And for the
2-handles, we use neighborhoods of the two truncated triangles which are formed
by gluing together the faces on the front of each tetrahedron in the diagram. It
is a simple exercise to confirm that the complement of the resulting embedded
manifold $M$ consists of a solid torus, and that the solid torus retracts onto a simple
closed geodesic curve, and that therefore this manifold possesses a valid hyperbolic
Mom-2 structure. Each of the 1-handles in this Mom-2 meets three of the 2-
handles, counting multiplicity; therefore we can conclude that $m003$ contains a
Mom-2 $(M, T, \Delta)$ where $M$ is obtained by gluing together two ideal three-sided
dipyramids. From Theorem 5.1 and the comments following it we know this must
be either $m125$ or $m203$. Further investigation with SnapPea shows that it must in
fact be $m125$.

It is instructive to get another view of this Mom-2 by constructing a cusp diagram
for this manifold. Specifically, consider the triangulation induced on $T$ by the given
ideal triangulation of $m003$. The two ideal tetrahedra in $m003$ will appear as eight
triangles, the four ideal triangles will appear as twelve edges, and the two edges
will appear as four vertices. The resulting cusp diagram is shown in figure 6; keep
in mind this is a diagram of a torus, so the edges of the parallelogram are identified
with one another. (The labels inside each triangle indicate which of the ideal
simplices contributes that triangle to the cusp diagram.) The highlighted edges in
the cusp diagram are those that correspond to the 2-handles of the handle structure
$\Delta$; in other words, they along with the four vertices of the diagram comprise $\Delta^1 \cap T$.

Example 7.3. As another example in this vein, consider the manifold $N=m017$.
This manifold has first homology group $\mathbb{Z} + \mathbb{Z}/7$, so again it is not a knot complement
in $S^3$, but for brevity’s sake we only present a cusp diagram here. In figure 7,
the corners of the three ideal hyperbolic tetrahedra which make up $m017$ can be
seen. And again, the highlighted edges in the cusp diagram correspond to two
faces of those tetrahedra which provide the 2-handles for an internal Mom-2 in
this manifold. Note that we can determine from the cusp diagram alone that

\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{The two ideal tetrahedra making up the manifold m003.}
\end{figure}
Figure 6. The cusp diagram for m003, with the components of the Mom-2 highlighted.

Figure 7. The cusp diagram for m017, with the components of a Mom-2 highlighted.

the 1-handles of this Mom-2 meet four and two 2-handles respectively, counting multiplicity, and that therefore in the resulting Mom-2 structure $(M, T, \Delta)$ the manifold $M$ is obtained by gluing together an ideal octahedron. As before, this implies that $M$ must be homeomorphic to the complement of the Whitehead link. Some further work with SnapPea confirms this: m017 is obtained by $(-7,2)$ Dehn surgery on either component of the link.

Example 7.4. Finally, we include the motivating example for this paper. Figure 8 shows the maximal cusp diagram of the 1-cusped manifold m011 as provided by Weeks’ SnapPea program. Unlike the previous cusp diagrams in this section, it also shows all the horoballs at hyperbolic distance at most 0.51 from the maximal horoball at infinity. The parallelogram shows a fundamental domain for the $\mathbb{Z} \oplus \mathbb{Z}$ action. Note that the ideal triangulation presented in this diagram is dual to the Ford decomposition of the manifold. In particular the 1-simplices of the triangulation are geodesics orthogonal to pairs of horoballs; these 1-simplices appear either
as edges in the figure joining the endpoints of the simplex in $S^2_{\infty}$, or as the vertical geodesics passing from the center of each horoball to the horoball at infinity.

Let $H_{\infty}$ denote the horoball at infinity. There are six horoballs in the diagram up to the $\mathbb{Z} \oplus \mathbb{Z}$-action, labelled $1-, 1+, 2-, 2+, 3-, \text{ and } 3+$. This notation means that if $\gamma \in \pi_1(m011)$ takes horoball $n\pm$ to $H_{\infty}$, the horoball at infinity, then $H_{\infty}$ is transformed to one labelled $n\mp$. The geodesic from $n-$ to $H_{\infty}$ is oriented to point into $H_{\infty}$ and hence the geodesic from $n+$ to $H_{\infty}$ is oriented out of $H_{\infty}$. These orientations induce, via the $\pi_1(m011)$-action, the indicated orientations on the edges of the diagram. We explain, by example, the meaning of the edge labels. The edge 3 from 2+ to 2− corresponds to a geodesic $\sigma$ with the property that when 2+ is transformed to $H_{\infty}$, then 2− is transformed to 3+ and $\sigma$ is transformed to the vertical geodesic oriented from $H_{\infty}$ to 3+. (Had the edge been oriented oppositely, then 2− would have been transformed to 3−.) SnapPea did not provide the orientation information, however such information can be derived from the SnapPea data.

By staring at this diagram we can see how m011 contains a Mom-2. Let $V_0$ be the maximal horotorus neighborhood of the cusp, slightly shrunken. By expanding $V_0$, the expanded $V_0$ touches the (expanded) horoballs labeled 1, thereby creating a 1-handle denoted $E_1$. Let $V_1$ denote this expanded $V_0$. Further expansion creates $V_2$ which is topologically $V_1$ together with another 1-handle $E_2$, this 1-handle occurring between horoball 2 and $H_{\infty}$. The edge labelled 1 between horoballs 2− and 2+ corresponds to a valence three 2-handle which goes over $E_1$ once and $E_2$ twice.
Similarly the edge labelled 2 between horoballs 1− and 1+ gives rise to a valence three 2-handle going twice over $E_1$ and once over $E_2$. The parallelogram of Figure 8 can also be viewed as $\partial V_0$, with the centers of 1−, 1+, 2−, 2+ as the attaching sites of the 1-handles and the thick black lines corresponding to where the 2-handles cross over $\partial V_0$. 
References


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