THE 4-DIMENSIONAL LIGHT BULB THEOREM

DAVID GABAI

Abstract. For embedded 2-spheres in a 4-manifold sharing the same embedded transverse sphere homotopy implies isotopy, provided the ambient 4-manifold has no $\mathbb{Z}_2$-torsion in the fundamental group. This gives a generalization of the classical light bulb trick to 4-dimensions, the uniqueness of spanning discs for a simple closed curve in S^4 and $\pi_0(\mathrm{Diff}_0(S^2 \times D^2)/\mathrm{Diff}_0(B^4)) = 1$. In manifolds with $\mathbb{Z}_2$-torsion, one surface can be put into a normal form relative to the other.

1. Introduction

In his seminal work on immersions [Sm1] Steven Smale classified regular homotopy classes of immersions of 2-spheres into Euclidean space and more generally into orientable smooth manifolds. In [Sm2] he gave the regular homotopy classification of immersed spheres in $\mathbb{R}^n$ and asked:

Question 1.1. *(Smale, P. 329* [Sm2]*) Develop an analogous theory for imbeddings. Presumably this will be quite hard. However, even partial results in this direction would be interesting.*

This paper works in the smooth category and addresses the question of isotopy of spheres in 4-manifolds. In that context Smale's results [Sm1] show that two embeddings are homotopic if and only if they are regularly homotopic. Given that 2-spheres can knot in 4-space, isotopy is a much more restrictive condition than homotopy. Indeed, the author is aware of only one unconditional positive result and that was proved more than 50 years ago: A 2-sphere in a 4-manifold that bounds a 3-ball is isotopic to a standard inessential 2-sphere, [Ce1] p. 231, [Pa].

Recall that a *transverse sphere* G to a surface R in a 4-manifold is a sphere with trivial normal bundle that intersects R exactly once and transversely. The following are the main results of this paper.

Theorem 1.2. *Let M be an orientable 4-manifold such that $\pi_1(M)$ has no 2-torsion. Two embedded 2-spheres with common transverse sphere G are homotopic if and only if they are ambiently isotopic. If they coincide near G, then the isotopy can be chosen to fix a neighborhood of G pointwise.*

For fundamental groups with 2-torsion, the methods of this paper yield the following.

Theorem 1.3. *Let M be an orientable 4-manifold and R_1, R_0 be embedded spheres which coincide near the common transverse sphere G. Then R_1 can be put into a normal form with respect to R_0 via an isotopy fixing a neighborhood of G pointwise. (See Definition 5.23 for the*

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description.) Here R_1 has double tubes representing elements $\{[\lambda_1], \cdots, [\lambda_n]\}$ where the $[\lambda_i]$'s are distinct nontrivial 2-torsion elements and $R_1 = R_0$ if this set is empty. Any finite set of distinct 2-torsion elements gives rise to a R_1 in normal form with double tubes representing this set and two such R_1's are isotopic if the corresponding set of $[\lambda_i]$'s are equal.

Remark 1.4. Following the release of the first version of this paper, two groups independently established the necessity of the 2-torsion hypothesis. Hannah Schwartz [Sch] first found an explicit 4-manifold with a pair of homotopic but not topologically isotopic spheres with a common transverse sphere. Rob Schneiderman and Peter Teichner [ST] used the Freedman - Quinn obstruction of Theorem 10.5 [FQ] to give an obstruction to topological isotopy and in particular they can show that two R_1's are not topologically isotopic if the corresponding sets of $[\lambda_i]$'s are distinct.

Generalizations of these results to multiple pairs of spheres is given in §10. This generalization implies that if R_0, R_1 are embedded spheres with a common transverse sphere and $M_1 \to M$ is a finite cover such that $\pi_1(M_1)$ has no 2-torsion, then the preimages of R_1 are simultaneously isotopic to the preimages of R_0, though perhaps not equivariently.

Here are some applications.

Theorem 1.5. *A properly embedded disc in $S^2 \times D^2$ is properly isotopic to a fiber if and only if its boundary is standard.*

Theorem 1.6. *Two properly embedded discs D_0 and D_1 in $S^2 \times D^2$ that coincide near their standard boundaries are properly isotopic rel boundary if and only if they are homologous in $H_2(S^2 \times D^2, \partial D_0)$.*

Let $\operatorname{Diff}_0(X)$ denote the group of diffeomorphisms of the compact manifold X that are properly homotopic to the identity.

Corollary 1.7. $\pi_0(\operatorname{Diff}_0(S^2 \times D^2)/\operatorname{Diff}_0(B^4)) = 1$.

Remark 1.8. In words, modulo diffeomorphisms of the 4-ball, homotopy implies isotopy for diffeomorphisms of $S^2 \times D^2$.

The classical *light bulb* theorem states that a knot in $S^2 \times S^1$ that intersects a $S^2 \times y$ transversely and in exactly one point is isotopic to the standard vertical curve, i.e. a $x \times S^1$. The next result is the 4-dimensional version.

Theorem 1.9. *(4D-Lightbulb Theorem) If R is an embedded 2-sphere in $S^2 \times S^2$, homologous to $x_0 \times S^2$, that intersects $S^2 \times y_0$ transversely and only at the point (x_0, y_0), then R is isotopic to $x_0 \times S^2$ via an isotopy fixing $S^2 \times y_0$ pointwise.*

In 1985, under the above hypotheses, Litherland [Li] proved that there exists a diffeomorphism *pseudo-isotopic* to the identity that takes R to $x_0 \times S^2$ and proved the full light bulb theorem for smooth m-spheres in $S^2 \times S^m$ for $m > 2$. (There is an additional necessary condition in that case.) Another version of the light bulb theorem was proven in 1986 by Marumoto [Ma]. He showed that two locally flat PL m-discs in an n-sphere, $n > m$ with the same boundary are topologically isotopic rel boundary. Here we prove that theorem for discs in S^4 in the smooth isotopy category.

Theorem 1.10. *(Uniqueness of Spanning Discs) If D_0 and D_1 are discs in S^4 such that $\partial D_0 = \partial D_1 = \gamma$, then there exists an isotopy of S^4 taking D_0 to D_1 that fixes γ pointwise.*

Remark 1.11. The analogous result for 1-discs in S^4 is well known using general position. The result for 3-discs in S^4 implies the smooth 4D-Schoenflies conjecture.

This paper gives two proofs of the 4D-Light Bulb Theorem. The first proof has two steps. First we give a direct argument showing that R is isotopic to a *vertical sphere*, i.e. viewing $S^2 \times S^2$ as $S^2 \times S^1 \times [-\infty, \infty]$ where each $z \times S^1 \times \infty$ and each $z \times S^1 \times -\infty$ is identified with a point, then after isotopy R is transverse to each $S^2 \times S^1 \times t$ and intersects each such space in a single component. This involves an analogue of the normal form theorem of [KSS] and repeated use of $S^2 \times 0$ as a transverse 2-sphere. The second step invokes Hatcher's [Ha] theorem (the Smale conjecture: $\text{Diff}^+(S^3) \simeq SO(4)$) to straighten out these intersections.

The proof of Theorem 1.2, and hence a somewhat different one for $S^2 \times S^2$ makes use of Smale's results on regular homotopy of 2-spheres in 4-manifolds [Sm1]. We show that if R_0 is homotopic to R_1 and both are embedded surfaces, then the homotopy from R_0 to R_1 is *shadowed by tubed surfaces*, i.e. there is an isotopy taking R_0 to something that looks like R_1 embroidered with a complicated system of tubes together with parallel copies of the transverse sphere. Through various geometric arguments we show that these tubes can be reorganized and eventually isotoped away. The proof formally relies on the first proof of the Light Bulb theorem at the very last step, though we outline how to eliminate the dependence in Remark 8.2. The proof uses the fact that R_0 is a 2-sphere. The $\mathbb{Z}_2$-condition is used in Proposition 6.9.

Both arguments make use of the 4D-Light Bulb Lemma, which is the direct analogue of the 3D-version where one can do a crossing change using the transverse sphere.

More is known in other settings. In the topological category a locally flat 2-sphere in S^4 is topologically equivalent to the trivial 2-knot if and only if its complement has fundamental group $\mathbb{Z}$ [Fr], [FQ]. There are topologically isotopic smooth 2-spheres in 4-manifolds that are not smoothly isotopic, yet become smoothly isotopic after a stabilization with a single $S^2 \times S^2$ [AKMR], [Ak]. Topologically isotopic smooth 2-spheres in simply connected 4-manifolds are smoothly pseudo-isotopic by [Kr] and after finitely many stabilizations with $S^2 \times S^2$'s are smoothly isotopic by [Qu].

The paper is organized as follows. §2 recalls some classical uses of transverse spheres and proves the Light Bulb Lemma. The Light Bulb theorem is proven in §3. Basic facts about regular homotopy are recalled in §4. The definition of tubed surface, basic operations on tubed surfaces, the notion of shadowing a homotopy by basic operations and normal form for a surface are given in §5. The reader is cautioned that *tubes* are used in two contexts here; as tubes that follow curves lying in the surface and as tubes that follow arcs with endpoints in the surface. The latter fall into two types; *single* and *double* tubes. In §6 it is shown how to transform pairs of double tubes into pairs of single tubes. If there is no $\mathbb{Z}_2$-torsion, then in the end all but at most one of the double tubes remains and that one is homotopically inessential. If $\pi_1(M)$ has 2-torsion, then there may be additional double tubes representing *distinct* 2-torsion elements of $\pi_1(M)$. A crossing change lemma is proven in §7 enabling distinct tubes following curves in the surface to be disentangled. In §8 the proof of Theorems 1.2 and 1.3 is completed. An extension to higher genus surfaces is given in §9. In particular it is shown that a closed oriented surface in $S^2 \times S^2$ homologous to $0 \times S^2$, that intersects $S^2 \times 0$ transversely in one point, is isotopically standard. Applications and questions are given in §10.

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2. The 4-dimensional light bulb lemma

Unless said otherwise, all manifolds in this paper are smooth and orientable and immersions are self-transverse.

Definition 2.1. A *transverse sphere* G to the immersed surface R is a sphere with trivial normal bundle that intersects R transversely in a single point.

All transverse spheres in this paper are embedded. The following is well known. We give the proof as a warm up to the light bulb lemma.

Lemma 2.2. *If R is an immersed surface with embedded transverse sphere G in the 4-manifold M, then the induced map $\pi_1(M \setminus R) \to \pi_1(M)$ is an isomorphism. If R is a sphere, then the induced maps $\pi_1(M \setminus R \cup G) \to \pi_1(M)$ and $\pi_1(M \setminus G) \to \pi_1(M)$ are isomorphisms.*

Proof. Surjectivity is immediate by general position. If γ is a loop in $M \setminus R$ bounding the singular disc $D \subset M$, then after a small perturbation we can assume that D is transverse to R. Tubing off intersections with copies of G shows that the map is also injective. For the second and third cases, we can assume that D is transverse to $R \cup G$. First use R to tube off intersections of D with G. This proves injectivity for the third case. (If R does not have a trivial normal bundle or is not embedded, then the resulting disc may have extra intersections with R.) Tubing with G eliminates all the $D \cap R$ intersections and so the induced map in the second case is also injective. □

The light bulb lemma basically says that in the presence of a transverse sphere one can do an ambient isotopy of a surface R as in shown in Figure 2.1, without introducing any self intersections.

Lemma 2.3. *(4D-Light Bulb Lemma) Let R be an embedded surface with transverse sphere G in the 4-manifold M and let $z = R \cap G$. Let α_0 and α_1 be two smooth compact arcs that coincide near their endpoints and bound the pinched embedded disc E that is transverse to R with $R \cap E = y$ and $E \cap G = \emptyset$. See Figure 2.2 a). Let f_t be an ambient isotopy of M taking α_0 to α_1 that corresponds to sweeping α_0 across E. Here f_t is fixed near $\partial\alpha_0$ and is supported in a small neighborhood of E. Suppose that $N(\alpha_0)$ is parametrized as $B^3 \times I$ and $R \cap N(E) = C \cup B$, where C is the disc containing y and $B \subset \text{int}(B^3) \times I$. If y and z lie*

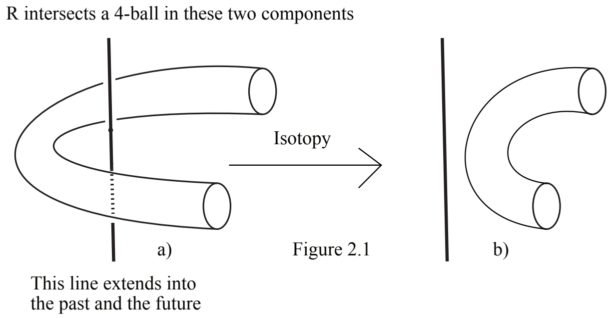


Figure 2.1

in the same component of $R \setminus B$*, then* R *is ambiently isotopic to* $g(R)$ *where* $g|R \setminus B = \mathrm{id}$ *and* $g|B = f_1|B$*. The ambient isotopy fixes* G *pointwise and the isotopy restricted to* R *is supported in* B*.*

If G *has a non trivial normal bundle with even Euler class, then the conclusion holds except for the assertion that the ambient isotopy fixes* G*.*

If the Euler class is odd, then under the additional hypothesis that B *is a union of unknotted and unlinked annuli parallel to* α_0*, the above conclusion holds with the additional modification* $g(B) = f_1(B)$*. In general* $g|N(\alpha_0)$ *is the composition of the standard isotopy taking* $N(\alpha_0)$ *to* $N(\alpha_1)$ *followed by the non trivial element of* $SO(3)$ *along* $N(\alpha_1)$*.*

Remarks 2.4. i) After an initial isotopy of R supported near $N(\alpha_0)$ we can assume that it is of the form $L \times I$ where L is a link in $\mathrm{int}(B^3) \times 0$.

ii) The hypothesis does not hold if B separates y from z in R.

Proof. Since y lies in the same component of z we can tube off E with a copy of G to obtain a disc D that coincides with E near ∂E and $D \cap (R \cup G) = \emptyset$. Since G has a trivial normal bundle, there exists a framing of the normal bundle of D that coincides with that of E near ∂E. See Figure 2.2. Therefore, we can isotope B to $f_1(B)$ by sweeping across D rather than E. This isotopy is supported in a neighborhood of D that is disjoint from a neighborhood of G.

When G has a nontrivial normal bundle with Euler class n, then $|D \cap G| = n$ and so the ambient isotopy taking $N(\alpha_0)$ to $N(\alpha_1)$ does not fix G. This isotopy is the composition of the standard one followed by n full twists along $N(\alpha_1)$. Since $\pi_1(SO(3)) = \mathbb{Z}_2$, the twisting can be isotopically undone when n is even. When n is odd the twisting can be isotoped to a single full twist. If the tubes in B are unknotted and unlinked, then they can be isotoped so that $g(B) = f_1(B)$ where g differs from f_1 by a Dehn twist along each of the tubes. □

3. The light bulb theorem for $S^2 \times S^2$

Theorem 3.1. *If* R *is a 2-sphere in* $S^2 \times S^2$*, homologous to* $x_0 \times S^2$*, transverse to* $S^2 \times y_0$*,* $R \cap (S^2 \times y_0) = (x_0, y_0)$ *and coincides with* $x_0 \times S^2$ *near* (x_0, y_0)*, then* R *is smoothly isotopic to* $x_0 \times S^2$ *via an isotopy fixing a neighborhood of* $S^2 \times y_0$ *pointwise.*

Definition 3.2. A *light bulb* in $S^2 \times S^2$ is a smooth 2-sphere transverse to a $S^2 \times y_0$ and intersects $S^2 \times y_0$ in a single point. View $S^2 \times S^2$ as a quotient of $S^2 \times (S^1 \times [-\infty, \infty])$ where each $x \times S^1 \times -\infty$ and $x \times S^1 \times \infty$ are identified with points and y_0 is identified with $(z_0, 0) \in S^1 \times [-\infty, \infty]$. We say that the light bulb R is *vertical* if it is transverse to each

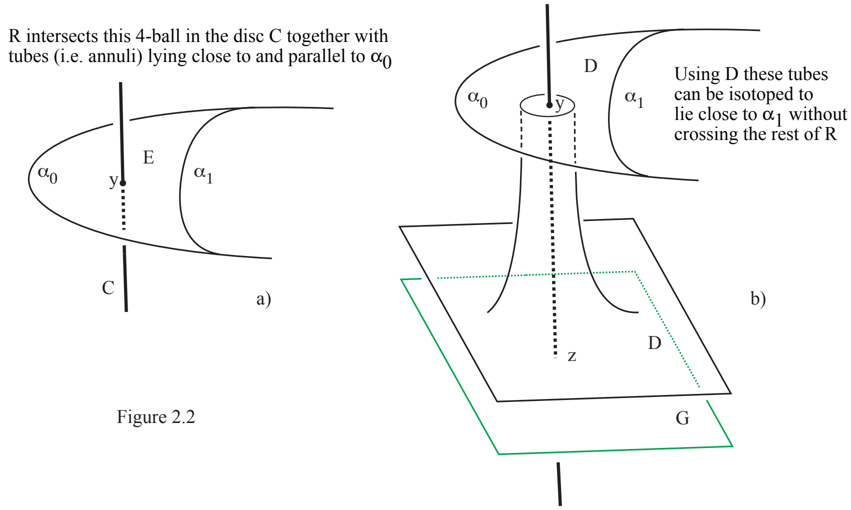


Figure 2.2

$S^2 \times S^1 \times u$, for $u \in [-\infty \times \infty]$. Let G^{std} denote the sphere $S^2 \times z_0 \times 0$ and R^{std} denote the sphere $x_0 \times S^1 \times [-\infty, \infty] \subset S^2 \times S^2$.

To prove the light bulb theorem it suffices to assume that R and R^{std} coincide in some neighborhood U of $(x_0, z_0, 0)$.

Step 1. The light bulb R is isotopic to a vertical light bulb by an isotopy fixing a neighborhood of G^{std} pointwise.

Step 1A. We can assume that R coincides with R^{std} within $S^2 \times (z_0 - \epsilon, z_0 + \epsilon) \times [-\infty, \infty] \cup (S^2 \times S^1 \times [-\infty, 10)) \cup (S^2 \times S^1 \times (10, \infty]$.

Proof. This follows from the fact that R intersects a neighborhood of $S^2 \times z_0 \times 0$ as does R^{std} and a small regular neighborhood of G^{std} is naturally ambiently isotopic to $S^2 \times (z_0 - \epsilon, z_0 + \epsilon) \times [-\infty, \infty] \cup (S^2 \times S^1 \times [-\infty, 10)) \cup S^2 \times S^1 \times (10, \infty])$. □

From now on we will take U to be the neighborhood of $(S^2, z_0, 0)$ given in the statement of Step 1A. Note that U is the complement of $S^2 \times [z_0 + \epsilon, z_0 - \epsilon] \times [-10, 10]$, where $S^1 = [z_0 + \epsilon, z_0 - \epsilon] \cup (z_0 - \epsilon, z_0 + \epsilon)$.

Step 1B. Via an isotopy fixing $R \cap U$, R can be isotoped to be transverse to each $S^2 \times S^1 \times u$ except for $u = -9, -6, 6, 9$. As u increases, p local minima (with respect to u) appear at $u = -9$, p saddles appear at $u = -6$, $R \cap S^2 \times S^1 \times u$ is connected for $u \in (-6, 6)$, q saddles appear when $u = 6$ and q local maxima appear when $u = 9$.

Proof. This is the analogy of the normal form of [KSS] in our setting, stated in the smooth category. Here is a brief outline. In the usual manner R can be isotoped so that it is transverse to each $S^2 \times S^1 \times u$ except for $u = -9, 0, 9$ where local minima, saddles, local maxima respectively appear. Up to smoothing of corners, the local minima (resp. maxima) correspond to the appearance of discs and the saddles correspond to the appearance of bands. After further isotopy we can assume that the bands are disjoint from each other, so for δ small, $R \cap S^2 \times S^1 \times \delta$ is the result of doing band sums to $R \cap S^2 \times S^1 \times -\delta$.

If p (resp. q) is the number of local minima (resp. maxima), then since $\chi(R) = 2$ the total number of saddles is $p + q$. Since R is connected there exist p bands such that the result of only doing band sums along these bands yields a connected curve. Push these bands to $S^2 \times S^1 \times -6$ and push the remaining bands to level to $S^2 \times S^1 \times 6$. $\square$

In what follows we let C_u denote the *core curve* i.e. the component of $R \cap S^2 \times S^1 \times u$ which is transverse to $S^2 \times z_0 \times u$, $u \neq -6, 6$. Define $C_{-6} = \lim_{t \to -6_-} C_t$. We abuse terminology by calling a *core curve* such a curve C without specifying u. After band sliding we can assume that all the bands at $u = -6$ have one end that attaches to the core curve.

In summary, up to smoothing corners, we can assume that $R \cap S^2 \times S^1 \times [-10, -5]$ appears as follows. For $u \in [-10, -9)$, $R \cap S^2 \times S^1 \times u$ is the standard core curve $x_0 \times S^1 \times u$. At $u = -9$, discs $D_1, \cdots, D_p$ appear. Let $c_1, \cdots, c_p$ denote their boundary curves. The surface $R \cap S^2 \times S^1 \times (-9, -6)$ is the product $(C \cup c_1 \cup \cdots \cup c_p) \times (-9, -6)$. Here we again abuse notation by denoting a c_i without specifying its u level. At $u = -6, p$ bands $b_1, \cdots, b_p$ appear where b_i connects C and c_i. Again $R \cap S^2 \times S^1 \times (-6, -5]$ is a product where each u section is parallel to $R \cap S^2 \times S^1 \times -6$ with the relative interiors of the bands removed.

By a vertical isotopy push the bands $b_2, \cdots, b_p$ up to level -5 and the disc D_1 to level -8. Let $\pi : S^2 \times S^1 \times [-\infty, \infty] \to S^2 \times S^1$ be the projection. To complete the proof of Step 1 we will show that after isotopy $\pi(b_1) \cap \pi(D_1) \subset \pi(\partial D_1)$. It follows that b_1 can be pushed to level -8 and its critical point can be cancelled with the one corresponding to D_1. Step 1 then follows by induction and the usual turning upside down argument to cancel the saddles at $u = 6$ with the maxima at $u = 9$.

Step 1C. There exist pairwise disjoint discs $E_1, \cdots, E_p \subset S^2 \times S^1 \times -6$ spanning $c_1, \cdots c_p$ such that for all i, $\pi(\text{int}(E_i)) \cap \pi(b_1) = \emptyset$ and $E_i \cap C \cup U = \emptyset$.

Proof. To start with, for $i = 1, \cdots, p$, let $E_i = D_i$. A given E_i projects to one intersecting $\pi(b_1)$ in finitely many interior arcs. View b_1 as a band starting at c_1 and sequentially hitting the various E_i's before attaching to C. Again we abuse notation by suppressing the fact that we should be talking about projections. Starting at the last intersection of b_1 with an E_i, sequentially isotope the E_i's to remove arcs of intersection at the cost of creating two points of intersection of an E_i with C. This type of argument was used in [Gi] and [Go]. Next by following C but avoiding the arc $b_1 \cap C$ tube off these intersections with parallel copies of $S^2 \times z_0$ to obtain the desired set of discs which we still call $E_1, \cdots E_p$. See Figure 3.1 $\square$

Remark 3.3. For the purposes of visualization one can ambiently isotope R via level preserving isotopy supported in $S^2 \times S^1 \times [-9.5, -5.5]$ so that the discs E_i become small and round and b_1 becomes a straight band connecting C and c_1, which is disjoint from the interior of the E_i's. Furthermore, up to rounding corners, possibly complicated discs $D_2, \cdots, D_p$ appear at level -9, a possibly complicated disc D_1 appears at level -8 and vertical annuli $c_1 \times [-8, -6], c_2 \times [-9, -6], \cdots, c_p \times [-9, -6]$ emanate from the ∂D_i's. At level -6 R appears as in the first sentence.

Step 1D. We can assume that $\pi(D_1) \cap \pi(E_i) = \emptyset$ for $i > 1$.

Proof. Let π_{-8} denote the projection of $S^2 \times S^1 \times -6$ to $S^2 \times S^1 \times -8$ fixing the first two factors. By construction $\partial D_1 = c_1$ and D_1 is disjoint from $C \cup U$ as well as the c_i's for $i > 1$.

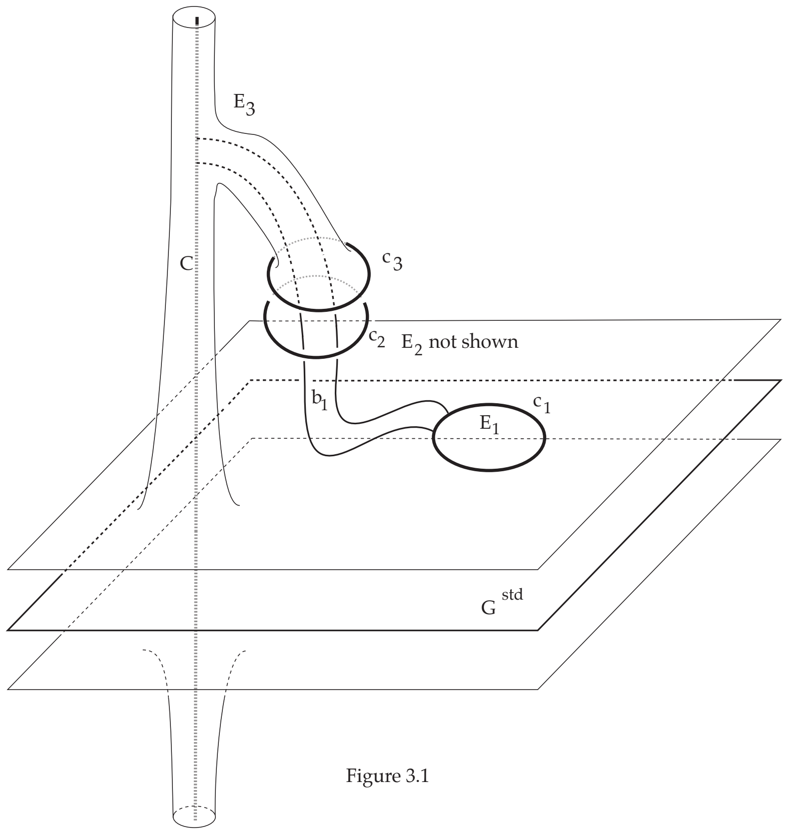


Figure 3.1

Let F_i denote $\pi_{-8}(E_i)$. We show how to isotope D_1 off F_2. Let $\alpha_0 \subset S^2 \times S^1 \times -8$ be an arc transverse to $\mathrm{int}(F_2)$. Then $N(\alpha_0) = D^2 \times I \times I$ where $D^2 \times I \times 1/2 \subset S^2 \times S^1 \times -8$. Next isotope D_1 so that $N(F_2) \cap D_1 \subset (1/2D^2) \times I \times 1/2$ and $D_1 \cap (D^2 \times I \times 1/2)$ is of the form $L \times I \times 1/2$, where L is a union of circles. Here we identify α_0 with $t_0 \times I \times 1/2$ where $t_0 \notin 1/2D^2$. Let $\alpha_1 \subset S^2 \times S^1 \times -8$ be an arc disjoint from $R \cup F_2 \cup \cdots \cup F_p$ whose ends coincide with those of α_0, such that $\alpha_0 \cup \alpha_1$ bounds a pinched disc $E \subset S^2 \times S^1 \times -8$ that only intersects R in a point of ∂F_2. Using the fact that $R \setminus D_1$ is connected and intersects G^{std} transversely once it follows from the Light Bulb Lemma 2.3 that R is isotopic to the surface obtained by isotoping $D_1 \cap N(\alpha_0)$ across E into $N(\alpha_1)$ via an isotopy supported in $D_1 \cap \mathrm{int}(N(\alpha_0))$. This isotopy moves D_1 off of F_2 without introducing new intersections with other F_i's. See Figure 3.2. Step 1D now follows by induction. $\square$

To summarize the situation at the moment: At level -9 discs $D_2, \cdots, D_p$ appear, at level -8 disc D_1 appears, vertical annuli $c_1 \times [-8,-6], c_2 \times [-9,-6], \cdots, c_p \times [-9,-6]$ emanate from the ∂D_i's, a band connects the core to $c_1 \times -6$ disjoint from $\mathrm{int}(E_1)$ and by Step 1D, for $i > 1$, $\pi(E_i) \cap \pi(D_1) = \emptyset$

We can therefore isotope $D_2, \cdots, D_p$ and hence $\cup_{i>1} c_i \times [-9,-6]$ so that $c_2, \cdots, c_p$ are *far away* from D_1, E_1 and b_1. This means that $\pi(c_2), \cdots, \pi(c_p)$ lie in a 3-ball $B \subset S^2 \times S^1$ that intersects C in a connected unknotted arc and B is disjoint from $\pi(D_1), \pi(E_1)$ and $\pi(b_1)$.

Step 1E. Cancel the critical points corresponding to D_1 and b_1 without introducing new ones, thereby completing Step 1.

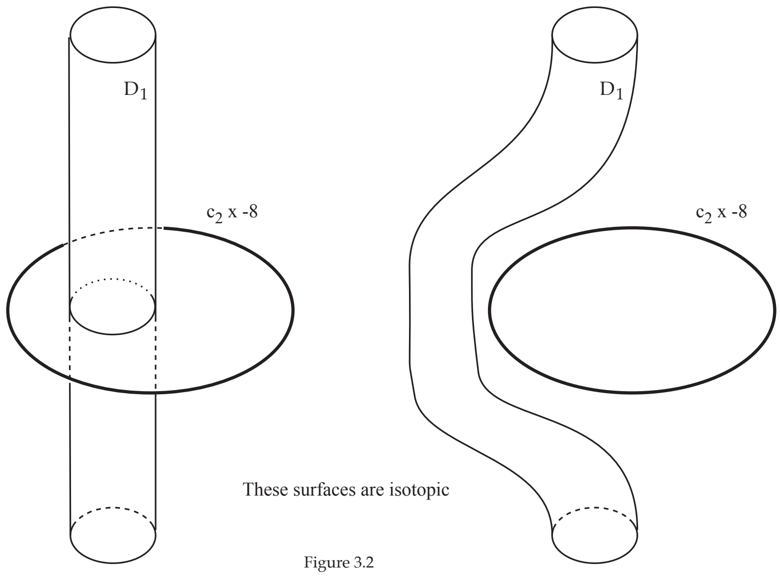


Figure 3.2

Proof. Note that $(S^2 \times S^1 \times -6) \setminus (C \cup B \cup S^2 \times z_0 \times -6)$ is diffeomorpic to $\mathbb{R}^3$. Therefore, the discs $\pi(D_1)$ and $\pi(E_1)$ are isotopic rel c_1 via an isotopy disjoint from $\pi(C \cup B) \cup (S^2 \times z_0)$ since spanning discs for the unknot in $\mathbb{R}^3$ are unique up to isotopy. After the corresponding isotopy of D_1, supported in $S^2 \times S^1 \times -8$ it follows from Remark 3.3 that $\pi(b_1) \cap \pi(D_1) \subset \pi(\partial D_1)$. Therefore, b_1 can be pushed down to level -8, hence the critical points corresponding to b_1 and D_1 can be cancelled. $\square$.

Remark 3.4. The content of Steps 1B-1E is that excess critical points of a boundary standard $I \times I \subset (S^2 \times I) \times [-10, 10]$ can be cancelled rel ∂. Cancelling critical points of submanifolds of product manifolds, to the extent possible is a long sought after goal. E.g. under suitable hypothesis in higher dimensions results have been obtained in [Ro], [Sh] and [Pe], the latter in the topological category. In dimension-4, Scharlemann [Sc] showed that a smooth 2-sphere in $\mathbb{R}^4$ with at most four critical points is smoothly isotopically standard.

Remark 3.5. Abby Thompson pointed out that the above arguments work for higher genus surfaces to eliminate critical points of index 0 and 2. So if genus$(R) = g$, then R can be isotoped so that $2g$ bands appear at $u = -6$ and there are no other critical levels.

From the point of view of $S^2 \times -6$ these bands can be twisted and linked. See §9.

Step 2. A vertical light bulb R homologous to R^{std} that agrees with R^{std} near G^{std} is isotopic to R^{std} via an isotopy fixing a neighborhood of G^{std} pointwise.

Remarks 3.6. 1) It is easy to construct a vertical lightbulb homologous to $[R^{std}] + n[S^2 \times z_0 \times 0]$ by first starting with R^{std}, removing a neighborhood of $(x_0, z_1, 0)$ and replacing it by one that *sweeps across* $S^2 \times z_1$ n times while $u \in (-\epsilon, \epsilon)$ where $z_1 \neq z_0$.

Proof The proof of Step 1 shows that we can assume that R coincides with R^{std} away from $S^2 \times S^1 \times [-10, 10]$ and coincides with R^{std} near $S^2 \times z_0 \times [-\infty, \infty]$. Thus R is standard outside a submanifold W of the form $S^2 \times [0, 1] \times [-10, 10]$ and within W corresponds to a smooth path of embedded smooth paths $\rho_t : D^1 \to S^2 \times I$ for $t \in [-10, 10]$, where $\rho_{-10}(D^1) = \rho_{10}(D^1) = (x_0, I)$ and ρ_t is fixed near the endpoints of D^1. By identifying D^1 with (x_0, I) we can assume that $\rho_{-10} = \rho_{10} = \text{id}$. Note that R^{std} corresponds to the identity path.

By the covering isotopy theorem, ρ_t, extends to a path $\phi_t \in \text{Diff}(S^2 \times I, \text{rel}(\partial S^2 \times I))$ with $\phi_{-10} = \text{id}$. We first show that such a path can be chosen so that $\phi_{10} = \text{id}$. By uniqueness of regular neighborhoods we can first assume that restricted to some D^2 neighborhood of x_0, in polar coordinates, $\phi_{10}(r, \theta, s) = (r, \theta + h(s)2\pi, s)$ for some $h : [0, 1] \to \mathbb{R}$ with $h(0) = 0$. Since $[R] = [R^{std}] \in H_2(S^2 \times S^2)$ it follows that $h(1) = 0$ and hence after further isotopy, that $\phi_{10}|D^2 \times I = \text{id}$. Since $S^2 \times I \setminus (\text{int}(D^2) \times I) = B^3$, we can assume that $\phi_{10} = \text{id}$, by [Ce2] or [Ha].

Thus ϕ_t is a closed loop in $\text{Diff}(S^2 \times I, \text{rel}\, \partial(S^2 \times I))$ which by Hatcher [Ha] is homotopically trivial since $\pi_1(\Omega(O(3)) = \pi_2(O(3)) = \pi_2(\mathbb{R}(P^3)) = 0$. Here we are using formulation (8) (see the appendix of [Ha]) of Hatcher's theorem which asserts that $\text{Diff}(D^1 \times S^2 \,\text{rel}\, \partial)$ is homotopy equivalent to $\Omega(O(3))$. Restricting this homotopy to ρ_t gives the desired isotopy of ρ_t to id. □

Theorem 3.7. *Let D denote $x_0 \times D^2 \subset S^2 \times D^2$. A properly embedded disc $D_0 \subset S^2 \times D^2$ that coincides with D near ∂D is isotopic to D rel ∂D if and only if it is homologous to D in $H_2(S^2 \times D^2, \partial D)$.*

Proof. Homologous is certainly a necessary condition. Let $M = S^2 \times D^2 \cup d(S^2 \times D^2) = S^2 \times S^2$ be obtained by doubling $S^2 \times D^2$ with $d(S^2 \times D^2)$ denoting the other $S^2 \times D^2$. This $d(S^2 \times D^2)$ can be viewed as a regular neighborhood $N(G)$ of $G = d(S^2 \times 0)$. Let R denote the sphere $D_0 \cup d(x_0 \times D^2)$ and R^{std} denote $D \cup d(x_0 \times D^2)$. G is a transverse sphere to the homologous spheres R and R^{std}. By Theorem 1.9 there is an isotopy of M fixing $N(G)$ pointwise taking R to R^{std}. Restricting to $S^2 \times D^2$ yields the desired isotopy. □

Conjecture 3.8. *The space of light bulbs is not simply connected.*

4. Regular homotopy of embedded spheres in 4-manifolds

The main result of this section is essentially Theorem D of [Sm1]. First recall [Sp] that a smooth *immersion* $f : M \to N$ is a smooth map of maximal rank at each $x \in M$. A smooth *regular homotopy* $F : M \times I \to N$ is a smooth map such that each F_t is an immersion.

Theorem 4.1. *(Smale (1957)) Two smooth embedded spheres in an orientable 4-manifold are regularly homotopic if and only if they are homotopic.*

The next well-known proposition follows by considering a generic regular homotopy, e.g. see [FQ] P.19. First we recall the basic definitions, [FQ].

Definition 4.2. Let S be a smooth immersed self transverse surface in the smooth 4-manifold Z. A *finger move* is the operation of regularly homotoping a disc in S along an embedded arc to create a pair of new transverse self intersections. A *Whitney move* is a regular homotopy supported in a neighborhood of a framed Whitney disc to eliminate a pair of oppositely signed self intersections. By an *isotopy* of S we mean a regular homotopy through self transverse surfaces. In particular, no new self intersections are either created or cancelled.

Proposition 4.3. *Let A and B be smooth embedded surfaces in the smooth 4-manifold Z. If A is regularly homotopic to B, then up to isotopy, the regular homotopy can be expressed as the composition of finitely many* finger moves, Whitney moves *and* isotopies. □

Remark 4.4. It is well known by the usual reordering argument that if A is regularly homotopic to B, then the regular homotopy can be chosen to consist of finger moves followed by Whitney moves in addition to intermediate isotopies.

5. Shadowing regular homotopies by tubed surfaces

In this section we show that if $f_0 : A_0 \to M$ is an embedding of a smooth surface with embedded transverse sphere G into a smooth 4-manifold M and $f_t : A_0 \to M$ is a generic *regular homotopy* supported away from G, then f_t can be *shadowed* by a *tubed surface*. Roughly speaking there is a smooth *isotopy* $g_t : A_0 \to M$ with $g_0(A_0) = f_0(A_0)$ such that when f_t is self transverse, $g_t(A_0)$ is approximately $f_t(A_0)$ with tubes connecting to copies of G. As $f_t(A_0)$ undergoes a finger or Whitney move, $g_t(A_0)$ changes by isotopy that adds tubes that connect to copies of G or modifies existing tubes. In particular, if $f_1(A_0) = A_1$ is an embedding, then A_0 is isotopic to A_1 with tubes connecting to copies of G. Sections §6 - §8 are about how to eliminate or normalize these tubes by isotopy. In this paper there are different types of tubes. Tubes may follow arcs in a surface as in the proof of Lemma 2.3, but they may follows paths in M away from A_1.

This section is motivated by the following lemma.

Lemma 5.1. *Let R be a connected embedded smooth surface in the smooth 4-manifold M. If R has an embedded transverse sphere G and R_1 is obtained from R by a finger move, then R is isotopic to a surface R_2 consisting of R_1 tubed to two parallel copies of G.*

Proof. Let $z = R \cap G$, $x, y \in R \setminus G$ and κ a path from y to x with $\text{int}(\kappa) \cap (R \cup G) = \emptyset$. Let $\sigma \subset R \setminus y$ be an embedded path from x to z and R_t a regular homotopy starting at R corresponding to a finger move along κ that is supported very close to κ. Then R_1 has two points of self intersection and a Whitney move along the obvious Whitney disc E undoes the finger move, up to isotopy supported in the neighborhood of the finger.

Let D be a small disc transverse to R with $D \cap R = x$. Let T be the disc disjoint from R which is the union of a tube that starts at ∂D and follows σ and then attaches to a parallel copy G' of $G \setminus \text{int}(N(z))$ disjoint from G. The tube should lie very close to σ and G' should be very close to G. Let R_2 be the embedded surface obtained from R_1 by removing two discs D', D'' and attaching parallel copies of T as in Figure 5.1. The key observation is that R_2 is isotopic to R via an isotopy supported near the union of T and the 4-ball B which is the support of the finger move. This isotopy is essentially the following one. If $F = I \times I \times 0 \cup I \times I \times \epsilon \cup 0 \times I \times [0, \epsilon]$ and H the closed complement of F in $\partial(I \times I \times [0, \epsilon])$, then up to rounding corners, F is isotopic to H via an isotopy supported in $I \times I \times [0, \epsilon]$. In our setting the product between the two copies of T corresponds to $I \times I \times [0, \epsilon]$ and, after a small isotopy, $N(E) \cap R_2$ corresponds to F. Now isotope F to the corresponding H. The resulting surface is readily seen to be isotopic to R via an isotopy supported in B. □

Remarks 5.2. Reversing the above isotopy gives an isotopy R'_t from R to R_2. Both R_t and R'_t start at R, the former ends at an immersed surface and the latter at an embedded one R'_1 obtained from the immersed one by removing some discs near the double points and tubing

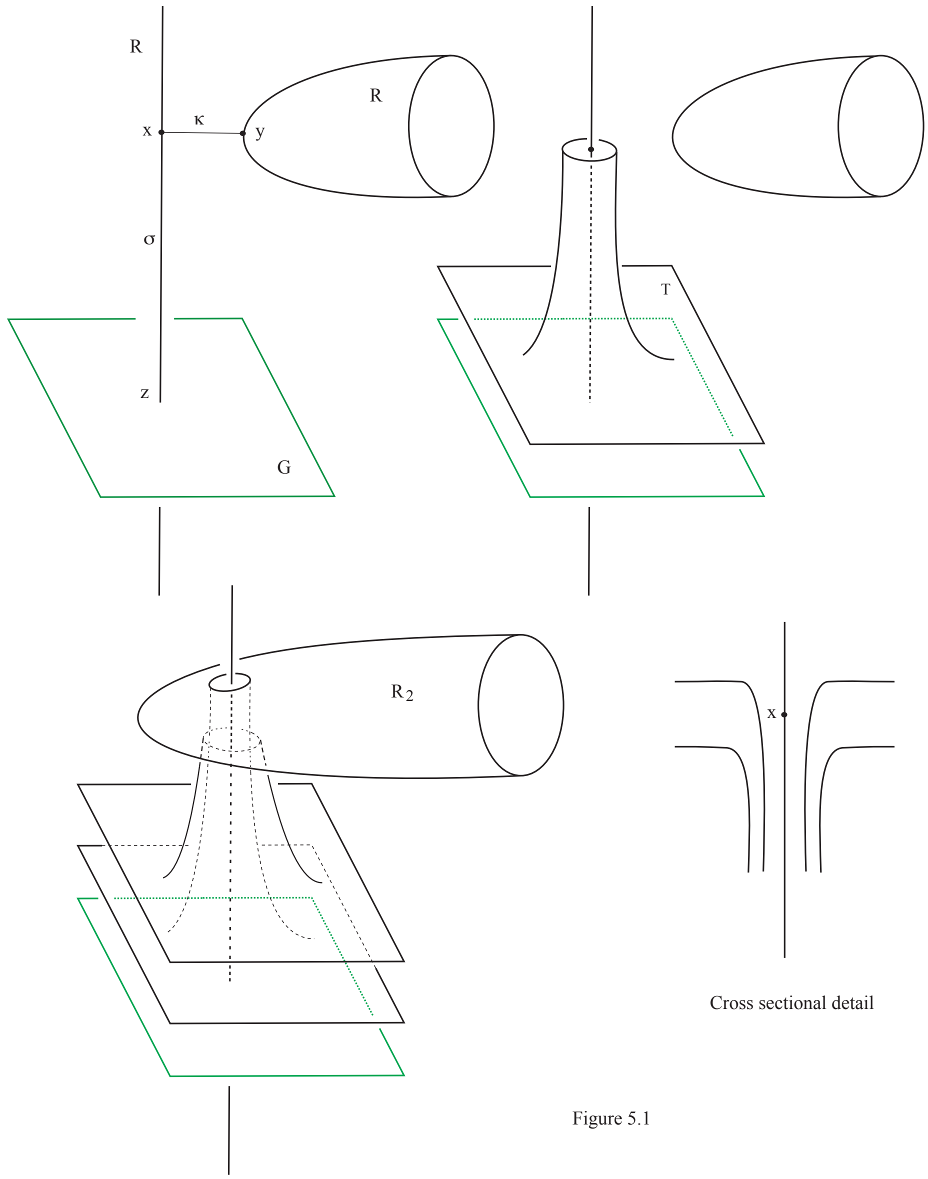


Figure 5.1

off with copies of G. In what follows we will have a regular homotopy R_t that starts at an embedded surface R. We will construct an isotopy R_t' starting at R such that, except at times near finger and Whitney moves, the immersed surface will be closely approximated by an embedded surface R_t' not counting a multitude of discs, which are essentially tubes that connect to parallel copies of G. For reasons of organization, we define a *tubed surface* as the data needed to define such a surface with tubes, rather than the surface itself. Unlike the model case of Lemma 5.1, the tubes may follow intersecting paths in the surface or embedded paths away from the surface. Note that if a tube T_1 (resp. T_2) follows path $\kappa_1 \subset R_t'$ (resp. $\kappa_2 \subset R_t'$) and $\kappa_1 \cap \kappa_2 \neq \emptyset$, then $T_1 \cap T_2 = \emptyset$, provided that one of the tubes is closer to R_t' than the other.

If R_1 is embedded, then the resulting tubed surface R_1', (or more precisely what we call it's *realization*) will look like R_1 together with a jumble of tubes connecting to parallel copies of

G. Most of this paper is about how to clean up the mess, i.e. to show that under appropriate hypotheses this surface is actually isotopic to R_1 or some normal form. The rest of the chapter is organized as follows. We first give a definition of *tubed surface*, second describe how to construct an embedded surface from tubed surface data, third describe moves on the data that give isotopic surfaces, fourth use all this to show how to shadow a regular homotopy and fifth describe tubed surfaces that are in *normal form*.

Remarks 5.3. Given an embedded path $\phi \subset M$ with $\phi \cap R = \partial\phi$ and whose ends are orthogonal to the possibly disconnected surface R, then we can tube $R \setminus \text{int}(N(\partial\phi))$ by attaching an annulus T that follows ϕ. Up to isotopy supported in $N(\phi)$ there are two ways to do this if R is oriented and the new surface maintains the orientation induced from R. That is because $\pi_1(SO(3)) = \mathbb{Z}_2$ and we can insist by construction that for $t \in I, T \cap B^3 \times t$ is an equator where $N(\phi) = B^3 \times I$. While the resulting two surfaces are equivalent as unparametrized surfaces, we keep the distinction since we may want to tube up other things such as links. In general there are four ways up to isotopy. Thus we have the next definition which enables us to keep track of how to attach tubes to pairs of circles and more generally to attach pairs of tubes to pairs of Hopf bands.

Definition 5.4. A *framed embedded path* is a smooth embedded path $\tau(t), t \in [0,1]$ in the 4-manifold M with a framing $\mathcal{F}(t) = (v_1(t), v_2(t), v_3(t))$ of its normal bundle. Let $(C(0), x(0))$ consist of a smooth embedded circle $C(0)$, with base point $x(0)$, lying in the normal disc to τ through $\tau(0)$ that is spanned by the vectors $(v_1(0), v_2(0))$ with $x(0)$ lying in direction $v_1(0)$. Define $(C(t), x(t))$ a smoothly varying family having similar properties for each $t \in [0,1]$. Call the annulus $(C(t), x(t))$, $t \in [0,1]$ the *cylinder connecting* $C(0)$ and $C(1)$. It should be thought of as lying very close to τ.

As a warmup to the following long definition, the reader is encouraged to look at Figures 5.8 and 5.9 which show how different types of tubes can arise in the course of an isotopy and thus the need for an elaborate definition. See Figures 5.2 - 5.5 which show some of the data in the definition of a *tubed surface* and exhibit realizations of this data

Definition 5.5. A *tubed surface* $\mathcal{A}$ in the 4-manifold M consists of

i) a generic self transverse immersion $f : A_0 \to M$, where A_0 is a closed surface based at z_0 with A_1 denoting $f(A_0)$. The preimages $(x_1, y_1), \cdots, (x_n, y_n)$ of the double points are pairwise ordered. A_0 is called the *underlying* surface and A_1 the *associated* surface to $\mathcal{A}$.

ii) An embedded transverse 2-sphere G to A_1, with $A_1 \cap G = z = f(z_0)$.

iii) For each $i = 1, \cdots, n$, an immersed path $\sigma_i \subset A_0$ from x_i to z_0. See Figure 5.2 for views in A_0 and A_1.

iv) immersed paths $\alpha_1, \cdots, \alpha_r$ in A_0 with both endpoints at z_0 and for each $i = 1, \cdots, r$, pairs of points (p_i, q_i) with $p_i \in \alpha_i$ and $q_i \in A_0$ and a framed embedded path, $\tau_i \subset M$ from $f(p_i)$ to $f(q_i)$ with $\text{int}(\tau_i) \cap (G \cup A_1) = \emptyset$. See Figure 5.4.

v) pairs of immersed paths $(\beta_1, \gamma_1), \cdots, (\beta_s, \gamma_s)$ in A_0 where β_i goes from z_0 to b_i and γ_i goes from g_i to z_0 and framed embedded paths $\lambda_i \subset M$ from $f(b_i)$ to $f(g_i)$ with $\text{int}(\lambda_i) \cap (G \cup A_1) = \emptyset$. See Figure 5.5.

Curves of the form $\sigma_i, \alpha_j, \beta_k, \gamma_l$ are called *tube guide* curves and the τ_p and λ_q curves are called *framed tube guide* curves. The union of all of these curves is the *tube guide locus*. The $\sigma_i, \alpha_j, \beta_k, \gamma_l$ curves are required to be self transverse and transverse to each other with interiors disjoint from the z_0, x_i, q_j, b_k, g_l points and disjoint from the p_j points except where required in iv). At points of intersection and self intersection of these curves, except z_0, one curve is determined to be *above* or *below* the other curve. The various points $z_0, x_i, y_j, p_k, q_l, b_m, g_n$ are all distinct.

The curves τ_i and λ_j are pairwise disjoint, disjoint from G and intersect A_1 only at their endpoints. Additional conditions on the framings of the τ_i and λ_j curves will be given in Definition 5.7. They dictate the placement of the $C(0)$ and $C(1)$ curves. This ends the definition of a tubed surface.

Remark 5.6. The data i) - iii) are what's needed to create a tubed surface arising from a finger move as in Lemma 5.1. Data iv) and v) are needed to describe tubed surfaces arising from Whitney moves. Crossings of tube guide curves may occur in preparation for Whitney moves and in the process of transforming pairs of double tubes to pairs of single tubes in §6.

We now show how a tubed surface gives rise to an embedded surface.

Construction 5.7. Associated to the tubed surface $\mathcal{A}$ construct an embedded surface A, called the *realization* of $\mathcal{A}$ as follows. For each i, remove from A_1 the image of a small D^2 neighborhood of y_i. Attach to $f(\partial D^2)$ a disc $D(\sigma_i)$ consisting of a tube $T(\alpha_i)$ that follows $f(\sigma_i)$ and connects to a slightly pushed off copy of $G \setminus \text{int}(N(z))$. See Figure 5.2. These copies of G are sufficiently close to G so that the closed product region between each of them and G is disjoint from all the framed tube guide curves. If $u \in \sigma_i \cap \sigma_j$, $u \neq z_0$, and σ_i lies above σ_j at u, then near $f(u)$, construct $T(\alpha_j)$ to lie closer to A_1 than does $T(\alpha_i)$. With abuse of notation, this allows for the case $i = j$. See Figure 5.3. Let $\hat{A}$ be the embedded surface thus far constructed.

In similar manner associated to the path α_i is a 2-sphere $P(\alpha_i)$ with $P(\alpha_i) \cap A_1 = \emptyset$, consisting of two parallel copies of $G \setminus \text{int}(N(z))$ connected by a tube $T(\alpha_i)$ that follows the path $f(\alpha_i)$. Again these copies of G are sufficiently close to G that the closed product region between each of them and G is disjoint from all the framed tube guide curves. Next attach a tube $T(\tau_i)$ following the framed embedded path τ_i from $C(0) = P(\alpha_i) \cap \partial N(\tau_i)$ to $C(1) = \hat{A} \cap \partial N(f(q_i))$. Note that the previous condition implies that $T(\tau_i)$ attaches to the tube part of $P(\alpha_i)$. Here we assume that τ_i approaches $f(p_i)$ normally to $\hat{A}$ and is parametrized by $[-1/4, 1]$ and framed so that restricting to $[0, 1]$, $C(0)$ (resp. $C(1)$) is in the plane spanned by $v_1(0)$ and $v_2(0)$ (resp. $v_1(1)$ and $v_2(1)$) as in Definition 5.4. This assumption is the condition on the framing on τ_i that is required but not explicitly stated at the end of Definition 5.5. The tube $T(\tau_i)$ is called a *single tube*. See Figure 5.4. Let $\hat{A}'$ the embedded surface constructed at this stage.

Next for each i, construct discs $D(\beta_i)$ and $D(\gamma_i)$ consisting of slightly pushed off copies of $G \setminus \text{int}(N(z))$ tubed very close to and respectively along $f(\beta_i)$ and $f(\gamma_i)$ with boundary lying in discs normal to $\hat{A}'$ at $f(b_i)$ and $f(g_i)$. Roughly speaking the rest of the construction of A from $\hat{A}'$ proceeds as follows. Appropriately sized 4-balls $N(f(b_i))$ and $N(f(g_i))$ have the property that $\partial N(f(b_i))) \cap (\hat{A}' \cup D(\beta_i))$ and $\partial N(f(g_i)) \cap (\hat{A}' \cup D(\gamma_i))$ are Hopf links. Connect these links by tubes that parallel λ_i such that $\partial N(f(g_i)) \cap \hat{A}'$ (resp. $\partial N(f(b_i)) \cap \hat{A}'$) connects to $\partial N(f(b_i)) \cap D(\beta_i)$ (resp. $\partial N(f(g_i)) \cap D(\gamma_i)$). See Figure 5.5.

More precisely, delete $\text{int}(N(f(b_i)))$ from $D(\beta_i)$ and continue to call $D(\beta_i)$ the disc that remains. Next remove $\text{int}(1/2N(f(b_i)))$ from $\hat{A}'$ and let $C(0) = \partial((1/2N(f(b_i))) \cap \hat{A}')$. Also

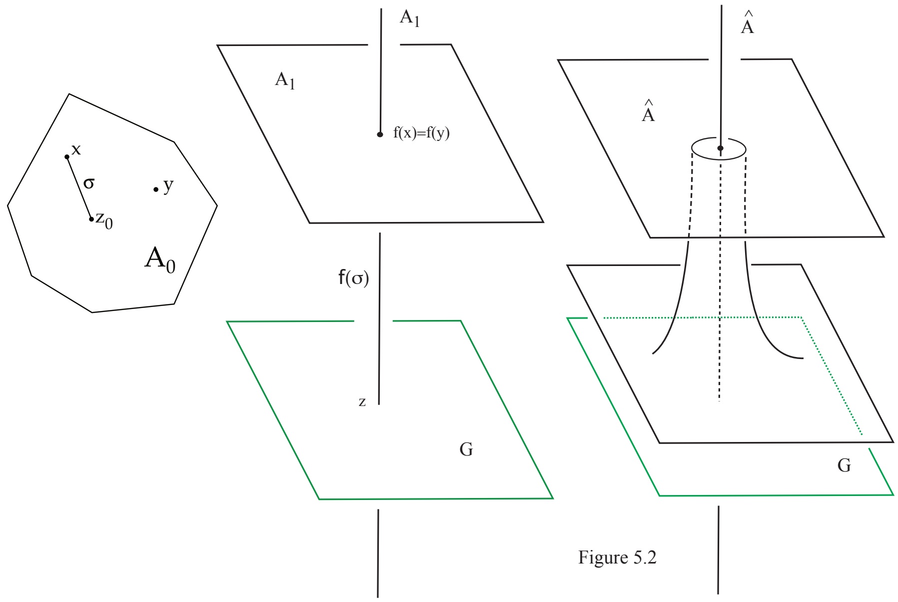
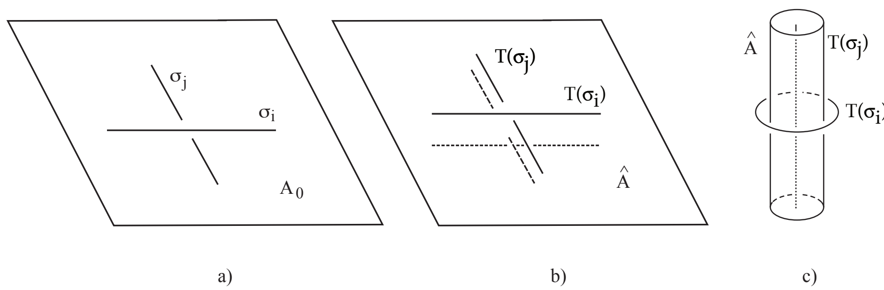


Figure 5.2



Creating tubes associated to σ_i and σ_j. Figures b) and c) are local 3-dimensional slices of $\hat{A}$.

Figure 5.3

remove $\text{int}(N(f(g_i)))$ from $\hat{A}'$ and $\text{int}(1/2N(f(g_i)))$ from $D(\gamma_i)$ and call $D(\gamma_i)$ what remains. Now bend $D(\gamma_i)$ near $\partial D(\gamma_i)$ in the direction of λ_i and then let $C(1) = \partial D(\gamma_i)$. See Figure 5.5 b). Suppressing the epsilonics, $1/2N(f(g_i))$ and $1/2N(f(b_i))$ are *half radius* 4-balls about $f(b_i)$ and $f(g_i)$ and with respect to that scale, the tube of $D(\beta_i)$ is very close to $f(\beta_i)$ and the tube of $D(\gamma_i)$ is very close to $f(\gamma_i)$ together with a short segment of λ_i.

We assume that λ_i approaches $\hat{A}'$ in geodesic arcs near $f(g_i)$ and $f(b_i)$, and in the two 3-planes spanned by these arcs and $\hat{A}'$, it approaches $\hat{A}'$ orthogonally. We assume that λ_i is parametrized by $[0,1]$ and framed so that $C(0)$ is in the plane spanned by $v_1(0)$ and $v_2(0)$ as in Definition 5.4. Also $x(0) = C(0) \cap \beta_i$ and $v_1(0)$ points towards $x(0)$. Again $C(1) = \partial D(\gamma_i)$ with $x(1)$ the point indicated in Figure 5.5 b) and assume that $C(1)$ lies in the plane spanned by $v_1(1)$ and $v_2(1)$ with $x(1)$ lying in the arc spanned by $v_1(1)$.

As in Definition 5.4 use λ_i to build a tube connecting $C(0)$ and $C(1)$. Using a tube that follows the path $x(t), t \in [0,1]$ connect $\partial D(\beta_i)$ to $\partial N(g_i) \cap \hat{A}'$ as in Figure 5.5 c). The union of these two tubes, a Hopf link $\times I$, is called a *double tube*. This completes the construction of the realization A of $\mathcal{A}$.

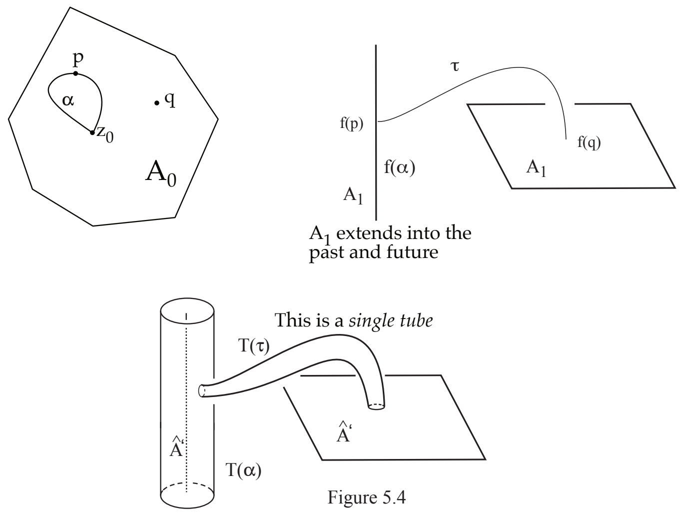


Figure 5.4

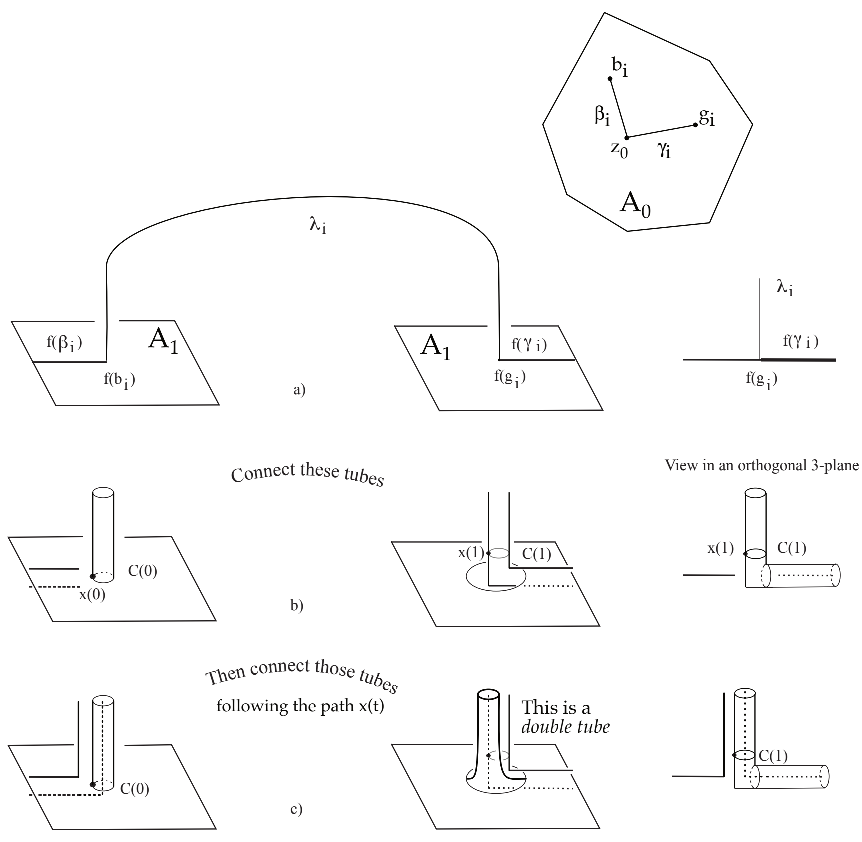


Figure 5.5

Remark 5.8. The single and double tubes do not *link* with other parts of the realization. In particular, except for the spots where they attach to and/or near the associated surface A_1, the single tubes and double tubes stay a uniformly bounded distance away from A_1 and the transverse sphere G. Further, the tubes following the $\sigma, \alpha, \beta, \gamma$ curves stay within this distance to the associated surface and the parallel copies of G also stay within this uniform distance to G.

We now describe operations on a tubed surface $\mathcal{A}$ that correspond to isotopies of the realizations.

Definition 5.9. We enumerate *tube sliding moves* on a tubed surface $\mathcal{A}$ corresponding to redefining the location and crossing information of tube guide curves in the underlying surface A_0.

i) Type 2), 3) Reidemeister moves on tube guide curves. See Figure 5.6 a).

ii) Reordering tube guide curves near z_0. See Figure 5.6 b).

iii) Sliding a tube guide curve across a double point. See Figure 5.6 c). There are two cases depending on whether or not the tube guide κ lies in the sheet through y_i or the sheet through x_i. In the former case we require that $\kappa \neq \sigma_i$.

iv) Sliding across a tube guide curve κ across a double tube. See Figure 5.6 d). Here $\kappa \neq \gamma_i$ (resp. β_i) can slide over b_i (resp. g_i).

v) Sliding tube guide curves across a single tube. Here a tube guide curve $\kappa \neq \alpha_i$ can slide across q_i and over p_i. Any tube guide curve can slide under p_i. See Figure 5.6 e).

Remark 5.10. Sliding σ_i across y_i has the self referential problem analogous to a handle sliding over itself. Similarly for sliding β_i (resp. γ_i, resp. α_i) across g_i (resp. b_i, resp. q_i).

Lemma 5.11. *If $\mathcal{A}$ and $\mathcal{A}'$ are tubed surfaces that differ by tube sliding, then their realizations A and A' are isotopic.*

Proof. We consider the effect on the realization of $\mathcal{A}$ by the various tube sliding moves. The under/over crossing data in A_0 reflects how close one tube is to A_1 compared with the another. As the Reidemeister 2), 3) moves respect this closeness it follows that they induce an isotopy from A to A'.

Next we consider reordering near z_0. Since G has a trivial normal bundle, there is an S^1 worth of directions that it can push off itself. These directions correspond to the directions that the image of tube guide curve $f(\kappa) \subset A_1$ can approach z. We can assume that the various parallel copies of $G \setminus \text{int}(N(z))$ are equidistant from G at angle that of the angle of approach of the various $f(\kappa)$'s. Let $D \subset A_1$ denote a disc which is a small neighborhood of the bigon that defines the reordering. Let K_i denote the disc consisting of a parallel copy G_i of $G \setminus \text{int}(N(z))$ together with its tube that follows the arc $f(\kappa_i) \cap D$. If κ_j is above κ_k as in Figure 5.6 b) and K_j' and K_k' are the discs resulting from the reordering, then there is an isotopy of A to A' supported on K_k where G_k is first pushed radially close to G, then rotated to the angle defined by $f(\kappa_k')$ and then pushed out. Here κ_k' is the reordered κ_k.

The proof that A is isotopic to A' for the operations of Figures 5.6 c i), d), e i) are all the same. Here we are sliding a tube in A that parallels a curve $f(\kappa) \subset A$ across a disc. In the case of Figure 5.6 c i) (resp. Figure 5.6 d)) that disc includes the disc $D(\sigma_i)$ (resp. $D(\gamma)$ or $D(\beta)$). In the case of Figure e i) that disc includes $P(\alpha_i)$ minus an open disc.

The proof that A is isotopic to A' for the operations of Figures 5.6 c ii) and e iii) are the same and are local operations. Here we are sliding a tube paralleling a curve $f(\kappa) \subset A$ across a small disc. The slid tube is very close to A, closer than other tubes in the vicinity.

The proof that A is isotopic to A' for the operation of Figure 5.6 e ii) requires Lemma 2.3 for we want Remark 5.8 to continue to hold. Let $T(\tau_i)$ (resp. $T(\kappa)$) denote the tube in

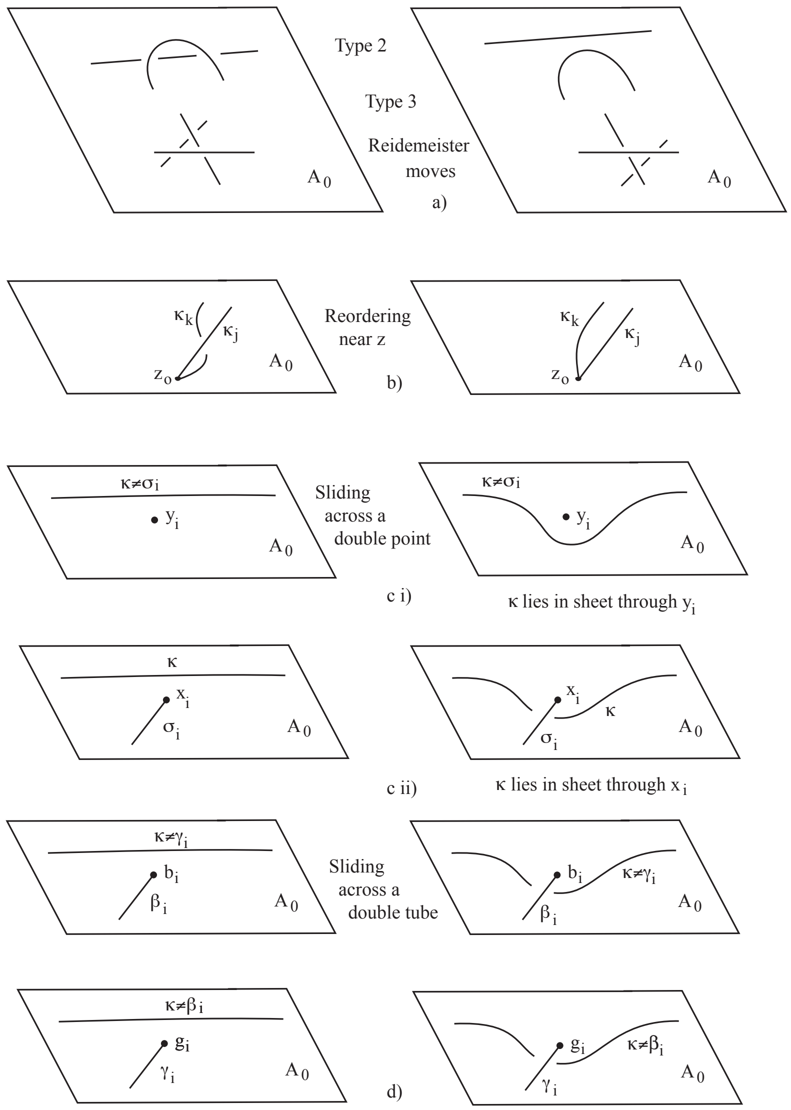


Figure 5.6

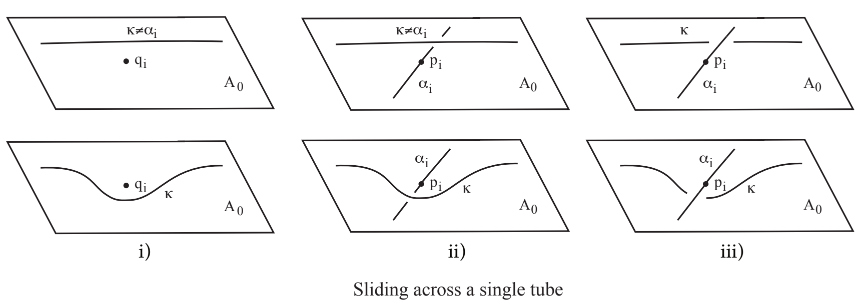


Figure 5.6 e)

A that parallels τ_i (resp. $\kappa \neq \alpha_i$). Sliding $T(\kappa)$ over $f(p)$ entangles $T(\tau_i)$ with $T(\kappa)$. The entangled $T(\tau_i)$ is the tube being isotoped using Lemma 2.3. Lemma 2.3 requires that there be a path, in the notation of that lemma, from y to z. This requires that $\kappa \neq \alpha_i$. $\square$

We now define operations on tubed surfaces corresponding to finger and Whitney moves.

Definition/Construction 5.12. Let A_1 be the associated surface to the tubed surface $\mathcal{A}$. To a generic finger move from A_1 to A_1' with corresponding regular homotopy from f to f' we obtain a new tubed surface $\mathcal{A}'$ said to be obtained from $\mathcal{A}$ by a *finger move*. By *generic* we mean that the support of the homotopy is away from all the framed tube guide curves and images of tube guide curves of $\mathcal{A}$. $\mathcal{A}'$ will have the same underlying surface A_0 as $\mathcal{A}$ and A_1' will be its associated surface. Let (x_1, y_1) and (x_2, y_2) be the new pairs of f' preimages of double points in A_0, where both $f'(x_1)$ and $f'(x_2)$ (resp. $f'(y_1)$ and $f'(y_2)$) lie in the same local sheet of A_1'. Let σ_1 and σ_2 be parallel embedded paths from x_1 and x_2 to z_0 transverse to the existing tube guide paths. The tube guide locus of $\mathcal{A}'$ consists of that of $\mathcal{A}$ together with σ_1 and σ_2 where all crossings of these σ_i's with pre-existing tube guide curves are under crossings. See Figure 5.7.

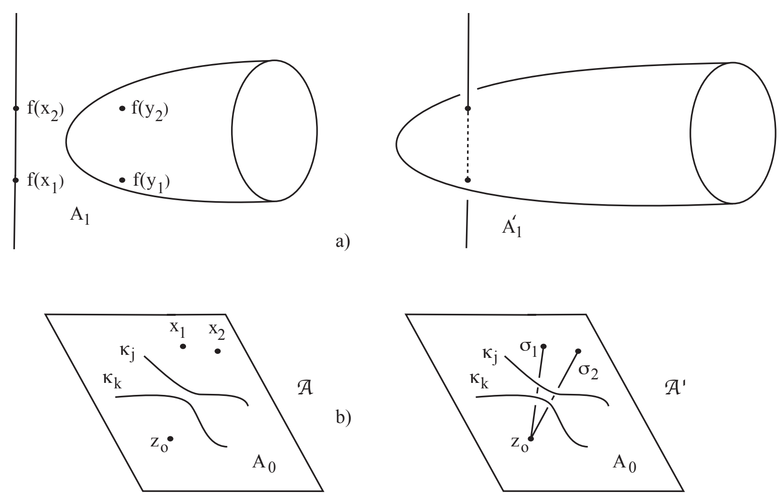


Figure 5.7

Remark 5.13. There is flexibility in the construction of $\mathcal{A}'$ from $\mathcal{A}$ in the choice of which pair of points are called x_i points and in the choice of the σ_i paths.

Lemma 5.14. *If $\mathcal{A}'$ is obtained from $\mathcal{A}$ by a finger move, then their associated realizations are isotopic.*

Proof. This lemma is a restatement of Lemma 5.1 in our setting. The proof is the same after recognizing that the support of the isotopy in the target is contained in the support of the finger move together with a small neighborhood of the product region between the discs $D(\sigma_1)$ and $D(\sigma_2)$, notation as in Definition 5.7. $\square$

Definition/Construction 5.15. Let A_1 be the associated surface to the tubed surface $\mathcal{A}$. A Whitney move from A_1 to A_1' corresponding to the regular homotopy from f to f' with Whitney disc w is said to be *tube locus free* if $\text{int}(w)$ is disjoint from the framed tube guide curves of $\mathcal{A}$ and ∂w intersects the images of tube guide curves of $\mathcal{A}$ only at double points of A_1. Let $(x_1, y_1), (x_2, y_2)$ denote the pairs of points in A_0 corresponding to these double points with notation consistent with that of Definition 5.5 and let E_1, E_2 denote the local discs involved in the Whitney move. We say that the Whitney move is *uncrossed* if both $f(x_1)$ and $f(x_2)$ lie in the same E_i and *crossed* otherwise. If w is an uncrossed Whitney disc, then we obtain the tubed surface $\mathcal{A}'$ as indicated in Figure 5.8 and if w is crossed, then $\mathcal{A}'$ is obtained as in Figure 5.9. Accordingly $\mathcal{A}'$ is said to be obtained from $\mathcal{A}$ by an *uncrossed* or *crossed* Whitney move.

Remark 5.16. An uncrossed Whitney moves gives rise to a single tube while a crossed Whitney move gives rise to a double tube. In the former case two σ curves become an α curve. In the latter case, the σ curves become β and γ curves.

Lemma 5.17. *If $\mathcal{A}'$ is obtained from $\mathcal{A}$ by a tube locus free Whitney move, then its realization is isotopic to that of $\mathcal{A}$.* □

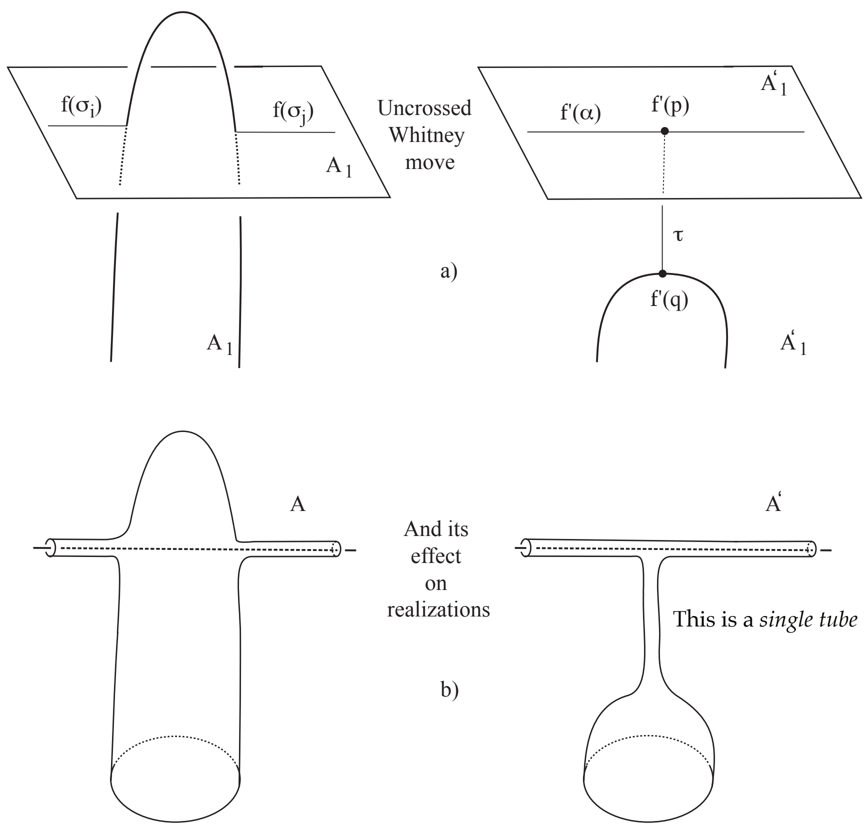


Figure 5.8

Definition 5.18. We define an *elementary tubed surface isotopy*, or *elementary isotopy* for short, on the tubed surface $\mathcal{A}$ as any of the following operations on $\mathcal{A}$.

a) The defining data changes smoothly without combinatorial change. In particular, at no time are there new tangencies or new intersections among the various objects.

b) tube sliding moves.

c) finger moves

d) tube locus free Whitney moves

Lemma 5.19. *If $\mathcal{A}$ and $\mathcal{A}'$ are tubed surfaces that differ by an elementary isotopy, then their realizations are isotopic.*

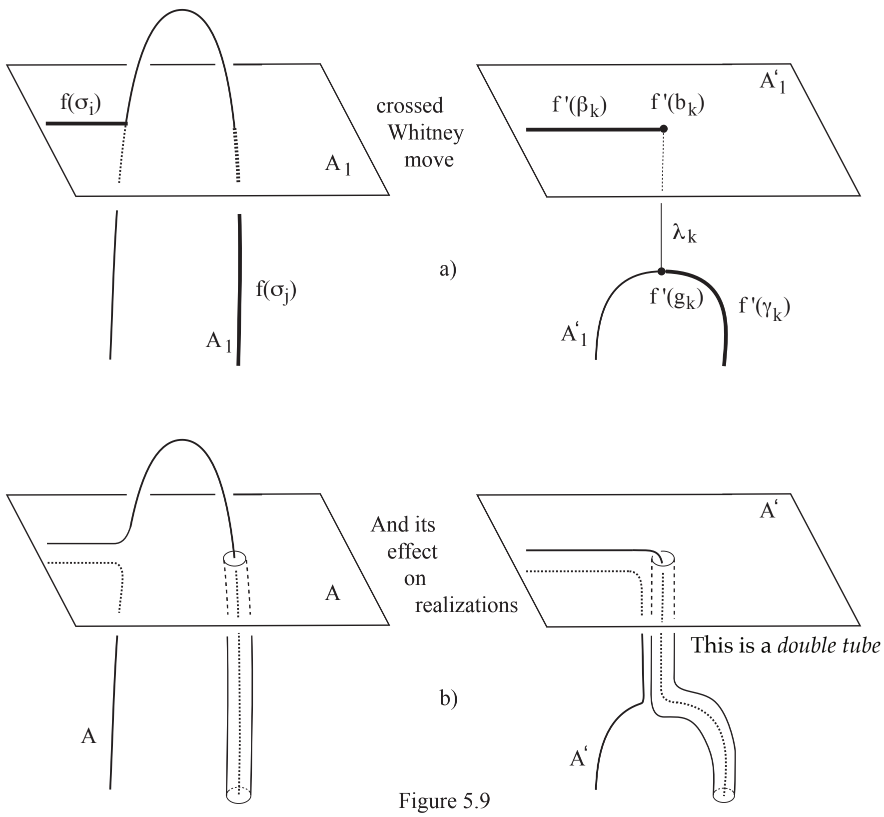


Figure 5.9

Proof. a) Smooth changes in the defining data induce smooth changes in the realizations.
b) This is Lemma 5.11.
c) This is Lemma 5.1.
d) This is illustrated in Figures 5.8 and 5.9. □

Definition 5.20. Let R_0 be an immersed surface in the smooth 4-manifold M with embedded transverse sphere G. Let $f_t : R \to M^4$ be a generic regular homotopy supported away from G which is a self-transverse immersion except at times $\{t_i\}$ where $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ and $i \notin \{0, m\}$. Let $R_0 = f_0(R), R_1, \cdots, R_m = f_1(R)$ be such that for $i = 1, \cdots, m-1$, R_i is a surface $f_s(R)$ for some $s \in (t_i, t_{i+1})$. We say that the regular homotopy f_t is *shadowed by tubed surfaces* if there exists a sequence $\mathcal{R}_0, \mathcal{R}_1, \cdots, \mathcal{R}_m$ of tubed surfaces such that for all i, R_i is the associated surface to $\mathcal{R}_i$ and for $i \neq m$, $\mathcal{R}_{i+1}$ is obtained from $\mathcal{R}_i$ by elementary isotopies. The tubed surfaces $\mathcal{R}_0, \mathcal{R}_1, \cdots, \mathcal{R}_m$ are called *f_t-shadow tubed surfaces.*

Theorem 5.21. *If $f_t : A_0 \to M$ is a generic regular homotopy with $f_0(A_0)$ an embedded surface, M a smooth 4-manifold, G a transverse embedded sphere to $f_0(A_0)$ and f_t is supported away from G, then f_t is shadowed by tubed surfaces.*

Proof. Define the tubed surface $\mathcal{A}_0$ as follows. A_0 is the underlying surface, $f_0 : A_0 \to M$ is the self transverse immersion with $f_0(A_0) = A_1^0$ the associated surface. The tube guide locus $= \emptyset$. Let $0 < t_1 < \cdots < t_{m-1} < 1$ be the non self transverse times of f_t. If there are no singular times in $[s, s']$ and a tubed surface $\mathcal{A}_s$ has been constructed whose associated surface $A_1^s = f_s(A_0)$, then an elementary isotopy of type a) transforms it to one with $A_1^{s'} = f_{s'}(A_0)$.

Thus we need only show how to shadow regular homotopies near non self transverse times. Now each non self transverse time corresponds to either a finger or Whitney move. Since $f_0(A_0)$ is embedded, t_1 is the time of a finger move. The shadowing of the initial finger move is given in the proof of Lemma 5.14. More generally, that lemma shows how to shadow any finger move. By induction we assume that the conclusion holds through time t, where $t_{k-1} < t < t_k$ and t_k is the time of a Whitney move.

Let $\mathcal{A}_{k-1}$ denote the tubed surface with underlying surface $A_1^{k-1} = f_t(A_0)$. By Lemma 5.19 we can assume that t is a time just before the Whitney move. Let w be a Whitney disc for the Whitney move. Being 2-dimensional we can assume that w is disjoint from the framed tube guide curves to $\mathcal{A}_{k-1}$. Suppose that w cancels the f_t images of the points $\{u, v, u', v'\} \subset A_0$, where $f_t(u) = f_t(v)$ and $f_t(u') = f_t(v')$. Here u and u' (resp. v and v') are the endpoints of disjoint arcs ϕ and ϕ' in A_0 which map to ∂w under f_t and $(u, v) = (x_i, y_i)$ and $(u', v') = (x_j, y_j)$, notation as in Definition 5.5. By switching ϕ and ϕ' and/or i and j if necessary, we can assume that the first equality is that of an ordered pair and the second is setwise.

Next we show that after tube sliding moves w becomes a tube locus free Whitney disc, i.e. no tube guide curve crosses $\operatorname{int}\phi$ or $\operatorname{int}\phi'$. Now $x_i \in \partial\phi$, so all the tube guide crossings with $\operatorname{int}(\phi)$ can be eliminated by sliding across the double point x_i. If $x_j \in \partial\phi'$, then we can clear $\operatorname{int}(\phi')$ of tube guide curves in a similar manner. If $\partial\phi' = (y_i, y_j)$ and the tube guide κ crosses $\operatorname{int}(\phi')$ we clear it from ϕ' as follows. Since $i \neq j$, it follows that $\kappa \neq \sigma_p$ for some $p \in \{i, j\}$. Apply a sequence of Reidemeister 2) moves supported in a small neighborhood of ϕ' to make κ adjacent to y_p and then slide it across the double point y_p.

Since w is now a tube locus free Whitney disc we can shadow the Whitney move by an uncrossed (if $u' = x_j$) or crossed (if $u' = y_j$) Whitney move. Thus $\mathcal{A}_k$ is obtained from $\mathcal{A}_{k-1}$ by a sequence of tube sliding moves and a tube locus free Whitney move. □

Remarks 5.22. i) If the regular homotopy f_t is of the form finger moves followed by Whitney moves, then the tubed surface following the finger moves can be chosen so that the curves σ_i are embedded and pairwise disjoint away from z_0.

ii) If $f_1(A_0)$ is embedded, then the final tubed surface has no σ_i curves.

iii) There is no restriction on the surface A_0. Below and in the next section we require that A_0 be a 2-sphere.

The rest of this section is relevant for a 4-manifolds M whose fundamental group has 2-torsion. See Figure 5.10.

Definition 5.23. We say that the tubed surface $\mathcal{A}$ is in *normal form* if in addition to the conditions of Definition 5.5 we have

a) The immersion $f : A_0 \to M$ is an embedding with associated surface $A_1 = f(A_0)$ and A_0 is a 2-sphere.

b) There are no α curves.

c) The paths $\beta_1, \gamma_1, \beta_2, \gamma_2, \cdots, \beta_n, \gamma_n$ are embedded and cyclicly arrayed in A_0 about the common endpoint z_0.

d) The framed embedded paths $\lambda_1, \lambda_2, \cdots, \lambda_n$ represent distinct nontrivial 2-torsion elements of $\pi_1(M)$.

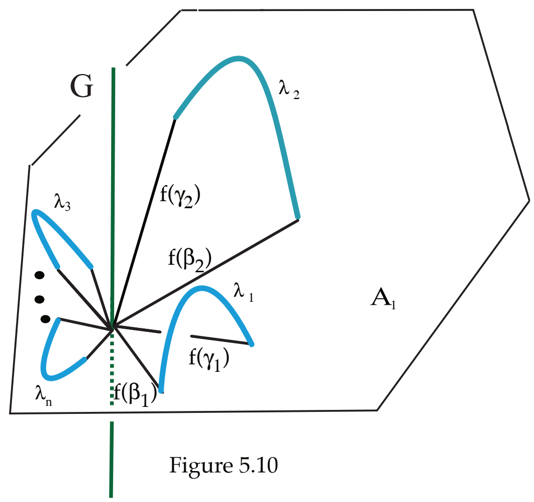


Figure 5.10

We say that the surface A is in *normal form* with respect to the embedded surface A_1 *representing* elements $[\lambda_1], \cdots, [\lambda_n]$ if A is the realization of $\mathcal{A}$ with data as in a)-d). We say two normal forms are *equivalent* if they represent the same set of elements.

The following result is immediate by our realization construction.

Proposition 5.24. *If A_1 is an embedded 2-sphere with transverse sphere G and given any finite set of distinct nontrivial 2-torsion elements of $\pi_1(M)$, then there exists an embedded 2-sphere A in normal form with respect to A_1 representing these elements of $\pi_1(M)$.* □

Lemma 5.25. *The isotopy class of the realization is independent of the cyclic ordering of the (β_i, γ_i)'s arrayed about $z_0 \in A_0$.*

Proof. Using the tube sliding operations any two adjacent pairs $(\beta_i, \gamma_i), (\beta_{i+1}, \gamma_{i+1})$ can be permuted. □

Remark 5.26. In §8 we will show that the isotopy class is also independent of the choice of framings on the framed tube guide curves. Thus equivalent normal form surfaces are isotopic, as stated in Theorem 1.3.

6. From double tubes to single tubes

In what follows $\mathcal{A}$ is a tubed surface in M with realization A whose associated surface A_1 is an embedded 2-sphere. In this section we show that $\mathcal{A}$ can be transformed to $\mathcal{A}'$ with isotopic realizations without changing A_1 so that, if $\pi_1(M)$ has no 2-torsion, then $\mathcal{A}'$ has at most one double tube, which is homotopically inessential. If $\pi_1(M)$ has 2-torsion, then $\mathcal{A}'$ may additionally have finitely many double tubes representing distinct elements of order two. This is done by appropriately replacing pairs of double tubes with pairs of single tubes.

Lemma 6.1. *Suppose that R_0 and R_1 are spheres with common transverse sphere G in the 4-manifold M. If R_0 and R_1 coincide near G and are homotopic in M, then they are homotopic via a homotopy whose support in M is disjoint from G, in particular the homotopy is basepoint preserving.*

Proof. Let $H : S^2 \times I \to M$ be a homotopy from R_0 to R_1. Let $E \subset R_0$ be a disc containing $R_0 \cap G$ that coincides with a disc of R_1. After an initial homotopy making H transverse to G we see that $H^{-1}(G)$ is 1-manifold with exactly one component ψ that goes from $S^2 \times 0$ to $S^2 \times 1$. By the 3-dimensional light bulb theorem ψ is unknotted, so after reparametrization of $S^2 \times I$, we can assume that H is a basepoint preserving homotopy. Next, replace H by another, also called H, such that there exists a disc $D \subset S^2$ where $H_t|D$ is independent of t with $H_0(D) = 1/2E$.

To complete the proof it suffices to show that if a sphere R_3 in $M \setminus G$ is homotopically trivial in M it is homotopically trivial in $M \setminus G$. The relevant R_3 is obtained from $R_0 \cup R_1$ by deleting $1/2E$ from each and gluing along the resulting boundaries. Let $\tilde{M}$ denote the universal covering of M and $\tilde{G}$ the preimage of G. By Lemma 2.2 the universal cover of $M \setminus G$ is $\tilde{M} \setminus \tilde{G}$, hence if $\tilde{R}_3$ denotes a lift of R_3 to $\tilde{M}$, it suffices to show that $\tilde{R}_3$ is homologously trivial in $\tilde{M} \setminus \tilde{G}$. Let $Z \subset \tilde{M}$ be a chain transverse to $\tilde{G}$ with $\partial Z = \tilde{R}_3$. Since $H_1(\tilde{G}) = 0, [Z \cap \partial N(\tilde{G})] = [(Z \cap \tilde{G}) \times \partial D^2] = 0 \in H_2(\partial N(\tilde{G}))$ and hence there exists Z' with $\partial Z' = \tilde{R}_3$ and $Z' \cap \tilde{G} = \emptyset$. □

Remark 6.2. If $\mathcal{A}$ is a tubed surface whose associated surface A_1 is an embedded sphere, then we can view double tubes as representing elements of $\pi_1(M)$ as follows. First A_1 itself can be viewed as the basepoint for $\pi_1(M)$. Recall Lemma 2.2. Next a double tube corresponds to a path λ_i in M from some $b_i \in A_1$ to some $g_i \in A_1$. Thus the double tube gives rise to an element of $\pi_1(M)$.

Definition 6.3. Let $\mathcal{A}$ be a tubed surface with realization A. Let κ denote one of σ_i, β_j or γ_k and y the corresponding x_i, b_j or g_k. If we compress the tube in A that follows $f(\kappa)$ near $f(y)$ we obtain an immersed surface, one component of which is an embedded 2-sphere $P = P(\kappa)$, that is homotopic to the transverse sphere G. This sphere has an induced orientation that coincides with A away from the compressing disc. Define $\epsilon(P) = \epsilon(\kappa) = +1$ if $[P] = [G]$ and -1 otherwise. Here M, A and G are oriented so that $< A, G >= +1$.

Similarly compressing A near a point of τ_i gives rise to an embedded surface one component of which is an embedded 2-sphere $P(\alpha_i)$ isotopic to two oppositely oriented copies of G tubed together along α_i.

Lemma 6.4. *If $P = P(\kappa)$ is constructed as above and D is the compressing disc that splits off P and oriented to coincide with that of P, then $\epsilon(P) =< D', A >$ Here D' is D shrunk slightly to have boundary disjoint from A.* □

Lemma 6.5. *If $\mathcal{A}$ is a tubed surface, then for all i, $[P(\beta_i)] = -[P(\gamma_i)] = \pm[G] \in H_2(M)$ and for all j, $[P(\alpha_j)] = 0$.* □

Remark 6.6. Up to isotopy there are four framings on the framed embedded path λ_i, hence four ways of constructing a double tube from λ_i. See Figure 5.5 and Remark 5.3. Note that two give $1 = P(\beta_i) = -P(\gamma_i)$ while two give $-1 = P(\beta_i) = -P(\gamma_i)$.

Sign Convention: By switching β_i and γ_i, if necessary, we can assume that $\epsilon(\beta_i) = -1$ and $\epsilon(\gamma_i) = +1$. Orient β_i to point from z to b_i, λ_i to point from $f(b_i)$ to $f(g_i)$ and γ_i to point from g_i to z.

We next calculate $[A] \in \pi_2(M)$. Since A and G are 2-spheres, each distinct element of $\pi_1(M)$ gives rise to a distinct pair of geometrically dual spheres in $\tilde{M}$, the universal cover of M, that projects to the pair (A, G). Thus, $\pi_2(M) = \pi_2(\tilde{M}) = H_2(\tilde{M})$ which contains the group ring $\mathcal{H} = H_2(G)\pi_1(M)$ as a submodule. Since $\mathcal{A}$ is a tubed surface, $[A]$ lies in the coset $\mathcal{H} + [A_1]$. Since $\pi_1(A_1) = 0$, each λ_i determines a well defined element, $[\lambda_i] \in \pi_1(M)$. With the above conventions the triple $(\beta_i, \lambda_i, \gamma_i)$ gives rise to the element $[G][\lambda_i] - [G][\lambda_i]^{-1} \in \mathcal{H}$. On the other hand, each α_j, τ_j gives rise to the trivial element. We therefore have:

Lemma 6.7. $[A] = [A_1] + \sum_{i=1}^s [G][\lambda_i] - [G][\lambda_i]^{-1} \in \pi_2(M)$. □

Remarks 6.8. i) If A is homotopic to A_1, then $\sum_{i=1}^s [G][\lambda_i] - [G][\lambda_i]^{-1} = 0$. Therefore, if $[\lambda_i] = 1$ whenever $[\lambda_i]^2 = 1$ holds, then we can reorder the λ_i's so that $[\lambda_1] = [\lambda_2]^{-1}, \cdots, [\lambda_{2p-1}] = [\lambda_{2p}]^{-1}$ and $[\lambda_s] = 1$ if $s = 2p + 1$.

ii) In general we can reorder the λ_i's so that $[\lambda_1] = [\lambda_2]^{-1}, \cdots, [\lambda_{2p-1}] = [\lambda_{2p}]^{-1}$, and $[\lambda_{2p+1}], [\lambda_{2p+2}], \cdots, [\lambda_s]$ represent distinct 2-torsion elements of $\pi_1(M)$ with possibly $[\lambda_s] = 1$.

The following is the main result of this section. It's the crucial point in the proof of Theorem 1.2 where the no 2-torsion condition is used. It also essentially uses that A_1 is a 2-sphere.

Proposition 6.9. *Let $\mathcal{A}$ be a tubed surface in the 4-manifold M whose associated surface A_1 is an embedded sphere homotopic to the realization A of $\mathcal{A}$. Let G denote the transverse sphere to A_1. Then via an isotopy supported away from G, A is isotopic to the realization A' of a tubed surface $\mathcal{A}'$ with associated surface A_1 such that if $\pi_1(M)$ has no 2-torsion, then $\mathcal{A}'$ has at most one double tube, in which case the unique double tube is homotopically trivial. If $\pi_1(M)$ has 2-torsion, then $\mathcal{A}'$ may have $n \geq 0$ double tubes, each representing distinct 2-torsion elements of $\pi_1(M)$.*

Proof. By Remark 6.8 we can reorder the λ_i's so that $[\lambda_1] = [\lambda_2]^{-1}, \cdots, [\lambda_{2p-1}] = [\lambda_{2p}]^{-1}$ and $[\lambda_{2p+1}], [\lambda_{2p+2}] \cdots, [\lambda_s]$ are distinct 2-torsion elements of $\pi_1(M)$ with possibly $[\lambda_s] = 1$. Thus if $\pi_1(M)$ has no 2-torsion, then $s = 2p$ or $2p + 1$ in which case all but at most one of the double tubes are paired up. The remaining one, if it exists, is homotopically trivial..

We will show that an isotopy of A transforms $\mathcal{A}$ to $\mathcal{A}'$ with the same A_1 but with the double tubes λ_1, λ_2 eliminated. The proposition then follows by induction on the number of double tubes. To start with we consider another model for a double tube as shown in Figure 6.1.

Remarks 6.10. i) Fixing an orientation on M and G our sign conventions determine an orientation on A_1 and hence $A \cap A_1$ as well as the orientations on A near the ends of a double tube associated to λ_i. The orientation near g_i is induced from A_1 and the orientation on $T(\gamma_i)$ is determined from the fact that $\epsilon(\gamma_i) = 1$. Similarly, for the b_i end of the double tube, except that $\epsilon(\beta_i) = -1$. Therefore, by Remark 5.3, up to isotopy, there are two rather than four, ways of constructing a double tube associated to λ_i. Representatives are shown in Figures 6.1 a) and b).

ii) Figures 6.1 a) and b) each show the projection into the x, y, z plane of a neighborhood of a double tube associated to λ. This consists of tubes emanating from two discs D_b and D_g lying in A_1 that are respectively neighborhoods of $f(b)$ and $f(g)$, where λ connects $f(b)$ to $f(g)$. In the figure, D_b lies in the x, y plane and D_g lies in the x, t plane. Here D_g corresponds to the horizontal lines at the top of the subfigures. The shaded regions are projections of the tube from D_g into the x, y, z plane. The preimage of the interior of each shaded region

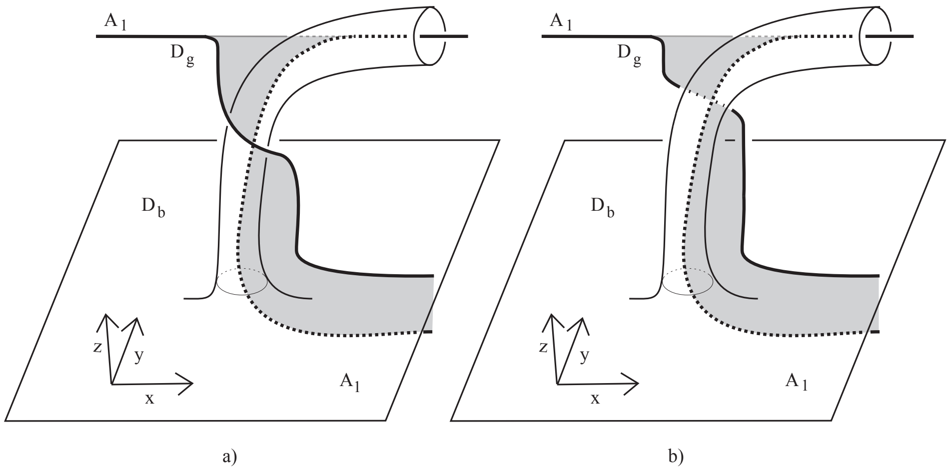


Figure 6.1

consists of two components, one in the future and one in the past. Except where it is twisted, this tube lies in the x, z, t plane. Its intersection with the x, y, z plane is the union of the thick solid and dashed lines.

Since A_1 has the transverse 2-sphere G it follows from Lemma 2.2 that the induced map $\pi_1(M\setminus(A_1 \cup G)) \to \pi_1(M)$ is an isomorphism. Since homotopy implies isotopy we can isotope λ_1 and λ_2 to be anti-parallel. I.e. there exists an embedded square D with opposite edges respectively on A_1 and λ_1 and λ_2. See Figure 6.2 a). Here E and F denote the components of $A_1 \cap N(D)$. Figure 6.2 b) shows how A *might* intersect $N(D)$. Figure 6.3 shows how to isotope the surface to effect a change of the framed embedded path corresponding to the non trivial element of $SO(3)$. Thus we can assume that A appears near λ_1 and λ_2 as depicted in Figure 6.2 b). Now $A \cap N(D)$ may fail to appear as in Figure 6.2 b) because the images of tube guide paths may cross the interior of $D \cap A_1$, however by doing tube sliding moves in a manner similar to those in the proof of Theorem 5.21 we can clear such curves from the neighborhood. Thus we can assume that $A \cap N(D)$ appears as in Figure 6.2 b).

Figure 6.4 shows an isotopy of A, supported in $N(D)$, that transforms a pair of double tubes as in Figure 6.2 b) into a pair of single tubes as shown in Figure 6.4 c). Call the *E-component* of $A \cap N(D)$ the one that intersects E and call the other the *F-component*. Again in Figure 6.4 the intersection of the E-component with the x, y, z-plane is drawn in thick, possibly dashed, lines. The isotopy from Figure 6.2 b) to Figure 6.4 a) is the composition of three isotopies, the first and third supported in the F-component and the second supported in the E component. The isotopy to Figure 6.4 b) is supported in the E component. The isotopy from Figure 6.4 b) to Figure 6.4 c) is as follows. Without changing the projection into the x, y, z plane, first push the tube emanating from E into the future and then isotope the tube emanating from F as indicated. $\square$

7. Crossing Changes

In this section we show that crossing changes involving distinct tube guide curves do not change the isotopy class of the realization of a tubed surface.

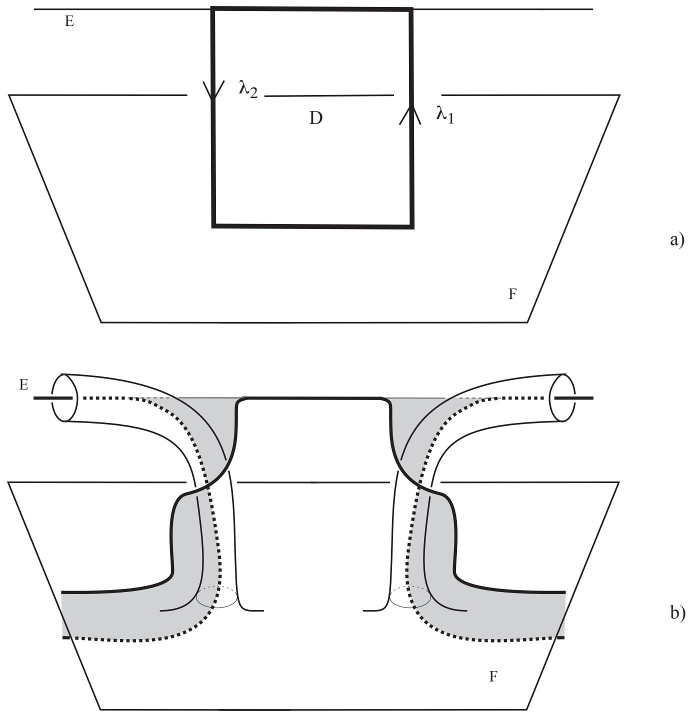


Figure 6.2

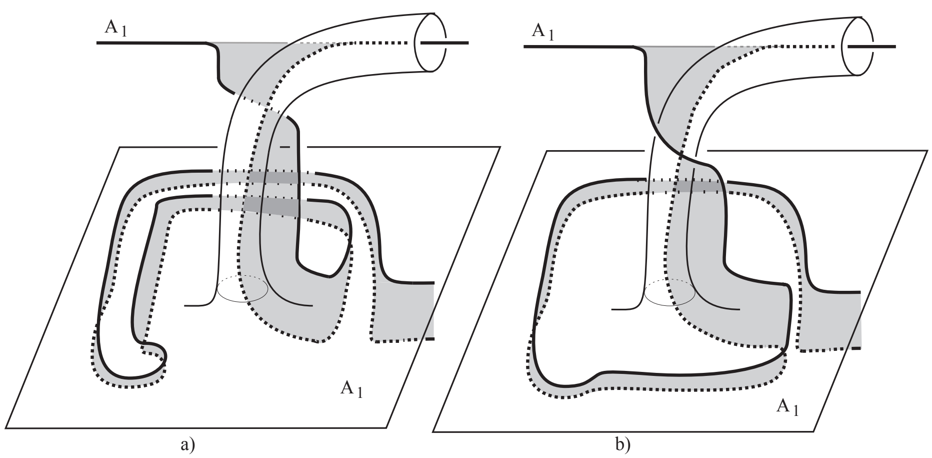


Figure 6.3

Lemma 7.1. *(Crossing Change Lemma) If the tubed surface $\mathcal{A}'$ is obtained from the tubed surface $\mathcal{A}$ by a crossing change involving either distinct tube guide paths or distinct components of $\alpha_i \setminus p_i$, then the corresponding realizations A' and A are isotopic via an isotopy supported away from the transverse surface G. See Figure 7.1 a).*

Proof. Since by Lemma 5.11 changing a tubed surface by type 2) and 3) Reidemeister moves does not change the isotopy class of the realization, it suffices to assume that the crossing is adjacent to z_0 as in Figure 7.1 b), e.g. see Figure 7.1 c). Let D be a small disc neighborhood of z_0 that contains the crossing and $D^2 = f(D)$. We can assume by Remark 5.8 that

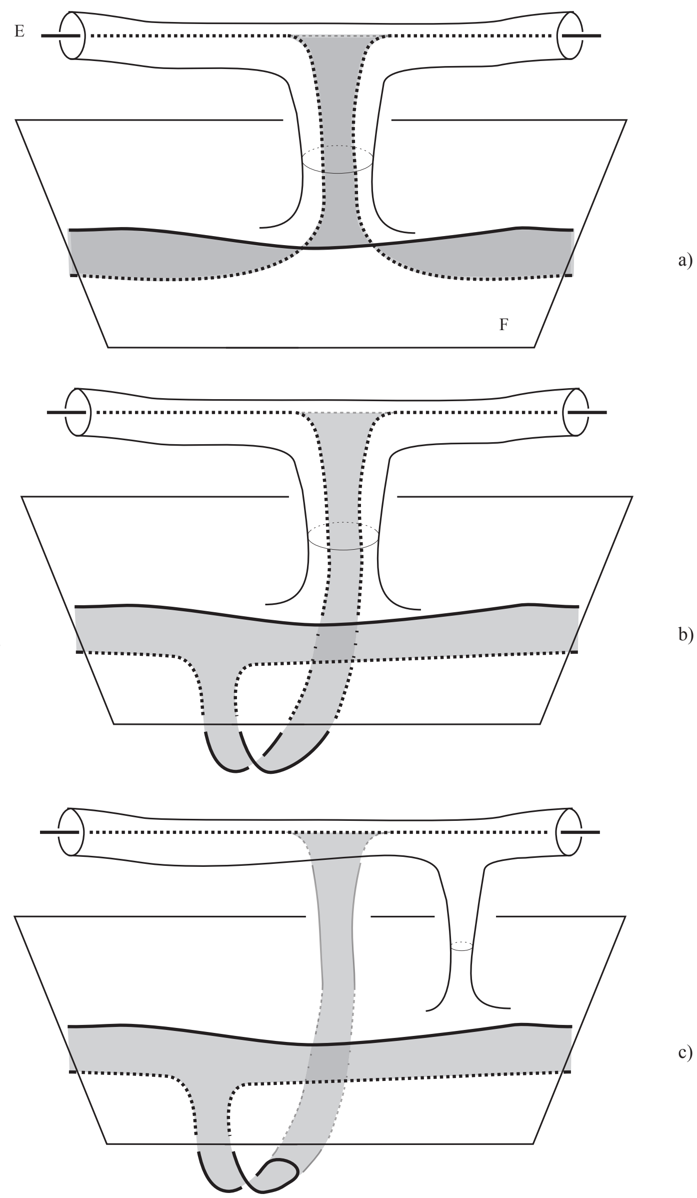


Figure 6.4

there are no framed embedded paths in $N(G) = D^2 \times G$. We shall see that except when Lemma 2.3 is invoked the isotopy from A to A' is supported in $N(G)$. Think of $N(G) = D^2 \times S^2 = D^2 \times S^1 \times [-\infty, +\infty]/\sim$, where each $x \times S^1 \times -\infty$ and each $x \times S^1 \times +\infty$ is identified to a point and with G being the S^2 fiber through the origin of D^2, $z = (0, t_0, 0)$ and $A_1 \cap N(G) = D^2 \times t_0 \times 0$.

The isotopy from A to A' is demonstrated in Figure 7.2. The various subfigures of Figure 7.2 show a 3-dimensional subset H of the form $D^2 \times [t_0 - \epsilon, t_0 + \epsilon] \times 0$. The sets K' and K seen in Figure 7.2 a) are projections of the components of $A \cap N(z)$ that contain the local tubes about $f(\kappa')$ and $f(\kappa)$. Here the *time coordinate* $[-\infty, \infty]$ is projected to 0 and the shaded regions are images of sheets from the past and future. Now consider the projection of $D^2 \times S^2$ to $D^2 \times t_0 \times 0$. Let δ denote the projection of K' to $D^2 \times t_0 \times 0$. The preimage W of δ in $D^2 \times S^2$ is an $I \times S^2$ which intersects A in three components; an annulus from K', a

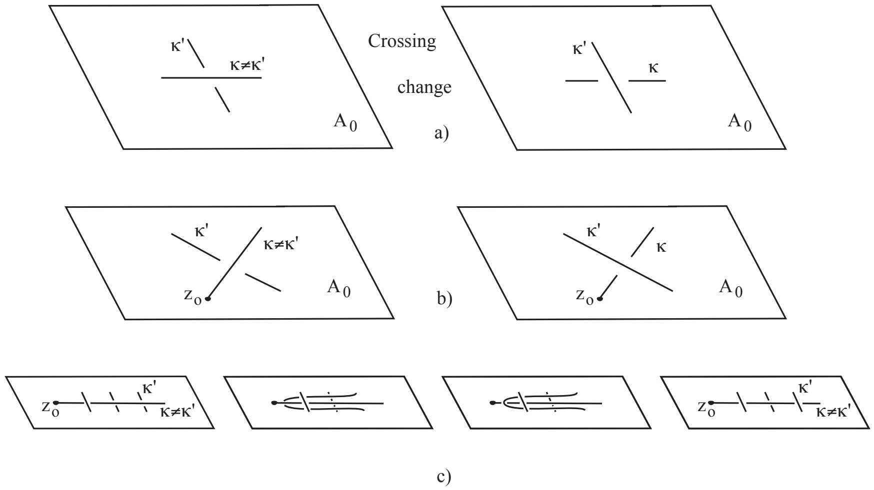


Figure 7.1

S^1 from K and the arc δ from A_1, all of which are shown in the Figure 7.3 a). The support of the isotopy, in the source, from this point on is within K'. The isotopy from Figures 7.2 a) to 7.2 b) is supported in W and can be viewed in more detail in Figure 7.3, where corresponding dark lines in Figures 7.2 a), b) and Figure 7.3 a), b) coincide. The dots on $W \cap K$ in Figure 7.3 are the intersection of the dark lines on K with W. The passage from Figure 7.2 b) to Figure 7.2 c) is the 4-dimensional light bulb move, Lemma 2.3, whereby a tube T appears to be crossing A at the point y. This requires that there be a path σ in A from y to z disjoint from T which in turn requires that $\kappa \neq \kappa'$. The isotopy from Figure 7.2 c) to d) simply squeezes the indicated tubes, and commutes with the previous one. The isotopy corresponding to Figures 7.2 d) and e) is essentially the reverse of that from 7.2 a) and 7.2 b). Here the projection of the $S^2 \times I$, corresponding to this isotopy, to $D^2 \times t_0 \times 0$ is an arc disjoint from the projection of K, so K is not in the way during that isotopy. □

8. Proof of Theorems 1.2 and 1.3

Suppose that the embedded spheres R and A_1 are homotopic with the common transverse sphere G. After an initial isotopy fixing G setwise, we can assume that they coincide near G. By Lemma 6.1 we can assume that the homotopy from R to A_1 is supported away from a neighborhood of G. It follows from [Sm1] that R is regularly homotopic to A_1 via a homotopy that is also supported away from a neighborhood of G. See §4. By Theorem 5.21 the homotopy from R to A_1 is shadowed by tubed surfaces. Thus there exists a tubed surface $\mathcal{A}$ with realization A, underlying surface A_0 and associated surface $A_1 = f(A_0)$ such that R is isotopic to A.

To prove Theorem 1.2 it suffices to show that A is isotopic to A_1 via an isotopy supported away from a neighborhood of G. To prove Theorem 1.3 it suffices to show that A can be isotoped into normal form with respect to A_1 and two surfaces A and A' in normal form with respect to A_1 are isotopic with support away from G if their framed tube guide curves represent the same set of 2-torsion elements. By Proposition 5.24 all finite sets of nontrivial 2-torsion elements can be realized.

By Proposition 6.9 we can assume that A has finitely many double tubes, each representing distinct 2-torsion elements of $\pi_1(M)$ at most one of which is homotopically trivial. In what

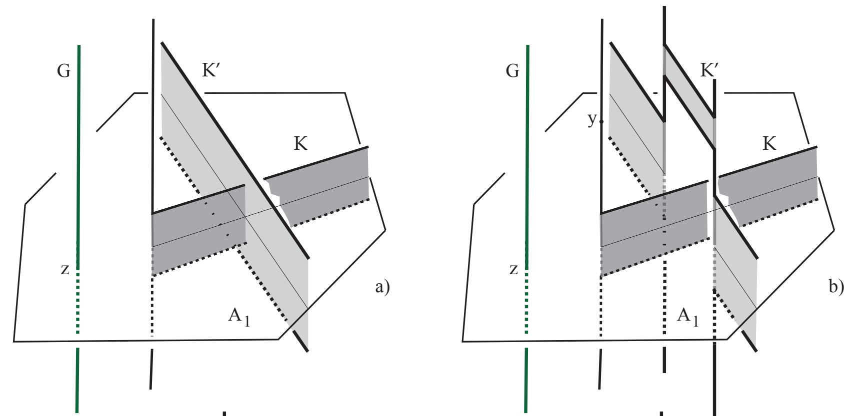
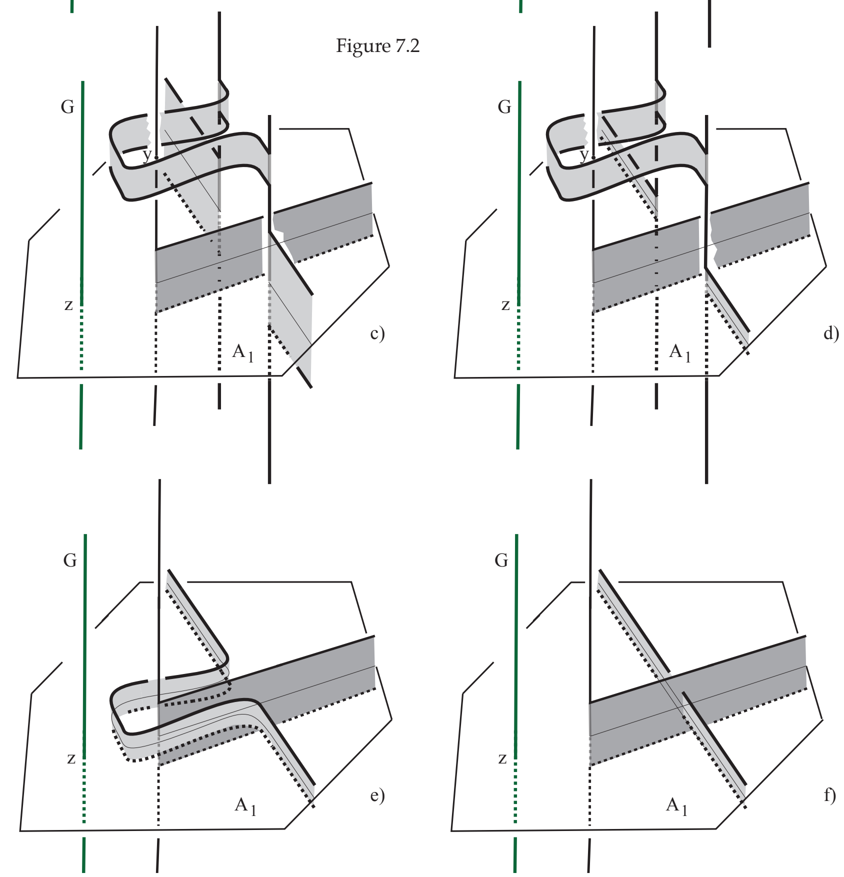


Figure 7.2



follows we assume for consistency of notation that there exists one such double tube and it's associated to (β_0, γ_0). Using the crossing change and tube sliding Lemmas 7.1 and 5.11 we can assume that each pair $(\alpha_1, q_1), \cdots, (\alpha_r, q_r), (\beta_0, \gamma_0), (\beta_1, \gamma_1), \cdots, (\beta_n, \gamma_n)$ lies in a distinct sector of A_0. This means there exists a neighborhood D_0 of $z_0 \in A_0$ parametrized as the unit disc in polar coordinates so that (α_i, q_i) lies in the subset of D_0 where $((i-1)/(r+n+1))2\pi < \theta < (i/(r+n+1))2\pi$ and (β_i, γ_i) lies in the region $((r+i)/(r+n+1))2\pi < \theta < ((r+i+1)/(r+n+1))2\pi$. We can further assume that $q_1 \in \partial D_0$. This requires that R_0 is a 2-sphere.

After r applications of the next lemma we can assume that $r = 0$ and that the remaining data defining $\mathcal{A}$ is unchanged.

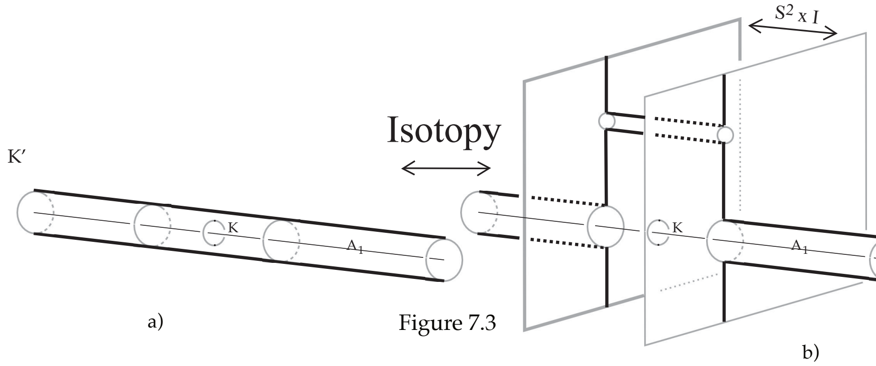


Figure 7.3

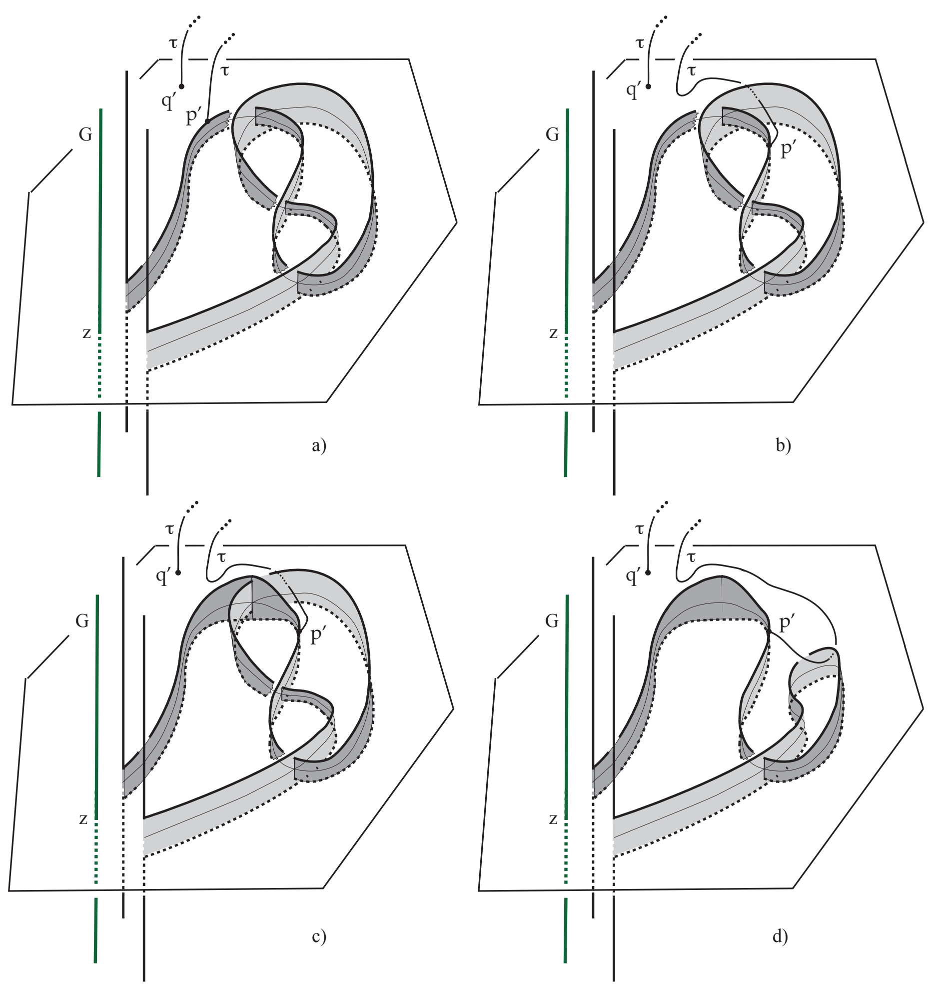


Figure 8.1

Lemma 8.1. *Let $\mathcal{A}$ be a tubed surface such that a neighborhood $D \subset A_0$ of z_0 is parametrized as the unit disc D in polar coordinates and $\alpha_1 \cup q_1$ is contained in the subset of D where $0 < \theta < \pi/2$ and that that region is devoid of all other tube guide curves and associated*

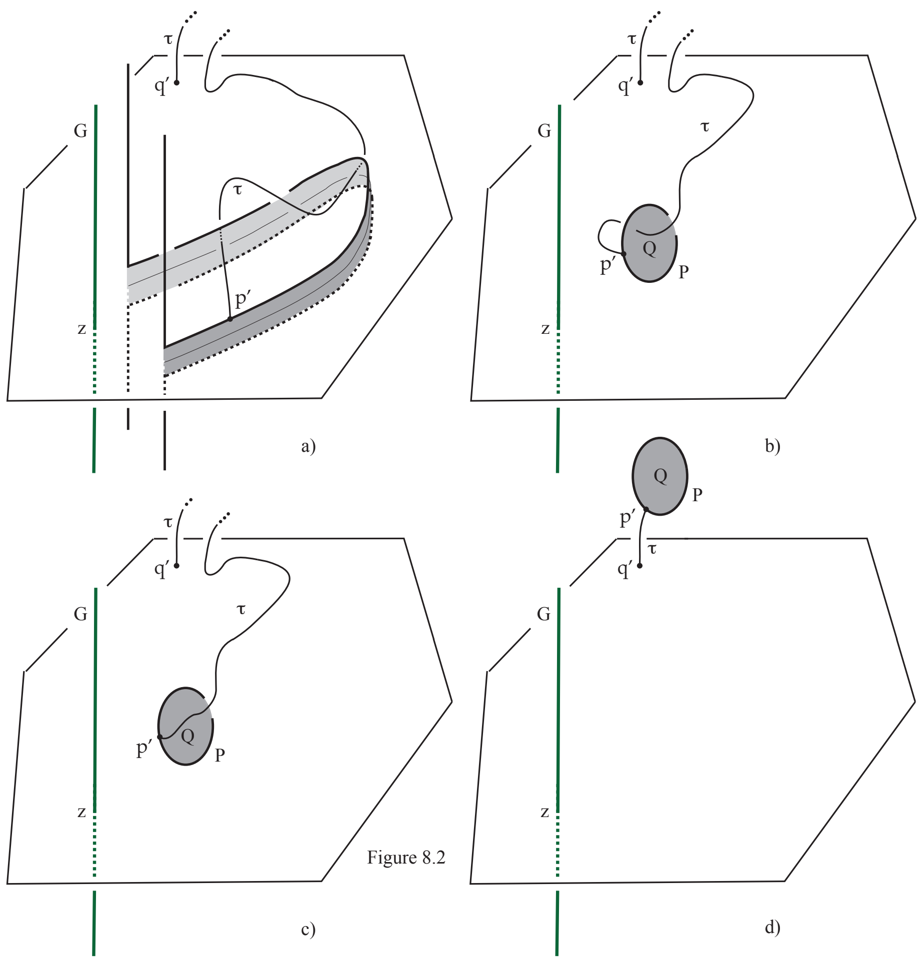


Figure 8.2

points. Assume also $q_1 \in \partial D$. *Then the tubed surface* $\mathcal{A}'$ *whose data consists of that of* $\mathcal{A}$ *with* $\alpha_1 \cup \tau_1 \cup q_1$ *deleted has realization* A' *isotopic to the realization* A *of* $\mathcal{A}$.

Proof. We first fix some terminology. To simplify notation $\alpha_1, p_1, q_1, \tau_1$ will be respectively denoted α, p, q, τ. $T(\alpha)$ will denote the tube about $f(\alpha)$ and $P(\alpha)$ will denote the 2-sphere consisting of two parallel copies of G tubed together along $T(\alpha)$. $T(\tau)$ will denote the tube about τ. So A is the surface obtained from A' by connecting $P(\alpha)$ to A' by the tube $T(\tau)$. We will let p' and q' denote the points respectively on $P(\alpha)$ and A' so that the ends of $T(\tau)$ connect to $\partial N(p')$ and $\partial N(q')$, where neighborhoods are taken in A. Here p' orthogonally projects to $f(p) \in A_1$, where $p \in \alpha \subset A_0$, and $q' = f(q)$. We let α_L, and α_R denote the components of α separated by p.

The first observation is that by isotopy extension p' can be isotoped to any point in $T(\alpha)$ at the cost of seemingly *entangling* τ with $T(\alpha)$. See Figures 8.1 a) and b). One cannot obviously use the light bulb lemma to remove the intersection of $\text{int}(\tau)$ with the projection of $T(\alpha)$ in Figure 8.1 b), since $T(\tau)$ separates $T(\alpha)$ from z.

By suitably moving p, the local arcs of α at a given crossing can lie in the different components α_L and α_R of $\alpha \setminus p$. Thus the proof of the crossing change lemma allows us to change this crossing as well as any other at the cost of entangling τ with $T(\alpha)$. This process is illustrated in Figures 8.1 a), b), c). Similarly, at the cost of further such entanglement we can perform the reordering move, Definition 5.9 ii) to arcs of α. Compare Figure 8.1 d) and Figure 8.2 a). Thus by crossing changes, Reidemeister 2), 3) moves and the tube sliding reordering move, we can assume that α has no crossings. It follows that $P(\alpha)$ can be isotoped to an unknotted 2-sphere P, that bounds a 3-ball Q disjoint from $A' \cup G$. See Figure 8.2 b). Further there exists a 4-ball B such that $Q \subset B$, $A' \cap B = \emptyset$, $B \cap G = \emptyset$ and $\tau \cap B$ is connected. While it can be avoided, we note that any entanglement with A_1 can be eliminated by Lemma 2.3. Since $\pi_1(B \setminus P) = \mathbb{Z}$ and $T(\tau)$ can be rotated about P it follows that via isotopy supported within B, A can be isotoped so that $Q \cap \text{int}(T(\tau)) = \emptyset$ as in Figure 8.2 c). It follows that A can be isotoped to A', thereby completing the proof. $\square$

We now assume that $r = 0$. Let $\mathcal{A}'$ be the tubed surface obtained from $\mathcal{A}$ with the data (β_0, γ_0) and λ_0 deleted. We now show that the realization A of $\mathcal{A}$ is isotopic to the realization A' of $\mathcal{A}'$. Since λ_0 can be homotoped rel endpoints into the associated surface A_1, Lemma 2.2 and Remark 5.8 imply that λ_0 can be isotoped to lie near the sector of $f(D_0)$ containing $f(\beta_0 \cup \gamma_0)$, via an isotopy supported away from $G \cup A_1$. Hence via isotopy supported away from G, A can be isotoped so that the double tube following λ_0 and $D(\beta_0) \cup D(\gamma_0)$ lie in a small regular neighborhood $N(G')$ of a parallel translate G' of G. Here $N(G') = G' \times D^2$, where $A_1 \cap N(G') = z' \times D^2$ and $N(G') \cap G = \emptyset$. Also A coincides with A' outside of $G' \times D^2$, A coincides with both A' and A_1 near $\partial N(G')$ and $[A \cap N(G')] = [z' \times D^2] \in H_2(N(G'), z' \times \partial D^2)$. By Theorem 3.7 A is isotopic to A' via an isotopy supported within $N(G')$. This completes the proof of Theorem 1.2. $\square$

Remark 8.2. With more work one can eliminate reliance on Theorem 3.7. By Lemma 7.1 and 5.11 we can assume that β_0 and γ_0 are pairwise disjoint. If each is embedded, then a direct argument allows for the elimination of this data from $\mathcal{A}$, via isotopy of A. If either β_0 or γ_0 are not embedded, then construct $\mathcal{A}'$ whose data equals that of $\mathcal{A}$ except that an adjacent sector contains β_{-1}, γ_{-1} which are disjoint and embedded and whose framed embedded path λ_{-1} is homotopically trivial. As noted above the realizations A and A' of $\mathcal{A}$ and $\mathcal{A}'$ are isotopic by a direct argument. The proof of Proposition 6.9 shows that the pair of double tubes associated to λ_0 and λ_{-1} can be transformed into a pair of single tubes and these tubes and their associated data can be eliminated via isotopy as in the proof of Theorem 1.2 without using Theorem 3.7.

We now complete the proof of Theorem 1.3. We have a tubed surface $\mathcal{A}$ whose data consists of $(\beta_1, \gamma_1) \cdots, (\beta_n, \gamma_n)$ and framed tube guide paths $\lambda_1, \cdots, \lambda_n$ representing distinct nontrivial 2-torsion elements of $\pi_1(M)$. As above we can assume that the (β_i, γ_i)'s lie in distinct sectors of $D_0 \subset A_0$. Let $\mathcal{A}'$ be a tubed surface with data $(\beta_1', \gamma_1'), \cdots, (\beta_n', \gamma_n')$ each lying in distinct sectors and where β_i', γ_i' are disjoint and embedded. Further each λ_i' is homotopic to λ_i, though its framing is allowed to be arbitrary.

We now show that the realizations A and A' of $\mathcal{A}$ and $\mathcal{A}'$ are isotopic. Using the idea of the previous remark we show how to replace $\beta_1, \gamma_1, \lambda_1$ by β_1', γ_1' and λ_1' without changing the isotopy class of the realization. The proof that A and A' are isotopic then follows by induction. First observe that there are adjacent sectors of D_0 containing the ray of angle 0, which are disjoint from (β_n, γ_n) and (β_1, γ_1). Let $\mathcal{A}''$ be the tubed surface whose data consists of that of $\mathcal{A}$ with $(\beta_{-1}, \gamma_{-1})$ and (β_0, γ_0) added to these sectors where for $i \in \{-1, 0\}$, β_i and γ_i are disjoint and embedded. Further the framed tube guide curves λ_{-1} and λ_0 represent the homotopy class of λ_1, with the framing given by λ_1'. See Figure 8.3. Now the underlying

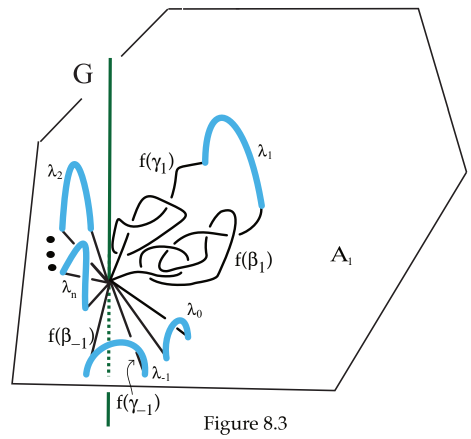


Figure 8.3

surface A'' of $\mathcal{A}''$ is isotopic to A since the proof of Proposition 6.9 shows how to transform the pair of double tubes associated to λ_{-1} and λ_0 to single tubes and these single tubes can be eliminated as usual. Similarly A'' is isotopic to A' since the pair of double tubes associated with λ_0 and λ_1 can be transformed into a pair of single tubes. These single tubes can then be eliminated as usual. Further, $(\beta_{-1}, \gamma_{-1})$ can now be rotated into the position of (β_1', γ_1') and λ_{-1} can be isotoped to λ_1'. By construction λ_{-1} has the same framing as that of λ_1'. It follows that A is isotopic to A'.

In summary, if R and A are embedded spheres in M with the common transverse sphere G, then R can be isotoped into normal form with respect to A as in Definition 5.23. If A' and A'' are in normal form with respect to A and represent the same set of elements, then Lemma 5.25 shows that after isotopy we can assume that for all i, λ_i' is homotopic to λ_i''. Thus after isotopy we can assume that A' and A'' are realizations of tubed surfaces $\mathcal{A}'$ and $\mathcal{A}''$ such that for all i, $(\alpha_i', \beta_i') = (\alpha_i'', \beta_i'')$ and $\lambda_i' = \lambda_i''$, however the framings of the λ_i's might differ. The previous paragraph shows that changing the framing of a λ_i'' does not change the isotopy class of the realization. It follows that A' and A'' are isotopic and hence equivalent normal forms are isotopic. By Proposition 5.24 any finite set of distinct nontrivial 2-torsion elements can be represented by a surface in normal form with respect to A. This completes the proof of Theorem 1.3. ☐

9. HIGHER GENUS SURFACES

In this section we give a partial generalization of our main result to higher genus surfaces, that is a full generalization for $S^2 \times S^2$.

Definition 9.1. Let S be an immersed surface in the 4-manifold M. We say that the embedded disc $D \subset M$ is a *compressing disc* for S if $\partial D \subset S$ and a section of the normal bundle to $\partial D \subset S$ extends to a section of the normal bundle of $D \subset M$.

Lemma 9.2. *If S is immersed in the 4-manifold M and $\alpha \subset S$ is an embedded curve with a trivial normal bundle in S and is homotopically trivial in M, then α bounds a compressing disc.*

Proof. First span α by an immersed disc D_0. Using boundary twisting [FQ] we can replace D_0 by D_1 that satisfies the normal bundle condition. Eliminate the self intersections of D_1 by applying finger moves. □

Lemma 9.3. *Let S be an orientable embedded surface in the 4-manifold M whose components have pairwise disjoint transverse spheres. Let $\alpha_1, \cdots, \alpha_k \subset S$ be pairwise disjoint simple closed curves, disjoint from the transverse spheres, such that for each component S' of S, $S' \setminus \{\alpha_1, \cdots, \alpha_k\}$ is connected. Suppose that for each i, α_i is homotopically trivial in the complement of the transverse spheres. Then there exist pairwise disjoint compressing discs $D_1, \cdots, D_k$ such that for each i, $D_i \cap S = \alpha_i$.*

Proof. Construct compressing discs $A_1, \cdots, A_k$ for the α_i's as in Lemma 9.2. These discs can be chosen to be disjoint from the transverse spheres. Using finger moves they can be made disjoint from each other. Finally use the transverse spheres to tube off unwanted intersections of the A_i's with S to create the desired D_i's. □

Definition 9.4. We say that the surface S_1 is obtained from S by *compressing* along D if $S_1 = S \setminus \text{int}(N(\partial D)) \cup D' \cup D''$ where D', D'' are two pairwise disjoint parallel copies of D.

Lemma 9.5. *Surfaces can be compressed along compressing discs. If S_1 is obtained by compressing the embedded surface S along the compressing disc D and $D \cap S = \partial D$, then S_1 is embedded.* □

Definition 9.6. We say that the surface $S \subset M$ is *G-inessential* if the induced map $\pi_1(S \setminus G) \to \pi_1(M \setminus G)$ is trivial.

The following is a generalization to higher genus surfaces of Theorem 1.2.

Theorem 9.7. *Let M be an orientable 4-manifold such that $\pi_1(M)$ has no 2-torsion. Two homotopic, embedded, G-inessential surfaces S_1, S_2 with common transverse sphere G are ambiently isotopic. If they coincide near G, then the isotopy can be chosen to fix a neighborhood of G pointwise.*

Proof. For each $i \in \{1, 2\}$ let $\alpha_1^i, \cdots, \alpha_k^i$ be a set of pairwise disjoint simple closed curves in S_i whose complement is a connected planar surface containing $S_i \cap G$. Let $D_1^i, \cdots, D_k^i$ be associated pairwise disjoint compressing discs with interiors disjoint from S_i and let T_i be the result of compressing S_i along these discs. Then T_i is a 2-sphere and S_i is obtained from T_i by attaching k tubes. Each tube $S^1 \times I$ extends to a solid tube $D^2 \times I$ which intersects T_i exactly at $D^2 \times 0$ and $D^2 \times 1$, which we call the *bases of the tube.* By construction, these tubes are pairwise disjoint. After a further isotopy we can assume there are k small pairwise disjoint 4-balls each of which intersects S_i in a single standard disc and each such disc contains the bases of a single solid tube.

Since S_i is G-inessential, it follows by the light bulb lemma that the solid tubes can be isotoped to 3-dimensional neighborhoods of tiny standard arcs with endpoints on T_i. Note that the induced ambient isotopy can be chosen to fix a neighborhood of T_i pointwise.

To complete the proof it suffices to show that T_1 and T_2 are homotopic and hence isotopic by Theorem 1.2. To see this, consider the lifts $\tilde{T}_1, \tilde{T}_2$ of T_1, T_2 to the universal covering $\tilde{M}$ of M which intersect a given lift $\tilde{G}$ of G. Since the S_i's are π_1-inessential and homotopic, their corresponding lifts $\tilde{S}_1, \tilde{S}_2$ are homotopic and hence homologous. It follows that $\tilde{T}_1$ and $\tilde{T}_2$ are homologous and hence homotopic and therefore so are T_1 and T_2. □

Applying to the case of $S^2 \times S^2$ we obtain:

Theorem 9.8. *Let R be a connected embedded genus-g surface in $S^2 \times S^2$ such that $R \cap S^2 \times y_0 = 1$. Then R is isotopically standard. I.e. it is isotopic to the standard sphere in its homology class, with g standard handles attached, which we denote by R_0. If R and R_0 coincide near $S^2 \times y_0$, then the isotopy can be chosen to fix a neighborhood of $S^2 \times y_0$ pointwise.* □

10. Applications and questions

We begin by stating the main result for multiple spheres.

Theorem 10.1. *Let M be an orientable 4-manifold such that $\pi_1(M)$ has no 2-torsion. Let $G_1, \cdots, G_n$ be pairwise disjoint embedded spheres with trivial normal bundles. Let $R_1, \cdots, R_n$ be pairwise disjoint embedded spheres transverse to the G_i's such that $|R_i \cap G_j| = \delta_{ij}$. Let $S_1, \cdots, S_n$ be another set of spheres with the same properties and coinciding with the R_i's near the G_i's. If for each i, R_i is homotopic to S_i, then there exists an isotopy of M fixing a neighborhood of the G_i's pointwise such that for all j, R_j is taken to S_j.*

Under corresponding hypotheses, the same conclusion holds when the R_i's and S_i's are G-inessential connected surfaces, where $G = \{G_1, \cdots, G_n\}$.

Proof. The methods of §9 reduce the general case to the case that all the R_i's and S_i's are spheres.

Proof by induction on n.

Step 1: R_1 is ambient isotopic to S_1 via an isotopy that fixes the G_i's pointwise.

Proof After a preliminary isotopy we can assume that R_1 and S_1 coincide near $R_1 \cap G$ and that the homotopy from R_1 to S_1 is supported away from a neighborhood of $\cup G_i$. Step 1 follows by applying Theorem 1.2 to the manifold $M \setminus \cup_{i=2}^n N(G_i)$. Note that the inclusion of $M \setminus \cup_{i=2}^n N(G_i) \to M$ induces a fundamental group isomorphism so the no 2-torsion condition is satisfied. □

Induction Step: Suppose that we have for $j < k$, $R_j = S_j$. There exists an isotopy of M fixing $\cup_{i=k}^n G_i$ pointwise and supported away from $\cup_{j=1}^{k-1}(G_j \cup S_j)$ such that R_k is taken to S_k.

Proof After a preliminary isotopy we can assume that R_k and S_k coincide near G_k and that R_k is homotopic to S_k via a homotopy supported away from $\cup_{j=1}^{k-1}(S_j \cup G_j)$. Next apply Step 1 to $R_k \subset M \setminus N(\cup_{j=1}^{k-1}(S_j \cup G_j))$. Again, the argument of Lemma 2.2 implies that the inclusion $M \setminus N(\cup_{j=1}^{k-1}(S_j \cup G_j)) \to M$ induces a fundamental group isomorphism, so the no 2-torsion condition is satisfied. $\square$

An analogous argument combined with the proof of Theorem 1.3 yields the corresponding result for manifolds with 2-torsion in their fundamental groups.

Theorem 10.2. *Let M be an orientable 4-manifold. Let $G_1, \cdots, G_n$ be pairwise disjoint embedded spheres with trivial normal bundles. Let $R_1, \cdots, R_n$ be pairwise disjoint embedded spheres transverse to the G_i's such that $|R_i \cap G_j| = \delta_{ij}$. Let $S_1, \cdots, S_n$ be another set of spheres with the same properties and coinciding with the R_i's near the G_i's. If for each i, R_i is homotopic to S_i, then there exists an isotopy of M fixing a neighborhood of the G_i's pointwise such that each R_i can be put into normal form with respect to S_i. Here each R_i has double tubes representing elements $\{[\lambda_1^i], \cdots, [\lambda_{n_i}^i]\}$, where for fixed i the $[\lambda_j^i]$'s are distinct nontrivial 2-torsion elements of $\pi_1(M)$ and $R_i = S_i$ if this set is empty. Finally, for each i any finite set of distinct 2-torsion elements gives rise to an R_i in normal form with double tubes representing this set and two sets of R_i's are isotopic if their corresponding sets of $[\lambda_j^i]$'s are pairwise equal.* $\square$

Definition 10.3. An essential simple closed curve in $S^2 \times S^1$ is said to be *standard* if it is isotopic to $x \times S^1$ for some $x \in S^2$.

Theorem 10.4. *Two properly embedded discs D_0 and D_1 in $S^2 \times D^2$ that coincide near their standard boundaries are isotopic rel boundary if and only if they are homologous in $H_2(S^2 \times D^2, \partial D_0)$.*

Proof. Homologous is certainly a necessary condition. In the other direction, after reparameterizing, we can assume that $\partial D_0 = x_0 \times S^1 \subset S^2 \times S^1$. Let $M = S^2 \times D^2 \cup d(S^2 \times D^2) = S^2 \times S^2$ be obtained by doubling $S^2 \times D^2$ with $d(S^2 \times D^2)$ denoting the other $S^2 \times D^2$. This $d(S^2 \times D^2)$ can be viewed as a regular neighborhood $N(G)$ of $G = d(S^2 \times 0)$. Let R_i denote the sphere $D_i \cup d(x_0 \times D^2)$ which we can assume is smooth for $i = 0, 1$. G is a transverse sphere to the homologous spheres R_0 and R_1. By Theorem 1.2 there is an isotopy of M fixing a neighborhood of G pointwise taking R_0 to R_1. Since R_0 and R_1 coincide in a neighborhood of $N(G)$ there is an isotopy of $S^2 \times D^2$ taking D_0 to D_1 that fixes a neighborhood of $S^2 \times S^1$ pointwise. $\square$

Theorem 10.5. *A properly embedded disc D in $S^2 \times D^2$ is properly isotopic to a D^2-fiber if and only if its boundary is isotopic to the standard vertical curve.*

Proof. After a preliminary isotopy we can assume that ∂D is the standard vertical curve $x_0 \times S^1$ which we denote by J. Let F be a D^2 fiber of $S^2 \times D^2$. Now $0 \to H_2(S^2 \times D^2) \to H_2(S^2 \times D^2, J) \to H_1(J) \to 0$ is split and exact, so the coset H mapping to the generator $[\partial F]$ of $H_1(J)$ equals Z and is represented by the classes $[F] + n[S^2 \times y_0]$, where $y_0 \in \partial D^2$. By properly isotoping D to D' where $\partial D' = J$ and so that the track of the homotopy restricted to the boundary is approximately $J \cup S^2 \times y_0$ it follows that $[D'] = [D] + [S^2 \times y_0] \in H_2(S^2 \times D^2, J)$. Therefore any class in H is represented by a disc properly isotopic to D. In particular after proper isotopy we can assume that $[D] = [F]$. After a further isotopy we can assume that D coincides with F near ∂D. The result now follows by Theorem 10.4. The other direction is immediate. $\square$

Recall that $\mathrm{Diff}_0(X)$ denotes the group of diffeomorphisms properly homotopic to the identity.

Corollary 10.6. $\pi_0(\mathrm{Diff}_0(S^2\times D^2)/\mathrm{Diff}_0(B^4))=1$.

Remark 10.7. This means that a diffeomorphism of $S^2\times D^2$ properly homotopic to the identity is isotopic to one that coincides with the identity away from a compact 4-ball disjoint from $S^2\times S^1$.

Proof. Let D denote a $x\times D^2$ and let $f: S^2\times D^2\to S^2\times D^2$ be properly homotopic to the identity. Since homotopic diffeormorphisms of $S^2\times S^1$ are isotopic [La], $f(\partial D)$ is isotopic to ∂D in $S^2\times S^1$ and hence isotopically standard. Next apply Theorem 10.5 to isotope f so that $f(D)=D$. After a further isotopy, using [Sm3], we can assume that $f|D=\mathrm{id}$ and after another that $f|N(D)=\mathrm{id}$. Since a diffeomorphism of $D^2\times S^1$ that fixes a neighborhood of the boundary pointwise is isotopic to the identity rel ∂ it follows that we can be further isotope f so that ∂f is also the identity. After another isotopy we can additionally assume that $f|N(\partial(S^2\times D^2))=\mathrm{id}$. Since the closure of what's left is a B^4, the result follows. □

The following is an immediate consequence of our main result.

Theorem 10.8. *(4D-Lightbulb Theorem) If R is an embedded 2-sphere in $S^2\times S^2$, homologous to $x_0\times S^2$, that intersects $S^2\times y_0$ transversely and only at the point (x_0,y_0), then R is isotopic to $x_0\times S^2$ via an isotopy fixing $S^2\times y_0$ pointwise.* □

Litherland [Li] proved that there exists a diffeomorphism pseudo-isotopic to the identity that takes R to $x_0\times S^2$.

Another version of the light bulb theorem was obtained in 1986 for PL discs in S^4 by Marumoto [Ma] where the isotopy is topological. He makes essential use of Alexander's theorem that any homeomorphism of B^n that is the identity on S^{n-1} is (topologically) isotopic to the identity. Here we prove a general form of the smooth version.

Theorem 10.9. *(Uniqueness of Spanning Surfaces) If R_0 and R_1 are smooth embedded surfaces in S^4 of the same genus such that $\partial R_0=\partial R_1=\gamma$, where γ is connected, then there exists a smooth isotopy of S^4 taking R_0 to R_1 that fixes γ pointwise.*

Proof. First consider the case that R_0 and R_1 are discs. After a preliminary isotopy of S^4 that fixes γ pointwise, we can assume that R_0 and R_1 coincide in an annular neighborhood of their boundaries. Now $S^4\setminus \mathrm{int}(N(\gamma))=S^2\times D^2$. Thus R_0 and R_1 restrict to properly embedded discs E_0 and E_1 in $S^2\times D^2$ that coincide near their boundaries.

Arguing as in the proof of Theorem 10.5 we can assume that after an isotopy of R_0, $[E_0]=[E_1]\in H_2(S^2\times D^2,\gamma)$ also holds. This isotopy fixes ∂R_0 pointwise but moves it's annular neighborhood. Here are more details. Let α_0 denote $\partial E_0\subset S^2\times S^1$. Let A_0 be the annulus bounded by α_0 and ∂R_0. An isotopy of α_0 induces an isotopy of A_0 fixing γ pointwise that extends to R_0 by isotopy extension. The resulting R_0' has annular boundary coinciding with that of R_1 and the class of the resulting E_0' in the coset $H\subset H_2(S^2\times D^2,\gamma)$ changes according to the number of times the isotopy of α_0 algebraically sweeps across the S^2-factor. Here H is as in the proof of Theorem 10.5.

It follows by Theorem 10.4 that E_0 can be isotoped to E_1 via an isotopy supported away from a neighborhood of $S^2\times S^1$.

The general case similarly follows using Theorem 9.8. □

Remark 10.10. By induction Marumoto [Ma] proved more generally that two locally flat PL m-discs in an n-sphere, $n > m$ with the same boundary are topologically isotopic rel boundary. Here is an outline of his argument for smooth discs in the n-sphere for the representative case $m = 2, n = 4$, where we use [Ce1], [Pa] to avoid his induction steps. Actually, the below argument works in all dimensions and codimensions since the same is true of [Ce1], [Pa] and the Alexander isotopy.

Start with D_0, D_1 where D_1 is the standard 2-disc in S^4 and $\partial D_0 = \partial D_1$. Then by [Ce1], [Pa] there is a diffeomorphism $f : S^4 \to S^4$ taking D_0 to D_1 fixing ∂D_0. We can assume that f fixes pointwise a neighborhood of ∂D_0. Next remove a small ball about a point in ∂D_0. After restricting and reparametrizing we obtain a map $g : B^4 \to B^4$ such that $g(E_0) = E_1$ where the E_i's are the restricted reparametrized D_i's. Here B^4 is the unit ball in $\mathbb{R}^4, \partial E_0$ is a straight properly embedded arc connecting antipodal points of ∂B^4 and $g|\partial B^4 = \mathrm{id}$. Finally apply the Alexander isotopy to obtain a topological isotopy of g to the identity which fixes ∂E_0 pointwise. □

More generally we have the following uniqueness of spanning discs in simply connected 4-manifolds.

Theorem 10.11. *If D_0 and D_1 are smooth embedded discs in the simply connected 4-manifold M such that $\partial D_0 = \partial D_1 = \gamma$, then there exists a smooth isotopy of M taking D_0 to D_1 fixing γ pointwise if and only if the mapped sphere $S = D_0 \cup_\gamma D_1$ is inessential in M.*

Proof. If D_0 and D_1 are isotopic, then the isotopy sweeps out a contracting ball for S. Conversely, after an initial isotopy of D_1 we can assume that it coincides with D_0 near γ and that the interior of the mapped 3-ball B defining the contraction of S intersects γ algebraically zero. Indeed, the second isotopy in the proof of Theorem 10.9 enables modification of the intersection number. These intersections can be eliminated using immersed Whitney discs. Next surger γ to obtain the simply connected manifold N so that D_0 and D_1 give rise to homotopic spheres R_0 and R_1 with common transverse sphere G, that coincide near their intersection with G. By Theorem 1.2, R_0 and R_1 are isotopic via an isotopy fixing G pointwise and hence D_0 and D_1 are isotopic rel boundary. □

Remark 10.12. In a similar manner, using Theorems 10.1 and 9.7, one can obtain uniqueness theorems for certain surfaces spanning simple closed curves in closed 4-manifolds with no 2-torsion in their fundamental groups.

One can ask the following parametrized form in the smooth category.

Question 10.13. *For $i = 1, 2$ let $f_i : D^k \to S^4$ be smooth embeddings such that $f_1|\partial D^k = f_2|\partial D^k$. Is there a smooth isotopy $F : S^4 \times I \to S^4$ such that $F_0 = \mathrm{id}_{S^4}, F_t(f_1(x)) = f_1(x)$ for $x \in \partial D^k$ and $t \in [0, 1]$ and for $y \in D^k, F_1(f_2(y)) = f_1(y)$?*

Remark 10.14. For $k \leq 3$ the unparametrized version implies the parametrized one by [Ce3] for k=3 and [Sm3] for $k = 2$ with the $k = 1$ case being elementary. The point of this question is to link various theorems, conjectures and questions.

Case k=1: This is the theorem *homotopy implies isotopy for curves in 4-manifolds.*

Case k=2: This is Theorem 10.9.

Case k=3: This implies the Schoenflies conjecture. Indeed the Schoenflies conjecture is equivalent to a positive resolution of the question after allowing lifting of the f_i's to some finite branched covering of S^4 over $\partial(f_i(D^3))$. See [Ga].

Case k=4: This is the question of connectivity of $\text{Diff}_0(B^4, \partial)$.

Question 10.15. *Does Theorem 10.1 hold without the G-inessential condition? What if G-inessential is replaced by π_1-inessential?*

The following are special cases of the long standing questions of whether a sphere R in $\mathbb{CP}^2$ homologous to $\mathbb{CP}^1$ is equivalent up to isotopy or diffeomorphism to the standard $\mathbb{CP}^1$. See problem 4.23 [Ki].

Questions 10.16. *i) If R is a smooth sphere in $\mathbb{CP}^2$ that intersects $\mathbb{CP}^1$ once is R isotopically standard?*

ii) [Me] *Is $(\mathbb{CP}^2, R)$ diffeomorphic to $(\mathbb{CP}^2, \mathbb{CP}^1)$?*

Remark 10.17. In his unpublished 1977 thesis, Paul Melvin [Me] showed that blowing down $\mathbb{CP}^2$ along $\mathbb{CP}^1$ transforms R to a 2-knot T in S^4 and Gluck twisting S^4 along T yields S^4 if and only if $(\mathbb{CP}^2, R)$ is diffeomorphic to $(\mathbb{CP}^2, \mathbb{CP}^1)$. He gave a positive answer to ii) for 0-concordant knots.

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Department of Mathematics, Princeton University, Princeton, NJ 08544
E-mail address: gabai@math.princeton.edu