KNOTTED 3-BALLS IN S^4

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Abstract. The unknot U in S^4 has non-unique smooth spanning 3-balls up to isotopy fixing U. Equivalently there are properly embedded non-separating 3-balls in $S^1 \times B^3$ not properly isotopic to $\{1\} \times B^3$. More generally there exist non-separating 3-spheres in $S^1 \times S^3$ not isotopic to $\{1\} \times S^3$ and non trivial elements of $\mathrm{Diff}_0(S^1 \times S^3)$. Along the way we introduce barbell diffeomorphisms, implantations and twistings to construct and modify diffeomorphisms homotopic to the identity. We also introduce a 2-parameter calculus of embeddings of the interval into 4-manifolds and introduce a framed cobordism method as well as a direct method for showing that certain 2-parameter families are homotopically non trivial and diffeomorphisms are isotopically nontrivial. Extensions to higher dimensional manifolds are obtained.

1. Introduction

This paper introduces the study of knotted 3-balls in 4-manifolds, in particular the 4-sphere and $S^1 \times B^3$. Let S^4 be the unit sphere in $\mathbb{R}^5$. Define a *standard* 3-ball in S^4 to be a great 3-ball, i.e. a geodesic 3-ball with boundary a great 2-sphere. A *knotted* ball in S^4 means a smoothly-embedded 3-ball $\Delta_1 \subset S^4$ whose boundary is a great 2-sphere which is not isotopic keeping the boundary fixed, to a standard 3-ball Δ_0. The requirement that the boundary be constrained throughout the isotopy is necessary since any two embedded k-balls in the interior of a connected n-manifold are ambiently isotopic [Ce1] p. 231, [Pa]. The existence of knotted 3-balls in S^4 contrasts with the uniqueness of spanning discs for the unknot in S^3 and uniqueness for spanning discs for circles in S^4, [Ga1]. This paper works in the smooth category and unless otherwise said, all mappings are smooth.

We say N is a *reducing* 3-ball in $S^1 \times B^3$ if N is a properly-embedded submanifold, diffeomorphic to B^3 such that the complement $(S^1 \times B^3) \setminus N$ is connected. By properly-embedded we mean that $N \cap \partial(S^1 \times B^3) = \partial N$. A reducing 3-ball N is *knotted* if it is not properly isotopic to the linear reducing 3-ball, $\{1\} \times B^3$. All reducing 3-balls are properly homotopic to $\{1\} \times B^3$. The study of reducing 3-balls up to isotopy is equivalent to the study of such balls that coincide with $\{1\} \times B^3$ near the boundary since Allen Hatcher has proven that the space of non-separating embeddings of S^2 in $S^1 \times S^2$ has the homotopy-type of $S^1 \times O(3)$ [Ha2].

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Isotopy classes of reducing 3-balls in $S^1 \times B^3$ admit an abelian group structure coming from an operation similar to boundary connect-sum, that we call *concatenation* defined as follows. Starting with two reducing 3-balls in $S^1 \times B^3$ whose boundary is $\{1\} \times S^2$, one glues the two copies of $S^1 \times B^3$ together along $S^1 \times H$ where H is a hemisphere in ∂B^3. This produces a new 4-manifold canonically diffeomorphic to $S^1 \times B^3$ together with a new reducing 3-ball. We will see in Sections 3 and 9 that concatenation has inverses, i.e. it is a group, with the unit being the linear reducing sphere. A less abstract way to describe concatenation would be to take $f_1, f_2 : B^3 \to S^1 \times B^3$ and assume $B_1, B_2 \subset B^3$ are disjoint 3-balls. One can assume $f_i(p) \in S^1 \times B_i$ provided $p \in B_i$ and $f_i(p) = (1, p)$ otherwise. Define the sum of f_1 and f_2 to be equal to f_i on $B^3 \setminus B_j$ where $\{i, j\} = \{1, 2\}$. Isotopy classes of oriented 3-balls in S^4 where the embeddings are required to be linear on the boundary also have a group structure, defined in essentially the same way, and these groups are isomorphic. The key point is that the closed complement (the exterior) of the unknotted S^2 in S^4 is diffeomorphic to $S^1 \times B^3$.

We use the convention that if M is a manifold with boundary, then $\mathrm{Diff}(M \operatorname{fix} \partial)$ denotes the diffeomorphisms of M that are the identity on the boundary. Similarly, we use the notation $\mathrm{Diff}_0(M)$ to denote the subgroup of diffeomorphisms *homotopic* to the identity. We shall see in §3 and §9 that the diffeomorphism groups $\mathrm{Diff}_0(S^4)$, $\mathrm{Diff}(S^1 \times B^3 \text{ fix } \partial)$ and $\mathrm{Diff}_0(S^1 \times S^3))$ are abelian and act transitively respectively on 3-balls with common boundary, reducing balls with common boundary and reducing 3-spheres. (Actually, $\mathrm{Diff}(S^1 \times B^n \text{ fix } \partial)$ is a $(n+1)$-fold loop space compatible with the group multiplication [Bu1].) This leads to the theorem $S^1 \times B^3$ and equivalently the closed complement of the unknot in S^4, have up to isotopy, infinitely many distinct fiberings over S^1 as does $S^1 \times S^3$.

A diffeomorphism $\phi : S^1 \times B^3 \to S^1 \times B^3$ properly homotopic to the identity, gives rise to the 3-ball $\Delta_1 = \phi(\{1\} \times B^3)$ which is unknotted if and only if ϕ is properly isotopic to a map supported in a 4-ball. The group of isotopy classes of oriented 3-balls that are linear on their boundary is isomorphic to $\pi_0(\mathrm{Diff}(S^1 \times B^3 \text{ fix } \partial) / \mathrm{Diff}(B^4 \text{ fix } \partial))$. Similarly, $\mathrm{Diff}_0(S^1 \times S^3) / \mathrm{Diff}(B^4 \operatorname{fix} \partial)$ is isomorphic to the group of *reducing 3-spheres* in $S^1 \times S^3$. See Theorem 3.13 and Theorem 3.12.

The main result of this paper is a construction of an infinite family of linearly independent elements of $\pi_0(\mathrm{Diff}(S^1 \times B^3 \text{ fix } \partial) / \mathrm{Diff}(B^4 \text{ fix } \partial))$ with explicit constructions of the corresponding knotted 3-balls in $S^1 \times B^3$ and hence S^4. Furthermore, these diffeormorphisms extend to a linearly independent set in $\mathrm{Diff}_0(S^1 \times S^3) / \mathrm{Diff}(B^4 \operatorname{fix} \partial)$. The techniques of this paper also construct subgroups of $\pi_{n-3} \mathrm{Diff}(S^1 \times B^n \text{ fix } \partial)$ whenever $n \geq 3$.

Denote the component of the unknot in $\mathrm{Emb}(S^2, S^4)$ by $\mathrm{Emb}_u(S^2, S^4)$. A consequence of the above results is that $\mathrm{Emb}_u(S^2, S^4)$ does *not* have the homotopy type of the subspace of linear embeddings. The latter has the homotopy type of the Stiefel manifold $V_{5,3} = SO_5/SO_2$ while the former has a non-finitely-generated fundamental group. See Theorem 10.1.

We give a framework for approaching the smooth 4-dimensional Schönflies problem, describing the set of counter-examples as the fixed points of an endomorphism

$\pi_0 \operatorname{Emb}(B^3, S^1 \times B^3) \to \pi_0 \operatorname{Emb}(B^3, S^1 \times B^3)$. The endomorphism is given by lifting such an embedding to a non-trivial finite-sheeted covering space of $S^1 \times B^3$. The non-trivial elements of $\pi_0 \operatorname{Emb}(B^3, S^1 \times B^3)$ we construct in this paper all belong to the kernel of iterates of this endomorphism.

The paper is organized as follows. In Section §2 we compute the homotopy group $\pi_{n-2} \operatorname{Emb}(S^1, S^1 \times S^n)$ and show that it contains an infinitely-generated free abelian group provided $n \geq 3$, giving explicit generators θ_k, $k \geq 2$. This result extends the work of Dax [Da] who among other things computed $\pi_1(\operatorname{Emb}(S^1, S^1 \times S^3))$ in terms of certain cobordism groups and Arone-Szymik [AS], who show π_1 and π_2 of $\operatorname{Emb}(S^1, S^1 \times S^3)$ contain infinitely-generated free abelian groups. For $n = 3$ we describe other generators in terms of embeddings of tori $T : S^1 \times S^1 \to S^1 \times S^3$. In §3 we compute the three and five dimensional rational homotopy groups of $C_3[S^1 \times B^3]$ and use them to define the W_3 invariant of $\pi_2 \operatorname{Emb}(I, S^1 \times B^3)$ which takes values in a quotient of $\pi_5 C_3[S^1 \times B^3]$. We define $G(p, q)$ a 2-parameter family of $\operatorname{Emb}(I, S^1 \times B^3)$ and show that $W_3[G(p, q)]$ is equal to the class of the standard Whitehead product $t_1^p t_2^q [w_{23}, w_{13}]$. We also describe fibration sequences relating various embedding spaces and diffeomorphism groups and also show that $\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial)$ and $\operatorname{Diff}_0(S^1 \times S^3)$ respectively act transitively on reducing balls and spheres. In §4 we introduce geometric methods for working with 2-parameter families of $\operatorname{Emb}(I, M^4)$. In particular, we introduce a *bracket* operation that produces a 2-parameter family from two 1-parameter families that are null homotopic and have disjoint domain and range supports. In §5 we introduce the barbell map β of $S^2 \times D^2 \natural S^2 \times D^2$ which we call the *barbell neighborhood* $\mathcal{NB}$. β fixes $\partial \mathcal{NB}$ pointwise hence induces homotopically trivial diffeomorphisms of 4-manifolds called *implantations* when $\mathcal{NB}$ is embedded in a 4-manifold and β is pushed forward. We describe the implantations of $S^1 \times B^3 \operatorname{fix} \partial$ induced from the generating elements θ_k. We also compute $\beta(\Delta_0)$ of the standard separating 3-ball Δ_0. This is used in §6 to produce two parameter families $\hat{\theta}_k$ in $\operatorname{Emb}(I, S^1 \times B^3)$ arising from the θ_k's. We then show how to modify these families when twisting the implantation. In §7 we compute the class $[\hat{\theta}_k] \in \pi_2 \operatorname{Emb}(I, S^1 \times S^3)$ as the sum of elements of a $(k-1) \times (k-1)$-matrix A_k with entries a sum of $G(p, q)$'s. This matrix is skew symmetric and hence $[\hat{\theta}_k] = 0$. In §8 we show that the effect of twisting the θ_k implantation is to modify A_k by row and column operations. By twisting the θ_k implantations we produce the δ_k implantations, $k \geq 4$, whose homotopy classes are shown to be linearly independent by the W_3 invariant. Together with the triviality of the $[\hat{\theta}_k]$'s we conclude that the δ_k implantations in $\operatorname{Diff}_0(S^1 \times S^3)$ are linearly independent up to isotopy. In §9 we discuss the relation between knotted 3-balls and the smooth 4-dimensional Schoenflies conjecture. More applications are given in §10 and questions and conjectures are given in §11. In the Appendix we give a direct argument that $W_3[G(p, q)]$ is equivalent to the standard Whitehead product $t_1^p t_2^q [w_{23}, w_{13}]$ up to sign independent of p and q.

Independently, Tadayuki Watanabe [Wa2] has constructed an invariant

$$Z_1 : \pi_1 B \operatorname{Diff}(S^1 \times B^3 \text{ fix } \partial) / \pi_1 B \operatorname{Diff}(B^4 \text{ fix } \partial) \to \mathcal{A}_1(R[t^{\pm 1}])$$

and has shown it to be non-trivial on some diffeomorphisms created via his graph surgery construction.

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2. Embeddings of circles in $S^1 \times S^n$

In this section we describe a range of low-dimensional homotopy groups of $\mathrm{Emb}(S^1, S^1 \times S^n)$. These results were essentially known to Dax [Da], who used a Haefliger-style parametrized double-point elimination process to describe the low-dimensional homotopy groups of a variety of embedding spaces. Given an element of an embedding space $f \in \mathrm{Emb}(S^1, S^1 \times S^n)$ we will denote the path-component of $\mathrm{Emb}(S^1, S^1 \times S^n)$ containing f by $\mathrm{Emb}_f(S^1, S^1 \times S^n)$.

We begin with the least technical elements in Theorem 2.5, describing for $n \geq 3$, three epimorphisms:

$$W_0 : \pi_0 \,\mathrm{Emb}(S^1, S^1 \times S^n) \to \mathbb{Z}$$

$$W_1 : \pi_1 \,\mathrm{Emb}_f(S^1, S^1 \times S^n) \to \mathbb{Z}$$

$$W_2 : \pi_{n-2} \,\mathrm{Emb}_f(S^1, S^1 \times S^n) \to \Lambda_n^{W_0(f)}$$

The epimorphisms W_1 and W_2 are defined for all components of the embedding space. The group $\Lambda_n^{W_0}$ is defined as a quotient of the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$, and contains a free abelian subgroup of infinite rank. It can also contain 2-torsion.

For this definition we consider the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$ to be only a group. We define $\Lambda_n^{W_0}$ to be the quotient group, $\mathbb{Z}[t^{\pm 1}]$ modulo the subgroup generated by the relations

$$\langle t^k + (-1)^n t^{W_0 - 1 - k} = 0 \;\forall k, \quad t^0 = 0, t^{-1} = 0 \rangle.$$

The group $\Lambda_n^{W_0}$ is the free abelian group on the generators

$$G_{W_0} = \{t^k : k \in \mathbb{Z}, k \geq \frac{W_0 - 1}{2}, k \notin \{-1, 0, W_0, W_0 - 1\}\}$$

with the sole exception when n is even, $|W_0| > 2$ and W_0 is odd. In this case one has the same generating set G_{W_0}, but $t^{\frac{W_0-1}{2}}$ represents 2-torsion. The remaining t^k are free generators, i.e. $\Lambda_n^{W_0} \simeq \mathbb{Z}_2 \oplus \left(\oplus_{k > \frac{W_0-1}{2}, t^k \in G_{W_0}} \mathbb{Z} \right)$.

The definitions of the maps W_0, W_1 and W_2 will be elementary applications of basic transversality theory.

Definition 2.1. Let $\pi : S^1 \times S^n \to S^1$ be defined as $\pi(z, v) = z$. Given an embedding $f : S^1 \to S^1 \times S^n$, define $W_0(f) = deg(\pi \circ f) \in \mathbb{Z}$. This is the degree of the map $\pi \circ f : S^1 \to S^1$.

$$\begin{tikzcd} S^1 \arrow[rr, "\pi\circ f"] \arrow[dr, "f"'] & & S^1 \\ & S^1 \times S^n \arrow[ur, "\pi"'] & \end{tikzcd}$$

The value $W_0(f)$ only depends on the homotopy class of f. Provided $n \geq 3$, the homotopy class of f agrees with the isotopy class, by transversality. Thus

$$W_0 : \pi_0 \operatorname{Emb}(S^1, S^1 \times S^n) \to \mathbb{Z}$$

is a bijection. An embedding $f : S^1 \to S^1 \times S^n$ satisfying $\pi(f(z)) = z^n \ \forall z \in S^1$ would have $W_0(f) = n$.

Definition 2.2. Given $F : S^1 \to \operatorname{Emb}(S^1, S^1 \times S^n)$ we define

$$W_1(F) = deg(\hat{F}) \in \mathbb{Z}$$

where $\hat{F} : S^1 \to S^1$ is defined as $\hat{F}(z) = \pi(F(z)(1))$, i.e. we consider $F(z) \in \operatorname{Emb}(S^1, S^1 \times S^n)$ and we evaluate it at $1 \in S^1$.

We can consider W_1 to be a function

$$W_1 : \pi_1 \operatorname{Emb}_f(S^1, S^1 \times S^n) \to \mathbb{Z}.$$

As a thought experiment, argue that given $[F] \in \pi_1 \operatorname{Emb}(S^1, S^1 \times S^n)$ satisfying $W_1(F) = k$, one can assume $\pi(F(z)(1)) = z^k \ \forall z \in S^1$. More generally, one can show $\operatorname{Emb}(S^1, S^1 \times S^n) \simeq S^1 \times \operatorname{Emb}^*(S^1, S^1 \times S^n)$ where $\operatorname{Emb}^*(S^1, S^1 \times S^n)$ is the subspace of $\operatorname{Emb}(S^1, S^1 \times S^n)$ where $\pi \circ F(1) = 1$. From this perspective, W_1 is simply the induced map from the projection onto the first factor, i.e. $\operatorname{Emb}(S^1, S^1 \times S^n) \to S^1$.

An appealing way to think of the invariants W_0 and W_1 is via the inclusion $\operatorname{Emb}(S^1, S^1 \times S^n) \to \operatorname{Map}(S^1, S^1 \times S^n)$, i.e. we are including the embedding space in the space of all continuous functions from S^1 to $S^1 \times S^n$. Notice that W_0 and W_1 extend to invariants of $\pi_0 \operatorname{Map}(S^1, S^1 \times S^n)$ and $\pi_1 \operatorname{Map}(S^1, S^1 \times S^n)$ respectively. Moreover, as invariants of the homotopy-groups of $\operatorname{Map}(S^1, S^1 \times S^n)$ they are isomorphisms, since $\operatorname{Map}(S^1, S^1 \times S^n)$ splits as a direct product of two free loop spaces, $\operatorname{Map}(S^1, S^1 \times S^n) \equiv L(S^1) \times L(S^n)$. A simple general position argument tells us that the inclusion $\operatorname{Emb}(S^1, S^1 \times S^n) \to \operatorname{Map}(S^1, S^1 \times S^n)$ induces an epi-morphism on π_{n-2} and an isomorphism on π_k for $k < n-2$. The rough idea is that if one has a map from a k-dimensional manifold

$\phi : M \to \mathrm{Map}(S^1, S^1 \times S^n)$ one constructs the *track* of the map $\overline{\phi} : M \times S^1 \to M \times S^1 \times S^n$ given by $\overline{\phi}(p,z) = (p, \phi_p(z))$. By transversality, this map can be uniformly approximated by a smooth embedding if $2(k+1) < k+1+n$, i.e. $k \leq n-2$ (see for example Theorem 2.13 [Hir]). Such an approximation is no longer a track-type function on the nose, but given that the approximation is uniform in (at least) the C^1-topology, and the fact that diffeomorphisms of M form an open subset of the space of maps $\mathrm{Map}(M,M)$, one can apply a diffeomorphism of M (close to Id_M) to generate an approximation that is the track of an embedding. The first two non-trivial homotopy-groups of $\mathrm{Map}(S^1, S^1 \times S^n)$ are π_0 and π_1, both infinite cyclic. The next non-trivial homotopy-group is $\pi_{n-1} \mathrm{Map}(S^1, S^1 \times S^n) \simeq \mathbb{Z}$.

The next invariant W_2 has the form

$$W_2 : \pi_{n-2} \mathrm{Emb}_f(S^1, S^1 \times S^n) \to \Lambda_n^{W_0(f)}.$$

By the previous paragraph, it measures the lowest-dimensional deviation between the homotopy-types of $\mathrm{Emb}(S^1, S^1 \times S^n)$ and $\mathrm{Map}(S^1, S^1 \times S^n)$.

Definition 2.3. Let $C_2(M)$ denote the configuration space of pairs of distinct points in M,

$$C_2(M) = \{(p_1, p_2) \in M^2 : p_1 \neq p_2\}.$$

Denote the *cocircular pair* subspace of $C_2(S^1 \times S^n)$ by $\mathcal{CC} = \{(z_1, p_1), (z_2, p_2) \in C_2(S^1 \times S^n) : p_2 = p_1\}$. The cocircular pair subspace is $(n+2)$-dimensional, having co-dimension n in $C_2(S^1 \times S^n)$. Given $F : S^{n-2} \to \mathrm{Emb}(S^1, S^1 \times S^n)$, assume the induced map

$$\hat{F} : S^{n-2} \times C_2 S^1 \to C_2(S^1 \times S^n)$$

is transverse to $\mathcal{CC}$, where $\hat{F}(v, z_1, z_2) = (F(v)(z_1), F(v)(z_2))$. In such a situation we will associate $W_2(F) \in \Lambda_n^{W_0(f)}$.

Our polynomial will be akin to the transverse intersection number of $\hat{F}$ with $\mathcal{CC}$, but we include an enhancement into the definition. The set $\hat{F}^{-1}(\mathcal{CC})$ is Σ_2 invariant, and Σ_2 acts freely on $C_2(S^1)$. The invariant $W_2(F)$ will be a sum of monomials associated to the points of $\hat{F}^{-1}(\mathcal{CC})/\Sigma_2$. Given a point $q = (v, z_1, z_2) \in \hat{F}^{-1}(\mathcal{CC})$ we associate an element $\mathcal{L}_q(F) \in \mathbb{Z}[t^{\pm 1}]$ and define

$$W_2(F) = \sum_{[q] \in \hat{F}^{-1}(\mathcal{CC})/\Sigma_2} \mathcal{L}_q(F) \in \Lambda_n^{W_0}.$$

We define $\mathcal{L}_q(F) = \epsilon t^k$, where $\epsilon \in \{\pm 1\}$ is the local oriented intersection number of $\hat{F}$ with $\mathcal{CC}$ at q. Observe the map

$$S^n \times C_2(S^1) \ni (w, z_1, z_2) \longmapsto ((z_1, w), (z_2, w)) \in C_2(S^1 \times S^n)$$

is a diffeomorphism between $S^n \times C_2(S^1)$ and $\mathcal{CC}$. This is how we give $\mathcal{CC}$ its orientation. This map is also Σ_2-equivariant. The monomial degree k is computed via a pair of conventions. If $(z_1, z_2) \in C_2(S^1)$, let $[z_1, z_2]$ denote the counter-clockwise oriented arc in S^1 that starts at z_1 and ends at z_2. Similarly, given a point of $\mathcal{CC}$, $((z_1, p_1), (z_2, p_1))$, the *cocircular arc* with this boundary is denoted $[z_1, z_2] \times \{p_1\}$. When thinking of $S^1 \times S^n$ we refer to this as the *vertical* orientation. The monomial degree k is obtained by concatenating $F(v)([z_1, z_2])$ with the opposite-oriented cocircular arc in $S^1 \times S^n$ associated to $\hat{F}(v, z_1, z_2)$, and taking the degree of the projection to the S^1 factor of $S^1 \times S^n$. We depict an example in Figure 1, with the concatenation appearing in green. The unused portion of the vertical circle is in red. In this example $W_0(f) = 6$, and $\mathcal{L}_q(F) = \epsilon t^2$.

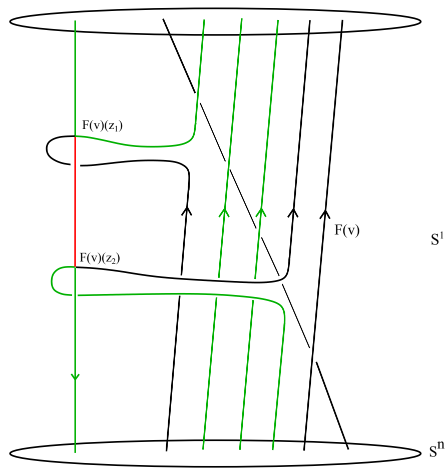


FIGURE 1. Example computation of monomial exponent, $\mathcal{L}_q(F) = \pm t^2$.

Notice that W_2 naturally factors as $W_2^* : \pi_{n-2} \operatorname{Emb}_f^*(S^1, S^1 \times S^n) \to \Lambda_n^{W_0(f)}$ after projecting-out the S^1 factor, using the product decomposition $\operatorname{Emb}(S^1, S^1 \times S^n) \simeq S^1 \times \operatorname{Emb}^*(S^1, S^1 \times S^n)$.

Given $(v, z_1, z_2) \in \hat{F}^{-1}(\mathcal{CC})$ then we also have $(v, z_2, z_1) \in \hat{F}^{-1}(\mathcal{CC})$ and one can check

$$\mathcal{L}_{(v,z_2,z_1)}(F) = (-1)^{n+1} t^{W_0 - 1} \overline{\mathcal{L}_{(v,z_1,z_2)}(F)}.$$

We use the notation $\bar{\cdot}$ to denote the $\mathbb{Z}$-linear mapping $\bar{\cdot} : \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[t^{\pm 1}]$ satisfying $\overline{t^k} = t^{-k}$. Thus $W_2(F)$ is well-defined for F. The relation $t^0 = 0$ in $\Lambda_n^{W_0}$ was chosen to ensure $W_2(F)$ is a homotopy-invariant of F. We use a compactification of configuration spaces to check homotopy-invariance.

Our manifold compactification of $C_2(S^1)$ is diffeomorphic to an annulus $S^1 \times [-1, 1]$. The boundary circles correspond to 'infinitesimal' configurations of pairs of points in S^1; one component where the direction vector from z_1 to z_2 agrees with the orientation of S^1, and the other being the reverse.

The Fulton-MacPherson compactified configuration space has the rather simple model of M^2 blown up along its diagonal $C_2[M] = Bl_{\Delta M} M^2$. Typically this is made formally precise by defining $C_2[M]$ to be the closure of the graph of a function [Si2], such as $\phi : C_2(M) \to S^k$ where $\phi(p, q) = \frac{p-q}{|p-q|}$, assuming $M \subset \mathbb{R}^{k+1}$. This compactification is functorial under embeddings of manifolds. The inclusion $C_2(M) \to C_2[M]$ is a homotopy-equivalence, i.e. $C_2[M]$ is diffeomorphic to M^2 remove an open tubular neighbourhood of the diagonal $\Delta M = \{(p, p) : p \in M\}$. There is a canonically-defined onto smooth map $C_2[M] \to M^2$, where the pre-image of ΔM is $\partial C_2[M]$, which is canonically isomorphic to the unit tangent bundle of M. The interior of $C_2[M]$ is mapped diffeomorphicly to $M^2 \setminus \Delta M$.

We now prove the homotopy-invariance of $W_2(F)$. Consider what happens in a homotopy of F. The boundary of $I \times S^{n-2} \times C_2[S^1]$ consists of the *temporal part* $(\partial I) \times S^{n-2} \times C_2[S^1]$ and the *annular part* $I \times S^{n-2} \times \partial C_2[S^1]$. The only monomial degrees that run off the annular part of the boundary are $t^{-1}, t^0, t^{W_0-1}, t^{W_0}$. For example, t^0 runs off the annular part if in our transverse family we have a tangent vector to our knot pointing in the vertical direction, oriented counter-clockwise. Similarly, t^{-1} can run off the boundary if we produce a tangent vector in the vertical direction, oriented clockwise. The monomials t^{W_0-1} and t^{W_0} are symmetric, after re-labelling the points of the domain $(z_1, z_2) \leftrightarrow (z_2, z_1)$.

Thus if we consider $W_2(F)$ to be an element of the quotient group $\Lambda_n^{W_0}$, it is a homotopy-invariant.

In Theorem 2.5 and Section 3 we need the notion of a half-ball.

Definition 2.4. Let $H^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}$ and define the half-ball $HB^n = B^n \cap H^n$. HB^n is a manifold with corners. As such, it is a stratified space with two co-dimension one strata, the *round boundary* $HB^n \cap \partial B^n$ and the *flat boundary* $HB^n \cap \partial H^n$. These two boundaries meet at the *corner (co-dimension two) stratum* $\{0\} \times S^{n-2}$.

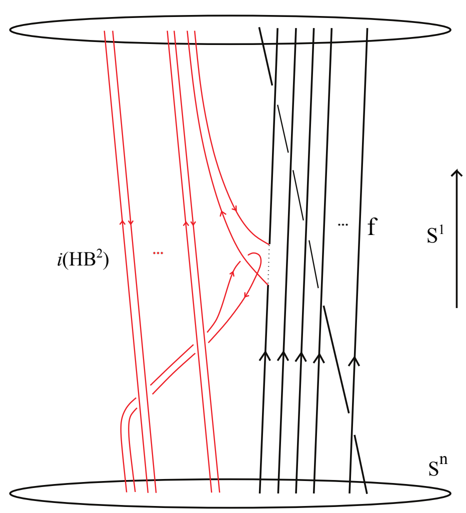


FIGURE 2. Constructing $\theta_{f,k}$.

Theorem 2.5. *Let $f \in \mathrm{Emb}(S^1, S^1 \times S^n)$. Provided $n \geq 3$ both W_1 and W_2 are epimorphisms. When $n = 3$ the map*

$$W_1 \times W_2 : \pi_1 \mathrm{Emb}_f(S^1, S^1 \times S^3) \to \mathbb{Z} \oplus \Lambda_3^{W_0}$$

is an epimorphism, i.e. $W_2^* : \pi_1 \mathrm{Emb}_f^*(S^1, S^1 \times S^3) \to \Lambda_3^{W_0}$ is an epi-morphism.

Proof. That W_1 is an epimorphism follows from the splitting $\mathrm{Emb}_f(S^1, S^1 \times S^n) \simeq S^1 \times \mathrm{Emb}^*_f(S^1, S^1 \times S^n)$, as it is the degree of the projection to the S^1 factor.

To argue that W_2 is an epimorphism, we start with the fixed degree W_0 near-linear embedding $f : S^1 \to S^1 \times S^n$, depicted in black in Figure 2.

Imagine an immersed half-ball $i : HB^2 \to S^1 \times S^n$ that is an embedding with the exception of a regular double point on the round boundary. We demand $i^{-1}(f(S^1))$ coincides with the flat boundary of HB^2. As in Figure 2 we demand that the map from the flat part of the boundary of $i(HB^2)$ to the vertical S^1 factor has no critical points. We further demand that the arc in HB^2 connecting the double points projects to a degree k map in the vertical S^1 factor, and that (as in Figure 2) the double-point occurs near the bottom of the flat boundary.

We modify the embedding f, creating a new immersed curve $S^1 \to S^1 \times S^n$ by replacing the arc $f(S^1) \cap i(\partial^f HB^2)$ with $i(\partial^r HB^2)$, i.e. we cut out the flat boundary of $i(HB^2)$ from $f(S^1)$ and replace it with the round boundary. This immersed curve is depicted in Figure 2 as the union of the solid black vertical broken curve (depicting f with the flat boundary removed), with the solid red curve (depicting the round boundary of $i(HB^2)$). Call this immersed curve $\tilde{\theta}_{f,k} : S^1 \to S^1 \times S^n$. Given that $n \geq 3$, we can assume the projection of $\tilde{\theta}_{f,k}$ to S^n is also an immersion, and embedding at all but the single double point.

The sum of the tangent spaces at the double-point of $\tilde{\theta}_{f,k}$ is 2-dimensional, so the orthogonal complement is $(n-1)$-dimensional, having an S^{n-2}-parameter family of unit normal vectors. Using a bump function, given a unit normal vector one can perturb one strand of $i(\partial^r HB^2)$ at the double-point, creating an embedded circle in $S^1 \times S^n$. This gives us our family of resolutions,

$$\theta_{f,k} : S^{n-2} \to \mathrm{Emb}_f(S^1, S^1 \times S^n).$$

The fact that the projection of $\tilde{\theta}_{f,k}$ to S^n was an embedding with the sole exception of the single double-point allows us to conveniently identify the cocircular points in our family $\theta_{f,k} : S^{n-2} \to \mathrm{Emb}_f(S^1, S^1 \times S^n)$, giving us $W_2(\theta_{f,k}) = t^k - t^{k-1}$. □

We have an involution of $\mathrm{Emb}(S^1, S^1 \times S^n)$ that negates the W_0 invariant. One description is the process that sends the embedding $f \in \mathrm{Emb}(S^1, S^1 \times S^n)$ to $z \longmapsto f(z^{-1})$. Call this embedding $\overline{f}$. Then we have $\overline{\theta_{f,k}} = \theta_{\overline{f},-k-1}$, i.e. the θ elements are symmetric about $-1/2$.

Another family $i : HB^2 \to S^1 \times S^n$ to consider is one where i is an embedding. We demand that $i(HB^2)$ intersects f along the flat boundary, and also at a single point along the round boundary – a regular double point. Consider the case where the projection of the embedding i to the S^1 factor is not onto, i.e. it is constrained to an interval in the S^1 factor. Then i connects one strand of f to adjacent strands. Let's say to the k-th strand (using the cyclic ordering) with $k \in \{0, 1, \cdots, W_0 - 1\}$, assuming $W_0 > 0$. Call the resolved family of knots $\gamma_{f,k} : S^{n-2} \to \mathrm{Emb}(S^1, S^1 \times S^n)$. Given that, we have $W_2(\gamma_{f,k}) = t^k - t^{k-1}$. Recall that $\Lambda_n^{W_0} = \mathbb{Z}[t^{\pm 1}]/\langle t^{-1}, t^0, t^k + (-1)^k t^{W_0-1-k}\ \forall k\rangle$. Thus $\{\gamma_{f,k} : k \in$

$\{0, 1, \cdots, W_0 - 1\}\}$ spans the same subspace of $\Lambda_n^{W_0}$ as the monomials $\{t, t^2, \cdots, t^{W_0-2}\}$, i.e. all the intermediate monomials that were not killed by the definining relations of $\Lambda_n^{W_0}$.

We give an alternative way to visualize elements of $\pi_1 \operatorname{Emb}_f(S^1, S^1 \times S^3)$ with $W_2 \neq 0$ by embedded tori, where f denotes the standard generator of $\pi_1(S^1)$, i.e. the $W_0 = 1$ component. Each generator will be represented by an embedded torus $T \subset S^1 \times S^3$ which contains the curve $\gamma_0 = S^1 \times \{y_0\}$. Such a torus T gives rise to an element z of $\pi_1 \operatorname{Emb}(S^1, S^1 \times S^3)$ by fibering T by parametrized smooth circles $\{\gamma_t | t \in [0,1]\}$ with $\gamma_0 = \gamma_1$. Once γ_0 is chosen, what really matters is which way to go around the torus. To do this and control W_1, we choose an oriented simple closed curve $\mu_w \subset T$, homotopically trivial in $S^1 \times S^3$, that intersects γ_0 transversely once at some point $w = (x_0, y_0) \in S^1 \times S^3$. The homotopy condition implies that that $W_1(z) = 0$ and the orientation informs us that γ_0 is required to spin about T so that w follows μ_w in the oriented direction. Denote by (T, μ_w) the represented element of $\pi_1 \operatorname{Emb}_f(S^1, S^1 \times S^3)$.

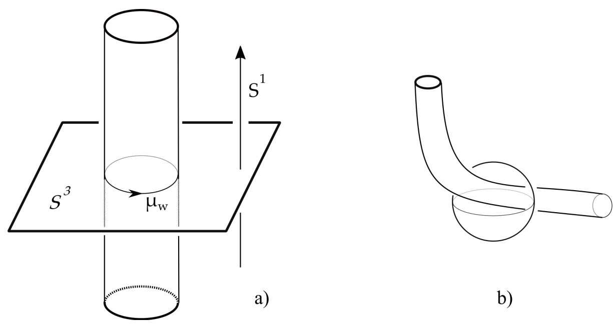


FIGURE 3. (a) Vertical torus in $S^3 \times S^1$. Vertical fibers represent trivial element in $\pi_1 \operatorname{Emb}(S^1, S^3 \times S^1)$, in the component with $W_0 = 1$.
(b) Sphere linking tube in 4-space.

The standard vertical torus T^*, shown in Figure 3(a) represents the trivial element of $\pi_1 \operatorname{Emb}(S^1, S^1 \times S^3)$. Figures 4 (a) and (b) describe embedded tori corresponding to t^2 and t^3 respectively. In our diagrams, $\mu_w \subset \{x_0\} \times S^3$, with S^3 being depicted horizontally as in Figure 3 (b). In a similar manner we obtain a torus corresponding to $t^n, |n| \geq 1$. Each of our tori is constructed by tubing T^* with an unknotted, unlinked 2-sphere as follows. Emanating from the boundary of a small disc on T^* the tube first links the sphere, then goes $n \in \mathbb{N}$ times around the S^1 factor before finally connecting to the 2-sphere. Figures 4 (a) and (b) show the projection of T to $S^1 \times (S^2 \times 0)$ where S^3 is identified with $S^2 \times [-\infty, \infty]$, where each component of $S^2 \times \{\pm\infty\}$ is identified to a point. By construction $T \subset S^1 \times (S^2 \times \{0\})$, except for where the tube links the 2-sphere. See Figure 3(b) for a detail. The crossing convention for the tube and sphere informs us that the part of the tube that projects to the right side of the 2-sphere lives a bit in the past (i.e. in $S^1 \times (S^2 \times [-1, 0))$ and the part of the tube on the left lives in the future. By construction, the 2-sphere bounds a 3-ball $B \subset S^1 \times (S^2 \times \{0\})$ that intersects the tube is a single simple closed curve. By either reversing the way the tube links the 2-sphere, or reversing the orientation on μ_w we obtain the inverse of the generator. See Figure 6(b).

Proposition 2.6 relates the generators $\alpha_{f,k}$ and $\theta_{f,k}$, described above. To make this proposition precise, and not simply up to a choice of sign, we need to provide an orientation to the parametrizing sphere S^{n-2} of our generators $\theta_{f,k} : S^{n-2} \to \mathrm{Emb}_f(S^1, S^1 \times S^n)$. The parametrizing sphere came up as the unit normal sphere to the sum of the tangent spaces at the double point. Given this proposition is only for $n = 3$, our S^{n-2} is a circle. We choose the orientation consistent with the rotation from above the double point (i.e. the positive vertical direction) to the *into the page* direction. This gives us the formula below.

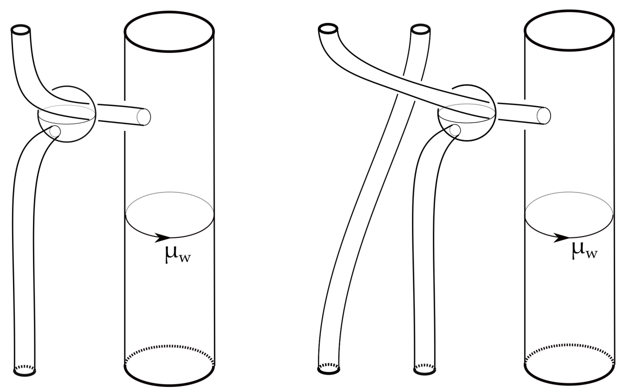


FIGURE 4. (a) Torus representing $\alpha_{f,1}$ with $W_0(f) = 1$, $W_2 = t^2 - t^0 = t^2 \neq 0$
(b) Torus representing $\alpha_{f,2}$ with $W_0(f) = 1$, $W_2 = t^3 - t^1 = t^3 \neq 0$

Proposition 2.6. $\alpha_{1,i} = \theta_{1,i+1} - \theta_{1,i}$.

Proof. The proof is by directly constructing a homotopy between representatives. We demonstrate it for $i = 1$ with the general case being similar. As above, we view $S^1 \times S^3$ as a quotient of $S^1 \times S^2 \times [-\infty, \infty]$. In what follows all the figures, except for f) live in $S^1 \times S^2 \times \{0\}$. Figure 5(a) depicts the constant loop $\kappa_t^a, t \in I$, with each κ_t^a representing the standard $W_0 = 1$ curve κ. Now Figure 5(b) also represents the $W_2 = 0$ loop κ_t^b.

Here $\kappa_t^b = \kappa_t^a$ for $t \in [0, 1/8] \cup [7/8, 1]$. During $t \in [1/8, 1/4]$ it sweeps to the curve shown in b) and stays there for $t \in [1/4, 3/4]$ before sweeping back to κ. Next we modify this loop to κ_t^c representing $-\theta_{1,1}$. Consider the half disc E_1. Here $\partial^f(E_1) \subset \kappa_{1/2}^b$ and $\partial^r(E_i)$ locally *links* $\kappa_{1/2}^b$. Define $\kappa_t^c = \kappa_t^b, t \in [0, 1/4] \cup [3/4, 1]$. During $t \in [1/4, 1/2]$, keeping endpoints fixed, $\partial^f(E_1)$ sweeps across E_1 to end at $\partial^r(E_1)$. The interior of each arc, for $t \in (1/4, 1/2)$ is pushed slightly into the future, i.e. into $S^1 \times S^2 \times s$ for $s > 0$. During $t \in [1/2, 3/4]$, keeping endpoints fixed $\partial^r(E_1)$ is pushed back to $\partial^f(E_1)$. Here for $t \in (1/2, 3/4)$, the interior of each arc is pushed into the past. We next modify this loop to κ_t^d representing $-\theta_{1,1} + \theta_{1,2}$ as shown in Figure 5(d). We have abused notation by calling one of the half discs of d) also E_1. Again, keeping endpoints fixed the arcs

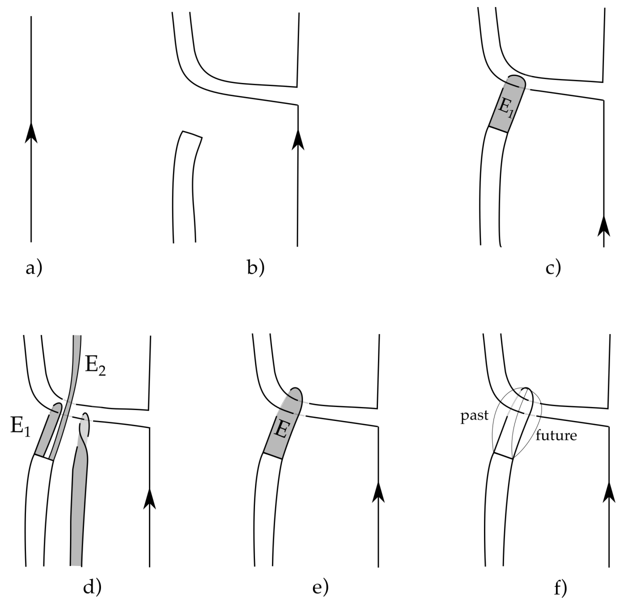


FIGURE 5. Associated to Proposition 2.6

$\partial^f(E_1) \cup \partial^f(E_2)$ sweep to $\partial^r(E_1) \cup \partial^r(E_2)$ and then back again with interiors of arcs in the first (resp. second) part of the motion in the future (resp. past). Note, that the twist in the half disc E_2, which lies in $S^1 \times S^2 \times 0$, gives rise to $\theta_{1,2}$ rather than its inverse. The loop κ_t^d is homotopic to κ_t^e where here the half disc E is used. Again, the homotopy is supported in $[1/4, 3/4]$ and κ_t^e has the feature that keeping endpoints fixed $\partial^f(E)$ sweeps, pushed slightly into the future, across E to $\partial^r(E)$, and then sweeps back to $\partial^f(E)$ again with interiors of arcs pushed slightly into the past. Thus Figure 5(f) also represents κ_t^e, with the sphere being the image of the track of $\partial^f(E)$ as it sweeps to $\partial^f(E)$ and back. Finally, it is readily checked that κ_t^e represents $\alpha_{1,1}$. □

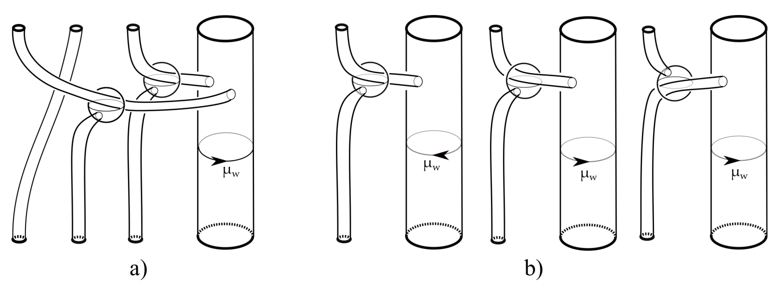


FIGURE 6. (a) Torus with $W_2 = t^2 + t^3$
(b) All three tori with $W_2 = -t^2$

We describe how to represent composition of generators when $f = \kappa$, the general case being similar. First some terminology. Let $p : T^* \to \mu_w$ be the vertical projection. By construction, each generator is obtained by removing a small disc D from $T^* \setminus \kappa$ and replacing it by a disc D'. Further each knotted disc lies in a small neighborhood of a 1-complex which itself lies in a neighborhood of $p^{-1}(\delta)$, for some interval $\delta \subset \mu_w$. Squeezing, expanding or rotating this interval and correspondingly modifying the discs D and D' does not change the based homotopy class of (T, μ_w) provided that the expanding or rotating is supported away from w. The composition of generators β_0 and β_1 is represented as follows. First find tori $(T_0, \mu_w), (T_1, \mu_w)$ constructed as above respectively representing β_0, β_1 so that T_0 coincides with T_1 near μ_w and the latter having a fixed orientation. Further, assume that each T_i is standard away from a neighborhood of $p^{-1}(\delta_i)$ where $\delta_0 \cap \delta_1 = \emptyset$ and δ_0 proceeds δ_1 when starting at $x \in \mu_w$. To obtain (T, μ_w) representing $\beta_0 * \beta_1$, modify T^* near both $p^{-1}(\delta_i), i = 0, 1$ according to T_0 and T_1. See Figure 6(a). To see that $\beta_0 * \beta_1$ is homotopic to $\beta_1 * \beta_0$ observe that the two tori representing these classes are isotopic via an isotopy fixing $\kappa \cup \mu_w$ pointwise. We conclude that any word in the generators is realizable by an embedded torus and $\pi_1 \operatorname{Emb}_f(S^1, S^1 \times S^3)$ is abelian.

Proposition 2.7. *Let $f : S^1 \to S^1 \times S^3$ be the standard vertical embedding with $W_0(f) = 1$. Let $p : S^1 \times S^3 \to S^1 \times S^3$ the m-fold cyclic cover, let $\{\alpha_i | i > 0\}$ denote the generators of $\pi_1(\operatorname{Emb}_f(S^1, S^1 \times S^3))$ as in Theorem 2.5 and let $\tilde{\alpha}_i$ denote the p^* pull back of α_i. Then $\tilde{\alpha}_n = 1$ if m does not divide n and $\tilde{\alpha}_n = m\alpha_{n/m}$ if m divides n.*

Proof. Represent α_n by the torus T_n as in §2. Then $\tilde{\alpha}_n$ is represented by $p^{-1}(T_n) = R_n$. Now T_n is constructed from the standard vertical torus T^* and an unknotted 2-sphere K by removing small discs D^T, D^K from T^* and K, and then adding a tube Y that starts at ∂D^T, links through K, goes n times about the S^1 direction before connecting to ∂D^K. Therefore R_n is obtained by removing m small discs from T^* and one from each of the preimages of K and then connecting their boundaries by the m preimages $Y_1, \cdots, Y_m$ of Y. I.e. R_n is obtained by removing m standard discs from T^* and replacing them by m other ones. Now assume that m divides n. Then the sphere K_i that Y_i links is also the sphere to which Y_i connects. Note that if D_i^T (resp. D_i^K) is the preimage of D^T (resp. D^K) whose boundary is tubed to Y_i, then $((T^* \cup K_i) \setminus (D_i^T \cup D_i^K)) \cup Y_i$ is a torus representing $\alpha_{n/m}$. After an isotopy of R_n supported near the D_i^T's we can assume that the projection $\pi : T^* \to \mu_w$ has the property that $\pi(D_1^T), \cdots, \pi(D_n^T)$ are disjoint intervals. (Recall that $\mu_w \subset T^*$ is an oriented loop intersecting each vertical S^1 fiber once.) It follows that R_n represents an element β of $\pi_1(\operatorname{Emb}(S^1, S^1 \times S^3))$ which corresponds to the standard vertical circle sweeping around T^* and going over one knotted disc at a time. Since there are m such discs it follows that $\beta = m\alpha_{n/m}$.

Now assume that m does not divide n. In that case each tube Y_i links a sphere distinct from the one to which it connects. Again isotope R_n near the D_i^T's so that the projection $\pi : T^* \to \mu_w$ has the property that $\pi(D_1^T), \cdots, \pi(D_n^T)$ are disjoint intervals. Again let β be the element represented by R_n. Here the discs swept over by β can be individually isotoped back to their D_i^T's without intersecting T^*. It follows that $\beta = 1$. $\square$

We return to the problem of determining if our invariants W_0, W_1 and W_2 are complete invariants of the low-dimensional homotopy groups of $\operatorname{Emb}(S^1, S^1 \times S^n)$.

Lemma 2.8. *The homotopy groups of $C_2(S^1 \times S^n)$ are:*

$$\pi_1 C_2(S^1 \times S^n) \simeq \mathbb{Z}^2$$

$$\pi_m C_2(S^1 \times S^n) \simeq \pi_m S^n \oplus \pi_m(S^n)[t^{\pm 1}]$$

for $m \geq 2$. The symbol $\pi_m(S^n)[t^{\pm 1}]$ denotes the Laurent polynomials with coefficients in the group $\pi_m(S^n)$, thus when $m = n$ it is the Laurent polynomial ring with integer coefficients.

The boundary of $C_2(S^1 \times S^n)$ can be canonically identified with $S^1 \times S^n \times S^n$ using our preferred trivialization of $T(S^1 \times S^n)$. Thus $\pi_1 \partial C_2(S^1 \times S^n) \simeq \mathbb{Z}$ and $\pi_n \partial C_2(S^1 \times S^n) \simeq \mathbb{Z}^2$. We compute the induced map on the above homotopy groups for the inclusion map $\partial C_2(S^1 \times S^n) \to C_2(S^1 \times S^n)$. To make sense of this map we need a common choice of basepoint. Identify $\partial C_2(S^1 \times S^n)$ with the unit sphere bundle of $S^1 \times S^n$. Our basepoint will be the direction vector pointing in the counter-clockwise direction of S^1, based at $(1, *)$ where $* \in S^n$ is any basepoint choice for S^n.

The induced map on π_1 is identified with the diagonal map $\Delta : \mathbb{Z} \to \mathbb{Z}^2$, $\Delta(t) = (t, t)$. The induced map on π_n is identified with $\mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}[t^{\pm 1}]$, which in matrix form is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 - t^{-1} \end{pmatrix}.$$

The above computation requires a choice of common basepoint in $\partial C_2(S^1 \times S^n)$ and $C_2(S^1 \times S^n)$, and is valid for any such choice.

These isomorphisms follow from the fact that the fiber bundle

$$Bl_*(S^1 \times S^n) \to C_2[S^1 \times S^n] \to S^1 \times S^n$$

has a section. There are several sections available: (1) using the trivialization of $T(S^1 \times S^n)$ or (2) using the antipodal map of S^1 or S^n or the combination of the two. All of these sections are homotopic. The section (1) is the only choice that allows for a common base-point in $C_2(S^1 \times S^n)$ and its boundary.

Theorem 2.9. *The invariant*

$$W_1 \oplus W_2 : \pi_1 \operatorname{Emb}_f(S^1, S^1 \times S^3) \to \mathbb{Z} \oplus \Lambda_3^{W_0}$$

is an isomorphism for all $f \in \operatorname{Emb}(S^1, S^1 \times S^3)$. Stated another way, $W_2^* : \pi_1 \operatorname{Emb}_f^*(S^1, S^1 \times S^3) \to \Lambda_3^{W_0}$ is an isomorphism for all $f \in \operatorname{Emb}_f^*(S^1, S^1 \times S^3)$, i.e. the components of $\operatorname{Emb}(S^1, S^1 \times S^3)$ have infinitely-generated, free-abelian fundamental groups.

We also prove an analogous theorem for $\operatorname{Emb}(S^1, S^1 \times S^n)$ with $n \geq 4$.

Theorem 2.10. *For $n \geq 4$, the first three non-trivial homotopy groups of the embedding space $\operatorname{Emb}(S^1, S^1 \times S^n)$ are given by the maps:*

1. *$W_0 : \pi_0 \operatorname{Emb}(S^1, S^1 \times S^n) \to \mathbb{Z}$ which is an isomorphism.*
2. *$W_1 : \pi_1 \operatorname{Emb}_f(S^1, S^1 \times S^n) \to \mathbb{Z}$ which is also an isomorphism, for any choice of path-component, i.e. $f \in \operatorname{Emb}(S^1, S^1 \times S^n)$.*
3. *$W_2 : \pi_{n-2} \operatorname{Emb}_f(S^1, S^1 \times S^n) \simeq \Lambda_n^{W_0}$ and it is also an isomorphism, for any choice of path component $f \in \operatorname{Emb}(S^1, S^1 \times S^n)$.*

As was described between Definitions 2.2 and 2.3, a general-position argument tells us the forgetful map $\mathrm{Emb}(S^1, S^1 \times S^n) \to \mathrm{Map}(S^1, S^1 \times S^n)$ is an isomorphism on all homotopy groups π_k for $k < n-2$, and an epi-morphism on π_{n-2}. Moreover, the space $\mathrm{Map}(S^1, S^1 \times S^n)$ splits as the product of two free loop-spaces $\mathrm{Map}(S^1, S^1 \times S^n) \simeq L(S^1) \times L(S^n)$, which proves claims (1) and (2) in Theorem 2.10. The primary role of Theorems 2.9 and 2.10 is the description of the first homotopy-group of the embedding space $\mathrm{Emb}(S^1, S^1 \times S^n)$ that differs from that of the mapping space $\mathrm{Map}(S^1, S^1 \times S^n)$. By Theorem 2.5 we know this happens in dimension $n-2$. The homotopy group $\pi_{n-2}\,\mathrm{Emb}_f(S^1, S^1 \times S^n)$ contains a large abelian subgroup, detected by the W_2 invariant. The purpose of Theorems 2.9 and 2.10 is to argue W_2 (and W_1 if $n = 3$) detects all non-trivial elements of $\pi_{n-2}\,\mathrm{Emb}_f(S^1, S^1 \times S^n)$, provided $n \geq 3$.

We will use a tool called *functor calculus* in the context of embedding spaces $\mathrm{Emb}(M, N)$ to prove Theorems 2.9 and 2.10. Although everything needed to prove these two theorems is present in the work of Dax [Da], we choose to use embedding calculus to situate the proof in a contemporary context. It should be noted that Theorems 2.9 and 2.10 are not essential to any of the results highlighted in the introduction.

The embedding calculus gives us a sequence of maps out of embedding spaces

$$\mathrm{Emb}(M, N) \to T_k\,\mathrm{Emb}(M, N)$$

where M is an m-dimensional manifold and N is an n-dimensional manifold. The k-th *evaluation map* $\mathrm{Emb}(M, N) \to T_k\,\mathrm{Emb}(M, N)$ is known to be $k(n-m-2)+1-m$-connected. This means that for any choice of path-component of $\mathrm{Emb}(M, N)$ the induced map $\pi_j\,\mathrm{Emb}(M, N) \to \pi_j T_k\,\mathrm{Emb}(M, N)$ is an isomorphism for $j < k(n-m-2)+1-m$ and an epimorphism for $j = k(n-m-2)+1-m$. This connectivity result is only valid provided $n \geq m+3$, i.e. it requires embeddings to be of co-dimension 3 or larger. In our case, $M = S^1$ is a 1-manifold and $N = S^1 \times S^{n-1}$, thus the k-th evaluation map is $k(n-3)$-connected. This tells us that we need only compute $\pi_{n-2}\left(T_2\,\mathrm{Emb}(S^1, S^1 \times S^n)\right)$ to verify Theorems 2.9 and 2.10.

Our invariant W_2 is *almost* defined on $T_2\,\mathrm{Emb}(S^1, S^1 \times S^n)$. Specifically, $T_2\,\mathrm{Emb}(S^1, S^1 \times S^n)$ is described as a homotopy pull-back of three familiar spaces in Corollary 4.3 of the paper of Goodwillie and Weiss [GW2]. Readers unfamiliar with homotopy pull-backs, or homotopy-limits of diagrams of the form $X \xrightarrow{f} Z \xleftarrow{g} Y$, see Definition 3.2.4 of the book [MV] which provides useful context. In short, such a homotopy pullback is denoted *holim*($X \xrightarrow{f} Z \xleftarrow{g} Y$). This is the space of triples

$$\{(x, \alpha, y) : x \in X, \alpha : [0,1] \to Z, y \in Y, \text{ s.t. } \alpha(0) = f(x), \alpha(1) = g(y)\}.$$

The element α is a continuous path between $f(x)$ and $g(y)$. We will describe $T_2\,\mathrm{Emb}(S^1, S^1 \times S^n)$ as a homotopy pullback of a diagram of three spaces, as in Corollary 4.3 of [GW2].

(1) $\mathrm{Map}(S^1, S^1 \times S^n)$, i.e. this is the space of continuous functions from S^1 to $S^1 \times S^n$.
(2) $\mathrm{Map}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2)$, this is the space of Σ_2-equivariant continuous functions from $(S^1)^2$ to $(S^1 \times S^n)^2$, where the Σ_2-action on the two spaces comes from permuting coordinates.

(3) $\mathrm{ivmap}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2)$, this is the space of *strictly isovariant maps*. This is a subspace of (2) where the maps f have the additional properties that $f^{-1}(\Delta_{S^1 \times S^n}) = \Delta_{S^1}$, i.e. the diagonal subspace of $(S^1)^2$ is the only subspace sent to the diagonal subspace of $(S^1 \times S^n)^2$, where $\Delta_M = \{(p,p) : p \in M\} \subset M^2$. The other condition is the derivative of f is fibrewise injective from the normal bundle of Δ_{S^1} (in $(S^1)^2$) to the normal bundle of $\Delta_{S^1 \times S^n}$ (in $(S^1 \times S^n)^2$).

The result of Goodwillie and Weiss [GW2] is that $T_2 \operatorname{Emb}(S^1, S^1 \times S^n)$ is the homotopy pullback of the diagram

$$\mathrm{ivmap}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2) \longrightarrow \mathrm{Map}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2) \longleftarrow \mathrm{Map}(S^1, S^1 \times S^n)$$

where the first map is set-theoretic inclusion. The second map $\mathrm{Map}(S^1, S^1 \times S^n) \to \mathrm{Map}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2)$ is given by *repetition* i.e. if $f \in \mathrm{Map}(S^1, S^1 \times S^n)$ then the equivariant map of pairs is given by $(z_1, z_2) \longmapsto (f(z_1), f(z_2))$.

Proposition 2.11. *The forgetful map* $T_2 \operatorname{Emb}(S^1, S^1 \times S^n) \to \mathrm{ivmap}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2)$ *induces an isomorphism on homotopy groups* π_k *for* $k \leq n-2$*, on all path components. Moreover, the space of isovariant maps* $\mathrm{ivmap}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2)$ *is homotopy-equivalent to the space of stratum-preserving* Σ_2*-equivariant maps* $C_2[S^1] \to C_2[S^1 \times S^n]$.

Proof. The forgetful map $holim(\ X \xrightarrow{f} Z \xleftarrow{g} Y\ ) \to X$ is the map that maps a triple $(x, \alpha, y) \longmapsto x$. As is described in Example 3.2.8 [MV], the fibre over a point $x_* \in X$ is $holim(\ \{x_*\} \xrightarrow{f} Z \xleftarrow{g} Y\ )$, and this can be identified with the homotopy-fibre of $g : Y \to Z$ over $f(x_*)$.

In our case we are interested in the forgetful map from the homotopy limit of

$$\mathrm{ivmap}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2) \longrightarrow \mathrm{Map}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2) \longleftarrow \mathrm{Map}(S^1, S^1 \times S^n)$$

to $\mathrm{ivmap}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2)$. Let's investigate the homotopy-groups of

$$\mathrm{hofib}_{W_0}\left(\mathrm{Map}(S^1, S^1 \times S^n) \to \mathrm{Map}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2)\right).$$

Given that this map is the repetition map, it is split. The splitting comes from restriction to the Σ_2-fixed subspaces of $(S^1)^2$ and $(S^1 \times S^n)^2$ respectively, i.e. the diagonals. This tells us that the homotopy-groups of hofib are the kernel of the induced maps $\pi_k \mathrm{Map}(S^1, S^1 \times S^n) \to \pi_k \mathrm{Map}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2))$. These groups are trivial when $k \leq n-2$, since $\pi_k \mathrm{Map}(S^1, S^1 \times S^n)$, with the exceptions of $k = 0$ or $k = 1$, in which case the repetition map is injective. Thus the map

$$\pi_{n-2} T_2 \operatorname{Emb}(S^1, S^1 \times S^n) \to \pi_{n-2}\left(\mathrm{ivmap}^{\Sigma_2}((S^1)^2, (S^1 \times S^n)^2)\right)$$

is always an isomorphism.

Regarding the claim that the space of strictly isovariant maps is homotopy-equivalent to the space of stratum-preserving Σ_2-equivariant maps $C_2[S^1] \to C_2[S^1 \times S^n]$, recall that a map of Fulton-Macpherson compactified configuration spaces descends to a map $(S^1)^2 \to (S^1 \times S^n)^2$. Given that our initial map was assumed to be Σ_2-equivariant, the induced map will be as well. Lastly, using the uniqueness of collar neighbourhoods theorem, one can assume all our maps $C_2[S^1] \to C_2[S^1 \times S^n]$ are fibrewise linear with respect to the distance parameter from the boundary – this is enough to guarantee our induced map $(S^1)^2 \to (S^1 \times S^n)^2$ is isovariant. Similarly, given a strictly isovariant map, one can lift it to a unique map of the Fulton-Macpherson compactified configuration spaces. $\square$

Proof. (of Theorem 2.9) The space $C_2[S^1]$ is simply an annulus, i.e. diffeomorphic to $S^1 \times [-1, 1]$. The space $C_2[S^1 \times S^3]$ is diffeomorphic to $(S^1 \times S^3) \times Bl_1(S^1 \times S^3)$, given by the map

$$C_2(S^1 \times S^3) \ni (z_1, p_1), (z_2, p_2) \longmapsto (z_1, p_1), (z_1 z_2^{-1}, p_1 p_2^{-1}) \in S^1 \times S^3 \times (S^1 \times S^3) \setminus \{1\}.$$

The blow-up $Bl_1(S^1 \times S^3)$ deformation-retracts to its 3-skeleton $S^1 \vee S^3$.

The boundary of $C_2[S^1 \times S^3]$ is canonically diffeomorphic to the unit tangent bundle of $S^1 \times S^3$. Due to the triviality of $T(S^1 \times S^3)$ we can think of the unit tangent bundle of $S^1 \times S^3$ as $S^1 \times S^3 \times S^3$.

The fundamental group $\pi_1 C_2[S^1 \times S^3]$ is free abelian on two generators, see Lemma 2.8. The natural set of generators are given by the winding numbers of the first and second points of the configurations about the S^1 factor of $S^1 \times S^3$. Consider the covering space of $C_2[S^1 \times S^3]$ corresponding to the homomorphism $\pi_1 C_2[S^1 \times S^3] \to \mathbb{Z}$ given by taking the difference between the two winding numbers. We denote this covering space by $\tilde{C}_2[S^1 \times S^3]$. By design, any stratum-preserving map $C_2[S^1] \to C_2[S^1 \times S^3]$ lifts to this covering space, $C_2[S^1] \to \tilde{C}_2[S^1 \times S^3]$. Since $C_2[S^1 \times S^3]$ fibers over $S^1 \times S^3$, this covering space does as well, but the fiber is the universal cover of $Bl_*(S^1 \times S^3)$, which could be described as $Bl_{\mathbb{Z} \times \{1\}}(\mathbb{R} \times S^3)$, giving

$$\tilde{C}_2[S^1 \times S^3] \simeq Bl_{\mathbb{Z} \times \{1\}}(\mathbb{R} \times S^3) \times S^1 \times S^3.$$

Consider a map $S^1 \times C_2[S^1] \to \tilde{C}_2[S^1 \times S^3] \equiv Bl_{\mathbb{Z} \times \{1\}}(\mathbb{R} \times S^3) \times S^1 \times S^3$. We can assume this map is *null* in the rightmost S^3 factor, as it is homotopic to a map that factors through a 2-dimensional domain. Thus we have reduced the computation of the fundamental group of this mapping space to understanding the space of stratum-preserving maps

$$C_2[S^1] \to Bl_{\mathbb{Z} \times \{1\}}(\mathbb{R} \times S^3) \times S^1.$$

If we take the degree of the projection to the S^1 factor we recover W_0. Given a family $S^1 \times C_2[S^1] \to Bl_{\mathbb{Z} \times \{1\}}(\mathbb{R} \times S^3) \times S^1$, the homotopy-class of the projection to the S^1 factor is determined by the W_0 and W_1 invariants.

Consider the projection $C_2[S^1] \to Bl_{\mathbb{Z} \times \{1\}}(\mathbb{R} \times S^3)$. By design, one boundary stratum is in the blow-up sphere corresponding to $k \times \{1\}$, and the other stratum is in the blow-up sphere corresponding to $(k + W_0) \times \{1\}$. The space $Bl_{\mathbb{Z} \times \{1\}}(\mathbb{R} \times S^3)$ has the homotopy type of an infinite wedge of 3-spheres, perhaps best thought of as the 3-skeleton of

$$\left(\mathbb{R} \times S^3\right) \setminus (\mathbb{Z} \times \{1\}) \simeq (\mathbb{R} \times \{-1\}) \cup \left(\sqcup_{i \in \mathbb{Z}} \{\frac{1}{2} + i\} \times S^3\right).$$

To describe the equivariance condition on our lift $C_2[S^1] \to Bl_{\mathbb{Z}\times\{1\}}(\mathbb{R} \times S^3)$, the relevant Σ_2-action on the target space is induced by the map $(t, p) \longmapsto (2j + W_0 - t, p^{-1})$. Choosing $j = 0$ gives us the same convention as in Theorem 2.5. This computation is done by considering the diffeomorphism $C_2(S^1 \times S^3) \to (S^1 \times S^3) \times ((S^1 \times S^3) \setminus \{1\})$. The involution of $C_2(S^1 \times S^3)$ in the Σ_2-action sends $(z_1, p_1), (z_2, p_2)$ to $(z_2, p_2), (z_1, p_1)$. Conjugating this involution by our identification gives us the map $(z_1, p_1), (z_2, p_2) \to (z_2^{-1} z_1, p_2^{-1} p_1), (z_2^{-1}, p_2^{-1})$, which, on the fiber lifts to the above map.

Homotopy classes of maps to wedges of spheres, via the Ponyriagin construction, are characterized by their intersection numbers with the points antipodal to the wedge point. Our maps are equivariant, so our framed points in the domain satisfy a symmetry condition. Our space $Bl_{\mathbb{Z}\times\{1\}}(\mathbb{R} \times S^3)$ equivariantly deformation retracts to the above 3-skeleton. The Σ_2-stabilizer is a single point if W_0 is even, and a pair of points $(\frac{W_0}{2}, \pm 1) \subset \{\frac{W_0}{2}\} \times S^3$ if W_0 is odd. Since the Σ_2-action on the domain is fixed-point free, all this tells us is the degree associated to this intermediate sphere is even. Thus, if W_2 is zero, we can equivariantly homotope our map so that its image is disjoint from the antipodal points to all the wedge points of the S^3 factors. We can therefore assume our map $C_2[S^1] \to \mathbb{R} \times \{-1\} \cup \left(\sqcup_{i \in \mathbb{Z}} \{\frac{1}{2} + i\} \times S^3\right)$ is homotopic to a map to the interval $[0, W_0] \times \{-1\}$. $\square$

Proof. (of Theorem 2.10) The proof roughly mimics Theorem 2.9. Unfortunately, the bundle $C_2(S^1 \times S^n) \to S^1 \times S^n$ is generally not trivial, so we do not have access to quite as simple an argument, but we take some inspiration from it.

As with Theorem 2.9 we need only consider equivariant stratum-preserving maps $C_2[S^1] \to C_2[S^1 \times S^n]$ to compute $\pi_{n-2} \operatorname{Emb}(S^1, S^1 \times S^n)$. So we consider a map

$$S^{n-2} \times C_2[S^1] \to C_2[S^1 \times S^n]$$

with $n \geq 4$.

The composite with the bundle projection map $C_2[S^1 \times S^n] \to S^1 \times S^n$ factors through the projection to S^1, and is given by the W_1 invariant. Thus our map lifts

$$S^{n-2} \times C_2[S^1] \to \tilde{C}_2[S^1 \times S^n].$$

As with the proof of Theorem 2.9 the fiber of the map $\tilde{C}_2[S^1 \times S^n] \to S^1 \times S^n$ can be identified with $Bl_{\mathbb{Z}\times\{1\}}(\mathbb{R} \times S^n)$, which is similarly identified with $\mathbb{R} \times \{-1\} \cup \left(\sqcup_{i \in \mathbb{Z}} \{\frac{1}{2} + i\} \times S^n\right)$.

Here our argument diverges from the proof of Theorem 2.9. While the action of Σ_2 on $C_2(S^1)$ is free, it has the invariant subspace of antipodal points on S^1.

By restricting to the subspace of antipodal points, we get a fibration from the space of stratum-preserving equivariant maps $C_2[S^1] \to C_2[S^1 \times S^n]$ to the space $Maps^{\Sigma_2}(S^1, C_2[S^1 \times S^n])$ of equivariant maps. This mapping space can be thought of as the space of maps $S^1 \to S^1 \times S^n$ where antipodal points are required to map to distinct points. By a transversality argument, any k-dimensional family of maps to the free loop space $L(S^1 \times$

S^n) can be perturbed to have this property, provided $k < n$. Thus through dimension $n-2$, this space has the same homotopy groups as $L(S^1 \times S^n) \simeq L(S^1) \times L(S^n)$, which are the homotopy groups of $\mathbb{Z} \times S^1$. i.e. this recovers our W_0 and W_1 invariants.

We are considering the fibration from the space of stratum-preserving equivariant maps

$$C_2[S^1] \to C_2[S^1 \times S^n]$$

to the space $Maps^{\Sigma_2}(S^1, C_2[S^1 \times S^n])$. The fiber is precisely the space of maps of an annulus $S^1 \times [0,1]$ to $C_2[S^1 \times S^n]$ that restrict to a fixed map on one boundary circle, and which send the other boundary circle to $\partial C_2(S^1 \times S^n)$. We lift this map to the fiber $\mathbb{R} \times \{-1\} \cup \left(\sqcup_{i \in \mathbb{Z}} \{\frac{1}{2} + i\} \times S^n \right)$. From this perspective we can see that the W_2 invariant is well-defined for an $(n-2)$-parameter family, and there are no further invariants. □

Theorem 2.12. *To each element of $\pi_1 \operatorname{Emb}(S^1, S^1 \times S^3)$, there is an explicitly constructible embedded torus that represents that element via the spinning construction.* □

3. Bundles of Embeddings and Diffeomorphism Groups

Rationally, the first three non-trivial homotopy groups of $\operatorname{Emb}(I, S^1 \times B^n)$ are in dimensions 0, $n-2$ and $2n-4$. In this section we construct invariants of these homotopy groups, specifically the W_2 and W_3 invariants,

$$W_2 : \pi_{n-2} \operatorname{Emb}(I, S^1 \times B^n) \to \mathbb{Z}[t^{\pm 1}]/\langle t^0 \rangle$$

$$W_3 : \pi_{2n-4} \operatorname{Emb}(I, S^1 \times B^n) \to \mathbb{Q}[t_1^{\pm 1}, t_3^{\pm 1}]/R$$

where R is the *hexagon relation*. Note that W_2 was defined in Section 2. We re-use the notation here as this also is an invariant of the 2^{nd} non-trivial homotopy group of an embedding space. We derive these invariants from a computation of the (rational) homotopy of configuration spaces in $S^1 \times B^n$. This allows us to detect diffeomorphisms of $S^1 \times B^3$ via the scanning construction $\pi_0 \operatorname{Diff}(S^1 \times B^3 \text{ fix } \partial) \to \pi_2 \operatorname{Emb}(I, S^1 \times B^3)$. To conclude the section we relate homotopy-type of $\operatorname{Diff}(S^1 \times B^n \text{ fix } \partial)$ to that of the space of co-dimension two unknots in $\operatorname{Emb}(S^{n-1}, S^{n+1})$, and the homotopy-type of $\operatorname{Emb}(B^n, S^1 \times B^n)$ via some simple fiber sequences.

The rational homotopy groups of the configuration spaces of points in Euclidean space were first described by Milnor and Moore [MM]. Their result is that the rational homotopy groups $\mathbb{Q} \otimes \pi_* C_k(B^{n+1})$ are isomorphic to the primitives of $H_*(\Omega C_k(B^{n+1}); \mathbb{Q})$ via a Hurewicz-style map. The generators of $\pi_n C_k(B^{n+1})$ we denote w_{ij}. The class w_{ij} has all k points stationary, with the exception of point j that orbits around point i. The Whitehead bracket operation $[\cdot, \cdot] : \pi_n X \times \pi_m X \to \pi_{m+n-1} X$ is the obstruction to a map $f \vee g : S^n \vee S^m \to X$, extending to $S^n \times S^m \to X$. We identify $S^n \vee S^m$ with all but the top-dimensional cell of $S^n \times S^m$, i.e. $S^n \vee S^m = S^n \times \{*\} \cup \{*\} \times S^m$. Thus the Whitehead product is the characteristic map of the top-dimensional cell $S^{n+m-1} \to S^n \vee S^m$ composed with $f \vee g$. The Whitehead bracket is bilinear, graded symmetric, i.e. $[y, x] = (-1)^{nm}[x, y]$ and it satisfies a Jacobi-like identity.

$$(-1)^{pr}[[f, g], h] + (-1)^{pq}[[g, h], f] + (-1)^{rq}[[h, f], g] = 0,$$

where $f \in \pi_p X, g \in \pi_q X, h \in \pi_r X$ with $p, q, r \geq 2$.

The theorem of Milnor and Moore implies $\mathbb{Q} \otimes \pi_* C_k(\mathbb{R}^{n+1})$ is generated by the w_{ij} classes, subject to the relations

- $w_{ii} = 0 \ \forall i$
- $w_{ij} = (-1)^{n+1} w_{ji} \ \forall i \neq j$
- $[w_{ij}, w_{lm}] = 0$ when $\{i,j\} \cap \{l,m\} = \emptyset$.
- $[w_{ij} + w_{il}, w_{lj}] = 0$ for all i, j, l.

The latter relation should be viewed a generalized 'orbital system' map $S^n \times S^n \to C_k(B^{n+1})$ where there is an earth-moon pair corresponding to points l and j respectively, orbiting around the sun corresponding to point i. This relation can be rewritten as

$$[w_{ij}, w_{jk}] - [w_{jk}, w_{ki}] = 0.$$

Thus we have the equality of the three cyclic permutations,

$$[w_{ij}, w_{jk}] = [w_{jk}, w_{ki}] = [w_{ki}, w_{ij}].$$

To compute the rational homotopy groups of $C_k[S^1 \times B^n]$ we start by considering the inclusion $C_k[B^{n+1}] \to C_k[S^1 \times B^n]$ induced by an embedding $B^{n+1} \to S^1 \times B^n$. This allows us to define classes $w_{ij} \in \pi_n C_k[S^1 \times B^n]$. The bundle $C_k(S^1 \times B^n) \to C_{k-1}(S^1 \times B^n)$ is split, with fiber the homotopy-type of a wedge of a circle and $(k-1)$-copies of S^n, we again have that the homotopy groups of $C_k(S^1 \times B^n)$ are rationally generated by the elements $t_l.w_{ij}$ where $\{t_1, \cdots, t_k\} \subset \pi_1 C_k[S^1 \times B^n]$ are the standard generators of the fundamental group, the free abelian group of rank k.

Proposition 3.1. *The n-th homotopy group of $C_k[S^1 \times B^n]$ has generators $t_l^m w_{ij}$ for $i, j, m \in \mathbb{Z}, l \in \{1, 2, \cdots, k\}$. The relations are*

- $w_{ii} = 0,$
- $w_{ji} = (-1)^{n+1} w_{ij},$
- $t_l.w_{ij} = w_{ij}$ *if* $l \notin \{i, j\},$
- $t_i.w_{ij} = t_j^{-1}.w_{ij}.$

The rational homotopy-groups of $C_k[S^1 \times B^n]$ are generated by the Whitehead products of the elements $t_l^m.w_{ij}$. These satisfy the relations

- $[w_{ij}, w_{lm}] = 0$ *if* $\{i, j\} \cap \{l, m\} = \emptyset,$
- $[w_{ij}, w_{jl}] = [w_{jl}, w_{li}] = [w_{li}, w_{ij}],$
- $t_l.[f, g] = [t_l.f, t_l.g].$

The above computation should be viewed as a slight rephrasing of the argument given in Cohen-Gitler [CG]. Observe

$$t_1^{\alpha_1} \cdots t_m^{\alpha_m}.w_{ij} = t_i^{\alpha_i - \alpha_j}.w_{ij}.$$

Thus we also have

$$t_1^{\alpha_1} \cdots t_m^{\alpha_m}[w_{ij}, w_{jl}] = [t_i^{\alpha_i - \alpha_j} w_{ij}, t_j^{\alpha_j - \alpha_l} w_{jl}].$$

As a $\mathbb{Q}[t_1^{\pm 1}, \cdots, t_k^{\pm 1}]$-module, we have (assuming $k \geq 3$)

$$\mathbb{Q} \otimes \pi_{2n-1} C_k[S^1 \times B^n] \simeq \bigoplus_{i<j<l} \mathbb{Q}[t_1^{\pm 1}, \cdots, t_k^{\pm 1}] / \langle t_i t_j t_l - 1 = t_m - 1 = 0 \ \forall m \notin \{i,j,l\} \rangle \oplus$$

$$\bigoplus_{\substack{i<j \\ 0 \leq l \text{ if } n \text{ even} \\ 1 \leq l \text{ if } n \text{ odd}}} \mathbb{Q}[t_1^{\pm 1}, \cdots, t_k^{\pm 1}] / \langle t_i t_j - 1 = t_m - 1 = 0 \ \forall m \notin \{i,j\} \rangle.$$

The top summands, i.e. for each $i < j < l$ are generated by the $[w_{ij}, w_{jl}]$ brackets, the bottom summands are generated by the $[w_{ij}, t_i^l w_{ij}]$ brackets.

We outline a general 'closure argument' that produce invariants of the homotopy groups $\pi_{n-2} \operatorname{Emb}(I, S^1 \times B^n)$ and $\pi_{2n-4} \operatorname{Emb}(I, S^1 \times B^n)$. For this computation the 2^{nd} stage of the Taylor tower suffices. We will use Dev Sinha's mapping-space model [Si1], analogously to how it is used in [BCSS].

Sinha's model for the k-th stage of the tower for $\operatorname{Emb}(I, M)$ is denoted AM_k. This consists of the stratum-preserving aligned maps $C_k[I, \partial] \to C'_k[S^1 \times B^n, \partial]$ where

- $C_k[I, \partial]$ is the subspace of $C_{k+2}[I]$ consisting of points of the form $(0, t_1, \cdots, t_n, 1)$ with $0 \leq t_1 \leq \cdots \leq t_n \leq 1$. In general $C_k[I, \partial]$ is the $(k+2)^{nd}$ Stasheff polytope, whose vertices correspond to the ways of bracketing a word with $(k+2)$-letters into a tree of nested binary operations, i.e. all the ways of expressing associativity in a word of length $(k+2)$. The edges are applications of the associativity rule. Thus the 3^{rd} Stasheff polytope is an interval. The 4^{th} Stasheff Polytope is the pentagon. In general it is a truncated simplex.
- $C_k[M, \partial]$ is the subspace of $C_{k+2}[M]$ consisting of points of the form $(b_0, p_1, \cdots, p_n, b_1)$ where $b_0, b_1 \in \partial M$ are the basepoints of the embedding space, i.e. provided we demand the maps in the embedding space $\operatorname{Emb}(I, M)$ sends $0 \longmapsto b_0$ and $1 \longmapsto b_1$.

To visualize $C_k[I, \partial]$, one first considers the *naive compactification* of $C_k(I)$, i.e. the simplex.

$$\overline{C_k(I)} = \{(t_1, \cdots, t_k) : 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

The space $\overline{C_k(I)}$ is the point-set topological closure of the path-component of $C_k(I)$ where the points are linearly ordered by $<$. To obtain the Stasheff polytope from the n-simplex, one iteratively truncates (blows up) strata corresponding to collisions of more than two points. Thus for $C_2[I, \partial]$ we blow up the $0 = t_1 = t_2$ stratum, since t_1 and t_2 are colliding, but they are also colliding with the initial point. Similarly we blow up the $t_1 = t_2 = 1$ stratum. Given that no other collisions occur, these are the only additional strata in the compactification. Similarly in $C_3[I, \partial]$, but now there are the blow-ups from $0 = t_1 = t_2$, $t_1 = t_2 = t_3$, $t_2 = t_3 = 1$, and the two relatively high co-dimension blow-ups $0 = t_1 = t_2 = t_3$ and $t_1 = t_2 = t_3 = 1$.

We will only be interested in the 2^{nd} and 3^{rd} stages of the embedding calculus in this paper, and given that the behaviour of our mappings will be fibrewise linear on the 'truncations' of $\overline{C_k[I]}$, we will often simply consider $T_k \operatorname{Emb}(I, S^1 \times B^n)$ to simply be stratum-preserving aligned maps $C_k[I] \to C'_k[S^1 \times B^n]$, i.e. suppressing extraneous combinatorial details to keep the technicalities light. Readers should be aware these additional constraints must always be considered, to ensure these simplified mapping spaces have the desired homotopy-type.

Elements of the 2^{nd} stage can be restricted to the faces of $C_2[I]$ giving loops in the three boundary sub-strata of $C_2'[S^1 \times B^n]$ corresponding to the collisions $t_1 = 0, t_1 = t_2$ and $t_2 = 1$. These sub-strata are diffeomorphic to $C_1'[S^1 \times B^n]$, i.e. the unit tangent bundle of $S^1 \times B^n$ which is diffeomorphic to $S^1 \times B^n \times S^n$. Given that elements of $\pi_{n-2}\Omega S^n$ are trivial, we can homotope any map $S^{n-2} \times C_2[I] \to C_2'[S^1 \times B^n]$ to standard linear maps on the boundary facets. These null-homotopies can be attached to the original map with domain $S^{n-2} \times C_2[I]$ to give us a map out of an adjunction. This new map is standard on the boundary.

In our case, we are only interested in the component of $\mathrm{Emb}(I, S^1 \times B^n)$ where the interval winds a net zero number of times about the S^1 factor. After the adjunction we have a map from a space diffeomorphic to B^n to the space $C_2'[S^1 \times B^n]$, and this map is constant on the boundary.

The important part of this construction is we used a choice of null-homotopies to construct the map $S^n \to C_2'[S^1 \times B^n]$. If we choose *different* null-homotopies, we should check how the resulting map may differ.

Proposition 3.2. (**Closure Argument 1**) *Given an element of $[f] \in \pi_{n-2}\mathrm{Emb}(I, S^1 \times B^n)$ we form the closure of the evaluation map $ev_2(f) \in \pi_{n-2}T_2\mathrm{Emb}(I, S^1 \times B^n)$ which is a map of the form*

$$\overline{ev_2}(f) : S^n \to C_2'[S^1 \times B^n].$$

The homotopy group $\pi_n C_2'[S^1 \times B^n]$ is isomorphic to a direct sum $\mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}^2$. One can think of this isomorphism via the splitting $C_2'[S^1 \times B^n] \simeq C_2[S^1 \times B^n] \times (S^n)^2$. There are three inclusions of $C_1'[S^1 \times B^n]$ into $C_2'[S^1 \times B^n]$ coming from the three facets $p_1 = \{1\} \times \{-1\}$, $p_1 = p_2$ and $p_2 = \{1\} \times \{1\}$ respectively, corresponding to the three facets of $C_2[I]$ via the stratum-preserving condition. These inclusions induce subgroups generated by $(t^0, 1, 1)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively. Thus there is a well-defined homomorphism

$$\pi_{n-2}\mathrm{Emb}(I, S^1 \times B^n) \to \mathbb{Z}[t^{\pm 1}]/\langle t^0 \rangle$$

by counting all monomials the Laurent-polynomial part of $\pi_n C_2'[S^1 \times B^n] \simeq \mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}^2$ other than t^0. This map is an epimorphism.

Proof. Consider the construction of the closure $ev_2(f)$. We attach null-homotopies to the three face maps of $ev_2(f) : S^{n-2} \times C_2[I] \to C_2'[S^1 \times B^n]$. We use the notation $C_2[I]_{1=2}$ to denote the $t_1 = t_2$ substratum of $C_2[I]$, and similarly $C_2'[S^1 \times B^n]_{1=2}$ will denote the substratum of $C_2'[S^1 \times B^n]$ where the two points collide. Similarly, $1 = 0$ will be our notation to indicate the first point is at its initial point in $C_2[I]$ and $C_2[S^1 \times B^n]$ respectively. We will use the *simplicial identifications* between $C_1[I]$ and the three boundary facets $(0 = 1, 1 = 2, 2 = 3)$, that is we identify $C_1[I]$ with $C_2[I]_{0=1}$ via the map $t \longmapsto (0, t)$, similarly we identify $C_1[I]$ with $C_2[I]_{1=2}$ via the map $t \longmapsto (t, t)$. We use the isomorphism $\pi_n C_2'[S^1 \times B^n] \simeq \mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}^2$ to talk about elements of $\pi_n C_2'[S^1 \times B^n]$. The generators of

$\mathbb{Z}[t^{\pm}]$ we denote t^i $\forall i$. The generators of the remaining $\mathbb{Z}^2$ factor are denoted α_1, α_2. Notice if we attach a different null-homotopy to the $i = i+1$ face of $C_2[I]$ we are modifying the homotopy-class of $\overline{ev_2}(f)$ by adding a multiple of:

- $(0,1,0)$, if $i = 0$,
- $(t^0, 1, 1)$, if $i = 1$,
- $(0,0,1)$, if $i = 2$.

Thus the closure is a well-defined element of a group isomorphic to $\mathbb{Z}[t^{\pm 1}]/\langle t^0 \rangle$, with generators $t^i w_{12}$. Using an argument similar to Theorem 2.5 we can argue these maps are epimorphisms. $\square$

We call the homomorphism from Proposition 3.2 the W_2-invariant,

$$W_2 : \pi_{n-2} \operatorname{Emb}(I, S^1 \times B^n) \to \mathbb{Z}[t^{\pm 1}]/\langle t^0 \rangle.$$

The subscript 2 indicates the domain is the 2^{nd} non-trivial (rational) homotopy-group of the space $\operatorname{Emb}(I, S^1 \times B^n)$. One can go a step further than Proposition 3.2 and argue that W_2 is an isomorphism. Roughly speaking, the idea is that given any based map $S^n \to C_2'[S^1 \times B^n]$, we reinterpret the function as having domain $B^{n-2} \times C_2[I]$ with the boundary collapsed. One then appends three copies of the standard null-homotopy of $ev_1(i)$ where i is the constant family $i : S^{n-2} \to \operatorname{Emb}(I, S^1 \times B^n)$. This gives us a map back from $Maps(S^n, C_2'[S^1 \times B^n])$ to $Maps(S^{n-2}, Maps(C_2[I], C_2'[S^1 \times B^n])$, which is an element of $Maps(S^{n-2}, T_2 \operatorname{Emb}(I, S^1 \times B^n))$ provided we began with something in the image of W_2, proving W_2 is injective.

We can perform the same kind of analysis for $[f] \in \pi_{2n-4} \operatorname{Emb}(I, S^1 \times B^n)$. These elements are detected by 3^{rd}-stage of the Embedding Calculus, which are maps of the form $C_3[I] \to C_3'[S^1 \times B^n]$. In general, when we restrict these maps to the boundary facets of $C_3[I]$, the resulting map $S^{2n-4} \times C_2[I] \to C_2'[S^1 \times B^n]$ may not be null-homotopic. That said, such maps are torsion. This allows us to perform the above construction rationally, i.e. if the order of the map $ev_2(f) : S^{2n-4} \times C_2[I] \to C_2'[S^1 \times B^n]$ is m, we can perform the same analysis to construct a closure of $ev_3(mf)$, thus $\frac{1}{m}\overline{ev_3}(mf)$ would be a well-defined rational-homotopy invariant of $[f]$, *provided* we mod-out by the boundary subgroups, in this case they come from the inclusions of the four facets of $C_3[I]$, $t_1 = 0$, $t_1 = t_2$, $t_2 = t_3$ and $t_3 = 1$. To do this we need to describe the change in homotopy-class to $\frac{1}{m}\overline{ev_3}(mf)$ due to attaching different null-homotopies to $ev_3(mf) : S^{2n-4} \times C_3[I] \to C_3'[S^1 \times B^n]$.

As we have seen $\pi_{2n-1} C_2'[S^1 \times B^n]$ is isomorphic to $\pi_{2n-1}(S^1 \vee S^n) \oplus \bigoplus_2 \pi_{2n-1} S^n$. Modulo torsion, the generators of $\pi_{2n-1}(S^1 \vee S^n)$ are the Whitehead products of elements $t^k w_{12}$ for $k \in \mathbb{Z}$. This gives us the result that $\pi_{2n-1} C_2'[S^1 \times B^n]$, mod torsion, is isomorphic to $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]/\langle t_1 t_2 - 1 = 0 \rangle$ as a module over the group-ring of the fundamental group. The generator of $\pi_{2n-1} C_2'[S^1 \times B^n]$ corresponding to a monomial $t_1^{\alpha} t_2^{\beta}$ is $t_1^{\alpha} t_2^{\beta} w_{12}$. By attaching a homotopy-class of maps $S^{2n-4} \times I \times C_2[I] \to C_2'[S^1 \times B^n]$ to a closed-off $S^{2n-4} \times C_3[I] \to C_3'[S^1 \times B^n]$ we change the homotopy class by adding:

(1) $[t_2^{\alpha} w_{23}, t_2^{\beta} w_{23}]$. This comes from the $t_1 = 0$ face. Thus the generator $t_1^{\alpha} w_{12}$ is mapped to $t_2^{\alpha} w_{23}$, and a Whitehead bracket $[t_1^{\alpha} w_{12}, t_1^{\beta} w_{12}]$ is mapped to $[t_2^{\alpha} w_{23}, t_2^{\beta} w_{23}]$.

(2) $[t_1^\alpha w_{12}, t_1^\beta w_{12}]$ to $[t_1^\alpha w_{13} + t_2^\alpha w_{23} + a_1 w_{21}, t_1^\beta w_{13} + t_2^\beta w_{23} + a_1 w_{21}]$. This comes from the $t_1 = t_2$ face map, i.e. the inclusion $C_2'[S^1 \times B^n] \to C_3'[S^1 \times B^n]$ that doubles the first point, i.e. $(p_1, p_2) \longmapsto (p_1, \epsilon^+ p_1, p_2)$, where the perturbation $\epsilon^+ p_1$ is in the direction of the velocity vector. The integer a_1 is the degree of this velocity vector map. This map sends w_{12} to $w_{13} + w_{23} + a_1 w_{21}$, t_1 to $t_1 t_2$ and t_2 to t_2. The 2nd stage of the Taylor tower induces a null-homotopy of the velocity vector map, so we can assume $a_1 = 0$, but it is of interest that the following computation gives the same answer for $a_1 \neq 0$. Thus it sends $[t_1^\alpha w_{12}, t_1^\beta w_{12}]$ to $[t_1^\alpha w_{13} + t_2^\alpha w_{23} + a_1 w_{21}, t_1^\beta w_{13} + t_2^\beta w_{23} + a_1 w_{21}]$. Expanding this bracket using bilinearity we get

$$= \left(-t_1^{\alpha-\beta} t_3^{-\beta} + (-1)^{n-1} t_1^{\beta-\alpha} t_3^{-\alpha}\right)[w_{12}, w_{23}] + [t_1^\alpha w_{13}, t_1^\beta w_{13}] +$$

$$a_1\left((-1)^n t_3^{-\beta} + (-1)^{n-1} t_3^{-\beta} + t_1^{-\alpha} - t_1^{-\alpha}\right)[w_{12}, w_{23}]$$

where the latter row comes from collecting the terms involving a_1, and clearly these terms sum to zero.

(3) $[t_1^\alpha w_{12} + t_1^\alpha w_{13} + a_2 w_{23}, t_1^\beta w_{12} + t_1^\beta w_{13} + a_2 w_{23}]$. This is for the $t_2 = t_3$ facet. This corresponds to the map $C_2'[S^1 \times B^n] \to C_3'[S^1 \times B^n]$ that doubles the second point, i.e. $(p_1, p_2) \longmapsto (p_1, p_2, \epsilon^+ p_2)$. This map sends w_{12} to $w_{12} + w_{13} + a_2 w_{23}$, t_1 to t_1 and t_2 to $t_2 t_3$. Thus $[t_1^\alpha w_{12}, t_1^\beta w_{12}] \longmapsto [t_1^\alpha w_{12} + t_1^\alpha w_{13} + a_2 w_{23}, t_1^\beta w_{12} + t_1^\beta w_{13} + a_2 w_{23}]$. Like the previous case, this simplifies to

$$= \left(-t_1^\alpha t_3^{\alpha-\beta} + (-1)^{n+1} t_1^\beta t_3^{\beta-\alpha}\right)[w_{12}, w_{23}] + [t_1^\alpha w_{13}, t_1^\beta w_{13}] +$$

$$a_2\left(t_1^\beta - t_1^\beta + (-1)^n t_1^\alpha + (-1)^{n+1} t_1^\alpha\right)[w_{12}, w_{23}].$$

Again, the terms with a_2 cancel.

(4) $[t_1^\alpha w_{12}, t_1^\beta w_{12}]$. This is for the $t_3 = 1$ facet. This corresponds to the inclusion $C_2'[S^1 \times B^n] \to C_3'[S^1 \times B^n]$ that maps (p_1, p_2) to $(p_1, p_2, (1,0))$, thus it sends $w_{12} \longmapsto w_{12}$ and $t_1 \longmapsto t_1$, $t_2 \longmapsto t_2$, thus it acts trivially on $[t_1^\alpha w_{12}, t_1^\beta w_{12}]$.

Thus our invariant via closure $\frac{1}{m}\overline{ev_3}(mf)$ of $\pi_{2n-4} \operatorname{Emb}(I, S^1 \times B^n)$ takes values in

$$\mathbb{Q} \otimes \pi_{2n-1} C_3'[S^1 \times B^n]/R$$

where R is the subgroup generated by the above four inclusions. Notice (1) kills the summand corresponding to the w_{23} brackets, and (4) kills the summands corresponding to the w_{12} brackets. Using relation (1) and (4) we can simplify (2) and (3) into relations between w_{13} brackets and brackets of the form $[w_{12}, w_{23}]$, giving us the proposition below.

Proposition 3.3. (**Closure Argument 2**) *Given an element of $[f] \in \pi_{2n-4} \operatorname{Emb}(I, S^1 \times B^n)$ such that $ev_2(f) : S^{2n-4} \to T_2 Emb(I, S^1 \times B^n)$ is null, we form the closure of the evaluation map $ev_3(f) : S^{2n-4} \to T_3 \operatorname{Emb}(I, S^1 \times B^n)$ which is a based map of the form*

$$\overline{ev_3}(f) : S^{2n-1} \to C_3'[S^1 \times B^n].$$

The homotopy-class of this map, as a function of the homotopy-class $[f]$ is well-defined modulo a subgroup we call R. Using the notation of Proposition 3.1, R is generated by the torsion subgroup of $\pi_{2n-1}C_3'[S^1 \times B^n]$ together with the elements

$$\left(t_1^{\alpha-\beta}t_3^{-\beta} - t_1^{\alpha}t_3^{\alpha-\beta} + (-1)^{n-1}\left(t_1^{\beta}t_3^{\beta-\alpha} - t_1^{\beta-\alpha}t_3^{-\alpha}\right)\right)[w_{12}, w_{23}]\ \forall \alpha, \beta \in \mathbb{Z},$$

$$[t_2^{\alpha}w_{23}, t_2^{\beta}w_{23}]\ \forall \alpha, \beta,$$

$$[t_1^{\alpha}w_{12}, t_1^{\beta}w_{12}]\ \forall \alpha, \beta,$$

$$[t_1^{\alpha}w_{13}, t_1^{\beta}w_{13}] + \left(t_1^{\alpha-\beta}t_3^{-\beta} + (-1)^n t_1^{\beta-\alpha}t_3^{-\alpha}\right)[w_{12}, w_{23}]\ \forall \alpha, \beta.$$

Since $\pi_{2n-4}T_2\operatorname{Emb}(I, S^1 \times B^n)$ is torsion, there is a homomorphism, called the **closure operator**

$$\pi_{2n-4}\operatorname{Emb}(I, S^1 \times B^n) \to \mathbb{Q}[t_1^{\pm 1}, t_3^{\pm 1}]/\langle t_1^{\alpha-\beta}t_3^{-\beta} - t_1^{\alpha}t_3^{\alpha-\beta} = (-1)^n\left(t_1^{\beta}t_3^{\beta-\alpha} - t_1^{\beta-\alpha}t_3^{-\alpha}\right)\ \forall \alpha, \beta \in \mathbb{Z}\rangle$$

given by mapping $f \longmapsto \frac{1}{m}\overline{ev_3}(mf)$.

Proof. The relations are given in the comments preceding the Proposition. Relations (1) and (4) kill $[t_2^{\alpha}w_{23}, t_2^{\beta}w_{23}]$ and $[t_1^{\alpha}w_{12}, t_1^{\beta}w_{12}]$ respectively. Using Relations (1) and (4) we can simplify relations (2) and (3) to 3-term relations, both expressing $[t_1^{\alpha}w_{13}, t_1^{\beta}w_{13}]$ in the $\mathbb{Z}[t_1^{\pm}, t_2^{\pm}]$-linear span of $[w_{12}, w_{23}]$. Comparing the two gives the relation

$$\left(t_1^{\alpha-\beta}t_3^{-\beta} - t_1^{\alpha}t_3^{\alpha-\beta} + (-1)^{n-1}\left(t_1^{\beta}t_3^{\beta-\alpha} - t_1^{\beta-\alpha}t_3^{-\alpha}\right)\right)[w_{12}, w_{23}] = 0.$$

□

It is important to note that in Proposition 3.3 we are allowing the attachment of distinct null-homotopies on all four boundary facets of $S^{2n-4} \times C_3[I]$. This is because the elements of the 3^{rd} stage of the Taylor tower restrict to potentially different maps on the four faces.

One way of interpreting the closure argument is we are studying the map between the stages of the Taylor tower, and using it to construct invariants of the homotopy groups of $\operatorname{Emb}(I, S^1 \times D^n)$. The homotopy-fiber of the map between the stages $T_k \to T_{k-1}$ is called the k-th *layer*, denoted L_k, and its homotopy-type was characterized by Michael Weiss (see [GKW]) as a certain space of sections. In our case it has the interpretation of an iterated loop space of a homotopy limit of a cubical diagram of configuation spaces. One can turn the long exact sequence $\cdots \to \pi_i L_k \to \pi_i T_k \to \pi_i T_{k-1} \to \cdots$ into a short exact sequence centered on $\pi_i T_k$. This tells us that $\pi_i T_k$ is an extension over a subgroup of $\pi_i T_{k-1}$ by a quotient group of $\pi_i L_k$. The quotient group of $\pi_i L_k$ has been the subject of much study. In her thesis [Ko1] Kosanovic computed an explicit map from the quotient group of $\pi_i L_k$ to the kernel of $\pi_i \operatorname{Emb}(I, M) \to \pi_i T_{k-1}$ for the smallest i with $\pi_i L_k$ non-trivial for the case of the embedding spaces of the form $\operatorname{Emb}(I, M)$ with M a 3-manifold. She has recently [Ko3] generalized this to the case $dim(M) \geq 3$. This can be viewed as a systematic procedure to express our 'closures' as linear combinations of Whitehead products.

The map that applies the rationalized closure operator to $[f] \in \pi_{2n-4}\operatorname{Emb}(I, S^1 \times B^n)$ we denote W_3, i.e.

$$W_3([f]) = \frac{1}{m}\overline{ev_3}(mf) \in \mathbb{Q} \otimes \pi_{2n-1}C_3'[S^1 \times B^n]/R.$$

By design, if $ev_2(f) \in \pi_{2n-1}T_2\operatorname{Emb}(I, S^1 \times B^n)$ is null, the invariant W_3 lives in the lift

$$W_3([f]) = \overline{ev_3}(f) \in \pi_{2n-4}\operatorname{Emb}(I, S^1 \times B^n)/R.$$

Proposition 3.4 concerns the structure of this group.

Proposition 3.4. *Let $\bar{W}$ denote the subspace of $\pi_{2n-1}(C_3(S^1 \times B^n))$ generated by the Whitehead products $t_1^p t_2^q[w_{13}, w_{23}], p, q \in \mathbb{Z}$. Let W be the quotient of $\bar{W}$ by the subspace K generated by $(t_1^p t_2^q - t_1^q t_2^{q-p} + (-1)^n(t_1^p t_2^{p-q} - t_1^q t_2^p))[w_{13}, w_{23}]$. Then the induced map from W to $\pi_{2n-1}(C_3(S^1 \times B^n))/R$ is an isomorphism. The quotient $\pi_{2n-1}(C_3(S^1 \times B^n))/R$ is a direct sum of a free-abelian group and a 2-torsion group.*

Proof. The map $\bar{W} \to \pi_{2n-1}C_3(S^1 \times B^n)/R$ being an isomorphism follows from Proposition 3.3 after noting that $t_1^m t_3^n[w_{12}, w_{23}] = t_1^m t_3^n[w_{23}, w_{31}] = (-1)^n t_1^{m-n} t_2^{-n}[w_{13}, w_{23}]$. Consider the relator

$$\left(t_1^{\alpha-\beta} t_3^{-\beta} - t_1^{\alpha} t_3^{\alpha-\beta} + (-1)^{n-1}\left(t_1^{\beta} t_3^{\beta-\alpha} - t_1^{\beta-\alpha} t_3^{-\alpha}\right)\right)[w_{12}, w_{23}] = 0.$$

If we replace the indices (α, β) by $(\alpha - \beta, -\beta)$ the above relator becomes

$$\left(t_1^{\alpha} t_3^{\beta} - t_1^{\alpha-\beta} t_3^{\alpha} + (-1)^{n-1}\left(t_1^{-\beta} t_3^{-\alpha} - t_1^{-\alpha} t_3^{\beta-\alpha}\right)\right)[w_{12}, w_{23}] = 0,$$

suppressing the Whitehead bracket we rewrite the polynomial relator as

$$t_1^{\alpha} t_3^{\beta} + (-1)^{n-1} t_1^{-\beta} t_3^{-\alpha} = t_1^{\alpha-\beta} t_3^{\alpha} + (-1)^{n-1} t_1^{-\alpha} t_3^{\beta-\alpha}.$$

Notice the map $(\alpha, \beta) \longmapsto (-\beta, -\alpha)$ is an involution of $\mathbb{Z}^2$, while the map $(\alpha, \beta) \longmapsto (\alpha - \beta, \alpha)$ has order 6, moreover if we conjugate the latter by the former we get $(\alpha, \beta) \longmapsto (\beta, \beta - \alpha)$, which is the inverse of the mapping $(\alpha, \beta) \longmapsto (\alpha - \beta, \alpha)$ i.e. these two mappings generate the dihedral group of the hexagon. One can also see this directly, by observing the orbit of $(1, 0)$ under this group action is the planar hexagon

$$\{(1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1)\}.$$

This allows us to quickly write-out the consequences of our relator.

$$t_1^{\alpha} t_3^{\beta} + (-1)^{n-1} t_1^{-\beta} t_3^{-\alpha} = t_1^{\alpha-\beta} t_3^{\alpha} + (-1)^{n-1} t_1^{-\alpha} t_3^{\beta-\alpha} \tag{1}$$

$$= t_1^{-\beta} t_3^{\alpha-\beta} + (-1)^{n-1} t_1^{\beta-\alpha} t_3^{\beta} \tag{2}$$

$$= t_1^{-\alpha} t_3^{-\beta} + (-1)^{n-1} t_1^{\beta} t_3^{\alpha} \tag{3}$$

$$= t_1^{\beta-\alpha} t_3^{-\alpha} + (-1)^{n-1} t_1^{\alpha} t_3^{\alpha-\beta} \tag{4}$$

$$= t_1^{\beta} t_3^{\beta-\alpha} + (-1)^{n-1} t_1^{\alpha-\beta} t_3^{-\beta} \tag{5}$$

Since the map $(\alpha, \beta) \longmapsto (\alpha - \beta, \alpha)$ is of order 6 and conjugate to a rotation, all of its orbits have six elements with the exception of the origin. Thus the orbits of our dihedral group can be trivial, as in the case of $(0, 0)$, or they have at least six elements. The orbits have only six elements provided any of the following equations hold $\alpha \pm \beta = 0$, $\alpha = 0$, $\beta = 0$, $2\alpha = \beta$ or $2\beta = \alpha$. This could be thought of as any of the vertices of the hexagon, or mid-points of the edges of the hexagon. Observe at the origin $(\alpha, \beta) = 0$ the hexagon relation is trivial, thus for the $(\alpha, \beta) = 0$ orbit, the quotient group is free abelian of rank one.

In the case of a 12-element orbit, no generator is mentioned more than once in the relators, so the quotient is a free abelian group of rank $7 = 12 - 5$. For the 7 generators one can take the vertices of the hexagon, i.e. the orbit of (α, β) under $(\alpha, \beta) \longmapsto (\alpha - \beta, \alpha)$, plus one generator obtained by taking a vertex of the hexagon and applying $(\alpha, \beta) \longmapsto (-\beta, -\alpha)$ to it.

In the case of a 6-element orbit the quotient is isomorphic to either $\mathbb{Z}^4$ or $\mathbb{Z}^3 \oplus \mathbb{Z}_2$, depending on which 6-element orbit one considers, and the parity of n. For example, consider the case $\alpha = 0$, then our relations are

$$t_1^0 t_3^\beta + (-1)^{n-1} t_1^{-\beta} t_3^0 = t_1^{-\beta} t_3^0 + (-1)^{n-1} t_1^0 t_3^\beta \tag{6}$$
$$= t_1^{-\beta} t_3^{-\beta} + (-1)^{n-1} t_1^\beta t_3^\beta \tag{7}$$
$$= t_1^0 t_3^{-\beta} + (-1)^{n-1} t_1^\beta t_3^0 \tag{8}$$
$$= t_1^\beta t_3^0 + (-1)^{n-1} t_1^0 t_3^{-\beta} \tag{9}$$
$$= t_1^\beta t_3^\beta + (-1)^{n-1} t_1^{-\beta} t_3^{-\beta}. \tag{10}$$

Thus for n odd (still in the $\alpha = 0$ case) this quotient is isomorphic to $\mathbb{Z}^4$, while for n even, it is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}_2$. Similarly if we take $2\alpha = \beta$ we get quotient $\mathbb{Z}^3 \oplus \mathbb{Z}_2$ if n is odd, and $\mathbb{Z}^4$ if n is even. Notice these two cases suffice as our relations are invariant under the rotations of the hexagon. We could further deduce the $2\alpha = \beta$ case using the mirror reflections of the hexagon. In this case, the symmetry does not preserve our system of equations, it preserves them after changing the parity of n. □

Remark 3.5. For n odd the coefficients of defining relations in K have the form
$t_1^p t_2^q + t_1^p t_2^{p-q} = t_1^q t_2^p + t_1^q t_2^{q-p}$.

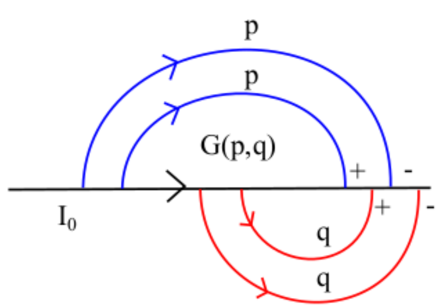


FIGURE 7. Chord diagram for $G(p, q)$

Example 3.6 is a computation of $W_3(G(p, q))$. An alternative computation, given in greater detail appears in Appendix Section 12. The techniques we use here are expanded upon in [BG2]. We begin by giving a careful definition of the homotopy-class $G(p,q) : S^{2n-4} \to \mathrm{Emb}(I, S^1 \times B^n)$.

We interpret the chord diagram in Figure 7 as defining an immersion $I \to S^1 \times B^n$ with four regular double-point pairs that we will resolve to create families of embeddings. Two of the double points are decorated in blue, and the other two are decorated in red. The chord decorations p, q indicate that the 'shortcut' loop $S^1 \to S^1 \times B^n$, when projected to the S^1 factor has degree p or q respectively. The immersion is represented in Figure 8 (left).

Resolving this immersion would give us a map

$$\hat{G}(p,q) : S^{n-2} \times S^{n-2} \times S^{n-2} \times S^{n-2} \to \mathrm{Emb}(I, S^1 \times B^n).$$

We pre-compose $\hat{G}(p,q)$ with the map $\Delta : S^{n-2} \times S^{n-2} \to S^{n-2} \times S^{n-2} \times S^{n-2} \times S^{n-2}$ given by $\Delta(v,w) = (M(v), v, M(w), w)$ where $M : S^{n-2} \to S^{n-2}$ is a map with $deg(M) = -1$. The choice of degree is governed by the signs in our chord diagram. The composite $\hat{G}(p,q) \circ \Delta$, when restricted to $S^{n-2} \vee S^{n-2}$ is null, giving us, after a small homotopy of $\hat{G}(p,q) \circ \Delta$, a commutative diagram

$$\begin{tikzcd}
S^{n-2} \times S^{n-2} \arrow[rr, "\hat{G}(p,q)\circ\Delta"] \arrow[dr] & & \mathrm{Emb}(I, S^1 \times B^n) . \\
& S^{n-2} \times S^{n-2} / S^{n-2} \vee S^{n-2} \equiv S^{2n-4} \arrow[ur, "G(p,q)"'] &
\end{tikzcd}$$

We proceed computing the homotopy-class of $\frac{1}{m}\overline{ev_3}(m \cdot G(p,q))$ in steps. We will see that $ev_2(G(p,q))$ is null-homotopic, thus $m = 1$. In general, one can prove an analogous theorem to Proposition 3.2, computing $\pi_{2n-4} T_2 \mathrm{Emb}(I, S^1 \times B^n)$ precisely. The exponent of this group can be shown to be equal to $|\pi_{2n-2} S^n|$. The group $\pi_{2n-2} S^n$ is one of the stable homotopy groups of spheres, and is known to be finite. When needing to compute the order of $ev_2(f)$ precisely, one constructs the closure as in Proposition 3.2, giving the map $\overline{ev_2}(f) : S^{2n-2} \to C_2'[S^1 \times B^n]$. The homotopy group $\pi_{2n-2} C_2'[S^1 \times B^n]$ can be described in terms of homotopy-groups of spheres via the homotopy-equivalence $C_2'[S^1 \times B^n] \simeq S^1 \times (S^1 \vee S^n) \times (S^n)^2$ giving

$$\pi_{2n-2} C_2'[S^1 \times B^n] \simeq \pi_{2n-2} S^n [t^{\pm 1}] \oplus \bigoplus_2 \pi_{2n-2} S^n.$$

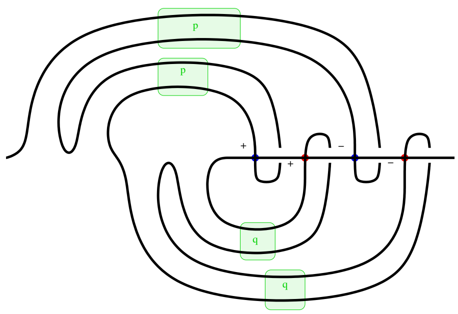


FIGURE 8. Immersion for $G(p,q)$ (left) and one resolution (right)

The homotopy-class of a map $S^{2n-2} \to (S^1 \vee S^n) \times (S^n)^2$ is determined by the homotopy classes of the projections: (a) $S^{2n-2} \to S^1 \vee S^n$ and (b) (two maps) $S^{2n-2} \to S^n$. By the Pontriagin construction, the latter two maps are determined by framed cobordism classes of $(n-2)$-manifolds in $\mathbb{R}^{2n-2}$, taking the pre-image of any point that is not the base-point of the sphere. The projection $S^{2n-2} \to S^1 \vee S^n$ is determined by a framed cobordism class of a countable collection of *disjoint* $(n-2)$-manifolds in $\mathbb{R}^{2n-2}$. A simple way to construct these manifolds is to take the cohorizontal manifolds, i.e. fix a unit direction $\zeta \in B^n$. Define $t^i Co_1^2$ consisting of pairs of points $(p_1, p_2) \in C_2[\mathbb{R}^1 \times B^n]$ such that the displacement vector $t^i.p_2 - p_1$ is a positive multiple of ζ. Then given $S^{2n-2} \to C_2'[S^1 \times B^n]$ we lift the map to the universal cover of $C_2'[S^1 \times B^n]$, interpreted as the submanifold of $C_2'[\mathbb{R}^1 \times B^n]$ such that the points have disjoint $\mathbb{Z}$-orbits. We take the pre-image of $t^i Co_1^2$. This manifold family (as a function of i) is precisely what we need to detect the Laurent polynomial associated to the homotopy-class of the projection $S^{2n-4} \to S^1 \vee S^n$.

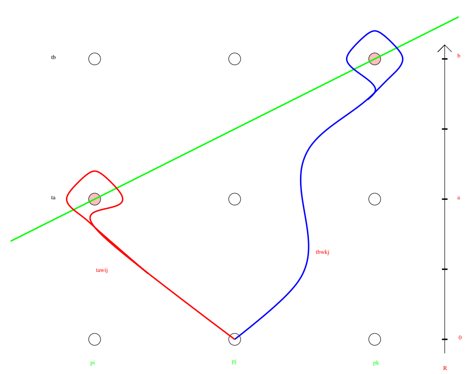


FIGURE 9. Collinear manifolds intersecting $\pi_n C_k[S^1 \times B^n]$ generators.

Rationally the class $\overline{ev_3}(G(p,q)) \in \pi_{2n-1} C_3'[S^1 \times B^n]$ is a linear combination of Whitehead products, by Proposition 3.1. To determine the linear combination, we use an idea from [BCSS], where it was shown that certain collinear (trisecant) manifolds can detect Whitehead products in $\pi_{2n-1}C_3(\mathbb{R}^{n+1})$. The key property of the collinear manifolds is they (a) they suffice to detect the generators of $\pi_n C_3[\mathbb{R}^{n+1}]$, and (b) there is more than one path-component to these manifolds, allowing one to go further and detect Whitehead products $[w_{12}, w_{23}]$. This allowed the authors in [BCSS] to express the type-2 Vassiliev invariant of knots as a linking number of trisecant manifolds, and further as a count of quadrisecants. Consider $Col^1_{\alpha,\beta}$ to be the submanifold of $C_3[\mathbb{R} \times B^n]$ where the points $(p_2, t^\alpha.p_1, t^\beta.p_3)$ sit on a straight line in $\mathbb{R} \times B^n$ in that linear order. Similarly, define $Col^3_{\alpha,\beta}$ to be the submanifold of $C_3[\mathbb{R} \times B^n]$ such that $(t^\alpha.p_1, t^\beta.p_3, p_2)$ sit on a straight line, in that linear order. These two manifolds are disjoint, moreover the former manifold detects the homotopy-class $t_2^\alpha w_{12}$, and the latter detects $t_2^\beta w_{32}$, thus the preimage of the disjoint pair $(Col^1_{\alpha,\beta}, Col^3_{\alpha,\beta})$ by the map $[t_2^\alpha w_{12}, t_2^\beta w_{23}] : S^{2n-1} \to C_3[S^1 \times B^n]$ is a 2-component Hopf link in S^{2n-1}. This is a non-trivial framed cobordism class of disjoint manifold pairs, as the linking number of the two components is ± 1. Moreover, this linking number is zero if we use the map $[t_2^p w_{12}, t_2^q w_{23}]$ provided $(p, q) \neq (\alpha, \beta)$.

The invariant W_3 is therefore computable via the linking numbers of the pre-images of the collinear manifolds, i.e. one computes the linking numbers of the pre-images of $Col^1_{\alpha,\beta}$ and $Col^3_{\alpha,\beta}$ via the lift of $\overline{ev_3}(G(p,q)) : S^{2n-1} \to C_3'[S^1 \times B^n]$ to the universal cover of $C_3'[S^1 \times B^n]$. This is the coefficient of $t_1^\alpha t_3^\beta [w_{12}, w_{23}]$ in $W_3(G(p,q))$.

Unfortunately the above is a relatively delicate visualisation task. See [BCSS] for examples of how one can directly compute linking numbers of trisecant manifolds. We use a variation of an argument of Misha Polyak [Po]. Polyak gave a direct argument showing that the quadrisecant formula of [BCSS], itself a linking number of trisecant manifolds, can be turned into a Polyak-Viro formula, i.e. a count involving only cohorizontal manifolds. Interestingly, Polyak's argument is done using maps out of 4-point configuration spaces, while ours uses submanifolds of 3-point configuration spaces.

There is cobordism of the manifold pair $(Col^1_{\alpha,\beta}, Col^3_{\alpha,\beta})$ and $(t^\alpha Co^1_2 - t^{\alpha-\beta} Co^1_3, t^{\beta-\alpha} Co^3_1 - t^\beta Co^3_2)$. A cobordism between these two families is given by the parabolic spline family. Specifically, fix a direction vector $\zeta \in \mathbb{R}^{n+1}$. Let V_ζ be the orthogonal compliment to ζ in $\mathbb{R}^{n+1}$. Let $L \subset V_\zeta$ be a line, and $Q : L \to \mathbb{R}.\zeta$ be a quadratic function whose second derivative with respect to arc-length on L is given by ϵ. Thus when $\epsilon = 0$ the graph of the function Q can be any line in $\mathbb{R}^{n+1}$ except those parallel to the ζ direction. Thus as a family parametrized by $\epsilon \in [0, \infty]$ we have a cobordism between $(Col^1_{\alpha,\beta}, Col^3_{\alpha,\beta})$ and $(t^\alpha Co^1_2 - t^{\alpha-\beta} Co^1_3, t^{\beta-\alpha} Co^3_1 - t^\beta Co^3_2)$.

There is an important technical issue here as the family is not disjoint when $\epsilon = \infty$, as it allows for triple points in the ζ direction. This does not pose a problem for us, since generically we can assume in our family $S^{2n-4} \times C_3[I] \to C_3[S^1 \times B^n]$ the direction vectors of collinear triples form a closed co-dimension 1 subset of S^n, i.e. the compliment is open and dense, thus generically we can choose ζ to be disjoint from this set. So we can compute the coefficient of $t_1^\alpha t_3^\beta [w_{12}, w_{23}]$ in $W_3(G(p,q))$ as the linking number of the pre-image of the above pairs, for $\overline{ev_3}(G(p,q))$.

Our strategy then, given $f : S^{2n-4} \to \mathrm{Emb}(I, S^1 \times B^n)$ is to first construct the framed cobordism class representing $ev_2(f) : S^{2n-4} \times C_2[I] \to C_2'[S^1 \times B^n]$ in the language of cohorizontal manifolds. This allows us to explicitly construct a null-cobordism corresponding to a null-homotopy of $ev_2(f)$. We then move on to study $ev_3(f) : S^{2n-4} \times C_3[I] \to C_3'[S^1 \times B^n]$. The null-cobordisms associated to $ev_2(f)$ can now be attached to the boundary of $ev_3(f)^{-1}(t^k Co_i^j)$, giving us a framed cobordism representation of $\overline{ev_3}(f)$, where $f = G(p,q)$.

Example 3.6. $W_3(G(p,q)) = t_1^p t_2^q [w_{23}, w_{31}] = t_1^{p-q} t_3^{-q} [w_{12}, w_{23}]$.

By careful choice of the immersion defining $G(p,q)$ we can arrange that there is a unique parameter value in S^{2n-4} where the embedding lives in $\mathbb{R}^3 \times \{0\}$ and the associated planar diagram has eight regular double-points. Moreover, we can ensure the locus of parameters in S^{2n-4} such that the associated embedding has double points is the wedge of two embedded copies of B^{n-2} in S^{2n-4}, parallel to the coordinate axis, i.e. considering S^{2n-4} to be $B^{n-2} \times B^{n-2}$ with the boundary collapsed. The resolution with the eight regular double points is depicted in Figure 8. Four of the double points persist along the red coordinate axis (i.e. a copy of B^{n-2}) and the remaining four persist along the blue coordinate axis. Considering the evaluation map $ev_2 : S^{2n-4} \times C_2[I] \to C_2[S^1 \times B^n]$, the submanifold of $S^{2n-4} \times C_2[I]$ mapping to cohorizontal points is depicted in Figure 10. This manifold is the disjoint union of four embedded spheres, diffeomorphic to $\sqcup_4 S^{n-2}$. There are two such spheres along the blue coordinate axis, consisting of the sphere where the 2^{nd} point is above the 1^{st}, and the sphere where the reverse is true; the 1^{st} point is above the 2^{nd}. Similarly there are two such spheres corresponding to the red coordinate axis. These sphere are essentially linearly-embedded in $S^{2n-4} \times C_2[I]$, having disjoint convex hulls, i.e. they are unlinked. Moreover, these spheres bound four disjointly-embedded balls, $\sqcup_4 B^{n-1} \to S^{2n-4} \times C_2[I]$.

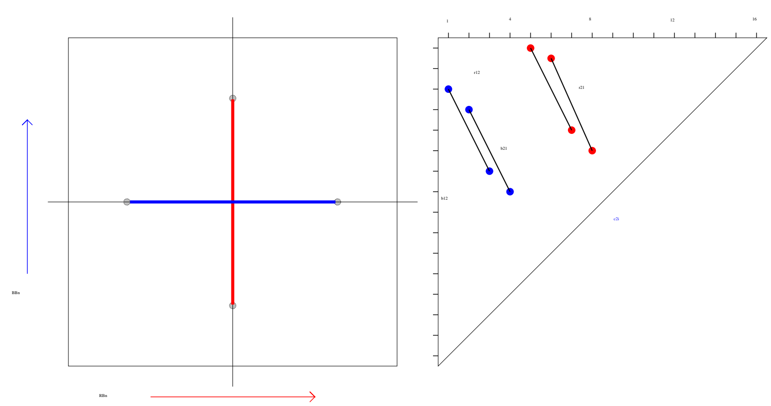


FIGURE 10. Preimages of cohorizontal manifolds $ev_2(G(p,q)) : S^{2n-4} \times C_2[I] \to C_2[S^1 \times B^n]$

Our 'units' for $C_2[I]$ in Figure 10 are the indices $\{1,2,\cdots,16\}$ for the cohorizontal points, along the parametrization of the embedded interval, suitably rescaled. The four spheres in the preimage are trivially framed, thus the disjoint balls they bound gives a null-cobordism. Thus Figure 10 depicts the framed cobordism classes of $\widetilde{ev_2}(f)^{-1}(t^i Co_1^2)$ and $\widetilde{ev_2}(f)^{-1}(t^i Co_2^1)$ for all i. The bounding discs can be thought to be depicted in the black arcs connecting the red and blue cohorizontal points, but these also trace out the cohorizontal points through the end homotopies. The bounding discs for the Co_1^2 manifolds determine null-homotopies of the maps $S^{2n-4}\times C_2[I]\to C_2[S^1\times B^n]$, while we assert that the corresponding bounding discs for the Co_2^1 manifolds determined by the null-homotopy are as depicted.

In Figure 11 we depict projections of the manifolds $\overline{ev_3}(G(p,q))^{-1}(t^l Co_i^j)\subset S^{2n-1}$. The domain of $\overline{ev_3}(G(p,q))$ is a copy of S^{2n-1} but we think of this sphere as a ball with its boundary collapsed to a point. The 'ball' being $B^{2n-4}\times C_3[I]$ with four triangular cylinders $B^{2n-4}\times C_2[I]\times I$ attached, due to the null-homotopies. The projection in Figure 11 is to the $C_3[I]$ (union triangular cylinders) factor. We use $0\le t_1\le t_2\le t_3\le 1$ as our coordinates for $C_3[I]$. In the figure t_1 and t_3 are the planar coordinates, with t_2 pointing out of the page. One therefore obtains Figure 11 from Figure 10 by considering how the cohorizontal points from Figure 10 induce cohorizontal points on the boundary of Figure 11. One then fills in the interior arcs: for example if there is a Co_1^2 point on the $t_2=t_3$ boundary facet of $C_3[I]$, then there will be an entire straight line parallel to the t_3-axis. Thus the Co_1^3 interior arcs will be orthogonal to the page in this projection. One then appends the null-cobordisms to the boundary facets of $C_3[I]$.

For example, the manifold labelled $t^p Co_3^1$ consists of three parts in the figure. There are two arcs parallel to the coordinate t_2-axis, these are represented by the short arcs between the nearby pairs of blue points. In $B^{2n-4}\times C_3[I]$ this represents two disjoint $(n-1)$-balls. There are also two longer diagonal arcs labelled $t^p Co_3^1$, one overcrossing $t^{-q}Co_2^3$ and one undercrossing. The overcrossing indicates the null-cobordism coming from the attachment on the $t_2=t_3$ face of $B^{2n-4}\times C_3[I]$, while the undercrossing represents the null-cobordism attached on the $t_1=t_2$ face.

Pairwise these linking numbers are all zero, with the sole exception of the pair (Co_3^1, Co_2^3), which gives $t_1^{p-q}t_3^{-q}[w_{12},w_{23}]$. We have suppressed the diagrams for the linking numbers of the preimages of the pairs (Co_3^1, Co_1^3), (Co_2^1, Co_1^3), and (Co_2^1, Co_2^3), as their computations are analogous.

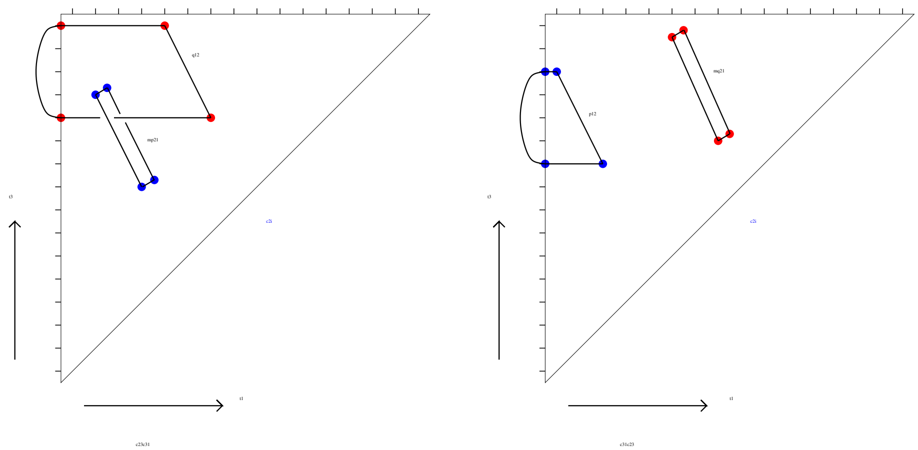


FIGURE 11. Linking cohorizontal manifold preimages for $\overline{ev_3}(G(p,q)) : S^{2n-1} \to C_3'[S^1 \times B^n]$

Proposition 3.7. *The homotopy group $\pi_{n-3}\,\mathrm{Diff}(S^1 \times B^n \text{ fix } \partial)$ is abelian, for all $n \geq 3$.*

Proof. A diffeomorphism of $S^1 \times B^n$ can be isotoped canonically so that its support is contained in $S^1 \times \epsilon B^n$ where $1 \geq \epsilon > 0$, i.e. we can effectively rescale the diffeomorphism radially in the B^n factor. Using conjugation by a translation in the B^n factor, one can show that up to isotopy, diffeomorphisms of $S^1 \times B^n$ can be assumed to have support in $S^1 \times U$ where U is any open subset of B^n. □

Let $\mathrm{Diff}_0(S^1 \times S^n)$ denote the subgroup of $\mathrm{Diff}(S^1 \times S^n)$ of elements homotopic to the identity. This is a subgroup of index 8 in $\mathrm{Diff}(S^1 \times S^n)$ and index two in the subgroup acting trivially on $H_*(S^1 \times S^n)$.

Theorem 3.8. *Any two elements of $\mathrm{Diff}_0(S^1 \times S^n)$, the group of diffeomorphisms homotopic to the identity, commute up to isotopy.*

Proof. Any element of $\mathrm{Diff}_0(S^1 \times S^n)$ is isotopic to one that fixes a neighborhood of $S^1 \times \{y_0\}$ pointwise. Thus commutativity follows as in the proof of the first part of Proposition 3.7. □

Let $\mathrm{Emb}(HB^i, B^n)$ denote the space of smooth embeddings $HB^i \to B^n$ that restricts to the standard inclusion $x \longmapsto (x, 0)$ on the round boundary $\partial^r HB^i = HB^i \cap \partial B^i$. Denote the corresponding framed embedding space by $\mathrm{Emb}^{fr}(HB^i, B^n)$. This space consists of pairs (f, ν) where $f \in \mathrm{Emb}(HB^i, B^n)$ and ν is a trivialization of the normal bundle to f that restricts to the canonical trivialization on $HB^i \cap \partial B^i$. Both $\mathrm{Emb}(HB^i, B^n)$ and $\mathrm{Emb}^{fr}(HB^i, B^n)$ are contractible spaces, the proofs are analogous to the homotopy classification of collar neighbourhoods. The role these embedding spaces play is as the total spaces of fiber bundles. The half-ball is defined in Definition 2.4.

The first fiber bundle to consider is $\mathrm{Emb}(HB^i, B^n) \to \mathrm{Emb}_u(B^{i-1}, B^n)$ where u denotes the unknot component of $\mathrm{Emb}(B^{i-1}, B^n)$, i.e. the path-component of the linear embedding. This bundle is in principle useful, but the fiber is embeddings of HB^i into B^n which are fixed on their boundary, which is not a very familiar space. By taking the derivative $\frac{\partial}{\partial x_1}$ along the boundary B^{i-1} we see this space fibers over $\Omega^{i-1}S^{n-i}$ with fiber homotopy-equivalent to $\mathrm{Emb}(B^i, S^{n-i} \times B^i)$. This latter space is the space of embeddings $B^i \to S^{n-i} \times B^i$ which restricts to the inclusion $p \longmapsto (*, p)$ on ∂B^i, where $* \in S^{n-i}$ is some preferred point. These fiber bundles were used in an analogous way by Cerf [Ce2] in his appendix, Propositions 5 and 6.

Alternatively we can form the bundle

$$\mathrm{Emb}(B^i, S^{n-i} \times B^i) \to \mathrm{Emb}(HB^i, B^n) \to \mathrm{Emb}_u^+(B^{i-1}, B^n)$$

where the base space consists of embeddings $B^{i-1} \to B^n$ together with a normal vector field along the embedding i.e. $\frac{\partial}{\partial x_1}$. The fiber of this bundle is technically the embeddings of HB^i into B^n which agree with the standard inclusion (and its derivative) along ∂HB^i. This fiber has the same homotopy type as $\mathrm{Emb}(B^i, S^{n-i} \times B^i)$.

Similarly, we have the corresponding bundles for the framed embedding spaces

$$\mathrm{Emb}^{fr}(B^i, S^{n-i} \times B^i) \to \mathrm{Emb}^{fr}(HB^i, B^n) \to \mathrm{Emb}_u^{fr}(B^{i-1}, B^n).$$

Given that the total space is contractible, this allows us to describe the unknot component of these embedding spaces as classifying spaces.

Lemma 3.9.

$$B\,\mathrm{Emb}^{fr}(B^i, S^{n-i} \times B^i) \simeq \mathrm{Emb}_u^{fr}(B^{i-1}, B^n)$$
$$B\,\mathrm{Emb}(B^i, S^{n-i} \times B^i) \simeq \mathrm{Emb}_u^{+}(B^{i-1}, B^n)$$

We take a moment to unpack some of the underlying geometric ideas involved in the lemma.

There is an isomorphism of homotopy groups

$$\pi_k\,\mathrm{Emb}(B^i, S^{n-i} \times B^i) \to \pi_{k+1}\,\mathrm{Emb}_u^+(B^{i-1}, B^n)$$

moreover, this map has an explicit geometric description. To do this, we need the exact fiber of the bundle $\mathrm{Emb}(HB^i, B^n) \to \mathrm{Emb}_u^+(B^{i-1}, B^n)$. This is the space of embeddings of HB^i into B^n which agrees with the standard inclusion $p \longmapsto (p, 0)$ and its derivative on the full boundary of HB^i. Denote this space by $\mathrm{Emb}_\partial(HB^i, B^n)$. Serre's homotopy-fiber construction tells us that $\mathrm{Emb}_\partial(HB^i, B^n)$ is homotopy-equivalent to

$$HF = \{\alpha : [0,1] \to \mathrm{Emb}(HB^i, B^n) \text{ s.t. } \alpha(0) = *, \alpha(1) \in \mathrm{Emb}_\partial(HB^i, B^n)\}.$$

In the above, $*$ denotes the basepoint of $\mathrm{Emb}(HB^i, B^n)$, i.e. the standard inclusion $p \longmapsto (p, 0)$. The homotopy-equivalence between $HF \to \mathrm{Emb}_\partial(HB^i, B^n)$ is given by associating $\alpha(1)$ to α. The homotopy-equivalence between HF and $\Omega\,\mathrm{Emb}_u^+(B^{i-1}, B^n)$ is given by associating $\overline{\alpha}$ to α where $\overline{\alpha}(t) = \alpha(t)_{|B^{i-1}}$.

For the sake of those not familiar with these methods we describe the homotopy-equivalence directly. For this we need to adjust our model slightly. We replace the space

$\mathrm{Emb}(HB^i, B^n)$ with the homotopy-equivalent space of embeddings $H^i \to \mathbb{R}^n$ where the support is constrained to be in HB^i, i.e. the maps are the standard inclusion $p \longmapsto (p, 0)$ outside of HB^i. Similarly, $\mathrm{Emb}^+_u(B^{i-1}, B^n)$ would be the space of embeddings $\mathbb{R}^{i-1} \to \mathbb{R}^n$ with a normal unit vector field, the embeddings and normal vector required to be standard outside of B^{i-1}. From this perspective, the fiber $\mathrm{Emb}(HB^i, B^n) \to \mathrm{Emb}^+_u(B^{i-1}, B^n)$ is the space of embeddings $H^i \to \mathbb{R}^n$ where the support is not only contained in HB^i, but the embedding and its derivative in the normal direction is required to be standard on ∂H^i. Observe the explicit deformation-retraction of $\mathrm{Emb}(HB^i, B^n)$ to a point, given by associating to $f \in \mathrm{Emb}(HB^i, B^n)$ the path (as a function of t), $f_t \in \mathrm{Emb}(HB^i, B^n)$ where

$$f_t(x_1, x_2, \cdots, x_k) = f(x_1 - t, x_2, \cdots, x_k) + (t, 0, \cdots, 0).$$

Thus the map $\mathrm{Emb}_\partial(HB^i, B^n) \to \Omega\,\mathrm{Emb}^+_u(B^{i-1}, B^n)$ in this model is the one that associates to $f \in \mathrm{Emb}_\partial(HB^i, B^n)$ the path $f_t \in \mathrm{Emb}^+_u(B^{i-1}, B^n)$ given by

$$f_t(x_2, \cdots, x_k) = f(1 - t, x_2, \cdots, x_k) + (t, 0, \cdots, 0).$$

The vector field being $\frac{\partial f}{\partial x_1}(1 - t, x_2, \cdots, x_k)$. So it would be reasonable to call this map *slicing the embedding*.

We mention a few elementary consequences of Lemma 3.9.

$$SO_n \simeq \mathrm{Emb}^{fr}_u(B^0, B^n) \simeq B\,\mathrm{Emb}^{fr}(B^1, S^{n-1} \times B^1).$$

For embeddings with 1-dimensional domains we have

$$\mathrm{Emb}^+_u(B^1, B^n) \simeq B\,\mathrm{Emb}(B^2, S^{n-2} \times B^2).$$

The space $\mathrm{Emb}^+_u(B^1, B^n)$ is a bundle over $\mathrm{Emb}_u(B^1, B^n)$, and this space is equal to $\mathrm{Emb}(B^1, B^n)$ when $n \geq 4$. The fiber of this bundle is ΩS^{n-2}, provided $n \geq 4$. This bundle is known to be trivial. One trivialization can be expressed as a splitting at the fiber $\mathrm{Emb}^+(B^1, B^n) \to \Omega S^{n-2}$. To construct it, use the null-homotopy of the Smale-Hirsch map [Bu1] $\mathrm{Emb}^+(B^1, B^n) \to \Omega S^{n-1}$. Given that the normal vector field is orthogonal to the Smale-Hirsch map, one can use the holonomy on S^{n-1} to homotope the normal vector field canonically to a map orthogonal to the x_1-axis, giving the map $\mathrm{Emb}^+(B^1, B^n) \to \Omega S^{n-2}$, and the splitting

$$\mathrm{Emb}^+(B^1, B^n) \simeq \mathrm{Emb}(B^1, B^n) \times \Omega S^{n-2}.$$

Substituting $i = n$ into Lemma 3.9 we get the theorem of Cerf [Ce1]

Theorem 3.10.

$$\mathrm{Emb}_u(B^{n-1}, B^n) \simeq B\,\mathrm{Diff}(B^n \textit{ fix } \partial).$$

Raising the codimension by one, and provided $n \geq 2$ we get the identification

$$\mathrm{Emb}^+_u(B^{n-2}, B^n) \simeq B\,\mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1}).$$

When $n \geq 2$, $\pi_0\,\mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1})$ is a monoid under concatenation, isomorphic as a monoid to $\pi_1\,\mathrm{Emb}^+_u(B^{n-2}, B^n)$, therefore with inverses. When $n \geq 3$, $\pi_0\,\mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1})$ is a commutative monoid under concatenation, therefore an abelian group, isomorphic to $\pi_1\,\mathrm{Emb}^+_u(B^{n-2}, B^n)$.

The space $\mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1})$ has a concatenation operation, which could also be thought of as an action of the operad of $(n-1)$-discs. The space $\mathrm{Emb}_u^+(B^{n-2}, B^n)$ similarly has a concatenation operation with one less degree of freedom. It can be encoded as an action of the operad of $(n-2)$-discs. These discs actions turn the two sets $\pi_0 \mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1})$ and $\pi_1 \mathrm{Emb}_u^+(B^{n-2}, B^n)$ into commutative monoids with the concatenation operation, provided $n \geq 3$. Moreover, one can see that the concatenation operation and concatenation of loops are the same operation on $\pi_1 \mathrm{Emb}_u^+(B^{n-2}, B^n)$. Thus, the isomorphism $\pi_0 \mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1}) \simeq \pi_1 \mathrm{Emb}_u^+(B^{n-2}, B^n)$ is an isomorphism of groups, which must be abelian. Lastly, notice that $\mathrm{Emb}_u^+(B^{n-2}, B^n)$ fibers over $\mathrm{Emb}_u(B^{n-2}, B^n)$ with fiber $\Omega^{n-2}S^1$, provided $n \geq 4$. So for all n we have a homotopy-equivalence

Corollary 3.11.
$$\mathrm{Emb}_u(B^{n-2}, B^n) \simeq B\,\mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1}).$$

Since the concatenation operation on $\mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1})$ turns $\pi_0 \mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1})$ into a group, it makes sense to consider the fiber bundle

$$\mathrm{Diff}(B^{n+1} \text{ fix } \partial) \to \mathrm{Diff}(S^1 \times B^n \text{ fix } \partial) \to \mathrm{Emb}(B^n, S^1 \times B^n).$$

Every embedding $B^n \to S^1 \times B^n$ that is standard $p \longmapsto (1, p)$ on ∂B^n is the fiber of some trivial smooth fiber bundle $S^1 \times B^n \to S^1$ with fiber B^n, by Lemma 3.9 and isotopy extension. Thus $\mathrm{Diff}(S^1 \times B^n \text{ fix } \partial)$ acts transitively on $\mathrm{Emb}(B^n, S^1 \times B^n)$. We record the observation.

Theorem 3.12. *The group* $\mathrm{Diff}(S^1 \times B^n$ *fix* $\partial)$ *acts transitively on* $\mathrm{Emb}(B^n, S^1 \times B^n)$*. Moreover every reducing ball* $B^n \to S^1 \times B^n$ *is the fiber of some smooth fiber bundle* $S^1 \times B^n \to S^1$*.*

Alternatively attach a $S^{n-1} \times D^2$ to obtain S^{n+1} where the reducing ball is now an n-ball Δ_1 with boundary a standard $(n-1)$-sphere. By the Cerf - Palais theorem there is a diffeomorphism of this sphere taking Δ_1 to a standard n-ball fixing $\partial\Delta_1$ pointwise.

Theorem 3.13. *The group* $\mathrm{Diff}(S^1 \times S^n)$ *acts transitively on the non-separating n-spheres in* $S^1 \times S^n$*. Moreover, every non-separating n-sphere is the fiber of a fiber bundle* $S^1 \times S^n \to S^1$*.*

Proof. Provided $n < 3$ this is classical. When $n \geq 3$ observe that complementary to a non-separating sphere there is an embedding $S^1 \to S^1 \times S^n$ that intersects the sphere precisely once and transversely. Since $dim(S^1 \times S^n) \geq 4$, we can isotope our embedding to be equal to $S^1 \times \{*\}$ and similarly isotope our non-separating sphere. If we drill a neighbourhood of $S^1 \times \{*\}$ out of $S^1 \times S^n$ we have constructed $S^1 \times B^n$, and our non-separating sphere is converted to a reducing ball. The result follows from Theorem 3.12. $\square$

Let S^1_0 denote a fixed $S^1 \times \{x_0\} \subset S^1 \times S^n$. Let $\mathrm{Emb}_0(S^1 \times B^n, S^1 \times S^n)$ (resp. $\mathrm{Emb}_0(S^1, S^1 \times S^n)$) denote the component of $\mathrm{Emb}(S^1, S^1 \times S^n)$ that contains the standard inclusion to $S^1 \times N(x_0)$ (resp. $S^1 \times \{x_0\}$), where $N(x_0) \subset S^n$ is a closed regular neighborhood of $x_0 \in S^n$. Note that the restriction map $\mathrm{Emb}_0(S^1 \times B^n, S^1 \times S^n) \to \mathrm{Emb}_0(S^1, S^1 \times S^n)$ is a fiber-bundle with fiber having the homotopy-type of one path-component of the free loop space of SO_n. The map $\operatorname{Diff}(S^1 \times S^n \operatorname{fix} N(S^1_0)) \to \operatorname{Diff}(S^1 \times B^n \operatorname{fix} \partial)$ given by deleting the interior of $N(S^1_0)$ is a homotopy-equivalence.

Lemma 3.14. *The locally trivial fiber bundle* $\operatorname{Diff}(S^1 \times S^n$ *fix* $N(S^1_0)) \to \operatorname{Diff}_0(S^1 \times S^n) \to \operatorname{Emb}_0(S^1 \times B^n, S^1 \times S^n)$ *induces a long exact homotopy sequence whose final terms are*

$$\cdots \to \pi_1 \operatorname{Emb}_0(S^1, S^1 \times S^n; S^1_0) \xrightarrow{p} \pi_0 \operatorname{Diff}(S^1 \times B^n \operatorname{fix} \partial) \xrightarrow{\phi} \pi_0 \operatorname{Diff}_0(S^1 \times S^n) \to 0$$

Here p is induced by isotopy extension and ϕ is induced by extension which is the identity on the complementary $S^1 \times B^n$. □

Most of the rest of the paper will be spent showing that there is a homomorphism $\pi_0 \operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) \to \pi_2 \operatorname{Emb}(I, S^1 \times B^3)$ which when composed with W_3 gives rise to a homomorphism, of the same name, $W_3 : \pi_0 \operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) \to (\pi_5(C_3(S^1 \times B^3))/\text{torsion})/R$ such that $W_3 \circ p$ is trivial. Furthermore, there exists an infinite set of elements of $\pi_0 \operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial)$ whose W_3 images are linearly independent. From this we will obtain the following result whose proof will be completed in §8. A sharper form of this result is given as Theorem 8.5.

Theorem 3.15. *Both the group* $\pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$ *and* $\pi_0(\operatorname{Diff}_0(S^1 \times S^3)) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$ *contain an infinite set of linearly independent elements.*

4. 2-Parameter Calculus

This section introduces techniques for working with 1 and 2-parameter families of embeddings of the interval into a 4-manifold.

4.1. **Spinning.** We start by setting conventions for the operation we call *spinning* that other authors call double point resolution. Spinning about arcs is an operation that generates $\pi_1^D \operatorname{Emb}(I, M; J_0)$, the *Dax subgroup*, i.e. the subgroup represented by loops that are homotopically trivial in $\operatorname{Maps}(I, M; J_0)$ [Ga2]. Here J_0 is an oriented properly embedded $[0,1]$ in the oriented 4-manifold M with 1_{J_0} a fixed parametrization.

Definition 4.1. Let Q be an oriented 2-sphere in M with $Q \cap J_0 = \emptyset$. Given an embedding $\lambda : [0,1] \to \operatorname{int}(M)$ with $\lambda \cap J_0 = \lambda(0)$ and $\lambda \cap Q = \lambda(1)$, we obtain a based loop α_t in $\operatorname{Emb}(I, M; J_0)$ by using λ to drag J_0 around Q. Let $\gamma_0 \subset J_0$ be a small arc containing $\lambda(0)$. α_t is defined so that for $t \in [0, .25]$, $\gamma_t := \alpha_t(\gamma_0)$ is a small arc containing $\lambda(4t)$, where $\gamma_{.25}$ is an embedded arc in Q. During $[.25, .75]$, keeping endpoints fixed, γ_t rotates around Q. I.e. view $Q = S^1 \times [0,1]$ with $S^1 \times 0$ and $S^1 \times 1$ identified to points and for $t \in [.25, .75], \gamma_t = \theta_t \times [0,1]$ for monotonically increasing θ_t. Finally during $t \in [.75, 1]$, γ_t returns to γ_0 following the reverse of λ. Corners are rounded so that each α_t is smooth. The local picture of spinning about Q is shown in Figure 12. The direction of the spinning is determined by the rule that (motion of γ_t, orientation of γ_t) gives the orientation of Q. Any loop in $\operatorname{Emb}(I, M; J_0)$, isotopic to one constructed as above is called a *λ-spinning of J_0 about Q*. The spinning that goes about Q in the opposite direction is called a $-\lambda$-spinning of J_0 about Q.

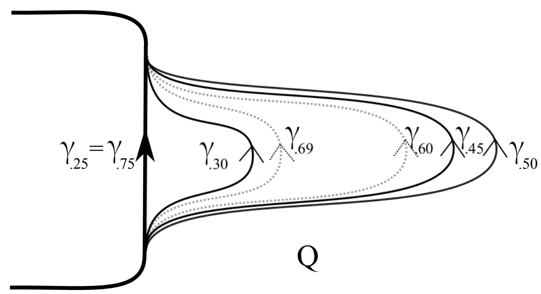


FIGURE 12.

Lemma 4.2. *Spinning depends only on the orientation of J_0, the orientation of Q and the relative path homotopy class of λ, i.e. if $\lambda_v \subset \mathrm{Maps}(I, M)$ is a path homotopy of $\lambda_0 := \lambda$ to λ_1, then we require that* $\mathrm{int}(\lambda_v) \cap Q = \emptyset$. □

If λ is an embedded arc from J_0 to the oriented arc $\tau \subset M \setminus J_0$, then let B be a 3-ball normal to τ at $\lambda(1)$ oriented so that (orientation of τ, orientation of B)=orientation of M. B is called a normal 3-ball. Let $Q = \partial B$ oriented with the outward first boundary orientation and $\lambda' = \lambda \setminus \mathrm{int}(B))$. Define the λ *spinning about* τ to be the λ'-spinning about Q. *Chord diagram* notation for this spinning and its inverse are shown in Figure 13a). The sign denotes whether this is a positive or negative spinning. *Band/lasso* notation is shown in Figure 13b). The *band* $= \cup\{\gamma_t | t \in [0, 1/4]\}$ with λ' being the core of the band and a *lasso* is a circle in Q containing, up to a small isotopy and rounding corners, $\gamma_{1/4}$. $\alpha_{1/4}$ and $\alpha_{3/4}$ are shown in Figure 13d) and $\alpha_{1/2}$ in Figure 13c). The positive (resp. negative) spinning corresponding to the band β and lasso κ will be denoted $\sigma(\beta, \kappa)$ (resp. $-\sigma(\beta, \kappa)$). We call $\beta \cap J_0$ the *base of the band* and $\beta \cap \kappa$ the *top of the band*. We orient the *core of the band* to point from the base to the top.

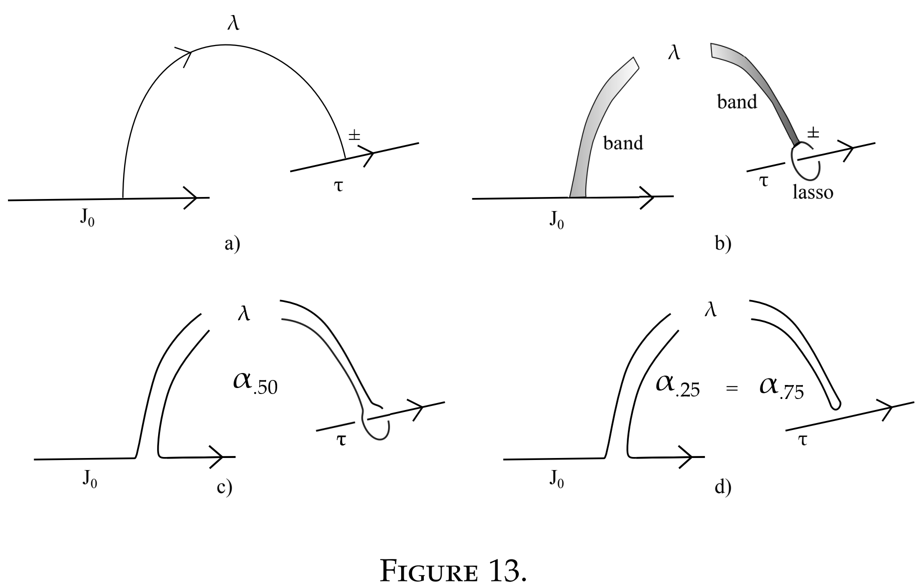


FIGURE 13.

For this section and the next we view $V = S^1 \times B^3$ as $D^2 \times S^1 \times [-1, 1]$ with the product orientation. Let I_0 be a properly embedded arc in V. When $\tau \subset I_0$, or very close to it, the λ of Figure 13) will be replaced by $n \in \mathbb{Z} = \pi_1(S^1 \times B^3; I_0)$, where I_0 is viewed as the basepoint. Unless said otherwise all charts of V will be of the form $(D^2 \times [0,1]) \times [-1,1]$, where the S^1-direction is the $[0,1]$-direction. Spinnings will almost always be about oriented arcs τ in a $D^2 \times I \times 0$ with the band $\subset D^2 \times S^1 \times 0$. Here the normal ball B intersects $D^2 \times I \times 0$ in a 2-disc called the *lasso disc* with the *lasso* its boundary. We call B the *lasso 3-ball* and Q the *lasso sphere*. In this paper, given a lasso, the lasso disc, sphere and 3-ball will be clear from context. The spinning will be denoted L/H (resp. H/L) if the homotopy from $\gamma_{1/4}$ to $\gamma_{3/4}$ first goes into the past (resp. future) and then into the future (resp. past).

We now give an oriented intersection theoretic way to decide whether or not a spinning of I_0 about an oriented arc τ is positive or negative. First, orient the band β to be coherent with its intersection with I_0 as in Figure 14 and then orient the lasso disc to be coherent with β as in Figure 14.

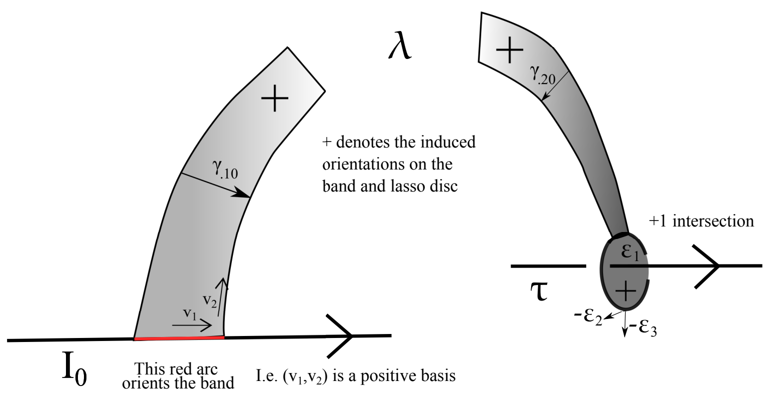


FIGURE 14.

Lemma 4.3. *The spinning is positive if and only if the spinning is L/H (resp. H/L) and the oriented intersection number of τ with the lasso disc is +1 (resp. -1).*

Proof. Let $(\epsilon_1, \cdots, \epsilon_4)$ denote the standard orientation of $D^2 \times I \times [-1,1]$. We can assume that ϵ_1 defines the orientation of τ so that the orientation of a normal ball B is given by $(\epsilon_2, \epsilon_3, \epsilon_4)$. We can also assume that the band and lasso appear as in Figure 14), i.e. so that $-\epsilon_3$ is an outward normal to B and an orienting vector for $\alpha_{.5}$ is $-\epsilon_2$. Therefore, $Q = \partial B$ is oriented by (ϵ_2, ϵ_4) and if the spinning is L/H, then the motion vector for $\alpha_{1/2}$ is ϵ_4 when the tangent vector to $\alpha_{1/2}$ is $-\epsilon_2$. It follows that the lasso disc has intersection +1 with τ. The H/L case follows similarly. $\square$

Lemma 4.4. *The spinnings of Figure 15 a) - d) and e) - f) are homotopic in $\Omega\,\mathrm{Emb}(I, M; I_0)$. Furthermore, the homotopy from a) to b) is supported in the union of a small 4-ball that contains the bands and the 3-balls spanning the spinning spheres Q and Q'. The homotopies from b) to d) are supported in a small neighborhood of the union of the bands and the sphere Q. The homotopy from e) to f) is supported in a neighborhood of the arc. Note that going from e) to f) the p changes sign, the arc changes orientation and the +/- changes to -/+.*

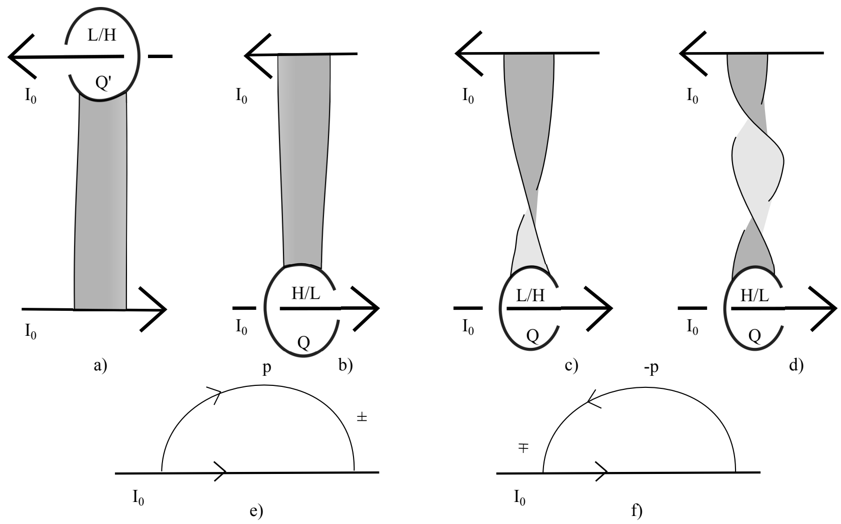


FIGURE 15.

Proof. The assertions regarding a) to b) are immediate. To go from b) to c) first choose coordinates so that a neighborhood of the top of the band as well as the lasso are contained in the $x-y$ plane and near the top of the band, the core of the band lies in the y-axis. Next rotate Q to take the lasso to its reflection in the y-axis. This is achieved by a rotation of the $x-t$ plane. Finally, a rotation in the $z-t$ plane isotopes the band back into $D^2 \times I \times 0$. Using a different rotation we could have obtained the opposite crossing. Apply this homotopy twice to go from c) to d). Using Lemma 4.3 the assertions regarding e) and f) follows from that of a) and b). $\square$

Remark 4.5. This lemma implies that the core of a band determines the isotopy class of the band up to possibly adding a half twist. Thus the isotopy class of the band is determined by the core arc and the orientation on the lasso disc.

Lemma 4.6. *Let λ_1, λ_2 be parallel oriented embedded arcs from I_0 to I_0 whose positive endpoints intersect in a subarc τ as in Figure 16 a). Let σ_1 (resp. σ_2) denote the spinning corresponding to λ_1 (resp. λ_2) with that of σ_2 being oppositely signed. Then, the loop α_t in $\mathrm{Emb}(I, S^1 \times B^3; I_0)$ which is the simultaneous spinning of σ_1 and σ_2 is homotopically trivial via a homotopy whose support lies in a small neighborhood of a parallelizing band between λ_1 and $\lambda_2 \setminus N(\tau)$.*

More generally, let α be the two spinnings as in Figure 16 b). Here the spinnings are expressed in band/lasso notation. Except for neighborhoods of the base arcs, where they differ by a half twist, the bands β_1 and β_2 are parallel, connecting to parallel lassos and parallel lasso spheres Q_1 and Q_2. Then, α is homotopic to 1_{I_0} via a homotopy supported in the union of a 4-ball U about the non parallel parts of the bands as in Figure 16 c) and a small neighborhood of the parallelism between the spheres and the bands. $\square$

Definition 4.7. The pair of spinnings defining α above is called a *parallel cancel pair*. Here β_1 and β_2 are carried by a *branched band surface*. The arcs in β_1 and β_2 where the bands diverge are called the *branch loci*. A homotopy as in Lemma 4.6 is called the *undo homotopy*.

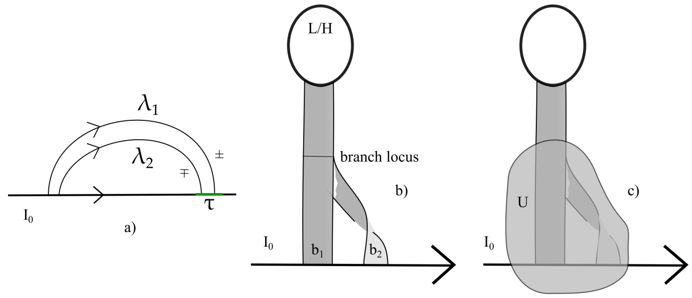


FIGURE 16.

Definition 4.8. Let γ_t be a λ-spinning about the 2-sphere Q. Suppose that $Q = \partial B, B$ a 3-ball and $E \subset B$ a properly embedded 2-disc with ∂E transverse to $\gamma_t, t \in [.25, .75]$ as in Definition 4.1. Let Q_1, Q_2 be the result of compressing Q along E. Then γ is homotopic to a concatenation $\gamma_1 * \gamma_2$ of two spinnings along nearby λ's, where γ_i spins about $Q_i, i = 1, 2$. The operation of replacing γ by $\gamma_1 * \gamma_2$ is called *splitting*. The reverse operation is called *zipping*. See Figures 16d) and e).

Remarks 4.9. i) The support of the homotopy from γ to $\gamma_1 * \gamma_2$ can be taken to be a small neighborhood of $\lambda \cup Q \cup E$.

ii) Suppose that γ is presented by the band β and lasso κ. Suppose that Q, D, B denote the lasso sphere, disc and 3-ball. Let β_1 and β_2 be obtained from β by removing a small neighborhood of its core and let κ_1 and κ_2 be the components of ∂D_1 and ∂D_2, where D_1 and D_2 are obtained from D by removing a small neighborhood of a properly embedded arc $e \subset D$ which intersects β at its core. Then $\sigma(\beta_1, \kappa_1) * \sigma(\beta_2, \kappa_2)$ is a splitting of $\sigma(\beta, \kappa)$ and we call $(\beta_1, \kappa_1), (\beta_2, \kappa_2)$ a splitting of (β, κ). They are spinnings about Q_1 and Q_2, the components of $\partial(B \setminus \text{int}(N(E))$, where $E \subset B$ is a properly embedded 2-disc such that $E \cap D = e$. Here we require that the spinning across Q_1 and Q_2 be L/H (resp. H/L) if the spinning across Q is L/H (resp. H/L). Note that σ is homotopic to $\sigma(\beta_1, \kappa_1) * \sigma(\beta_2, \kappa_2)$ by a homotopy supported in a small neighborhood of $\beta \cup Q \cup E$.

iii) As an application, suppose that $\tau_0, \tau_1, \cdots, \tau_n$ are the local edges of a 1-complex $K \subset M$ emanating from the vertex v. If σ_0 is a λ-spinning about a subarc of τ_0 close to v, then σ_0 splits to a concatenation $\sigma_1 * \cdots * \sigma_n$ via a homotopy supported away from K. Figure 17 a) and b) demonstrates the argument for $n = 2$.

Definition 4.10. An *abstract chord diagram* C in the manifold M consists of

a) an oriented properly embedded arc I_0

b) finitely many pairwise disjoint ordered pairs of distinct points $p(C)$ in $I_0, (x_1, y_1), \cdots,$ (x_n, y_n) with $p(C)$ linearly ordered by the orientation on I_0

c) For each $1 \leq i \leq n, g_i \in \pi_1(M; I_0)$

d) For each $1 \leq i \leq n, \eta_i \in \pm 1$.

Remarks 4.11. 1) An abstract chord diagram C in a 4-manifold gives rise to an $\alpha(C) \in \Omega\,\text{Emb}(I, M; I_0)$, well defined up to homotopy, which is a concatenation of spinnings, by choosing pairwise disjoint embedded paths λ_i from x_i to y_i representing g_i. Call

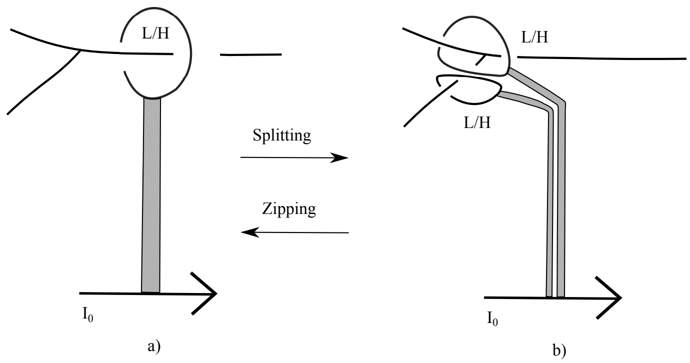


FIGURE 17.

such paths *chords*. An abstract chord diagram together with chords is called a *realization* and sometimes just called a *chord diagram*. Since spinnings commute as elements of $\pi_1 \operatorname{Emb}(I, M; I_0)$, this element is unaffected by modifying the relative location of the pairs of points (x_i, y_i), however most of the abstract chord diagrams of interest in this paper represent the trivial element, up to homotopy and we are not free to move these points in general. We will be considering the following types of modifications.

Definition 4.12. We have the following operations on abstract chord diagrams.

i) (reversal) $(x_i, y_i, g_i, \pm)$ is replaced by $(y_i, x_i, g_i^{-1}, \mp)$

ii) (exchange) If $a_i, b_j \in p(C)$ are adjacent in the linear ordering where $a_i \in \{x_i, y_i\}$ and $b_j \in \{x_j, y_j\}, i \neq j$, then replace $a_i \in \{x_i, y_i\}$ by b_j and $b_j \in \{x_j, y_j\}$ by a_i.

iii) (sliding) As input, $y_i \in p(C)$ is adjacent to an $x_j \in p(C)$ with $i \neq j$. As output the chord $(x_i, y_i, g_i, \pm)$ is replaced by three chords $(x_i, y_i', g_i, \pm)$, $(x_r, y_r, g_i * g_j, \sigma_r)$, $(x_s, y_s, g_i * g_j, \sigma_s)$. Here y_i' is moved to the other side of x_j, Also x_r, x_s, x_i are order adjacent as are y_s, y_j, y_r. *Sign Rule*: $\sigma_r = \pm$ if (resp. $\sigma_s = \pm$) the interval between y_i and y_r (resp. y_i and y_s) contains none or both of x_j, y_j, otherwise and σ_r (resp. σ_s) $= \mp$. See Figure 18.

iv) (isotopy) Points are moved isotopically in I_0 without any collisions.

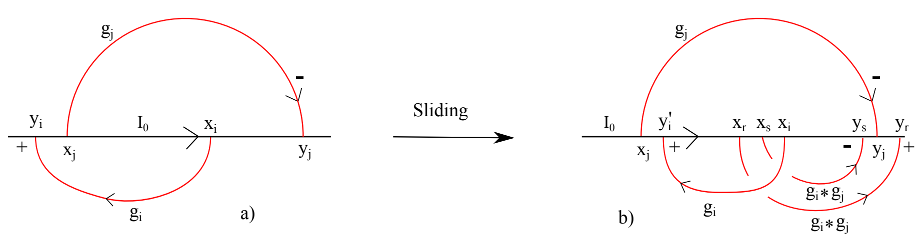


FIGURE 18.

Remark 4.13. Let τ_j denote a chord from x_j to y_j. A sliding is the result of splitting the $(x_i, y_i, g_i, \pm)$ spinning near x_j, then isotoping the resulting lasso that links τ_j along τ_j and finally do a second splitting near y_j. Note that the support of this isotopy is disjoint from τ_j.

Definition 4.14. We say the abstract chord diagrams C on I_0 and C' on I_1 are *combinatorially equivalent* if their data agrees up to reversals and isotopy.

Lemma 4.15. *If the abstract chord diagrams C and C' defined on I_0 are combinatorially equivalent, then α_C is homotopic to $\alpha_{C'}$.* □

Lemma 4.16. *(Change of Basepoint) Let I_0 and I_1 be path homotopic embedded arcs in the 4-manifold M. An abstract chord diagram C_0 on I_0 induces a combinatorially equivalent chord diagram C_1 on I_1. If $\alpha_j := \alpha_{C_j} \in \Omega\,\mathrm{Emb}(I, M, I_j), j = 0, 1$ then $\alpha_1 = \gamma * \alpha_0 * \gamma^{-1}$ is well defined up to homotopy and in particular is independent of the path isotopy γ from I_0 to I_1.*

Proof. Since I_0 and I_1 are path homotopic, there is a canonical isomorphism between $\pi_1(M, I_0)$ to $\pi_1(M, I_1)$ and hence C_0 induces a chord diagram C_1 on I_1 well defined up to combinatorial equivalence and hence α_1 is well defined up to homotopy. Alternatively, we can assume that I_0 and I_1 agree in a small neighborhood U of their initial point and $p(I_0) \subset U$. It follows that the homotopy class of α_1 is independent of the conjugating γ. □

4.2. **Factorization.** In this subsection we introduce techniques for constructing and working with 2-parameter families of embeddings. We will define the *bracket* of two null homotopic loops in $\mathrm{Emb}(I, M; J_0)$ whose domain and range supports are disjoint and will show that such brackets are well defined in $\pi_2\,\mathrm{Emb}(I, M; J_0)$, anticommute and are bilinear. Our fundamental example is the $G(p, q)$ family in $\mathrm{Emb}(I, S^1 \times B^3; I_0)$ depicted in Figure 19 a) which is the *bracket* of the 1-parameter families B_p and R_q defined by the blue and red cords. The p or q means that the indicated cord goes p or q times about the S^1.

Definition 4.17. Let M be an oriented 4-manifold and $1_{J_0} : [0, 1] \to M$ a proper embedding with image J_0. Let $\alpha : [0, 1] \to M$ be an embedding. We define the *domain support of* $\alpha := \mathrm{D}^{\mathrm{supp}}(\alpha) = \mathrm{cl}\{s \in I | \alpha(s) \neq 1_{J_0}(s)\}$ and the *range support of* $\alpha := \mathrm{R}^{\mathrm{supp}}(\alpha) = \mathrm{cl}\{\alpha(I) \setminus J_0\}$. If $f : X \to \mathrm{Emb}(I, M; J_0)$, then define the *domain support of* $f := \mathrm{D}^{\mathrm{supp}}(f) = \mathrm{cl}(\cup_{x \in X} \mathrm{D}^{\mathrm{supp}}(f_x))$ and the *range support of* $f := \mathrm{R}^{\mathrm{supp}}(f) = \mathrm{cl}(\cup_{x \in X} \mathrm{R}^{\mathrm{supp}}(f_x))$. Define the *parameter support of* f to be $\mathrm{cl}\{x \in X | f_x \neq 1_{J_0}\}$.

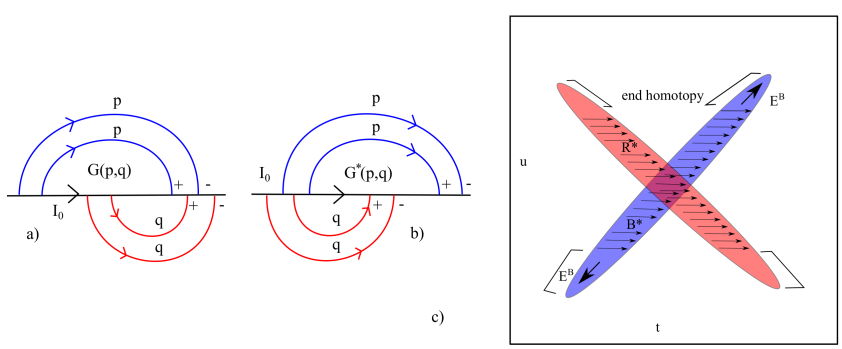


FIGURE 19.

Example 4.18. The domain support of B_p (resp. R_q) are four small intervals that contain the endpoints of the blue (resp. red) cords while the range support of B_p (resp. R_q) is contained in a small neighborhood of the blue (resp. red) cords. Note that both B_p and R_q are null homotopic and have disjoint domain and range supports. (B,R) is an example of a *separable pair* which we now define.

Definition 4.19. Let $B, R \in \Omega\,\mathrm{Emb}(I,M;J_0)$, the based loop space. We say that the ordered pair (B,R) is *separable* if
i) B and R have disjoint domain supports
ii) B and R have disjoint range supports
iii) B and R are null homotopic

Definition 4.20. If $F^B, F^R : X \to \mathrm{Emb}(I,M;J_0)$ are such that for all $x \in X, F^B_x, F^R_x$ have both disjoint domain and range supports, then define $F = F^B \circ F^R : X \to \mathrm{Emb}(I,M;J_0)$ by $F_x(s) = F^B_x(s)$ (resp. $F^R_x(s)$) if $s \in \mathrm{D}^{\mathrm{supp}}(F^B_x)$ (resp. $\mathrm{D}^{\mathrm{supp}}(F^R_x)$) and $F_x(s) = 1_{J_0}(s)$ if $s \notin \mathrm{D}^{\mathrm{supp}}(F^B_x) \cup \mathrm{D}^{\mathrm{supp}}(F^R_x))$. F is called the *adjunction* of F^B and F^R.

Definition 4.21. Let (B,R) be a separable pair with parameter supports within $[3/8, 5/8]$. Define the *bracket* of (B,R) to be $F \in \Omega\Omega\,\mathrm{Emb}(I,M;J_0)$, the adjunction of the elements $F^B, F^R \in \Omega\Omega\,\mathrm{Emb}(I,M;J_0)$ defined as follows.

i) For $u \in [.25, .75], F^B_{t,u}(s) = B_{t+u-.5}(s)$ when $s \in \mathrm{D}^{\mathrm{supp}}(B)$ and $1_{J_0}(s)$ otherwise. $F^R_{t,u}(s) = R_{t+.5-u}(s)$ when $s \in \mathrm{D}^{\mathrm{supp}}(R)$ and $1_{J_0}(s)$ otherwise.

ii) For $u \in [0, 1/4], F^B_u \in \Omega\,\mathrm{Emb}(I,M;J_0)$, is a null homotopy of $F^B_{.25}$ such that $F^B_{t,u} = 1_{J_0}$ if $(t,u) \notin [1/8, 3/8] \times [1/8, 1/4]$. F^R_u, is a null homotopy of $F^R_{.25}$ such that $F^R_{t,u} = 1_{J_0}$ if $(t,u) \notin [5/8, 7/8] \times [1/8, 1/4]$.

iii) for $u \in [3/4, 1], F^B_{t,u} = F^B_{t-.5,1-u}$ when $(t,u) \in [5/8, 7/8] \times [3/4, 7/8]$ and 1_{J_0} otherwise. $F^R_{t,u} = F^R_{t+.5,1-u}$ when $(t,u) \in [1/8, 3/8] \times [3/4, 7/8]$ and 1_{J_0} otherwise.

We let $[F]$ denote the class in $\pi_2(\mathrm{Emb}(I,M;J_0))$ represented by F. The homotopies F^B_u, F^R_u, $u \in [1/8, 1/4]$ and $u \in [3/4, 7/8]$ are called *end homotopies*. B and R are called the *midlevel loops* of the adjunction F.

Example 4.22. The blue (resp. red) region of Figure 19 c) contains the parameter support of the 2-parameter family F^{B_p} (resp. F^{R_q}) arising from B_p (resp. R_q). The arrows are meant to suggest that as we horizontally traverse the blue or red region, for $u \in [1/4, 3/4]$, we see a conjugate of B or R.

Remark 4.23. If the range supports of B and R are disjoint, then F is homotopically trivial.

Lemma 4.24. *(Independence of End Homotopies) If $F, G \in \Omega\Omega\,\mathrm{Emb}(I,M;I_0)$ are separable such that $F_u = G_u$ for all $u \in [1/4, 3/4]$, then $[F] = [G] \in \pi_2(\mathrm{Emb}(I,M;I_0))$.*

Proof. It suffices to consider the case that $F^R = G^R$. The map F, schematically shown in Figure 19, is homotopic to the one in Figure 20 a), which is homotopic to the one in Figure 20 b), which is homotopic to the map G. □

Proposition 4.25. *The class $[F] \in \pi_2\,\mathrm{Emb}(I,M;I_0)$ is determined by (B,R).*

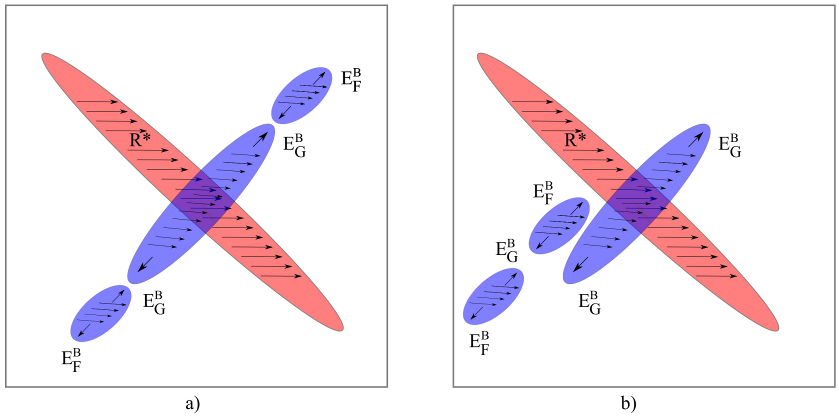


FIGURE 20.

Proof. Let F and G have the common (B,R). Since the domain and range supports of B and R are disjoint, we can independently reparametrize F^B and F^R so that F_u and G_u coincide for $u \in [1/4+\epsilon, 3/4-\epsilon]$. The Independence lemma implies that they represent the same element of $\pi_2(\mathrm{Emb}(I,M;I_0))$. □

Definition 4.26. We denote by $[(B,R)]$ the class in $\pi_2(\mathrm{Emb}(I,M;I_0))$ induced from (B,R). We say that the elements $B,R \in \Omega(\mathrm{Emb}(I,M;I_0))$ are *separable* if their domain and range supports, as in Definition 4.17 are disjoint. We call (B,R) a *separable pair*. We say that (B_0,R_0) and (B_1,R_1) are *separably homotopic* if there exist homotopies $B_s, R_s s \in [0,1]$, such that for each s, B_s and R_s are separable.

Lemma 4.27. *If B_0 and R_0 are homotopically trivial and (B_0,R_0) and(B_1,R_1) are separably homotopic, then $[(B_0,R_0)] = [B_1,R_1]$.* □

Proposition 4.28. *(bilinearity) If (B,R) is a separable pair, separably homotopic to $(B_1 * B_2 * \cdots * B_m, R_1 * \cdots * R_n)$ where each B_i and R_j is homotopically trivial and $*$ denotes concatenation, then $[B,R] = \sum_{i,j}[(B_i,R_j)]$.*

Proof. It suffices to consider the case that n= 2 and $m = 1$. The result will then follow from induction. First notice that the support of a concatenation is the union of the supports of the terms so that each of (B,R_1) and (B,R_2) is a separable pair. By the independence of end homotopies we can assume that the end homotopy for R is a concatenation of those of R_1 and R_2. Since R is a concatenation, we can assume that the usual support of $R_1 \subset [0,.4] \subset [0,1]$, while support $R_2 \subset [.6,1] \subset [0,1]$. Thus we can assume that (B,R) appears as in Figure 21 a) and we can split B into two copies of itself as in Figure 66 b) and so $[(B,R)] = [(B,R_1)] + [(B,R_2)]$. □

Lemma 4.29. *We have the following equalities in $\pi_2 \mathrm{Emb}(I,M;I_0)$ where $\bar{B}$ denotes the reversal of B.*

$$[(B,R)] = [(R,\bar{B})] = [(\bar{R},B)] = -[(R,B)] = [(\bar{B},\bar{R})]$$

Proof. Start with (B,R) as in Figure 22 a) and then rotate clockwise to obtain Figure 22 b) which represents the same element of $\pi_2 \mathrm{Emb}(I,M;I_0)$. Finally, homotopically

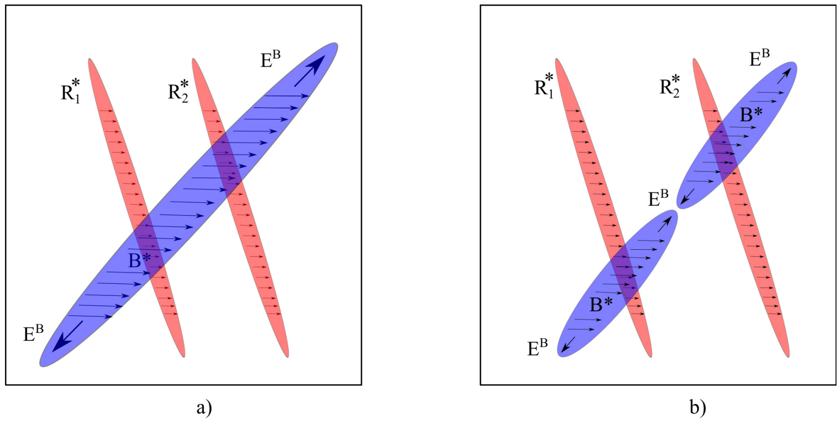


FIGURE 21.

reparametrize the rotated F^B and F^R to obtain Figure 22 c) which depicts $(R, \bar{B})$. Since B and R have disjoint supports in both domain and range, this reparameterization can be done independently on F^B and F^R subject to the condition that during the homotopy all points that are both blue and red lay in the original $u \in [1/4, 3/4]$ region. For this reason the arrows can be made horizontal in only one direction. This argument applied to a counterclockwise rotation shows that $[(B, R)] = [\bar{R}, B]$. Next note that the standard homotopy from 1_{I_0} to $R * \bar{R}$ is through loops that have domain and range supports disjoint from that of B and hence $0 = [(1_{I_0}, B)] = [(R * \bar{R}, B)] = [(R, B)] + [(\bar{R}, B)]$ thereby establishing the third inequality. The final equality follows from the earlier ones. $\square$

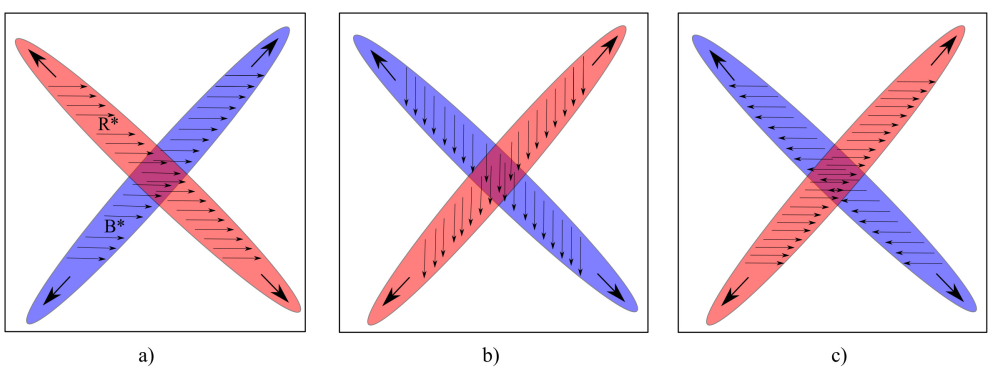


FIGURE 22.

Example 4.30. If $B, R \in \Omega\,\mathrm{Emb}(I, M; I_0)$ are defined by abstract chord diagrams C_B, C_R such that $p(C_B) \cap p(C_R) = \emptyset$, then (B, R) is separable. That is because the domain support of B is contained in $p(C_B)$ and the range support is contained in a small neighborhood of the chords and we can assume that the chords of C_B and C_R are disjoint. Our fundamental example is the separable pair denoted $G(p, q)$ defined by the chord diagram shown in Figure 19 a), where C_B (resp. C_R) is defined by the blue (resp. red) chords. It's class in $\pi_2 \operatorname{Emb}(I, M; I_0)$ is called the (p,q)-*primitive class*. A second basic example is the separable pair denoted $G^*(p,q)$ shown in Figure 19 b) and called the *symmetric* $G(p,q)$.

Definition 4.31. We say that the abstract chord diagrams C_1 and C_2 are *disjoint* if $p(C_1) \cap p(C_2) = \emptyset$. The *sum* of disjoint chord diagrams is the union of the data of the diagrams.

Lemma 4.32. *Let B_i and R_i be represented by pairwise disjoint abstract chord diagrams C_{B_i} and C_{R_i} for $i = 0, 1$.*

i) (reversal) If C_{B_1} is obtained from C_{B_0} by a reversal, then (B_0, R_0) is separably homotopic to (B_1, R_0). If B_0 and R_0 are homotopically trivial, then $[(B_0, R_0)] = [(B_1, R_0)]$.

ii) (exchange) If C_{B_1} is obtained from C_{B_0} by an exchange along an interval whose interior is disjoint from $p(C_R)$, and both B_0, R_0 are homotopically trivial, then $[(B_0, R_0)] = [(B_1, R_0)]$.

iii) (sliding) If C_{B_1} (resp. C_{R_1}) is obtained from C_{B_0} (resp. C_{R_0}) by sliding a chord over either a blue or red chord, then $[(B_1, R_0)] = [(B_0, R_0)]$ (resp. $[(B_0, R_1)] = [(B_0, R_0)]$.

iv) (addition/cancellation) If C_{B_1}, C_{B_0}, C_{R_0} are disjoint chord diagrams with C_{B_2} the sum of C_{B_1} and C_{B_0} and $[(B_1, R_0)] = 0$, then $[(B_2, R_0)] = [(B_0, R_0)]$. The similar statement holds with R and B switched.

v) (isotopy) If C_{B_t} (resp. C_{R_t}) is an isotopy of C_{B_0} to C_{B_1} (resp. C_{R_0} to C_{R_1}) and for every t, $p(C_{B_t}) \cap p(C_{R_t}) = \emptyset$, then $[(B_1, R_1)] = [(B_0, R_0)]$.

Proof. i) and v) are immediate. iv) follows from Proposition 4.28. Being the composite of an isotopy and two splittings, it follows from Remark 4.9 that the support of a sliding is disjoint from the chord being slid over. Therefore the homotopy from (B_0, R_0) to (B_1, R_0) is separable. This proves iii). To prove ii) consider the representative Figure 23 a) which shows a chord diagram with the exchange interval indicated. The chord diagram of Figure 23 b) has the same C_R, but the blue chord diagram C is given by a parallel pair of oppositely signed arcs representing the same group element. Furthermore, the intervals between their initial points and between their final points contains no points of $p(C_R)$. Apply the undo homotopy, Lemma 4.6 to α_C to reduce C to the trivial diagram. Since the domain and range support of this undo homotopy is disjoint from that of α_{C_R}, it follows that $[(\alpha_C, \alpha_{C_R})] = 0$. Using bilinearity we obtain Figure 23 c). We again use the undo homotopy to cancel a pair of parallel oppositely signed blue chords to obtain the diagram of Figure 23 d) which has a pair of chord diagrams isotopic to the exchanged pair. $\square$

Remark 4.33. A special case of iv) is when C_{B_0} contains a pair of parallel oppositely signed chords with the same group elements, $p(C_{R_0})$ is disjoint from the parallelism between their endpoints and C_{B_1} is the chord diagram with the parallel pair deleted, in which case $[(B_0, R_0)] = [(B_1, R_0)]$.

Definition 4.34. Let C_B and C_R be disjoint chord diagrams on I_0. We say that (C_B, C_R) is *combinatorially equivalent* to (C'_B, C'_R) if one can be transformed to the other through disjoint chord diagrams that change by reversals, isotopies and allowable exchange moves.

We have the following analogues of Lemmas 4.15, 4.16 which, in combination with Lemma 4.16 have similar proofs.

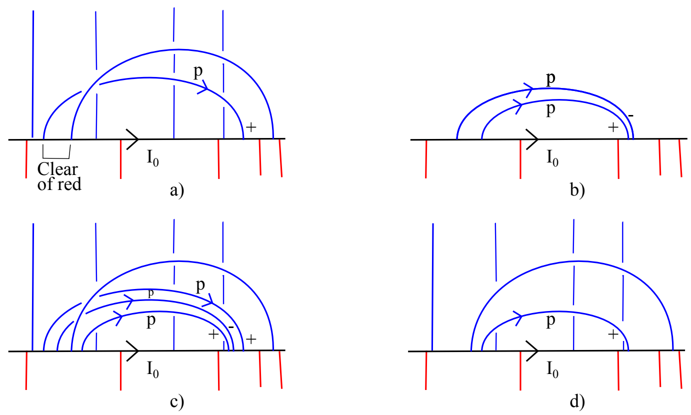


FIGURE 23.

Lemma 4.35. *If (C_B, C_R) and (C'_B, C'_R) are combinatorially equivalent, then $[(C_B, C_R)] = [(C'_B, C'_R)] \in \pi_2(\mathrm{Emb}(I, M; I_0))$.* □

Lemma 4.36. *(Change of Basepoint) Let I_0 and I_1 be path homotopic embedded arcs in the 4-manifold M. A pair of disjoint chord diagrams (C^0_B, C^0_R) on I_0 induces a pair (C^1_B, C^1_R) on I_1 unique up to combinatorial equivalence. Furthermore, $[(C^1_B, C^1_R)] \in \pi_2 \mathrm{Emb}(I, M; I_0)$ is obtained from $[(C^0_B, C^0_R)] \in \pi_2 \mathrm{Emb}(I, M; I_1)$ by a change of basepoint isomorphism which is independent of the path isotopy from I_0 to I_1.* □

Lemma 4.37. $[G^*(p, q)] = -[G(p, p-q)]$.

Proof. The argument is given in Figure 24. Figure 24 a) shows $G^*(p, q)$. One of the red chords is reversed to obtain Figure 24 b). That chord is slid over the adjacent blue chord to obtain Figure 24 c). A red chord is then reversed and the resulting pair of oppositely signed red chords with group element p are cancelled using the undo homotopy. □

4.3. **Fundamental Classes.** We have already introduced the primitive family $G(p, q)$. We now introduce the *elementary classes* $E(p, q)$ and *double classes* $D(p, q)$ and compute $[E(p, q)]$ in terms of $[G(p, q)]$'s and $[D(p, q)]$ in terms of $[E(p, q)]$'s.

Definition 4.38. Define the *standard elementary family* $E(p, q)$by the band/lasso diagram as in Figure 25. In words $E(p, q) = (B, R)$ where B (resp. R) is defined by the band β_b (resp. β_r) and lasso κ_b (resp. κ_r), where the base of β_b appears on I_0 before the base of β_r. Also κ_b (resp. κ_r) links the core of β_r (resp. κ_b) very close to its base and both spinnings are positive. When the top of the core of β_b (resp. β_r) is pushed slightly to lie on I_0, then the core represents p (resp. q) $\in \pi_1(S^1 \times B^3; I_0)$.

More generally, an $E(p, q)$ *family* is one of the form (B, R) where each of B and R are represented by a single band and lasso where the lasso of one links the core of the other very close to its base.

Remarks 4.39. i) Up to the ordering of their bases on I_0, twisting of the bands and $L/H, H/L$ designation an $E(p, q)$ family is isotopic to the standard $E(p, q)$ for appropriate p and q, hence up to sign and permutation of p and q such a family represents $[E(p, q)]$.

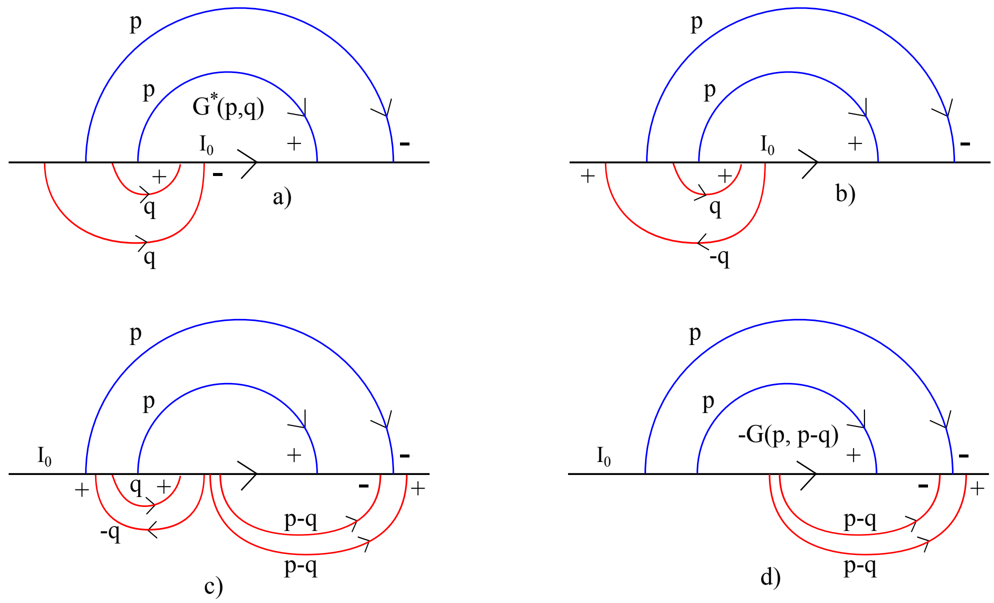


FIGURE 24.

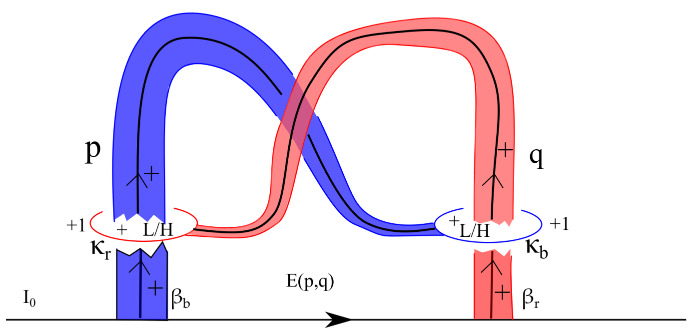


FIGURE 25.

ii) Recall that the sign of a spinning is determined by a rotation about an oriented sphere, which links an oriented arc τ. Here, the oriented arcs are the oriented cores of the bands β_b and β_r.

iii) To over emphasize, the order of the base of the bands on I_0 and the homotopy classes of their cores determine p and q. Given such bands, the sign of $[E(p,q)]$ is determined by the color of the first band, and the signs of the spinnings. The sign of a given spinning is determined by the $L/H, H/L$ data and the <band core, lasso> algebraic intersection number from Lemma 4.3. Thus the sign is given by the parity of the five possible differences from the standard model. In the standard model, shown in Figure 25, we have L/H with +1 intersection number in both cases. That figure also shows the oriented cores of the bands as well as the orientations of the bands and the lasso discs.

Lemma 4.40. $[E(p,q)] = -[G(-q,p)] + [G(p,-q)] = -[G^*(-q,p)] + [G^*(p,-q)]$.

Proof. We first isotope $E(p,q)$ as in Figure 26 a), then homotope it to a chord diagram as in Figure 26 b), where we use Lemma 4.3 to determine the signs of the chords. Use Lemma 4.32 i) to reverse the red chords and then 4.32 iii) to isotope the chords to obtain Figure 26 c). Figure 26 d) is the result of applying two exchanges, Lemma 4.32 ii). A representative of $[G(-q,p)]$ is shown in Figure 26 e). Note that it differs from the standard $G(-q,p)$ by a color change and replacing R by $\bar{R}$, i.e. two sign changes at the π_2 level. We use bilinearity to add 26 d) and e) and then use the undo homotopy Lemma 4.6 to cancel a pair of parallel red chords to obtain 26 f). Adding Figure 26 g) which is a $[-G(p,-q)]$ and then doing an undo homotopy to the red chords we obtain Figure 26 h) which is homotopically trivial, since another undo homotopy eliminates the blue chords. It follows that $[E(p,q)] + [G(-q,p)] - [G(p,-q)] = 0$.

The proof that the left hand side equals the right hand side follows similarly. Here we add a $-[G^*(p,-q)]$ as shown in Figure 26 i) to Figure 26 d) which gives Figure 26 j). Adding to that a $[G^*(-q,p)]$ as shown in Figure 26 k) gives a chord diagram representing the trivial element. It follows that $[E(p,q)] = -[G^*(-q,p)] + [G^*(p,-q)]$. ☐

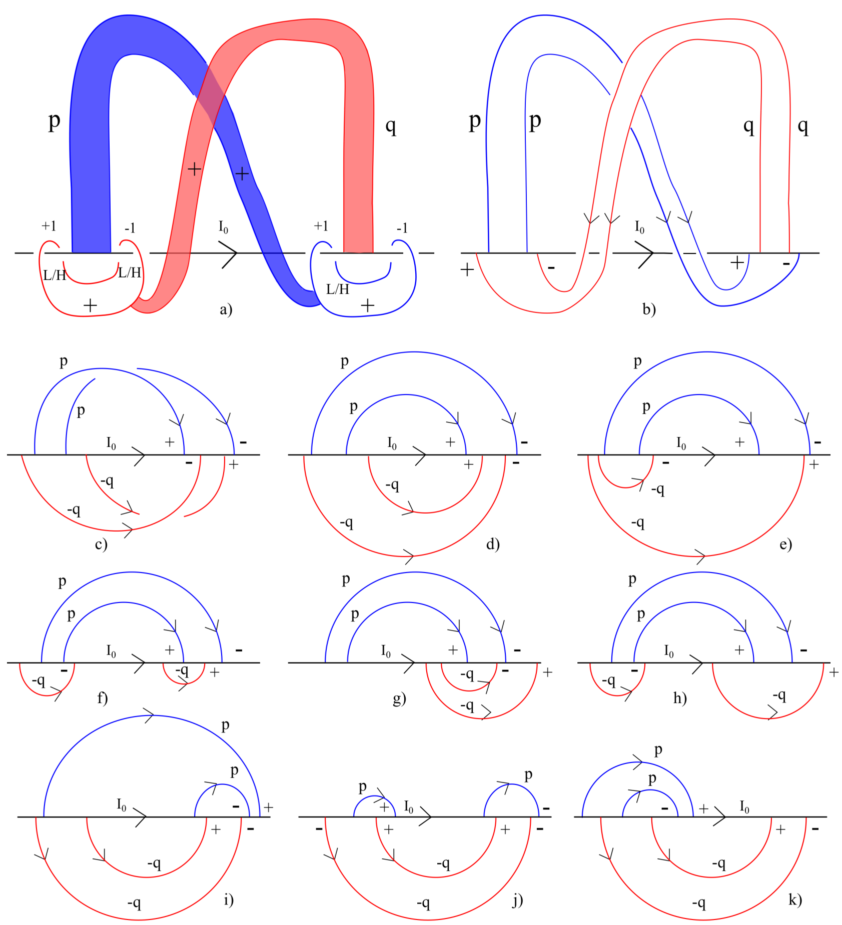


FIGURE 26.

Corollary 4.41. *i)* $[E(p,q)] = 0$ *if* $q = -p$

ii) $[E(p,q)] = -[E(-q,-p)]$
iii) (Hexagon Relation) $[G(p,q)] + [G(p,p-q)] = [G(q,p)] + [G(q,q-p)]$.

Proof. The first two conclusions are immediate. The third follows from the second equality of Lemma 4.40, Lemma 4.37 and a change of variables. $\square$

The next result follows from iii) and Proposition 3.4. See also Remark 3.5.

Theorem 4.42. *The induced map* $W_3 : G \to W$ *is an isomorphism, where* G *is the subgroup of* $\pi_2 \operatorname{Emb}(I, S^1 \times B^3; I_0)$ *generated by the* $G(p,q)$*'s and* W *is the quotient group defined in Proposition 3.4.* $\square$

Definition 4.43. Define the *standard double family* $D(p,q)$ in band/lasso notation as in Figure 27. In words, it is of the form (B,R) where B consists of the band β_b and lasso κ_r. R consists of a parallel cancel pair defined by the bands β_r, β_r' and lassos κ_r, κ_r'. Base(β_r), base(β_b), base(β_r') appear in order along I_0. κ_r, κ_r' link band(β_b) just above its base and κ_b links band(β_r), band(β_r') at the branch loci. When the top of the bands are pushed to lie on I_0 then β_b represents p and both β_r, β_r' represent q. Also, the spinning of $\sigma(\beta_b, \kappa_b)$ about the oriented cores of β_r, β_r' is positive. The spinning of $\sigma(\beta_r, \kappa_r)$ (resp. $\sigma(\beta_r', \kappa_r')$ about the oriented core of β_b is positive (resp. negative).

More generally, we call a family (B,R) a *$D(p,q)$ family* if up to isotopy it is represented by bands and lassos as a standard $D(p,q)$ family, up to switching the roles of B and R, $L/HH/L$ designations and twisting of the bands.

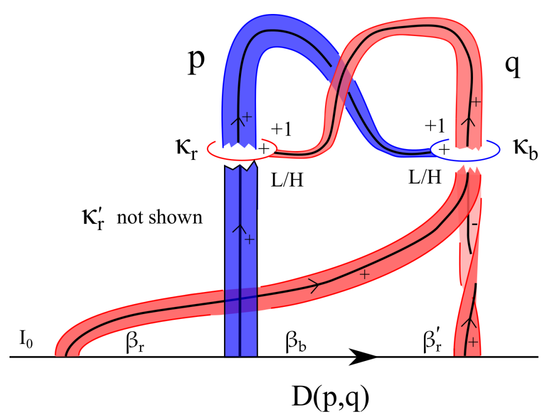


FIGURE 27.

Remarks 4.44. i) Up to sign and permutation of p and q a $D(p,q)$ family represents the class $[D(p,q)]$.

ii) The order of the base of the bands and the homotopy classes of their cores determine p and q. Given such bands, the sign of $[D(p,q)]$ is determined by the color of the first band on I_0, and the signs of the spinnings. Here, we need not consider the spinning from β_r', κ_r'since it is always opposite that of β_r, κ_r. If we use $L/H, H/L$ data and intersection numbers to determine the spinnings, then as before the final sign is given by the parity of the five possible differences from the standard model.

Lemma 4.45. $[D(p,q)] = -[E(q,p)] - [E(p,q)]$.

Proof. First note that $D(p,q)$ is a separable family. By construction R is a concatenation of $\sigma(\beta_r,\kappa_r)$ and $\sigma(\beta_r',\kappa_r')$ and each is homotopically trivial. Applying bilinearity to Figure 27 we obtain (B_1,R_1) and (B_2,R_2) of Figures 28 a), b). (B_1,R_1) is homotopic to the standard $E(q,p)$ after a blue/red switch, thus $[(B_1,R_1)] = -[E(q,p)]$. Similarly, the family in Figure 28 b) gives $-E(p,q)$. Here there is a single sign change from the standard $E(p,q)$ arising from $<\text{band}(\beta_b),\kappa_r'>= -1$. $\square$

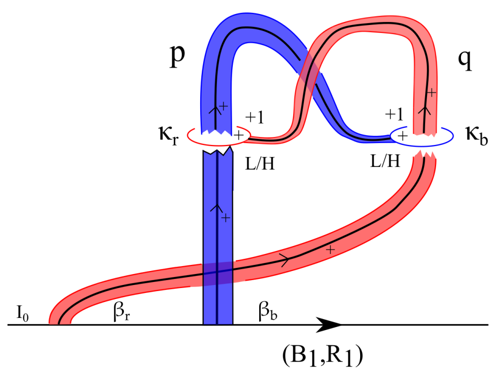


Figure 28.

Corollary 4.46. $[D(p,q)] = -[G(q,-p)] + [G(-q,p)] - [G(p,-q)] + [G(-p,q)]$.

Corollary 4.47. *i)* $[D(p,-p)] = 0$
ii) $[D(p,q)] = [D(q,p)]$
iii) $[D(-p,-q)] = -[D(p,q)]$

5. Barbell Diffeomorphisms

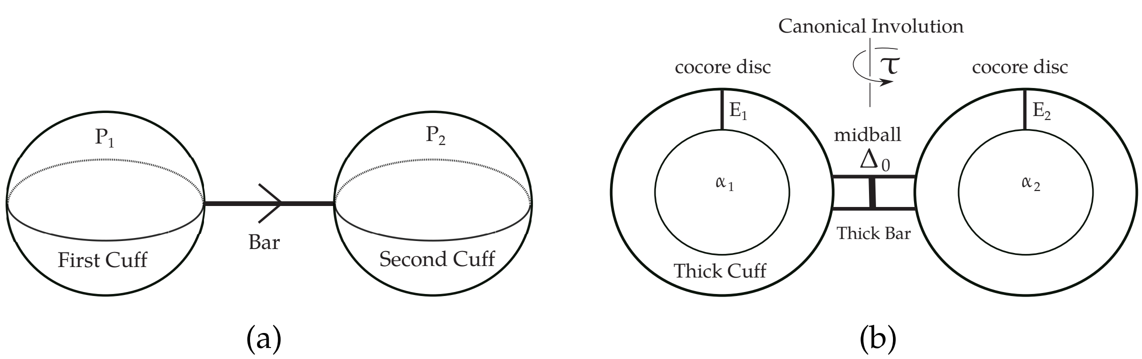


Figure 29. (a) A Barbell. (b) The $t=0, y=0$ slice of the model thickened barbell.

In §2 we defined the families $\{\alpha_k\}$ and $\{\theta_k\}$ each of which generated $\pi_1 \operatorname{Emb}(S^1, S^1 \times S^3; S^1_0)$, where S^1_0 is the standard vertical $S^1 \subset S^3$. In §3 we noted that isotopy extension applied to an $\alpha_k, k \geq 1$ or $\theta_k, k \geq 2$ gives rise to an element of $\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial)$

which is conjecturally isotopically nontrivial. In this section we explicitly identify these diffeomorphisms, the foundational observation being that each is supported in a single $S^2 \times D^2 \natural S^2 \times D^2 \subset S^1 \times B^3$. We call such a space a *thickened barbell* $\mathcal{NB}$, the *barbell* $\mathcal{B}$ itself, a spine, being the disjoint union of two oriented 2-spheres together with an oriented arc that joins them. See Figure 29(a). We will see that $\pi_0\left(\mathrm{Diff}(\mathcal{NB}\,\mathrm{fix}\,\partial)/\,\mathrm{Diff}(B^4\,\mathrm{fix}\,\partial)\right) = \mathbb{Z}$ and is generated by the *barbell map*. The barbell map is the result of applying isotopy extension to a loop of a pair of properly embedded arcs in B^4, where a subarc of the first arc is standardly spun around the second and then restricting to the closed complement of these arcs.

By embedding the barbell in another space M and pushing forward the barbell map we obtain elements of $\mathrm{Diff}(M\,\mathrm{fix}\,\partial M)$ an operation we call *barbell implantation*. Implantations yielding elements of $\mathrm{Diff}(S^1 \times B^3\,\mathrm{fix}\,\partial)$ induced from the θ_k's and α_k's will be described. Other implantations that we will prove to be isotopically nontrivial in both $\mathrm{Diff}(S^1 \times B^3\,\mathrm{fix}\,\partial)$ and $\mathrm{Diff}_0(S^1 \times S^3)$ modulo $\mathrm{Diff}(B^4\,\mathrm{fix}\,\partial)$ will be given in the next section. For $M = S^4$ we offer explicit candidates including the ones arising from the θ_k's via filling $S^1 \times B^3$ with an $S^2 \times D^2$. We will show how to construct the image of $\{x_0\} \times B^3 \subset S^1 \times B^3$ and $\{x_0\} \times S^3 \subset S^1 \times S^3$ under implantation, thereby exhibiting possibly knotted nonseparating 3-balls (resp. 3-spheres) in $S^1 \times B^3$ (resp. in $S^1 \times S^3$), inducing knotted 3-balls in S^4. Some of these will be shown to be non trivial in later sections. For S^4 we will show that some barbell maps are order at most 2. Finally, we note that barbell maps extend and generalize the graph surgery diffeomorphisms of Watanabe [Wa1].

We start by defining the *model barbell* and its thickening in $\mathbb{R}^4$, along the way establishing conventions and identifying various subspaces.

Definition 5.1. The *model barbell* $\mathcal{B}$ is the union two 2-spheres in $\mathbb{R}^3$ of radius $\frac{1}{4}$ respectively centered $(1,0,0)$ and $(2,0,0)$ together with the arc $[1.25, 1.75] \subset \mathbb{R} \times \{0\}^2$ called the *bar* that points from the first sphere P_1 to the second called P_2. These spheres are called the *cuffs* with P_1 (resp. P_2) being the *first* (resp. *second*) cuff. Construct the underlying space of a *model thickened barbell* $\subset \mathbb{R}^4$ by first taking an ϵ-neighborhood $N^3_\epsilon(\mathcal{B}) \subset \mathbb{R}^3$ and then taking the product with $[-\epsilon, \epsilon]$. Here $\epsilon > 0$ and is small. The space $N(P_i) := N^3_\epsilon(P_i) \times [-\epsilon, \epsilon]$ is called a *thick cuff* and the closed complement of the thick cuffs is called the *thick bar*. The properly embedded 2-disc $E_1 \subset \mathcal{NB}$ normal to P_1 and centered at $(1, 0, \frac{1}{4}, 0)$ is called the *cocore disc* to P_1. E_2 centered at $(2, 0, \frac{1}{4}, 0)$ is defined and named analogously. Let τ denote the *canonical involution* of $\mathcal{NB}$ induced from the involution on $\mathbb{R}^4$ corresponding to rotating the x, y plane by π at the point $(1.5, 0, 0, 0)$ and fixing the z, t coordinates. Define $\Delta_0 \subset \mathcal{NB}$ to be the properly embedded transverse 3-ball to the dual arc at $(1.5, 0, 0, 0)$ and call it the *midball*. See Figure 29(b). Let $\alpha_i = (i, 0, 0, t), t \in [-\epsilon, \epsilon], i = 1, 2$. Let $B_i = B^3 \times [-\epsilon, \epsilon]$ where B^3 is the closed complementary 3-ball region of $N^3_\epsilon(P_i)$. View α_i as the cocore of the 3-handle B_i.

We give $\mathcal{NB}$ the orientation induced from $\mathbb{R}^4$. We orient P_1 so that at $(1, 0, \frac{1}{4}, 0)$, $(\epsilon_2, -\epsilon_1)$ is a positive basis and orient E_1 so that at $(1, 0, \frac{1}{4}, 0)$, (ϵ_3, ϵ_4) is a positive basis. Here $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ is the standard positive basis for $\mathbb{R}^4$. Note that $\langle P_1, E_1 \rangle = 1$. Use τ to push forward the orientations on P_1, E_1 to ones on P_2, E_2. Use ϵ_1 to positively orient the bar and $(\epsilon_2, \epsilon_3, \epsilon_4)$ to orient Δ_0.

A *barbell* in a 4-manifold is a subspace diffeomorphic to the model barbell with oriented cuffs and bar such that the cuffs have trivial normal bundles. A *thick barbell* in a 4-manifold is an embedded $S^2 \times D^2 \natural S^2 \times D^2$ together with a barbell spine.

The following is standard.

Lemma 5.2.
- $H_2(\mathcal{NB}) \simeq \mathbb{Z}^2$ *with generators* $[P_1], [P_2]$.
- $H_2(\mathcal{NB}, \partial \mathcal{NB}) \simeq \mathbb{Z}^2$ *with generators* $[E_1], [E_2]$.
- $H_2(\mathcal{NB}, \partial E_1 \cup \partial E_2) \simeq \mathbb{Z}^4$ *with generators* $[P_1], [E_1], [P_2], [E_2]$. $\square$

Construction 5.3. We now define the *barbell* map $\beta : \mathcal{NB} \to \mathcal{NB}$. Let $W = \mathcal{NB} \cup B_1 \cup B_2$ and hence $\mathcal{NB}$ is the closed complement of $N(\alpha_1) \cup N(\alpha_2) \subset W$ where $N(\alpha_i) = B_i$. A loop λ_s in $\mathrm{Emb}(N(\alpha_1), W)$ based at $N(\alpha_1)$ that is fixed near $(\partial\alpha_1) \times B^3$ and avoids B_2 induces a diffeomorphism of $\mathcal{NB}\,\mathrm{fix}\,\partial\mathcal{NB}$ by isotopy extension whose proper isotopy class depends only on the based homotopy class of that loop. To define the path λ we construct two discs F_0, F_1, where F_1 is a parallel copy of E_1 having been pushed off in the ϵ_1 direction and F_0 is a disc that coincides with F_1 near its boundary and defined below. λ_s is a loop that first sweeps across F_0 and then sweeps back using F_1. In what follows for $X \subset \mathcal{B}$, let X_{t_0} denote $X \cap \{t = t_0\}$.

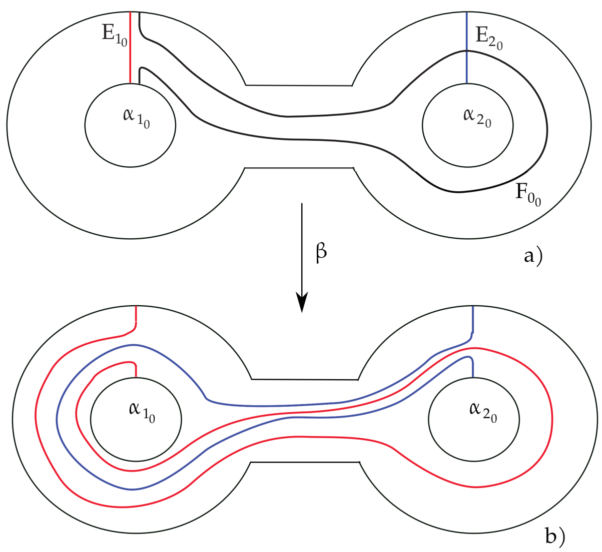


FIGURE 30. A slice of the barbell map.

We describe F_0 as follows. It will intersect each $\mathcal{NB}_t$ in an arc F_{0_t} whose ends coincide with that of F_{1_t}. Here F_{0_0} lies in the $y = 0$ plane and is shown in Figure 30 a). As t increases (resp. decreases) F_{0_t} slips to the $y > 0$-side (resp. $y < 0$-side) of ∂B_{2_t}. For t close to $\pm\epsilon$, F_{0_t} coincides with F_{1_t}.

The images of $\beta(E_1)_0, \beta(E_2)_0$ are shown in Figure 30 b). As t increases $\beta(E_1)_t$ (resp. $\beta(E_2)_t$) slips to the $y > 0$-side (resp. $y < 0$-side) of ∂B_{2_t} (resp. ∂B_{1_t}). Note that $\beta(E_i)$ coincides with E_i near its boundary.

Remarks 5.4. i) An orientation preserving diffeomorphism of $\mathcal{NB}$ supported in the thick bar that is a full right hand twist of the x, z-plane as one traverses the bar does not change the isotopy class of the barbell map, since it fixes $F_0 \cup F_1$ setwise, hence does not change the based isotopy class of the loop λ_s.

ii) The barbell map is the result of applying isotopy extension to a positive λ-spinning of α_1 about α_2 where λ is a straight arc from α_1 to α_2 and each α_i is oriented by ϵ_4. It follows from Lemma 4.4 that the barbell map is also the result of a negative spinning of α_2 about α_1.

Lemma 5.5. *In $H_2(\mathcal{NB}, \partial(E_1 \cup E_2))$,*
i) $\beta_*[E_1] = [E_1] + [P_2]$
ii) $\beta_*[E_2] = [E_2] - [P_1]$ □

The next result follows by construction.

Theorem 5.6. *$\tau \circ \beta \circ \tau$ is isotopic to β^{-1}.*

Theorem 5.7. *$\pi_0 \operatorname{Diff}(\mathcal{NB} \operatorname{fix} \partial)/\pi_0 \operatorname{Diff}(B^4 \operatorname{fix} \partial) \simeq \mathbb{Z}$ and is generated by the barbell map.*

Proof. Let $\psi \in \operatorname{Diff}(\mathcal{NB} \operatorname{fix} \partial)$. Since ψ fixes $\partial\mathcal{NB}$ pointwise it is orientation preserving. Its effect on $H_2(\mathcal{NB}, \partial(E_1 \cup E_2))$ is as follows.
a) $\psi_*([P_i]) = [P_i]$, $i = 1, 2$
b) $\psi_*([E_1]) = [E_1] + n[P_2]$
c) $\psi_*([E_2]) = [E_2] - n[P_1]$ for some $n \in \mathbb{Z}$.

Since ψ fixes $\partial\mathcal{NB}$ a) follows. Apriori, $\psi_*([E_1]) = n_1[E_1] + n_2[P_1] + n_3[E_2] + n_4[P_2]$. Since ψ fixes $\partial\mathcal{NB}$, $n_1 = 1$ and $n_3 = 0$. Now double $\mathcal{NB}$ to obtain $S^2 \times S^2 \sharp S^2 \times S^2$ and extend ψ to $\hat{\psi}$ so that $\hat{\psi}$ is the identity outside of $\mathcal{NB}$. In the double P_1 becomes a $S^2 \times \{*\}$ and the doubled E_1 becomes the 2-sphere $\hat{E}_1 = \{*\} \times S^2$, both in the first $S^2 \times S^2$ factor. Since $\hat{E}_1$ has trivial normal bundle as does P_1 and P_2 it follows that $n_2 = 0$. In a similar manner $\psi_*([E_2]) = [E_2] + m[P_1]$. Therefore if $n_4 = n$, then $0 = \langle E_1, E_2 \rangle = \langle \psi(E_1), \psi(E_2) \rangle$ and hence $m = -n$.

Therefore, $f := \beta^{-n} \circ \psi$ acts trivially on $H_2(\mathcal{NB}, \partial(E_1 \cup E_2))$ and hence for $i = 1, 2, f(E_i)$ is homotopic to E_i rel ∂E_i. After an application of Theorem 0.6 i) [Ga2] f can be isotoped so that $f|(E_1 \cup \partial\mathcal{NB}) = \mathrm{id}$ and then by Theorem 10.4 [Ga1], f can be further isotoped so that $f|(E_1 \cup E_2 \cup \partial\mathcal{NB}) = \mathrm{id}$ and hence $f = \mathrm{id}$ modulo $\operatorname{Diff}(B^4 \operatorname{fix} \partial)$. □

Definition 5.8. Let M be a properly embedded 3-manifold in the 4-manifold V. We say that Y is obtained from M by *embedded surgery* if there is a sequence $M = M_0, M_1, M_2, \cdots$, $M_n = Y$ such that M_i is obtained from the regular neighborhood $N(M_{i-1})$ by first attaching a single 4-dimensional handle embedded in V and then restricting to some relative boundary components.

Construction 5.9. We construct $\beta(\Delta_0) := \Delta_1$ by embedded surgery. To start with, Δ_0 is isotopic to the 3-ball Δ_b obtained from Δ_0 by two embedded 2-handle surgeries where the second 2-handle attachment is given by the cocore of the first as in Figure 31 a). More precisely, the first 0-framed 4-dimensional 2-handle σ is attached to link B_2 and intersect E_2 in a cocore C. Further if Δ_a is the result of this embedded surgery, then $\lambda_s \cap \Delta_a = \emptyset$ all s, where λ_s is as in Construction 5.3. The second 2-handle τ is a normal neighborhood of C and hence Δ_b is disjoint from E_2. Since $\beta(E_2)$ is obtained by removing a small subdisc $C' \subset C \subset E_2$ and replacing it by a disc C'', it follows that $\beta(\Delta_b) := \Delta_c$ is obtained by removing a solid torus, the unit normal bundle of C', and reimbedding it as the unit normal bundle of C''. Finally, Δ_1 is obtained by isotoping the attaching zone of $\beta(\tau)$ to lie in Δ_0. At the 3-dimensional level Δ_1 is obtained from Δ_0 by 0-surgery along a Hopf link.

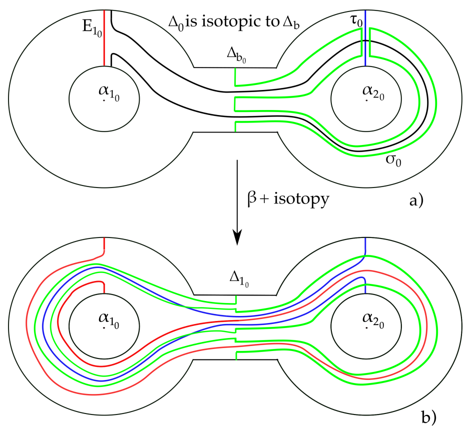


FIGURE 31. Construction of Δ_1 as seen from the $t = 0, y = 0$ slice.

Remark 5.10. Note that if $\Delta^1, \cdots, \Delta^n$ are n parallel copies of Δ_0 with Δ^1 closest to B_2, then $\beta(\cup\Delta^i)$ are n parallel copies of $\beta(\Delta_1)$. The corresponding 2-handles $\beta(\sigma^1), \beta(\sigma^2), \cdots, \beta(\sigma^n)$ (resp. $\beta(\tau^n), \beta(\tau^{n-1}), \cdots, \beta(\tau^1)$) *nest*, one inside the previous.

Definition 5.11. Let N be a 4-manifold and $\mathcal{B}_0$ a smoothly embedded barbel with framed cuffs. Let $N(\mathcal{B}_0)$ be an ϵ-regular neighborhood. There is an orientation preserving diffeomorphism $f : N(\mathcal{B}_0) \to \mathcal{N}\mathcal{B}$ that restricts to one from $\mathcal{B}_0$ to the model barbell $\mathcal{B}$, which essentially preserves normal fibers of the thickenings. Pulling back the barbell map induces a map $\beta_f : N \to N$, well defined up to isotopy. The operation that starts with a framed cuffed barbell $\mathcal{B}_0 \subset N$ and produces $\beta_f \in \text{Diff}(N)$ up to isotopy, is called *implantation*. Here β_f can be viewed as an element of $\text{Diff}(N)$ or $\text{Diff}(N \operatorname{fix} \partial)$.

Remarks 5.12. If N is oriented we can assume that the framing of the cuffs together with the orientation of the cuffs agrees with the orientation of N. Different framings of the bar will give rise to maps f and g that differ by a twist as in Remarks 5.4 i). That remark also shows that β_f is isotopic to β_g. Thus when N is oriented the isotopy class of β_f depends only on the embedding of the barbell and the framing of the cuffs. When the framing is implicit, then β_f depends only on the embedding.

By chasing orientations we obtain:

Lemma 5.13. *If $\mathcal{B}$ is a barbell in N, then reversing the orientation of N without changing the orientations on the barbell changes the implantion to its inverse.*

We now give a bundle theoretic interpretation of the barbell map. Consider the fiber bundle $\mathrm{Diff}(D^n \text{ fix } \partial) \to \mathrm{Emb}(\sqcup_2 B^n, B^n)$ from the group of diffeomorphisms of B^n to the space of embeddings of two copies of B^n into itself. To define this bundle, we fix two disjoint n-discs embedded in the interior of a fixed B^n. The bundle comes from restricting a diffeomorphism to the sub-discs. Not only it is a fiber-bundle, but it is null-homotopic, Proposition 5.3 [Bu2]. Thus there is an injection $\pi_k \mathrm{Emb}(\sqcup_2 B^n, B^n) \to \pi_{k-1} \mathrm{Diff}(B^n \text{ fix } \sqcup_2 B^n)$, moreover the co-kernel of this injection is isomorphic to $\pi_{k-1} \mathrm{Diff}(B^n \text{ fix } \partial)$.

$\mathrm{Emb}(\sqcup_2 B^n, B^n)$ has the homotopy-type of $S^{n-1} \times SO_n^2$. Thus we have

$$\pi_{n-1} \mathrm{Emb}(\sqcup_2 B^n, B^n) \simeq \mathbb{Z} \oplus \bigoplus_2 \pi_{n-1} SO_n.$$

The $\mathbb{Z}$ factor corresponding to $\pi_{n-1} S^{n-1}$ is what generates the barbell diffeomorphism. We have a short-exact sequence

$$0 \to \pi_{k-1} S^{n-1} \oplus \bigoplus_2 \pi_{k-1} SO_n \to \pi_{k-2} \mathrm{Diff}(B^n \text{ fix } \sqcup_2 B^n) \to \pi_{k-2} \mathrm{Diff}(B^n \text{ fix } \partial) \to 0$$

We consider $S^{n-1} \times B^2 \natural S^{n-1} \times B^2$ to be $(S^{n-1} \times I \natural S^{n-1} \times I) \times I$, i.e. as a trivial I-bundle over $S^{n-1} \times I \natural S^{n-1} \times I$. Then as a bundle over I, denote the group of fiber-preserving diffeomorphisms of $S^{n-1} \times B^2 \natural S^{n-1} \times B^2$ that are the identity on the boundary by $\mathrm{Diff}^I(S^{n-1} \times B^2 \natural S^{n-1} \times B^2)$. These diffeomorphisms are sometimes also called *horizontal diffeomorphisms*. By design, this group is the loop-space on $\mathrm{Diff}(B^n \text{ fix } \sqcup_2 B^n)$, thus we have shown there is a short exact sequence

Proposition 5.14.

$$0 \to \pi_{k-1} S^{n-1} \oplus \bigoplus_2 \pi_{k-1} SO_n \to \pi_{k-3} \mathrm{Diff}^I(S^{n-1} \times B^2 \natural S^{n-1} \times B^2 \textit{ fix } \partial) \to \pi_{k-2} \mathrm{Diff}(B^n \text{ fix } \partial) \to 0$$

valid for all $n \geq 3$ and all $k \geq 3$.

The barbell diffeomorphism (family) being the image of the $\pi_{n-1} S^{n-1}$ generator in $\pi_{n-3} \mathrm{Diff}^I(S^{n-1} \times B^2 \natural S^{n-1} \times B^2 \text{ fix } \partial)$, when $k = n$.

We now consider S^4 *implantation* where the cuffs P_1, P_2 of $\mathcal{B}$ are fixed unknotted and unlinked oriented 2-spheres that are permuted under an elliptic involution τ of S^4. We assume that $P_1 \cup P_2 \subset B$, a fixed oriented 3-ball, thereby determining the framings of $P_1 \cup P_2$ as in the model barbell. The bar is then determined up to isotopy by its relative homotopy class and hence by a word $w(x,y)$ in the rank-2 free group, the fundamental group of $S^4 \setminus (P_1 \cup P_2)$. See Figure 32. With generators as in the figure we will assume that w is reduced and if $w \neq 1$, then the first (resp. last) letter of w is $y^{\pm 1}$ (resp. $x^{\pm 1}$). For example, if x was the first letter, then it can be removed by rotating $N(P_1)$ fixing P_1 pointwise. We denote such a barbell by $\mathcal{B}_w$. It is well defined up to isotopy fixing $P_1 \cup P_2$ pointwise. We denote by β_w the corresponding diffeomorphism of S^4 which is also well defined up to isotopy.

Definition 5.15. We say that w is *inverse-xy-palendromic* if w is obtained by first taking its inverse and then everywhere switching x and y.

Example 5.16. $(yx^{-1})^n$ is inverse-xy-palendromic.

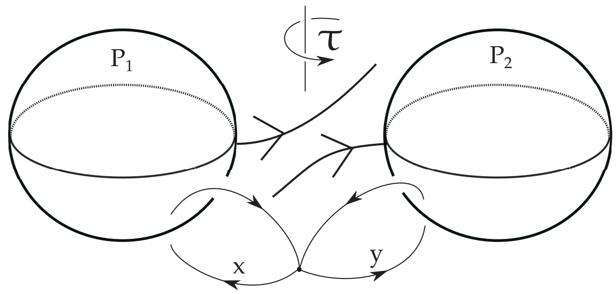


FIGURE 32. Generators for the fundamental group of a barbell complement in S^4

Proposition 5.17. *If w is inverse-xy-palendromic, then β_w^2 is isotopic to* id.

Proof. Since w is inverse-xy-palendromic, $\tau(\beta_w)$ is setwise isotopic to β_w via an isotopy that is supported in the bar. The time one map of the isotopy, $g : \tau(\mathcal{B}_w) \to \mathcal{B}_w$, is orientation preserving on the cuffs and reversing on the bar. It follows from Theorem 5.6 that β_w is isotopic β_w^{-1}. □

Conjecture 5.18. $\pi_0 \operatorname{Diff}_0(S^4) \neq 1$. *In particular, β_{yx} and $\beta_{yx^{-1}}$ are not isotopic to* id. *See Figure 33.*

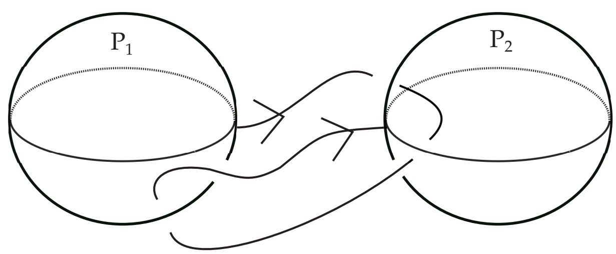


FIGURE 33. Is this a non trivial element of $\pi_0 \operatorname{Diff}_0(S^4)$?

Questions 5.19. *i) What are the relations in the subgroup of $\pi_0 \operatorname{Diff}_0(S^4)$ generated by the β_w's?*

ii) We can also consider implantations in S^4 arising from knotted or linked cuffs. When do these implantations give isotopically trivial diffeomorphisms? What if any are relations amongst these implantations?

Definition 5.20. Let $\mathcal{B}$ be a barbell in M. Define $a(P_1)$, the *associated sphere to P_1*, be the sphere obtained by orientation preserving tubing P_1 along the bar to a parallel copy of P_2.

Remark 5.21. Let α be an arc in a normal disc of P_2 from $a(P_1)$ to P_2. Then $N(a(P_1) \cup \alpha \cup P_2)$ is a barbell whose thickening is isotopic to $N(\mathcal{B})$. The next result shows that if $w \neq 1$, $\mathcal{B}_w$ has thickening with a knotted barbell spine.

Proposition 5.22. *Let $\mathcal{B}_w$ be a barbell in S^4. Then*

i) $a(P_1)$ is knotted if and only if $w \neq 1$ and hence if $w \neq 1$, then $\mathcal{B}_w$ is not isotopic to $\mathcal{B}_1$.
ii) If $w \neq 1$, then $\mathcal{B}_w$'s thickening has no barbell spine isotopic to $\mathcal{B}_1$.

Proof. i) If $w = 1$, then $a(P_1)$ is obviously unknotted. In general $a(P_1)$ is a ribbon 2-knot K whose fundamental group is $\langle x, y | xwyw^{-1} \rangle$. Since we can assume that w is a nontrivial reduced word that does not start (resp. end) with a $x^{\pm 1}$ nor end with a $y^{\pm 1}$ it follows that the relator is non trivial and hence the group is non cyclic.

ii) Every non separating embedded 2-sphere in $\partial \mathcal{N}\mathcal{B}_1$ is isotopically trivial. □

We now consider $S^1 \times X$ implantations where $X = S^3$ or B^3.

Construction 5.23. Let $\mathcal{B}_0 \subset S^1 \times X, X \in \{B^3, S^3\}$ a barbell with β_0 the implanted map. If $\Delta_0 := x_0 \times X$ is disjoint from the cuffs and transverse to the bar, then $\Delta_1 := \beta_0(\Delta_0)$ is constructed as follows. Let $h : (N(\mathcal{B}_0), \mathcal{B}_0) \to (\mathcal{N}\mathcal{B}, \mathcal{B})$ an orientation preserving diffeomorphism, where $\mathcal{N}\mathcal{B}$ is the thickened model barbell. We can assume that $f(\Delta_0 \cap N(\mathcal{B}_0))$ is $m \geq 0$ parallel 3-balls normal to the model bar. Construction 5.9 shows how to construct the image of these balls under the barbell map. Now pull back by h.

Remark 5.24. Δ_1 is obtained by embedded surgery to Δ_0 where the attaching zone of the 2-handles is the split union of m Hopf links. At the 3-dimensional level, Δ_1 is obtained by 0-surgery to the components of this link, thus its topology is unchanged.

Construction 5.25. (Barbells from Spinning) Let J be an embedded oriented S^1 in the oriented 4-manifold M. Construction 5.3 and Remark 5.4 show that the diffeomorphism of M induced by applying isotopy extension to a λ-spinning of a subarc J_1 of J about another subarc J_2 is the barbell map corresponding to a barbell $\mathcal{B} \subset M$ with P_1 linking J_1, P_2 linking J_2 and the bar given by λ. Applying this to θ_k and α_i, denoted in §2 as $\theta_{1,k}$ and $\alpha_{1,i}$, we obtain the barbells shown in Figure 34. To see this for θ_3, first consider Figure 34 c) which shows the basic construction and then send f to "infinity". We now view these barbells as subsets of $S^1 \times B^3$ and denote them by $\mathcal{B}(\theta_k)$ and $\mathcal{B}(\alpha_i)$. Let $\beta_{\theta_k}, \beta_{\alpha_k} \in \mathrm{Diff}_0(S^1 \times B^3 \,\mathrm{fix}\, \partial)$ denote the corresponding implantations.

Since S^4 is obtained by gluing an $S^2 \times D^2$ to $S^1 \times B^3$ elements of $\mathrm{Diff}(S^1 \times B^3 \,\mathrm{fix}\, \partial)$ extend to elements of $\mathrm{Diff}_0(S^4)$. For example, viewing $\mathcal{B}(\theta_3)$ in S^4 we obtain the barbell $\mathcal{B}_{S^4}(\theta_3)$ shown in Figure 34 d). Since this barbell is inverse-xy-palendromic its implantation is order at most 2.

Remark 5.26. Watanabe constructs [Wa1] various B^4-bundles over n-spheres to show, among other things, that $\pi_1 \mathrm{Diff}(B^4 \,\mathrm{fix}\, \partial)$ is non trivial. Specializing his construction to $n = 1$ and considering a bundle's gluing map, give possible nontrivial elements of $\mathrm{Diff}(B^4 \,\mathrm{fix}\, \partial)$. He notes that this construction applied to a 4-manifold M might yield nontrivial elements of $\pi_0 \mathrm{Diff}(M \,\mathrm{fix}\, \partial)$. Here we observe that these diffeomorphisms are compositions of barbell maps. We very briefly outline this in the simplest case when $M = S^4$.

Watanabe starts with an embedding α of a two vertex arrow graph into S^4 and then constructs the corresponding Y-link in S^4, see [Wa1] §4. We abuse notation by letting α denote this embedding when extended to a regular neighborhood. The Y-link has

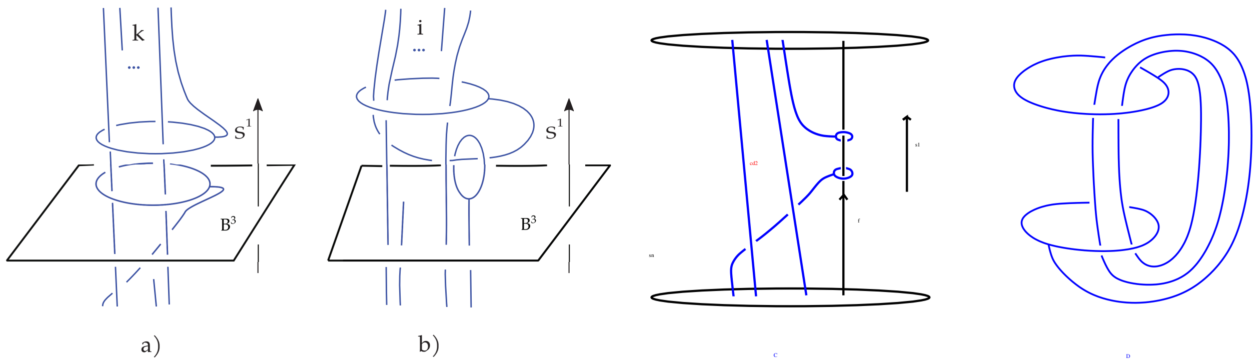


FIGURE 34. (a) The barbell $\mathcal{B}(\theta_k)$ induced from θ_k
(b) The barbell $\mathcal{B}(\alpha_i)$ induced from α_i
(c) The initial barbell induced from θ_3
(d) $\mathcal{B}(\theta_3)$ viewed as a subset of S^4

two components, one Type 1 and one Type 2. Call their respective regular neighborhoods V_1 and V_2. Watanabe's diffeomorphism of S^4, which we denote by ϕ_α, arises from *parametrized Borromean surgery* of Type 1 on $\alpha(V_1)$ and Type 2 on $\alpha(V_2)$. Roughly, $V_1 = B^4 \setminus \operatorname{int}(N(D_1) \cup N(D_2) \cup N(D_3))$ where D_1 and D_2 are 2-discs and D_3 a 1-disc, the union standardly embedded. $V_2 = B^4 \setminus \operatorname{int}(N(E_1) \cup N(E_2) \cup N(E_3))$ where E_1 is a 2-disc and E_2, E_3 are 1-discs and the union is standardly embedded, thus $V_2 = S^2 \times D^2 \natural S^2 \times D^2$ plus a 1-handle. Being a Y-link, the 1-handle of $\alpha(V_2)$ passes parallel to $\alpha(D_3)$ through $\alpha(V_1)$. Type 1 surgery effectively replaces D_3 by D_3' where $D_1 \cup D_2 \cup D_3'$ is the 4-dimensional Borromean rings viewed as a three component tangle in the B^4 defining V_1. Thus, Type 1 surgery reimbeds the 1-handle of $\alpha(V_2)$, hence reimbeds $\alpha(V_2) \subset S^4$. Let $\alpha' : V_2 \to S^4$ be this new embedding. Type 2 surgery on V_2 induces $\psi \in \operatorname{Diff}(V_2 \operatorname{fix} \partial)$ by an arc pushing diffeomorphism and ϕ_α is obtained by pushing forward ψ by α'. The track of this arc pushing is more or less a disc E_2' where $E_1 \cup E_2' \cup E_3$ is the 4-dimensional Borromean rings. Our assertion follows from the fact that ψ is the composition of two barbell maps. To see this expressed from our point of view, consider V_2 as $T \times [-1,1]$ where T is a $D^2 \times S^1$ with two open 3-balls B_1 and B_2 removed. Here, we have two properly embedded discs $F_0 \subset V_2$ and $F_1 \subset V_2$ that coincide near their boundaries and ψ is obtained by first pushing in along F_0 and then out along F_1. The F_1 is a $\sigma \times [-1,1]$, where $\sigma \subset T$ is a properly embedded arc and F_0 intersects each $T \times t$ in an arc F_{0_t}. F_{0_0} and $\sigma \times \{0\}$ are shown in Figure 35. F_{0_0} consists of two subarcs τ_0, τ_0' that go around the the second 3-ball $B_2 \times \{0\}$. As t increases (resp. decreases) τ_t slips over (resp. under) $B_2 \times \{t\}$ and τ_t' slips under (resp. over) $B_2 \times \{t\}$, both returning to coincide with a segment in $\sigma \times \{t\}$. This can be done so that for all t, $\operatorname{int}(\tau_t) \cap \operatorname{int}(\tau_t') = \emptyset$ and hence F_0 is embedded. With this description it is evident that ψ and hence ϕ_α, is isotopic to the composition of two barbell maps.

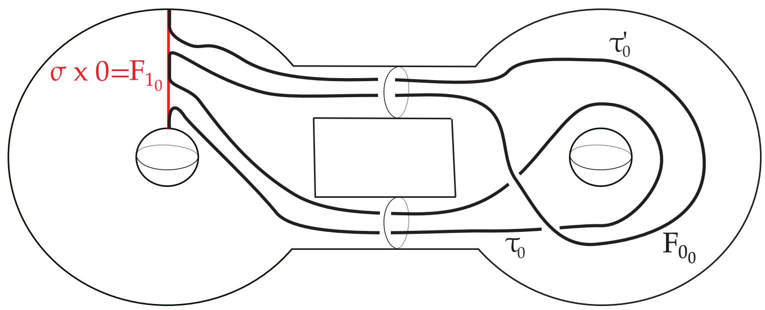


FIGURE 35. $t = 0$ slice of Watanabe's Type 2 Borromean surgery.

6. 1- and 2-parameter families arising from implantations

In the last section we introduced the barbell map $\beta : S^2 \times D^2 \natural S^2 \times D^2 \to S^2 \times D^2 \natural S^2 \times D^2$ and constructed $\Delta_1 = \beta(\Delta_0)$ where Δ_0 is the standard separating 3-ball. In this section we slice Δ_0 into 1- and 2-parameter families of discs and arcs. We will show how under barbell implantation into a 4-manifold M, where the bar is transverse to a 3-ball, these families produced loops and spheres in $\mathrm{Emb}(D^2, M)$ and $\mathrm{Emb}(I, M)$ respectively.

Recall that $\mathcal{NB} = V \times [-1, 1]$ where $V = D^2 \setminus \mathrm{int}(B_1 \cup B_2)$ where B_1 and B_2 are 3-balls. By scaling $[-1, 1]$ to $[0, 1]$, we henceforth identify $\mathcal{NB}$ with $V \times [0, 1]$. Let $D \subset V$ denote the separating 2-disc such that $\Delta_0 = D \times [0, 1]$. Recall that $\beta(V \times u) = V \times u$ for all $u \in [0, 1]$ it follows that $\Delta_1 = \cup_{u \in [0,1]} \beta(D \times u)$. The next result follows by slicing the construction of Δ_1.

Proposition 6.1. *$\{\beta(D \times u)\}_{u \in [0,1]}$ is isotopic to the family $\{D_u\}$ obtained as follows.*

i) For $u \in [0, .125]$, $D_u = D \times u$. See Figure 36 a).

ii) For $u \in [.125, .25]$ squeeze $D_{.125}$ to create a neck as in Figure 36 b).

iii) For $u \in [.25, .50]$ isotope the neck halfway around B_1, going from the $y > 0$ side to the $y < 0$ side. See Figure 36 c).

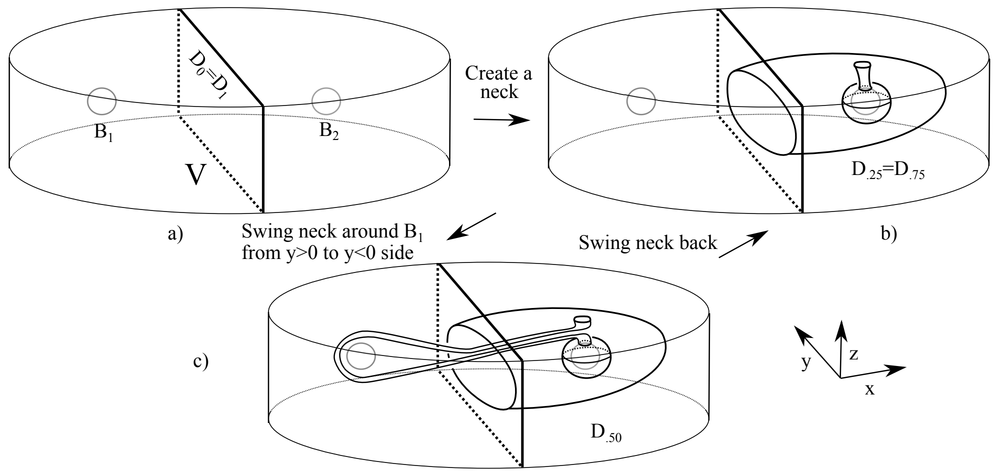
iv) For $u \in [.50, .75]$ isotope the neck back to its original position shown in Figure 36 b), continuing from the $y > 0$ side to the $y < 0$ side.

v) For $u \in [.75, 1]$ reverse the isotopy from $[0, .25]$. □

Remark 6.2. $\{\beta(D \times u)\}_{u \in [0,1]}$ is similarly isotopic to the family $\{E_u\}$ where we first create a neck connecting discs that surround B_1 instead of B_2. For that isotopy the neck swings around B_2 going from the $y < 0$ side to the $y > 0$ side. The neck of $E_{.50}$ is essentially seen in Figure 30 b) where the red arc is replaced by a cylinder. Notice that replacing the blue arc by a cylinder essentially gives the neck of $D_{.50}$.

We now describe another isotopic family $\{H_u\}$ that is neither B_1 nor B_2 centric. For that isotopy it is convenient to change the base disc D to the disc $H \subset V$ shown in Figure 37 a), via an isotopy $\phi_{D,H}(t)$ of V supported in a neighborhood of D.

First consider the disc $H_{.50}$ shown in Figure 37 c). This disc is obtained by removing four open discs from H_0, shown in Figure 37 a) and replacing them with two blue discs that surround B_2 and two red ones that surround B_1. The red (resp. blue) discs emanate from the first two sheets of the folded H_0 that are closest to B_1 (resp. B_2). To go from



Δ_1 as a 1-parameter family of discs

FIGURE 36.

Figure 37 c) to Figure 37 b)(resp. d)) isotope the blue necks down (resp. up) and the red necks up (resp. down). The region between the blue discs of $H_{.25}$ is a product which can be boundary compressed using a fold of H_0. A similar statement holds regarding the red discs, where the boundary compression starts at the other fold. These boundary compressions are done simultaneously and are supported away from the arc J_0. The result of these boundary compression is shown in Figure 37 e).

Proposition 6.3. *Under the change of basepoint map $\phi_{D,H}(1)$, the 1-parameter family D_u is isotopic to the following 1-parameter family H_u.*

i) For $u \in [0, .125]$, $H_u = H_0$.

ii) For $u \in [.125, .25]$, $H_{.125}$ is isotoped to the surface $H_{.25}$ shown in Figure 37 b) via the intermediary surface $H_{.18}$ shown in Figure 37 e). This isotopy fixes J_0 pointwise. The isotopy for $u \in [.125, .18]$ is supported near H_0. For $u \in [.18, .25]$ the isotopy restricted to the disc below (resp. above) J_0 is supported in the union of a regular neighborhood of H_0 and the right (resp. left) hand side of H_0.

iii) For $u \in [.25, .75]$ the blue necks are isotoped up and the red necks down to obtain Figure 37 d) with the intermediary $H_{.50}$ shown in Figure 37 c).

iv) For $u \in [.75, 1]$, the isotopy is analogous to the reverse of the H_0 to $H_{.25}$ isotopy. Two boundary compressions supported away from J_0 produce $H_{.82}$ which is the mirror in the $[0, 1]$ coordinate of $H_{.18}$.

Proof. A direct calculation gives an isotopy between $\{\phi_{D,H}(1)(D_u)\}$, which by abuse of notation we call $\{D_u\}$ and H_u that preserves each $V \times u$ slice. Alternatively, let e_1, e_2 be properly embedded arcs in V such that for $i = 1, 2$, $e_i \cap B_i \neq \emptyset$ and $e_i \cap (H_0 \cup D_0) = \emptyset$. The loops in $\operatorname{Emb}(N(e_1) \cup N(e_2), V)$ induced from D_u and H_u by isotopy extension are homotopic. It follows that $\{D_u\}$ and $\{H_u\}$ differ by an element of $\pi_1 \operatorname{Emb}(D^2, B^3 \operatorname{fix} \partial B^3)$, but that group is trivial by Cerf [Ce2] and Hatcher [Ha1]. $\square$

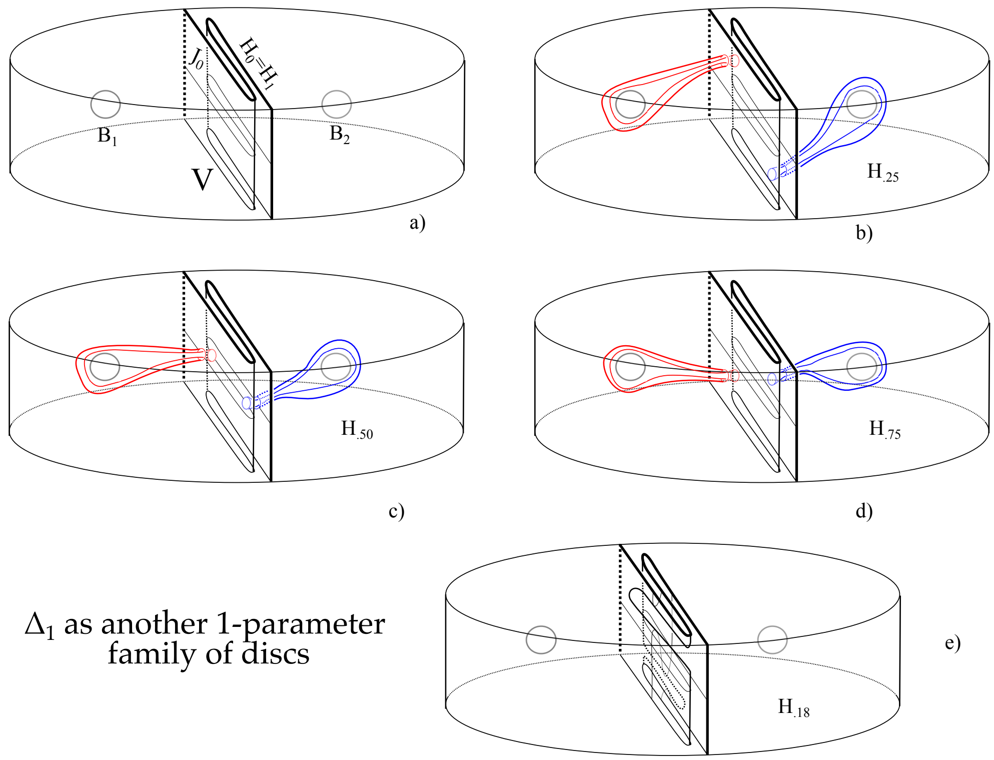


FIGURE 37.

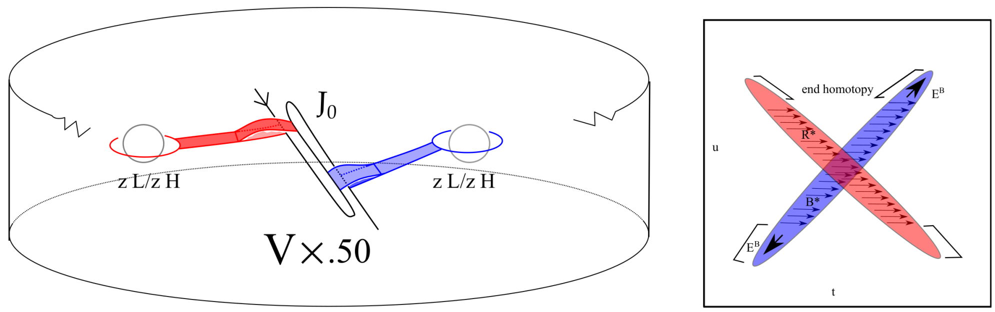
Since $V = D^2 \times [0,1] \setminus \mathrm{int}(B_1 \cup B_2)$, the $[0,1]$ factor induces a $[0,1]$-fibering on each $H \times u$ and hence expresses Δ_0 as a 2-parameter family of intervals. Pushing forward to H_u expresses Δ_1 as a 2-parameter family $\gamma_{t,u}$. Because $\beta(V \times u) = V \times u$ for all u, it follows that we can assume that $\gamma_{t,u}$ is supported in $V \times .50$.

The family $\gamma_{t,u} \in \Omega\Omega([0,1], V \times I; J_0)$ is a bracket as in Definition 4.21. To see this define H_u^B as the 1-parameter family which only involves the blue discs, so $H_{.50}^B$ is H_0 with two blue discs attached and the passage from $H_{.25}^B$ to $H_{.18}^B$ only involves the boundary compression of the blue discs, etc. In an analogous manner we define H_u^R. The associated 2-parameter families $\gamma_{t,u}^B, \gamma_{t,u}^R$ satisfy the conditions of Definition 4.21. The midlevel loops $B, R \in \Omega\,\mathrm{Emb}(I, S^1 \times B^3; I_0)$ are defined in band lasso notation as in Figure 38 a). For B (resp. R) only the blue (red) band and lasso are used. We call $\gamma_{t,u}$ the *barbell family* and all its defining information its *data*. We have the following result.

Proposition 6.4. *The barbell family $\gamma_{t,u}$ is supported in $V \times .50$ and is homotopic to the separable family (B, R) of Figure 38 a) described in band lasso notation. The notation zL/zH means that the spinnings go about the spheres first by going down in the z direction and then up. The support of $\gamma_{t,u}^B$ (resp. $\gamma_{t,u}^R$) in the parameter space is the blue (resp. red) region shown in Figure 38 b)* $\square$

Remark 6.5. Note that the end homotopy is homotopic to the undo homotopy.

We now describe how barbell implantation induces an $[\alpha_{t,u}] \in \pi_2\,\mathrm{Emb}(I, S^1 \times B^3; I_0)$. View $S^1 \times B^3$ as $D^2 \times S^1 \times [-1,1]$ with D^2 viewed as $[0,1] \times I$. Fixing $x_0 \in S^1$ and $I_0 =$



Δ_1 as a 2-parameter family of arcs a) b)

FIGURE 38.

$([0,1] \times .50) \times x_0 \times 0$, then $U := D^2 \times x_0 \times [-1,1]$ corresponds to the trivial 2-paraxmeter family by translating within U each $[0,1] \times s \times x_0 \times w$ to I_0. If $\psi \in \mathrm{Diff}(S^1 \times B^3 \,\mathrm{fix}\, \partial)$, then ψ induces a 2-parameter family $\alpha_\psi \in \Omega\Omega\,\mathrm{Emb}(I, S^1 \times B^3; I_0)$ by naturally translating each $\psi([0,1] \times s \times x_0 \times w)$ so that its end points coincide with that of I_0. Isotopic ψ's give rise to homotopic α_ψ's.

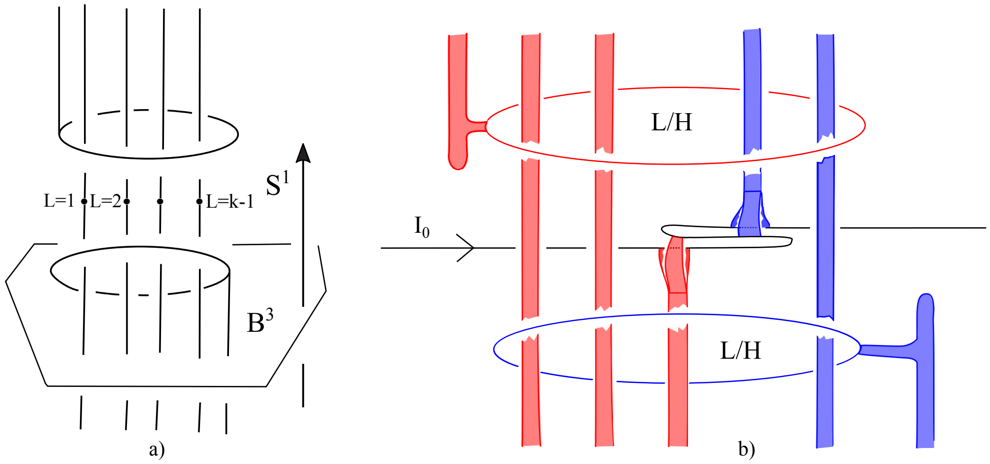
Construction 6.6. Let $f : \mathcal{NB} \to S^1 \times B^3$ be an embedding of the barbell neighborhood such that the bar is transverse to U and the cuffs are disjoint from U. We can assume that $f^{-1}(U) = \Delta_0^1 \cup \cdots \cup \Delta_0^n$, parallel copies of the midball and the $[0,1]$-fibering of U pulls back to the standard $[0,1]$-fibering of each Δ_0^i. Further, each $f(\Delta_i)$ is of the form $D_i \times x_0 \times [-.5,.5]$ where D_i is a subdisc of D^2 and that the projections of D_i to the I factor are pairwise disjoint. It follows that α_{β_f} is represented by the sum of n 2-parameter families $\alpha^i_{\beta_f}$, where $\beta_f \in \mathrm{Diff}_0(S^1 \times B^3 \,\mathrm{fix}\, \partial)$ is the implantation of f. Finally, to construct $\alpha^i_{\beta_f}$ push forward the data of the barbell family.

Definition 6.7. For $k \geq 2$ define the 2-parameter families $\hat\theta_k^1, \cdots, \hat\theta_k^{k-1}$ analogously to the family $\hat\theta_k^L$ shown in Figure 39 b) where $k = 5$ and $L = 3$. Here the red bands go around the S^1 just under L times while the blue bands go around the S^1 just under $k - L$ times. Here the color coding and support in the parameter space are as for the $\gamma_{t,u}$ family. Define $\hat\theta_k = \sum_{L=1}^{k-1} \hat\theta_k^L$. $\hat\theta_k$ is called the *symmetric* θ_k.

Lemma 6.8. $[\alpha_{\beta_{\theta_k}}] = \pm[\hat\theta_k] \in \pi_2\,\mathrm{Emb}(I, S^1 \times B^3; I_0)$.

Proof. $\alpha_{\beta_{\theta_k}}$ is obtained by applying Construction 6.6 to the barbell $\mathcal{B}(\theta_k)$. Each α^L appears qualitatively as $\hat\theta_k^L$ up to twisting of the bands near the lasso and whether the individual spinnings are L/H or H/L with differences uniform in L. Since any such difference changes the sign of the class in π_2 by Lemmas 4.4 and 4.29 the result follows. □

Definition 6.9. Let $f : \mathcal{B} \to M$ be an embedding of the barbell such that the cuffs bound disjoint 3-balls V_1 and V_2 and the bar is transverse to $V_1 \cup V_2$. A reembedding of the bar supported within a small neighborhood $N(V_1 \cup V_2)$ of $V_1 \cup V_2$ to obtain f_1 is called



The θ_k implantation, with k=5

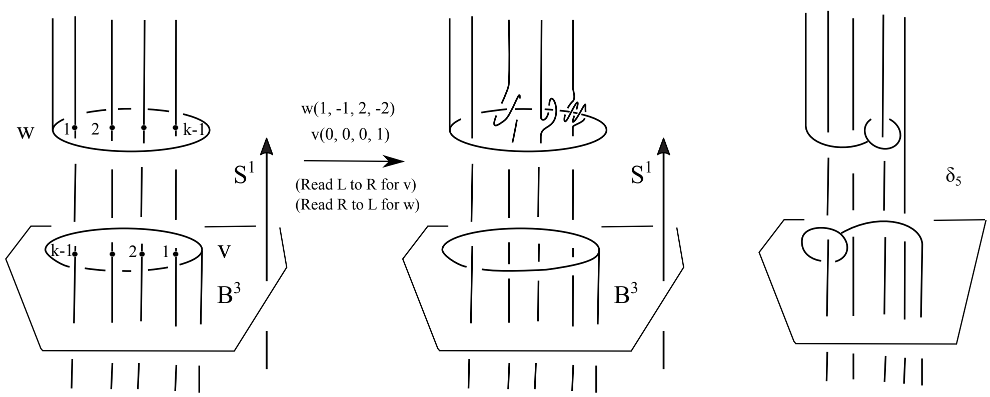
The L=3 family for the symmetric θ_5 implantation

FIGURE 39.

a *twisting* of f. Here the components of the bar that intersect both $N(V_1 \cup V_2)$ and the cuffs remain unchanged. β_{f_1} is called a *twisting* of the implantation β_f.

Remark 6.10. Up to isotopy, a twisting corresponds to replacing arcs σ_i that locally link a cuff once to ones that link $n_i \in \mathbb{Z}$ times.

Definition 6.11. (Twisting the $\hat{\theta}_k$ implantation) Ordering the points through the cuffs v and w as in Figure 40 a) we construct the $v(m_1, \cdots, m_{k-1}), w(n_1, \cdots, n_{k-1})$ implantation. This means that, the i'th (resp. j'th) arc through the P (resp. Q) cuff now locally links it m_i (resp. n_j) times. For example see Figure 40 b). The $\hat{\alpha}_{k-1}$ *implantation* is the $\hat{\theta}_k$ implantation twisted by $v(0, \cdots, 0, 1), w(1, 1, \cdots, 1)$. Define δ_k the $\theta_k(0, 0, \cdots, 0, 1)(0, 0, \cdots, 0, 1, 0)$ implantation. See Figure 40 c) for δ_5.



a) b) c)

Twisting the θ_k implantation, with k=5

FIGURE 40.

Construction 6.12. Figure 41 shows how to modify a 2-parameter family according to twisting an implantation. Note that modifying a band by a 2π-twist does not change the homotopy class of the 2-parameter family. Therefore, what matters is how many times the modified band goes through the cuff rather than how exactly it is reembedded.

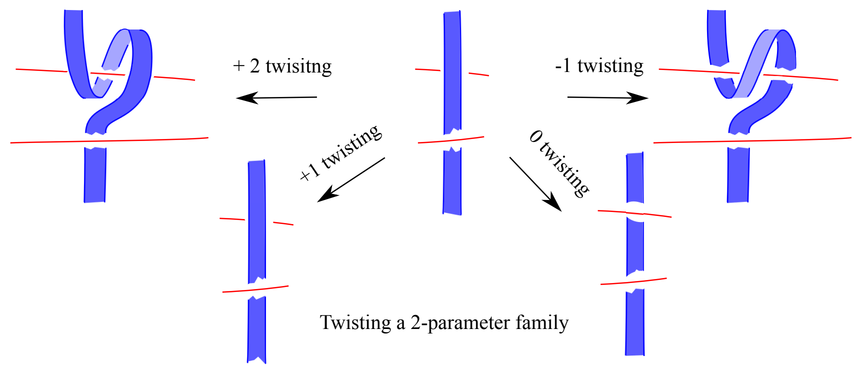


FIGURE 41.

Remark 6.13. In the next section we shall see that $\hat{\theta}_k$ gives rise to a $(k-1)\times(k-1)$ matrix F_k with values in $\pi_2\,\mathrm{Emb}(I, S^1\times B^3; I_0)$ such that the sum of the entries equals $[\hat{\theta}_k]\in\pi_2\,\mathrm{Emb}(I, S^1\times B^3; I_0)$. Further, at the π_2 level, a $v(m_1,\cdots,m_{k-1}), w(n_1,\cdots,n_{k-1})$ twisting has the effect of multiplying the i'th row of F_k by m_i and the j'th column by n_j.

7. FACTORING THE SYMMETRIC θ_k

In this section we investigate the 2-parameter family $\hat{\theta}_k : I^2 \to \mathrm{Emb}(I, S^1\times B^3, I_0)$ called the *symmetric* θ_k defined in Definition 6.7. We will show $\hat{\theta}_k^L$ factors into $(k-1)^2$ 2-parameter families, giving rise to a $(k-1)\times(k-1)$ matrix $F_k^L(p,q)$ with values in $\pi_2\,\mathrm{Emb}(I, S^1\times B^3; I_0)$ equal to those represented by the (p,q)'th family. Summing over L we obtain F_k which will be shown to be skew symmetric. It will follow that $[\hat{\theta}_k]=0\in\pi_2\,\mathrm{Emb}(I, S^1\times B^3; I_0)$ and hence $W_3(\ker(\mathrm{Diff}_0(S^1\times B^3 \text{ fix } f)\to \mathrm{Diff}_0(S^1\times S^3)))=0$.

As stated in previous sections we view $S^1\times B^3$ as $(D^2\times S^1)\times[-1,1]$ with the product orientation and I_0 denotes a fixed geodesic arc through the origin of $D^2\times x_0\times 0$.

Definition 7.1. Fix $k\geq 2$. For $1\leq L\leq k-1$ we define properly embedded arcs $J_k^L\subset D^2\times S^1\times 0$ path homotopic to I_0 whose ends also coincide with those of I_0. We first do a small initial isotopy of I_0 to obtain the arc J_0^0 of Figure 42 b). To obtain $J_k^L\subset D^2\times S^1\times 0$, send the two local top strands just under $k-L$ times positively around the S^1 and send the two local bottom strands just under L times negatively around the S^1 to obtain an embedded arc, as exemplified by J_5^3 or J_5^2 shown in Figure 42 c), d). The 20 strands at the bottom of those figures go around the S^1 to attach to the strands at the top.

For $1\leq L\leq k-1$ homotope each $\hat{\theta}_k^L$ to one based at J_k^L by shrinking the bands without modifying the lassos. For example, when $k=5$ and $L=3$, starting with the family shown in Figure 39 b) we obtain the one shown in Figure 43. Denote this 2-parameter as the separable pair (B_k^L, R_k^L), where B_k^L (resp. R_k^L) is defined by the blue (resp. red) bands and lassos.

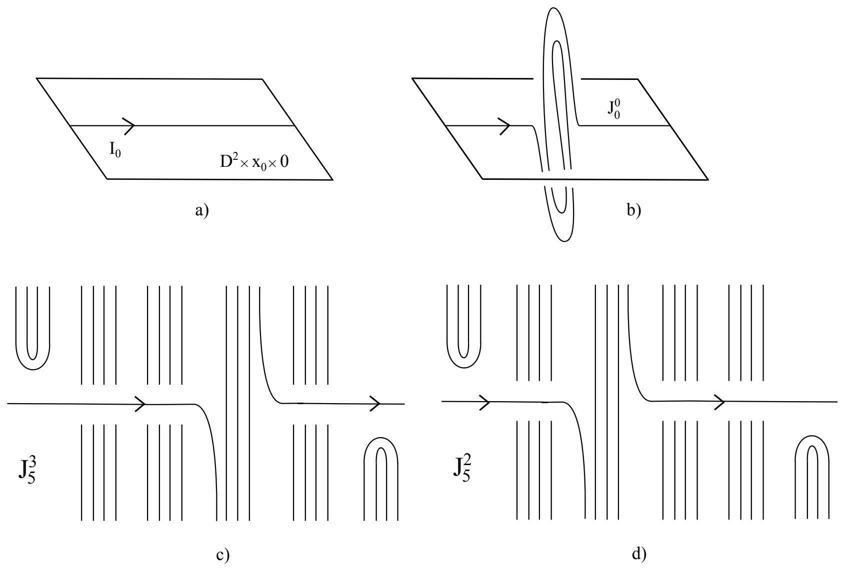


FIGURE 42.

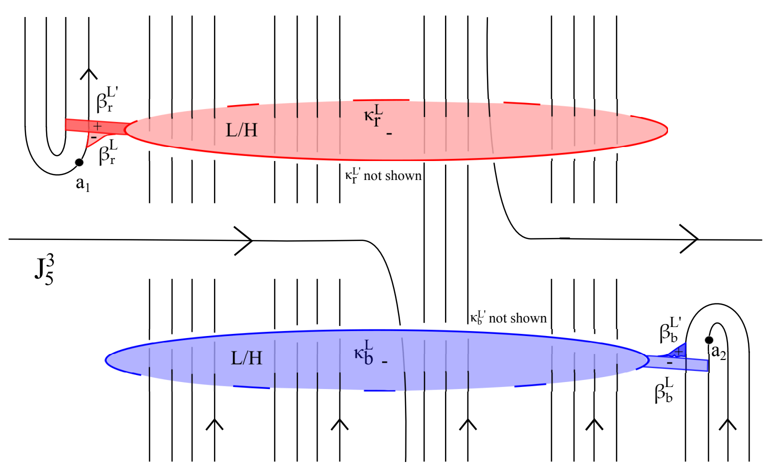


FIGURE 43.

Remarks 7.2. i) By Lemma 4.36, since J_k^L is path isotopic to I_0 and as we shall see (B_k^L, R_k^L) can be represented by an abstract chord diagram, it gives a well-defined element of $\pi_2 \operatorname{Emb}(I, S^1 \times B^3, I_0)$, and so $[\hat{\theta}_k] = \sum_{L=1}^{k-1}[(B_k^L, R_k^L)]$.

ii) Fix $k \geq 2$. To minimize notation we suppress most of our k-subscripts and abuse notation by introducing others.

Introduction to the rest of the chapter We will express B^L (resp. R^L) as a concatenation $B_1^L, \cdots, B_{k-1}^L$ (resp. $R_1^L, \cdots, R_{k-1}^L$) and then use bilinearity to produce the $(k-1)^2$ 2-parameter families representing the classes $F^L(p,q), 1 \leq p,q \leq k-1$, and so $[\hat{\theta}_k^L] = \sum F^L(p,q)$. The terms $F^L(p,q)$ define a $(k-1)\times(k-1)$ matrix. Summing over L we obtain the $(k-1)\times(k-1)$ matrix F_k whose sum equals $[\hat{\theta}_k]$. We shall see that $F^L(p,q) = -F^{k-L}(q,p)$ and hence F_k is skew-symmetric.

Definition 7.3. By the *red coloring* (resp. *blue coloring* of J^L we mean a red (resp. blue) half disc D_r (resp. D_b) in $D^2 \times S^1 \times 0$ such that ∂D_r consists of an arc in J^L and a complementary arc as in Figure 44 a) and similarly for D_b. These discs should be viewed as long and thin. With notation as in those figures, these discs color the arc $[a_1, a_3] \subset J^L$ red and the arc $[a_2, a_4] \subset J^L$ blue as in Figure 44 c). We call the *bottom* (resp. *top*) of D_r to be that part of the half disc near a_1 (resp. a_2). Similarly the *bottom* of D_b is near a_2 and the *top* of D_b is near a_3.

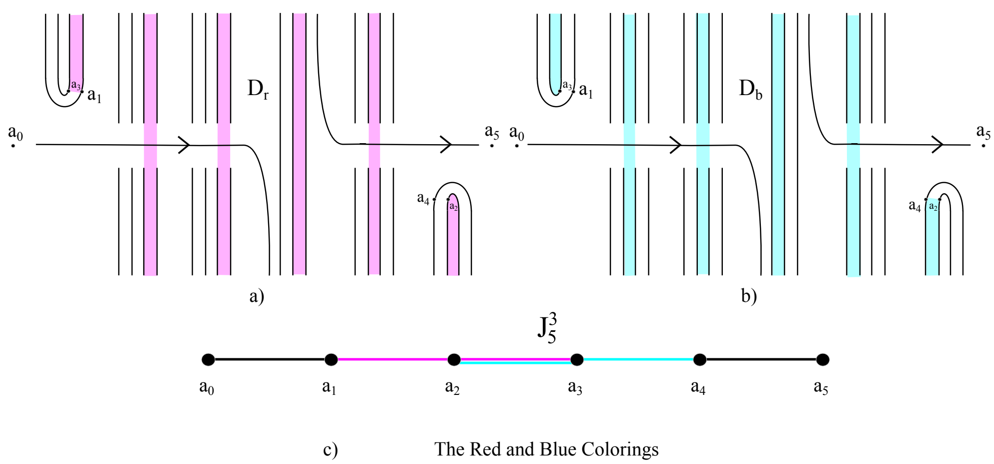


FIGURE 44.

Remark 7.4. These colorings are very helpful for keeping track of both location and orientation. Here is a first application.

Lemma 7.5. *$B^L, R^L \in \Omega\,\mathrm{Emb}(I, S^1 \times B^3, J_L)$ are homotopically trivial.*

Proof. While we already know this result it is useful to see it from the coloring point of view. We prove this for R^L with the B^L argument being similar. Following the red coloring, slide the base of the red bands until they reach D_r's top. From there it is apparent that these bands form a parallel cancel pair. See Figure 45. By Lemma 4.6 the support of the undo homotopy is within the union of a neighborhood of the bases and a neighborhood of the parallelism between the bands and the parallelism between the lassos spheres, thus there is no problem with the band linking the lassos. $\square$

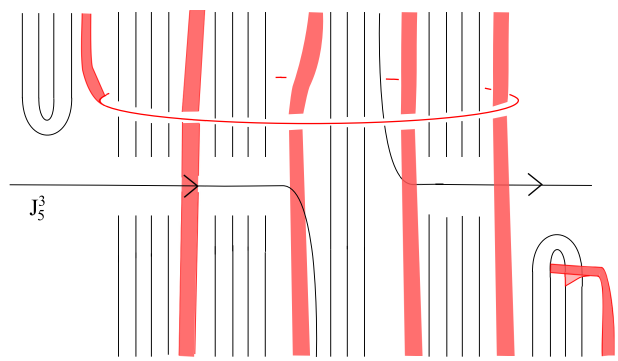


FIGURE 45.

Convention 7.6. We let $\beta_b^L, \beta_b^{L'}, \kappa_b^L, \kappa_b^{L'}$ denote the band and lassos defining B^L, where base(β_b^L) appears before base($\beta_b^{L'}$ along J^L. $\beta_r^L, \beta_r^{L'}, \kappa_r^L, \kappa_r^{L'}$ are defined analogously. The red coloring and blue colorings allow us to readily deduce which base comes first and to keep track of how a given one is oriented. This in turn enables us to orient the red and blue bands near their bases and then orient their lasso discs as indicated in Figure 43. By Remark 4.5 it suffices to draw only the core of the band provided we also know the orientation of the lasso disc. In what follows usually only the core of the band will be shown in the figures.

Lemma 7.7. *B^L (resp. R^L) is homotopic to the concatenation of the loops $B_1^L, \cdots, B_{k-1}^L$ (resp. $R_{L_1}, \cdots, R_{k-1}^L$) represented in band lasso form in Figure 46. Each of these loops is homotopically trivial.*

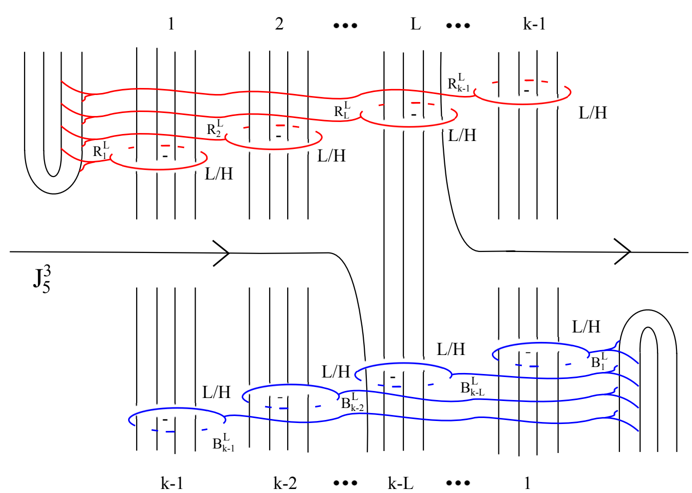


FIGURE 46.

Proof. The principle is that if a lasso encircles two arcs τ_1, τ_2, then the corresponding spinning about the union can be homotoped into a concatenation of spinnings of the same type (e.g. L/H or positive) about τ_1 and τ_2. The proof that the B_i^L's and R_j^L's are homotopically trivial loops follows as the proof of Lemma 7.5. □

Notation 7.8. B_i^L (resp. R_j^L) denotes loop represented by the pair of blue (resp. red) bands and lassos corresponding to linking about the i'th (resp. j'th) local pair of four strands, counted from the right (resp. left).

Remarks 7.9. i) This notation is chosen, since up to signs/orientations there is a symmetry ψ which takes J_k^L to J_k^{k-L} such that the bands and lassos defining B_i^L are taken to those defining R_i^{k-L}. This will induce the skew symmetry mentioned above.

Lemma 7.10. $[\theta_k^L] = \sum_{1\leq i,j\leq k-1}[(B_i^L, R_j^L)]$.

Proof. The homotopy from (B^L, R^L) to $(B_1^L * \cdots * B_{k-1}^L, R_1^L * \cdots * R_{k-1}^L)$ is separable, thus the result follows from bilinearity. □

Definition 7.11. Define $F^L(p,q) = [(B_p^L, R_q^L)]$.

Proposition 7.12. *If $p \geq k-L$ and $q \geq L$, then $F^L(p,q) = D(p,-q)$*

Proof. To minimize notation we suppress the L superscripts. Figure 47 shows a representative case. Observe that B_p is the concatenation of B_p^0 and B_p^1 where B_p^0 (resp. B_p^1) is given by a band lasso pair we call (β_b^0, κ_b^0) (resp. (β_b^1, κ_b^1)) where base(β_b^0) precedes base(β_b^1) on J and that (B_p, R_p) is separably homotopic to $(B_p^0 * B_p^1, R_q)$. The hypotheses on p and q are the conditions such that both the blue and red lassos can slide off of J following four of its local strands, so in particular each of B_p^0 and B_p^1 is homotopically trivial. Thus, by bilinearity $[(B_p, R_q)] = [(B_p^0, R_q)] + [(B_p^1, R_q)]$. Since the support of the undo homotopy of R_q is disjoint from both β_p^1 and its lasso sphere it follows that $[(B_p^1, R_q)] = 0$ and hence $F(p,q) = [(B_p^0, R_q)]$.

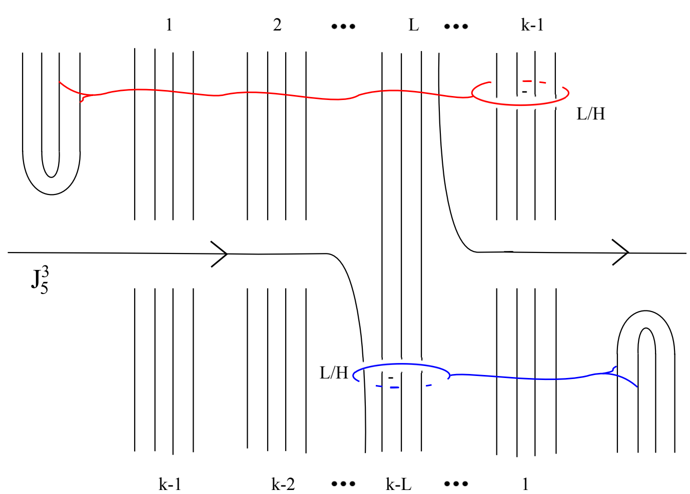


FIGURE 47.

To compute $F(p,q)$ slide the red bands so that their bases are near the top of the red coloring D_r and the red lasso discs link β_b^0 near its base. See Figure 48 a). Next, isotope β_b^0 and κ_b^0 so that κ_b^0 links the cores of the red bands at their branch loci. This is done in two steps. First, use the 4-strand isotopy to isotope κ_b^0 off of J and link the red bands. Then, isotope κ_b^0 to link the bands at their branch loci. Figure 48 b) shows κ_b^0 midway through the second step.

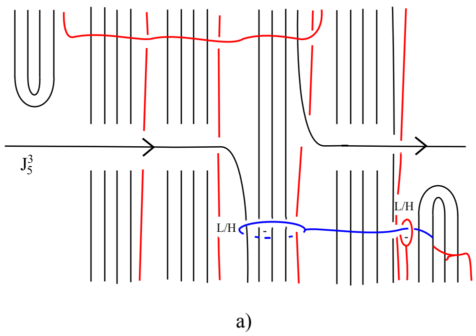


FIGURE 48.

We have therefore obtained a $\pm D(m,n)$ as in Definition 4.43. Since the red bases have been moved k times around the S^1 and its lasso $k-q$ times around the S^1, following the convention of Definition 4.43, $n=(k-q)-k=-q$. The blue lasso moves $-(k-p)$ times around S^1 to slide off of J and then moves k times around S^1 to link the red bands at their branch loci. Therefore, $m=-(k-p)+k=p$.

We now compute the sign. Following Remarks 4.44 the sign is determined the parity of how many of the following things deviate from the standard $D(m,n)$.

i) It is of standard (B,R) type
ii) The blue spinning is L/H
iii) The red spinning is L/H
iv) <red band core, blue lasso disc>$=-1$
v) <blue band core, red lasso disc>$=-1$

Since there are two changes the sign is $+1$. ☐

Proposition 7.13. *If $q<L$ and $p\geq k-L$* or *$q\geq L$ and $p<k-L$, then $F^L(p,q)=F^{L,s}(p,q)+F^{L,r}(p,q)$ where*

$F^{L,s}(p,q)=$

$D(p,-q)$ *if* $q<L$ *and* $p+q\geq k$
$D(p,-q)-D(p+q,-q)$ *if* $q<L$ *and* $p+q<k$
$D(-q,p)$ *if* $p<k-L$ *and* $p+q\geq k$
$D(-q,p)-D(-p-q,p)$ *if* $p<k-L$ *and* $p+q<k$.

$F^{L,r}(p,q)=$

$-D(p-q,q)$ *if* $q<L$ *and* $p+q\geq k$

$-D(p-q,q)+D(p,q)$ *if* $q<L$ *and* $p+q<k$
$-D(p-q,-p)$ *if* $p<k-L$ *and* $p+q\geq k$
$-D(p-q,-p)+D(-q,-p)$ *if* $p<k-L$ *and* $p+q<k$.

Proof. To minimize notation we delete the L superscript.

Case 1: $q<L$

Idea of Proof: As before we reduce to B_p being defined by a single band and lasso. Under the hypothesis only the blue lasso can slide off of J by following four strands. To address this, we homotope R_q into a concatenation of R_q^s and R_q^r each of whose lassos slides off J by following the red coloring. We define $F^s(p,q)=[(B_p,R_q^s)]$ and $F^r(p,q)=[(B_p,R_q^r)]$. Each of these has two subcases; $p+q\geq k$ and $p+q<k$. In the first case the blue and red lassos can slide off of J without crashing into each other. In the second case we factor B_p into B_p^0 and B_p^1 each of which can now slide off of J without crashing into the red bands and lassos. An argument similar to that of the previous proposition shows that each of $F^s(p,q), F^r(p,q)$ is one or two $\pm D(m,n)$'s depending on whether subcase 1 or 2 applies.

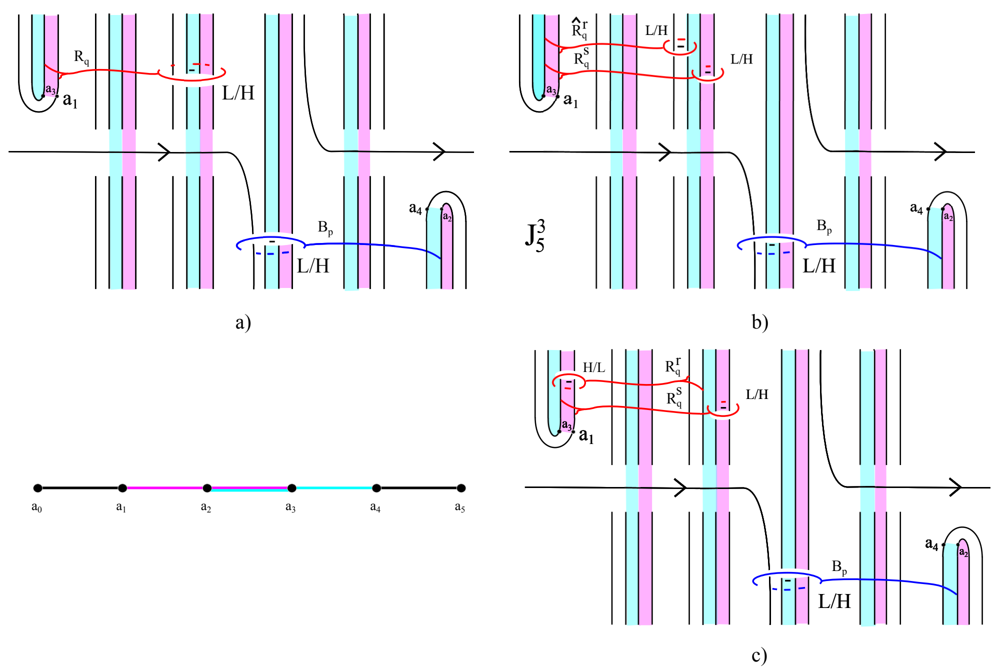


FIGURE 49.

Proof of Case 1: B_p is represented by (β_b,κ_b) and (β_b',κ_b') where the base of β_b appears before that of β_b' on J. We abuse notation by calling B_p the loop represented by just $(beta_b,\kappa_b)$, since as in the proof of the previous proposition we can use this B_p to compute $F(p,q)$. Figure 49 a) shows a representative example of a (B_p,R_q) for Case 1. Now homotope R_q into the concatenation of R_q^s and $\hat{R}_q^r$ as in Figure 49 b) and use Lemma 4.4 to reverse the direction of the band and lassos representing $\hat{R}_q^r$ to obtain R_q^r. See

Figure 49 c) We use the superscript r (resp. s) to denote *reversed* (resp. *standard*). Several loops in Figure 49, such as R^r_q, are represented by two bands and lassos whose bands are branched. For such a pair, the orientation of the lasso disc shown in Figure 49 is for the lasso whose base appears first on J. The homotopies of R_q to $R^s_q * \hat{R}^r_q$ and $\hat{R}^r_q$ to R^r_q are supported away from both the domain and range support of B_p, thus (B_p, R_q) (resp. $(B_p, \hat{R}^r_q)$) is separably homotopic to $(B_p, R^s_q * R^r_q)$ (resp. (B_p, R^r_q)). Each of the loops R^s_q and R^r_q are homotopically trivial for their lassos can be slid off J by following the red coloring. Therefore,

$$F(p,q) = [(B_p, R_q)] = [(B_p, R^s_q * \hat{R}^r_q)] = [(B_p, R^s_q)] + [(B_p, \hat{R}^r_q)] = [(B_p, R^s_q)] + [(B_p, R^r, q)].$$

Define $F^s(p,q) = [(B_p, R^s_q)]$ and $F^r(p,q) = [B_p, R^r_q)]$.

Computation of $F^s(p,q)$ This is very similar to that of the previous proposition. If $p+q \geq k$, then following the red coloring, slide the red lassos to link the core of the blue band near its base and then slide the red bands so that their bases are at the top of the red coloring. Next first slide the blue lasso off of J to link the red band cores and then isotope the blue lasso to link the red band cores at their branch loci. The same calculation as in Proposition 7.12 shows that $F^s(p,q) = D(p,-q)$.

If $p+q < k$, then slide the blue lasso to just before it crashes into the red lassos and bands. See Figure 50 a). Next homotope B_p into a concatenation of B^0_p and B^1_p as in Figure 50 b). The separable homotopy and bilinearity arguments imply that $[(B_p, R^s_q)] = [(B^0_p, R^s_q)] + [(B^1_p, R^s_q)]$. That $[(B^1_p, R^s_q)] = D(p,-q)$ follows as in the $p+q \geq k$ case. Now $[(B^0_p, R^s_q)]$ is also a $\pm D(m,n)$ with the following changes in comparison with that of $[(B^1_p, R^s_q)]$. Here we have <red band core, blue lasso>$= +1$ vs the -1 before, while the other four values determining the sign are the same as those for $[(B^1_p, R^s_q)]$. Therefore, we have a sign change. The previous argument also shows that $n = -q$. To compute m, note that the lasso of B^0_p is already linking the core of the red bands and a horizontal isotopy moves it to their branch loci while the B^1_p lasso had to travel an extra $-q$ times around the S^1 to get to the same point. It follows that $m = p+q$ and hence $[(B^0_p, R^s_q)] = -D(p+q,-q)$ and so $F^s(p,q) = D(p,-q) - D(p+q,-q)$. $\square$

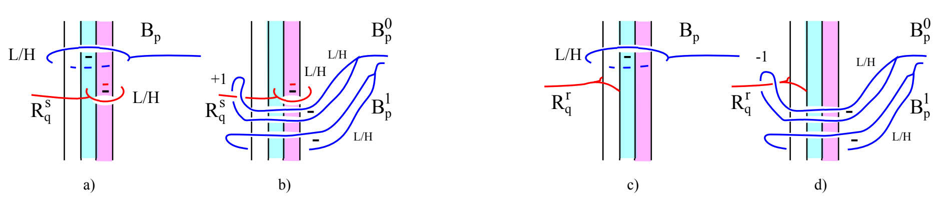


FIGURE 50.

Computation of $F^r(p,q)$: If $p+q \geq k$, then following the local four strands, isotope the blue lasso off of J to link the bands of the red lassos and then further isotope the blue lasso to link these bands at their branch loci. Next, follow the red coloring to isotope the red lassos off of J and link the blue band near its base. Finally, isotope the red bands, so that their bases are near the top of the red coloring. This is done in two steps. First isotope so that the bases contain the points a_1, a_3 and then follow the red coloring from its bottom to near its top. Simultaneously isotope the linking blue lasso to follow along. We now compute the values of this $\pm D(m,n)$. The blue lasso goes $-(k-p)$ times around the S^1 to slide off of J and then, in the aggregate, $k-q$ times about the S^1 while following the red bands. Therefore, $m = p - q$. The red lassos go k times around the S^1 while in the aggregate the bases go $k-q$ times about the S^1 so that $n = k-(k-q) = q$. We now compute the sign.

i) It is of standard (B,R) type
ii) The blue spinning is L/H
iii) The red spinning is H/L
iv) <red band core, blue lasso disc>$= +1$
v) <blue band core, red lasso disc>$= +1$

Since there is one sign change from the standard $D(m,n)$ the sign is -. Therefore $F^r(p,q) = -D(p-q,q)$.

If $p+q<k$, then slide the blue lasso to just before it crashes into the red bands. See Figure 50 c). Next homotope B_p into a concatenation of B_p^0 and B_p^1 as in Figure 50 d). The separable homotopy and bilinearity arguments imply that $[(B_p,R_q^r)] = [(B_p^0,R_q^r)] + [(B_p^1,R_q^r)]$. That $[(B_p^1,R_q^r)] = -D(p-q,q)$ follows as in the $p+q\geq k$ case.

Now $[(B_p^0,R_q^r)]$ is also a $\pm D(m,n)$. Here $n=q$ as before. To compute m, note that the lasso of B_p^0 is already linking the core of the red bands while the B_p^1 lasso had to travel an extra $-q$ times around the S^1 to get to essentially the same point. It follows that $m = (p-q)+q = p$. For the sign, note that there is a single change in comparison with the five values computed for $[(B_p^1,R_q^r)]$. Here <red band core, blue lasso disc>$= -1$. It follows that the sign is positive. Therefore, $[(B_p^0,R_q^s)] = D(p,q)$ and hence $F^r(p,q) = -D(p-q,q)+D(p,q)$. $\square$

Case 2: $p<k-L$

Proof of Case 2: R_q is represented by $(\beta_r,\kappa_r),(\beta_r',\kappa_r')$ where the base of β_r appears first. The argument of Case 1 shows that $[(B_p,\sigma(\beta_r,\kappa_r)] = 0$ and so we again abuse notation by letting R_q be represented by just (β_r',κ_r'). Figure 51 a) shows a representative example of a (B_p,R_q) for Case 2. Note the sign of the κ_r' lasso disc. In this case B_p is homotopic to a concatenation of B_p^s and $\hat{B}_p^r$ as shown in Figure 51 b). We reverse the direction of the band and lasso representing $\hat{B}_p^r$ as in Figure 51 c) to obtain B_p^r. As before we obtain $F(p,q) = [(B_p^s,R_q)] + [(B_p^r,R_q)]$ and define $F^s(p,q) = [(B_p^s,R_q)]$ and $F^r(p,q) = [(B_p^r,R_q)]$.

Computation of $F^s(p,q)$: As in Case 1 this will be one or two $\pm D(m,n)$'s depending on whether or not $p+q\geq k$. In both subcases the branched band will be blue, so following Definition 4.43 m will be computed using the red band and n from the blue

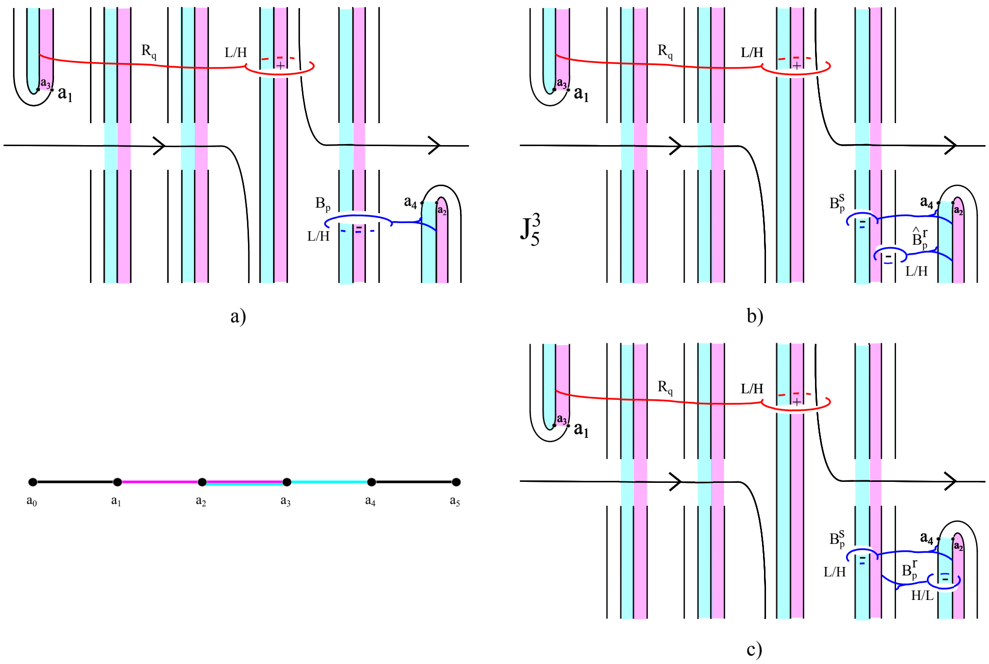


FIGURE 51.

bands. With that in mind the calculation follows the method of Case 1. When $p+q \geq k$ we have $m = -q$ and $n = p$. To compute the sign note that there are two cancelling changes in comparison with Case 1. Here there is a red/blue switch and also <blue band core, red lasso>$= +1$, in comparison with the earlier -1. Therefore the sign is $+$ and $F^s(p,q) = D(-q,p)$.

When $p+q < k$, then slide the red lasso to just before it crashes into the blue lassos and bands. See Figure 52 a). Next homotope R_q to a concatenation of R_q^0 and R_q^1 as in Figure 52 b). Here $[(B_p^s, R_q)] = [(B_p^s, R_q^0)] + [(B_p^s, R_q^1)]$. That $[(B_p^s, R_q^1)] = D(-q,p)$ follows as in the $p+q \geq k$ case. Using that, the method of Case 1 shows that $[(B_p^s, R_q^0)] = -D(-p-q,p)$. Therefore, $F^s(p,q) = D(-q,p) - D(-p-q,p)$. □

Computation of $F^r(p,q)$: Again there will be one or two $\pm D(m,n)$'s depending on whether or not $p+q \geq k$, the branched band will be blue and so m will be computed using the red band and n from the blue bands. With that in mind the calculation follows the method of Case 1. When $p+q \geq k$ we have $m = p-q$ and $n = -p$. We now compute the sign.

i) It is of (R,B) type
ii) The blue spinning is H/L
iii) The red spinning is L/H
iv) <red band core, blue lasso disc>$= +1$
v) <blue band core, red lasso disc>$= -1$

There are three changes so that the sign is - and hence $F^r(p,q) = -D(p-q,-p)$.

When $p+q<k$, then slide the red lasso to just before it crashes into the blue bands. See Figure 52 c). Next homotope R_q to a concatenation of R_q^0 and R_q^1 as in Figure 52 d). Again $[(B_p^r, R_q)] = [(B_p^r, R_q^0)] + [(B_p^r, R_q^1)]$. That $[(B_p^r, R_q^1)] = -D(p-q,-p)$ follows as in

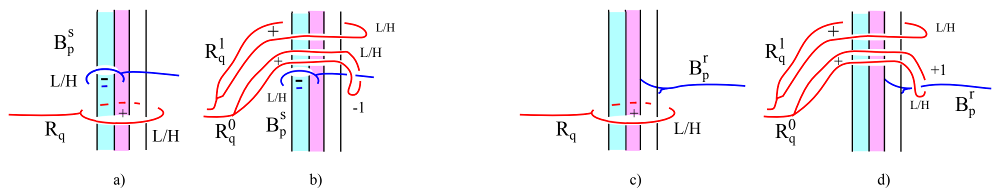


FIGURE 52.

the $p+q \geq k$ case. Using that, the method of Case 1 shows that $[(B_p^r, R_q^0)] = D(-q,-p)$. Therefore, $F^s(p,q) = -D(p-q,-p) + D(-q,-p)$. □

Proposition 7.14. *For $p < k-L$ and $q < L$, $F^L(p,q) = 0$.*

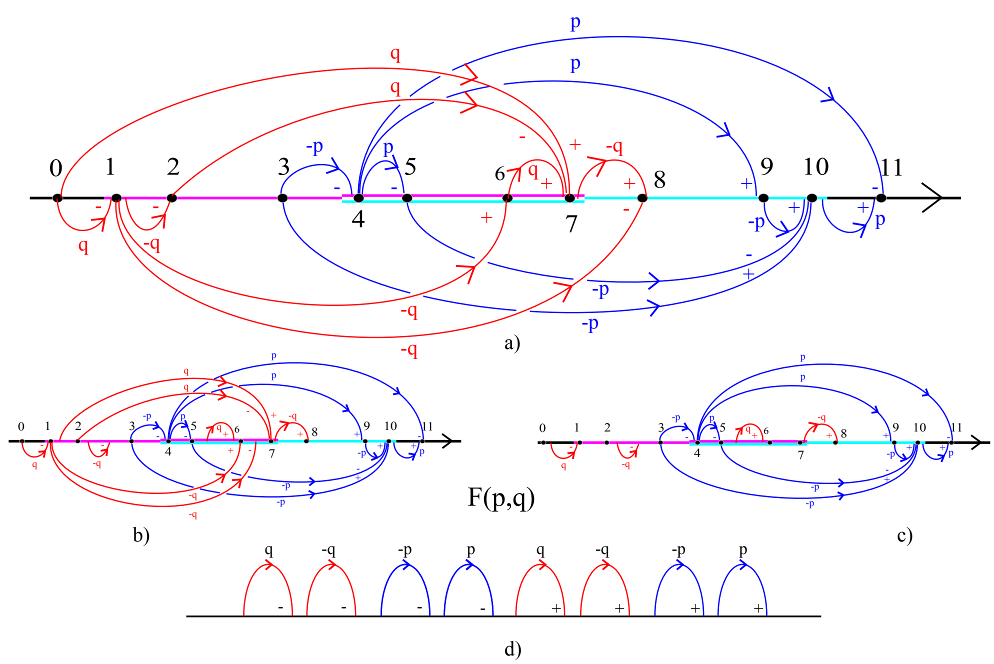


FIGURE 53.

Proof. Again we supress the L's.

Step 1: $F(p,q)$ is represented by the abstract chord diagram pairs of Figure 53 a), b), c) and d).

Proof: A representative (B_p, R_q) is shown in Figure 54 a). Figure 54 b) shows details of neighborhoods of the band bases and lasso discs. Each of the spinnings determined from the four bands and lassos is homotopic to a concatenation of four spinnings of arcs about arcs, via a homotopy that is supported in a small neighborhood of the relevant band and lasso ball. For example, (β_r, κ_r) corresponds to spinning an arc near the point labeled 1 about arcs near the points 0,8,6 and 2. The labeling of the eleven points in the diagram is induced by their ordering in J. See Figure 54 c). This data gives rise to the abstract chord diagram of Figure 53 a), where the chords are oriented from the low label to the high one. For each chord we need to compute group elements and signs. For example, β_r, κ_r gives rise to a λ spinning where λ is approximately a horizontal arc in $S^1 \times B^3$ from 1 to 0. If $D_r$denotes the oriented lasso disc bounded by κ_r, then near 0, $< J, D_r >= +1$, so by Lemma 4.5 the oriented chord from 1 to 0 has positive sign. By Lemma 4.4 reversing the direction changes the sign as indicated in Figure 53 a). To compute the group element we concatenate λ with the arc $[0,1] \subset J$ to obtain an oriented closed loop representing $-q \in \pi_1(S^1 \times B^3; J)$. Since we reversed the direction, the chord in Figure 53 a) is labeled q. Note that the colorings inform us that the short arcs between 9,5,3 and 11 as well as the one between 1 and 7 can be homotoped into J. It follows that all eight of the chords between these points are labeled $\pm q$.

Applying multiple applications of the Exchange Lemma 4.32 to the red chords gives Figure 53 b). Two applications of the undo homotopy eliminates two pairs of parallel oppositely signed red chords with the same group element, giving Figure 53 c). Similarly, blue exchanges followed by two applications of the undo homotopy gives Figure 53 d). □

Step 2: F(p,q)=0

Proof: Using bilinearity, the class of Figure 53 a) factors into four classes represented by the four chord diagrams $H(\pm p, \pm q)$, where $H(p,q)$ is the subdiagram consisting of the chords labeled p and q. The other diagrams are defined similarly. Notice that any diagram can be turned into another by an appropriate sign change of the group elements and exchange moves. By bilinearity $[H(p,q)]$ factors into four classes represented by the chord diagrams D_a, D_b, D_c, D_d respectively shown in Figures 55 a), b), c), d). The chord diagrams D_a, D_b, D_c are readily shown to respectively represent $G(p+q,p), -G(q,p)$ and $G^*(p,q)$. The chord diagram D_e is obtained from D_d by a chord sliding move. Applying bilinearity $[D_e]$ factors into $[D_e^0] + [D_e^1]$ where D_e^0 (resp. D_e^1) consists of the red chords and the blue chords labeled p (resp. $p+q$). Now $[D_e^0] = -E(p,-q)$ and $[D_e^1]$ is readily shown to be equal to $G(p+q,q)$. Therefore, $[D_d] = [D_e] = G(p+q,q) - E(p,-q)$ and hence $[H(p,q)]$
$= G(p+q,p) - G(q,p) + G^*(p,q) + G(p+q,q) - E(p,-q)$
$= G(p+q,p) + G(p+q,q) - G(q,p) - G(p,p-q) + G(q,p) - G(p,q)$
$= G(p,p+q) + G(p,-q) - G(p,p-q) - G(p,q)$.

Here the first equality followed from Lemmas 4.37 and 4.40 and the second from the symmetry relation, Lemma 4.41 iii). It follows that $[H(p,q)] + [H(p,-q)] = [H(-p,q)] + [H(-p,-q)] = 0$ and hence $F(p,q) = 0$. □

Definition 7.15. Define
$\mathrm{I}(p,q) = D(p,-q)$
$\mathrm{II}_b(p,q) = D(-q,p) - D(p-q,-p)$

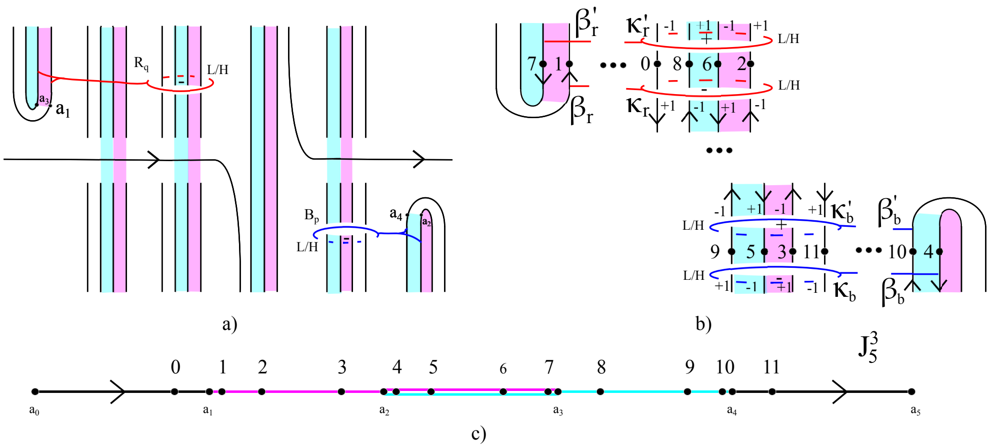


FIGURE 54.

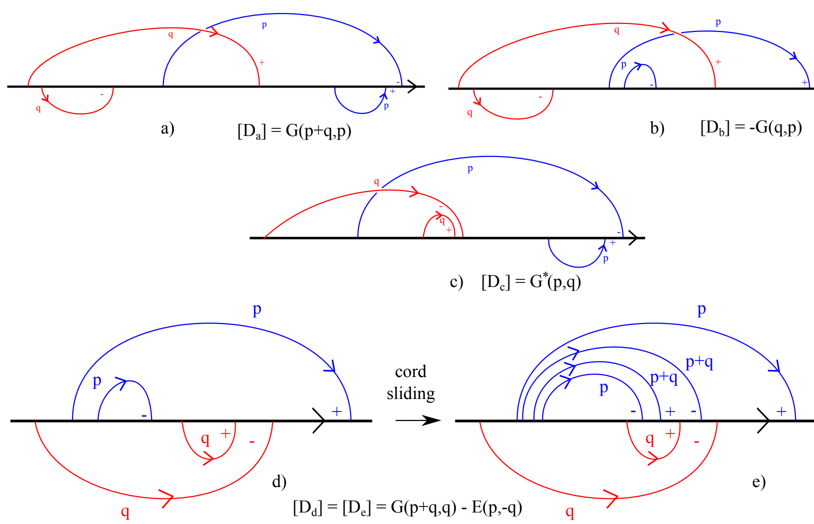


FIGURE 55.

$$\begin{aligned}
\mathrm{II}_{be}(p,q) &= D(-q,p) - D(-p-q,p) - D(p-q,-p) + D(-q,-p)\\
\mathrm{II}_{r}(p,q) &= D(p,-q) - D(p-q,q)\\
\mathrm{II}_{re}(p,q) &= D(p,-q) - D(p+q,-q) - D(p-q,q) + D(p,q).
\end{aligned}$$

Here the b (resp. r, resp. e) denotes blue (resp. red, resp. extra).

Definition 7.16. Define $F_k(p,q) = \sum_{L=1}^{k-1} F_k^L(p,q)$. Define A_k^L (resp. A_k) the $(k-1)\times(k-1)$ matrix with entries $F_k^L(p,q)$ (resp. $F_k(p,q)$).

Theorem 7.17. *1) If $p+q<k$, then*

$$F(p,q) = pII_{re}(p,q) + qII_{be}(p,q).$$

2) If $p+q\geq k$, then

$$F(p,q) = (k-p-1)II_b(p,q) + (k-q-1)II_r(p,q) + (p+q+1-k)I(p,q).$$

Proof. 1) Fix p and q. Figure 56 a) shows the points $1, 2, \cdots k-1 \subset \mathbb{R}$ with the red point, denoting q, the q'th on the left and the blue point the p'th on the right. Since $p+q<k$, the red point is strictly to the left. If $L \leq q$ (resp. $p \geq k-L$), then Proposition 7.13 applies for the case $p<k-L, p+q<k$ (resp. $q<L$ and $p+q<k$). When $q<L<k-p$, then Proposition 7.14 applies. □

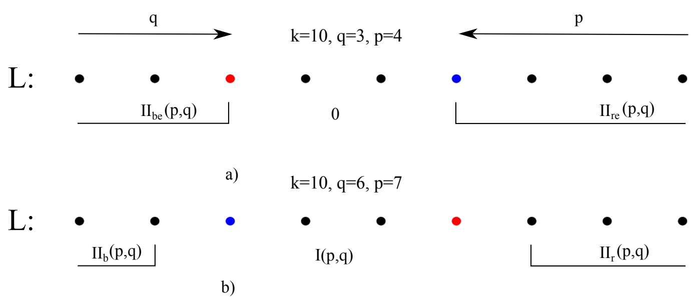


FIGURE 56.

2) Figure 56 b) shows the $p+q \geq k$ case. Here the blue point either coincides with or is to the left of the red one. If $L<k-p$ (resp. $L>q$), then Proposition 7.13 applies for the case $p<k-L$ (resp. $q<L$) where $p+q \geq k$. If $k-p \leq L \leq q$, then Proposition 7.12 applies. □

Corollary 7.18. *F_k is skew symmetric and hence $[\hat{\theta}_k] = 0 \in \pi_2 \operatorname{Emb}(I, S^1 \times B^3; I_0)$.*

Proof. Apply Corollary 4.47 to Theorem 7.17 to conclude that $F_k(p,q) = -F_k(q,p)$. □

8. TWISTING THE SYMMETRIC θ_k

In Definition 6.11 and Construction 6.12 we defined how to twist an implantation and construct its associated 2-parameter family. In particular, we defined the twisted implantations $\theta_k(v,w)$ where $v, w \in \mathbb{Z}^{k-1}$ with the δ_k implantations as a special case. In this section we first compute $[\hat{\theta}_k(v,w)] \in \pi_2 \operatorname{Emb}(I, S^1 \times B^3; I_0)$ and use that result to show that for $k \geq 4$ the 2-parameter families $\hat{\delta}_k$ have linearly independent W_3 invariants, thereby completing the proofs of Theorems 3.15 and 8.5.

Theorem 8.1. *$[\hat{\theta}_k(v,w)] = \sum_{1 \leq p,q \leq k-1} v_p w_q F_k(p,q) \in \pi_2 \operatorname{Emb}(I, S^1 \times B^3; I_0)$, where $v = (v_1, \cdots, v_{k-1})$ and $w = (w_1, \cdots, w_{k-1})$.*

Proof. Construction 6.6 showed how to construct the 2-parameter family $\hat{\theta}_k$ corresponding to the implantation θ_k as a sum of $k-1$ families in band lasso form. Construction 6.12 shows how to modify these families according to the twisting data. In particular, the modification corresponds to locally twisting the bands near where they pass through the lasso discs. Now fix $k \geq 2$. Recall that we shrunk the bands of these L families to obtain families the families $\hat{\theta}^L = (B^L, R^L), L = 1, 2, \cdots, k-1$ and then separably homotoped B^L (resp. R^L) to a concatenation of $B^L_1, \cdots, B^L_{k-1}$. Since the homotopies to the concatenations were supported away from where the strands passed through the blue and red lasso discs it follows that the effect of twisting is to modify the various B^L_i, R^L_j's as in

Figure 57, where the case $w_j = -2$ is shown. Note that the four parallel strands undergo a 4π twist in Figure 57 d). We can assume that there is no twisting since these twists can be transferred to the blue bands if $j \geq L$ and the red bands if $j < L$ and twisting bands by multiples of 2π does not change the homotopy class of the 2-parameter family. It follows that the L'th family of $\hat{\theta}(v, w)$ is homotopic to the family as described in Figure 58. The argument of Lemma 7.5 applies without change to these families and hence the result follows by Proposition 4.28. □

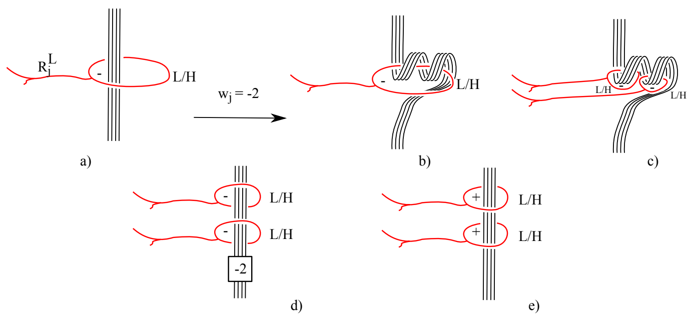


FIGURE 57.

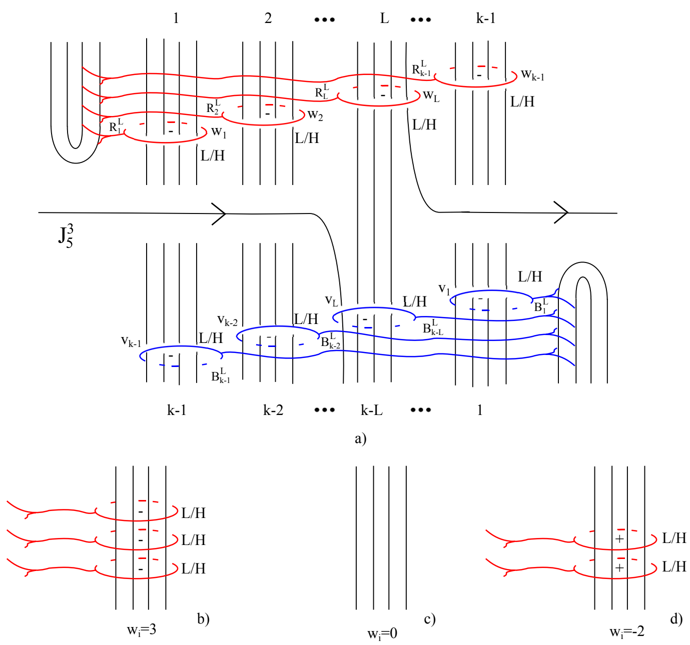


FIGURE 58.

Applying to the case of $v = (0, ..., 0, 1)$ and $w = (0, 0, \cdots, 0, 1, 0)$, i.e. zeros everywhere except for the $k - 1$'th entry of v and $k - 2$'nd entry of w we obtain:

Corollary 8.2. $[\hat{\delta}_k]$

$= (k-1)(G(k-2,k-1) - G(k-1,k-2) + G(1-k,2-k) - G(2-k,1-k))$
$-(-G(k-2,-1) + G(-(k-2),1) - G(1,-(k-2)) + G(-1,k-2))$
$\in \pi_2 \operatorname{Emb}(I, S^1 \times B^3; I_0)$.

Proof. By Theorem 8.1, Theorem 7.17 and Definition 7.15, $[\hat{\delta}_k] = F_k(k-1,k-2)$
$= 0 + 1\mathrm{II}_r(k-1,k-2) + (k-2)\mathrm{I}(k-1,k-2)$
$= D(k-1,-(k-2)) - D(1,k-2) + (k-2)D(k-1,-(k-2))$
$= (k-1)D(k-1,-(k-2)) - D(1,k-2)$
$= (k-1)(G(k-2,k-1) - G(k-1,k-2) + G(1-k,2-k) - G(2-k,1-k))$
$-(-G(k-2,-1) + G(-(k-2),1) - G(1,-(k-2)) + G(-1,k-2))$
by Corollary 4.46. □

Theorem 8.3. *$W_3(\hat{\delta}_4), W_3(\hat{\delta}_5), \cdots$ are linearly independent.*

Proof. Fix $n \geq 4$. By passing to the quotient of the target of W_3 by declaring everything of the form $G(p,q) = 0$ if $p \neq n-1$, then we see that $W_3([\hat{\delta}_n]) \neq 0$, but in that quotient $[\hat{\delta}_m] = 0$ for $m < n$. This implies that if we have a finite linear combination of $\hat{\delta}_m$'s with n the largest m, then that sum equals zero implies that the coefficient of $\hat{\delta}_n = 0$. □

Remark 8.4. $[\hat{\delta}_3] = 0 \in \pi_2 \operatorname{Emb}(I, S^1 \times B^3; I_0)$. This can be deduced from Corollary 8.2 and Corollary 4.41 iii).

Theorem 8.5. *Both $\pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$ and $\pi_0(\operatorname{Diff}(S^1 \times S^3) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$ are infinitely generated. In particular, the implantations $\beta_{\delta_4}, \beta_{\delta_5}, \cdots \subset \operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial)$ and their extensions to $\operatorname{Diff}_0(S^1 \times S^3)$ represent linearly independent elements.*

Proof. That these implantations represent linearly independent elements of $\pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial))$ follows from Corollary 8.2. Since a diffeomorphism of $S^1 \times B^3$ supported in a B^4 can be supported away of Δ_0 it follows that these elements are linearly independent in $\pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$. Let p and ϕ be as in Lemma 3.14. Since the β_{θ_k}'s represent generators for the image of p and by Theorem 7.18, all the $\hat{\theta}_k$'s are homotopically trivial, it follows that $W_3 \circ p \equiv 0$ and hence there is an induced homomophism $W_3 : \pi_0(\operatorname{Diff}_0(S^1 \times S^3)) \to (\pi_5(C_3(S^1 \times B^3))/\text{torsion})/R$ and under this homomorphism the extensions $\beta_{\delta_4}, \beta_{\delta_5}, \cdots$ are linearly independent. A similar argument shows that these elements are linearly independent in $\pi_0(\operatorname{Diff}(S^1 \times S^3) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$. □

Corollary 8.6. *There exist infinitely many distinct knotted non separating 3-spheres in $S^1 \times S^3$, i.e. there exists infinitely many isotopy classes of 3-spheres that are homotopic to $x_0 \times S^3$.* □

Corollary 8.7. *There exist infinitely many isotopically distinct fiberings of $S^1 \times S^3$ homotopic to the standard fibering.*

Applying Construction 5.9 we obtain the knotted 3-ball Δ_{δ_4} by six embedded 0-framed surgeries on Δ_0, the standard 3-ball in $S^1 \times B^3$. See Figure 59 b). That figure also shows the conjecturally knotted 3-ball arising from $\mathcal{B}(\alpha_1)$.

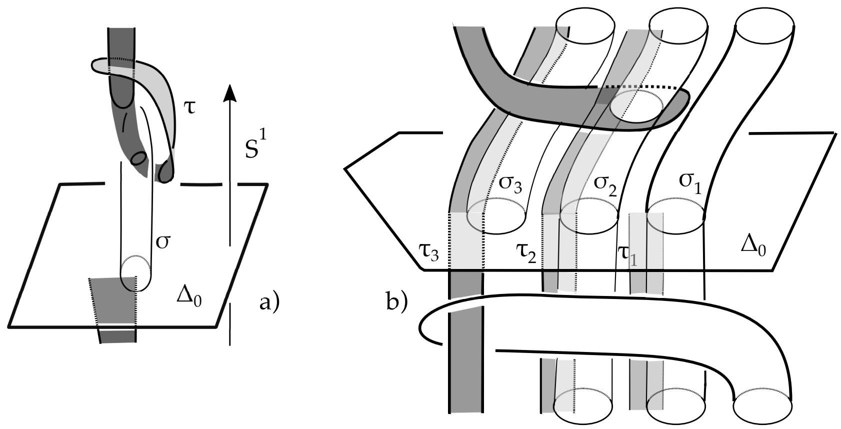


FIGURE 59. (a) Conjecturally knotted 3-ball arising from α_1. (b) Knotted 3-ball arising from δ_4

9. KNOTTED 3-BALLS AND THE SCHÖNFLIES CONJECTURE

In this section we detail the close connection between knotted 3-balls in S^4, and knotted 3-balls in $S^1 \times B^3$ and the relation between the Schönflies conjecture and virtual unknotting of 3-balls.

Definition 9.1. Let Δ_0 be a 3-ball in S^4. We say that the 3-ball Δ is *knotted* relative to Δ_0 if $\partial\Delta = \partial\Delta_0$ and Δ is not isotopic rel $\partial\Delta$ to Δ_0.

In what follows Δ_0 will denote a *standard* or *linear* 3-ball as defined in the introduction. It is to be fixed once and for all. Its boundary, the 2-unknot, will be denoted by U. Unless said otherwise, 3-balls in S^4 will all have boundary equal to U and knottedness is relative to Δ_0. Relative knottedness is essential for by [Ce1] p. 231, [Pa] any two smooth embedded 3-balls in the interior of a connected 4-manifold are smoothly ambiently isotopic.

We abuse notation by letting Δ_0 also denote $\{x_0\} \times B^3 \subset S^1 \times B^3$. Its use should be clear from context.

Definition 9.2. The properly embedded non-separating 3-ball $\Delta \subset S^1 \times B^3$ is *knotted* if it is not properly isotopic to Δ_0.

Remark 9.3. Equivalently Δ is knotted if it is properly homotopic to but not properly isotopic to Δ_0. Since any two non-separating 2-spheres in $S^1 \times S^2$ are isotopic [Gl], [Ha2], we can assume that $\partial\Delta = \partial\Delta_0$. By uniqueness of regular neighborhoods we can further assume that Δ coincides with Δ_0 near partial Δ and since $\mathrm{Diff}_0(S^2)$, the diffeomorphisms homotopic to id, is connected they have the same parametrization there. Finally, since $\pi_1(\mathrm{Emb}(S^2, S^1 \times S^2)) = 1$ up to paths in $SO(3)$ and translations in $S^1 \times S^2$, [Ha2] it follows that if Δ_2 and Δ_3 are isotopic and already coincide near $\partial\Delta_2$, then there is an isotopy fixing a neighborhood of the boundary pointwise. By [Ce2], since $\mathrm{Diff}(B^3 \text{ fix } \partial)$ is connected, Δ_0 has a unique parametrization up to isotopy.

Since $S^1 \times B^3$ is diffeomorphic to the closed complement of the unknot U in S^4 it follows that we can assume, up to isotopy fixing the boundary pointwise, that all 3-balls Δ with boundary Δ_0 coincide with Δ_0 near $\partial\Delta_0$. Also, that there exists a uniform neighborhood $N(U)$ of U such that $\Delta \cap N(U) = \Delta_0$.

Notation 9.4. Let $N(U)$ be a fixed regular neighborhood of the unknot U in S^4, with Δ_0 isotoped to be properly embedded in $S^4 \setminus \operatorname{int}(N(U))$. Fix a diffeomorphism $\psi : S^4 - \operatorname{int}(N(U)) \to S^1 \times B^3$ such that $\psi(\Delta_0) = \Delta_0$.

The following is immediate.

Proposition 9.5. *ψ induces a 1-1 correspondence between isotopy classes of knotted 3-balls in S^4 and knotted 3-balls in $S^1 \times B^3$.* ☐

Remark 9.6. It is important to remember that our correspondence is given by ψ, since by Theorem 3.12 $\operatorname{Diff}(S^1 \times B^3 \text{ fix } \partial)$ acts transitively on properly embedded 3-balls of $S^1 \times B^3$.

Theorem 9.7. *If Δ_0 and Δ_1 are properly embedded 3-balls in $S^4 \setminus \operatorname{int}(N(U))$ coinciding near their boundaries, then there exists an orientation preserving diffeomorphism $\phi : (S^4, \Delta_1) \to (S^4, \Delta_0)$ fixing $N(U)$ pointwise. Any 3-ball Δ_1 with boundary U restricts to a fiber of a fibration of $S^4 \setminus \operatorname{int}(N(U))$.*

Proof. i) This is [Ce1], [Pa] applied to 3-balls in S^4, together with uniqueness of regular neighborhoods and the fact that $\operatorname{Diff}_0(S^2)$ is connected. ☐

Theorem 9.8. *ψ induces isomorphisms between the following abelian groups.*
i) Isotopy classes of 3-balls in S^4 with boundary Δ_0
ii) isotopy classes of 3-balls in $S^1 \times B^3$ with boundary Δ_0
iii) $\pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$

Proof. Recall that by $\pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$ we mean isotopy classes of diffeomorphisms of $S^1 \times B^3$ fixing a neighborhood of $S^1 \times S^2$ pointwise modulo diffeomorphisms that are supported in a compact 4-ball.

To an element $[\phi] \in \pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$ we associate the 3-ball $\Delta = \phi(\Delta_0)$. It's isotopy class is well defined since $\operatorname{Diff}(B^4 \operatorname{fix} \partial)(\Delta_0) = \Delta_0$ up to isotopy. If $\Delta \subset S^1 \times B^3$ is a 3-ball with boundary Δ_0, then by Theorem 9.7 there exists a $\phi \in \operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial)$ with $\phi(\Delta_0) = \Delta$. If ϕ' is another such diffeormorphism and $\phi_0 = \phi \circ \phi'^{-1}$, then $\phi_0(\Delta_0) = \Delta_0$. By [Ce2] we can assume that after isotopy $\phi_0(\Delta_0) = \phi_0(\Delta_0)$ pointwise, and then also $\phi_0(N(\Delta_0)) = N(\Delta_0)$ pointwise. Thus ϕ is equivalent to ϕ' modulo $\operatorname{Diff}(B^4 \operatorname{fix} \partial)$. It follows that there is a 1-1 correspondence between ii) and iii).

Recall that the group structure on 3-balls in $S^1 \times B^3$ to be the one induced from $\pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$, so Δ_0 is the id and multiplication is given by concatenation. Use ψ to induce the group structure on isotopy classes of 3-balls in S^4 with boundary Δ_0 and hence the isomorphism between i) and ii).

Theorem 3.8 implies that these groups are abelian. ☐

Remark 9.9. Theorem 1.10 of [Ga1] proves the uniqueness of spanning 2-discs in S^4, i.e. two discs with the same boundary are isotopic rel boundary. The existence of knotted

3-balls in S^4, i.e. the non uniqueness of spanning 3-discs in S^4, negatively answers Question 10.13 of [Ga1] for $k = 3$. The second author also conjectured that knotted 3-balls exist [Ga3].

Corollary 9.10. *There exist fiberings of the unknot $U \subset S^4$ not isotopic to the linear fibering. The set of isotopy classes of fiberings forms an infinitely generated abelian group.* $\square$

The four dimensional smooth Schönflies conjecture is that smooth 3-spheres in the 4-sphere are smoothly isotopically standard. In 1959 Mazur [Ma1] showed that such spheres are topologically standard. More generally Brown [Br] and Morse [Mo] showed that locally flat 3-spheres in S^4 are topologically standard. The corresponding conjecture is known to be true smoothly for all dimensions not equal to four.

The following is essentially stated in [Ga1] as Remark 10.14.

Theorem 9.11. *The following are equivalent:*

i) The Schönflies conjecture is true.

ii) For every 3-ball Δ in S^4 with $\partial\Delta = \partial\Delta_0$, there exists $n \in \mathbb{N}$ such that the lift of Δ to the n-fold cyclic branched cover of S^4, branched over $\partial\Delta$ is isotopic to Δ_0 rel $\partial\Delta$.

iii) For every non-separating properly embedded 3-ball Δ in $S^1 \times B^3$, there exists $n \in \mathbb{N}$ such that the lift of Δ to the n-fold cyclic cover of $S^1 \times B^3$ is isotopically standard.

Proof. Let $\Sigma_0 \subset S^4$ be an unknotted 3-sphere. If $\Sigma_1 \subset S^4$ is an embedded 3-sphere that coincides with Σ_0 in a neighborhood of a 3-disc, then Σ_1 is isotopic to Σ_0 if and only if there exists an isotopy that also fixes a neighborhood of the 3-disc pointwise.

We now show that i) implies ii). Let $\Delta_0' \subset S^4$ be the linear 3-ball such that $\Delta_0 \cap \Delta_0' = \partial\Delta_0$ and $\Sigma_0 = \Delta_0 \cup \Delta_0'$ is a smooth 3-sphere. Let Δ be a 3-ball in S^4 that coincides with Δ_0 near $\partial\Delta_0$. By passing to a sufficiently high odd degree branched cover over $\partial\Delta_0$ we can assume that there are preimages $\tilde\Delta_0, \tilde\Delta, \tilde\Delta_0'$ of $\Delta_0, \Delta, \Delta_0'$ such that $\tilde\Delta_0$ coincides with $\tilde\Delta$ near $\partial\tilde\Delta_0$, $\mathrm{int}(\tilde\Delta) \cap \tilde\Delta_0' = \emptyset$ and $\tilde\Sigma_0 = \tilde\Delta_0' \cup \tilde\Delta_0$ is a smooth, necessarily unknotted, 3-sphere in S^4. Now $\tilde\Sigma_1 = \tilde\Delta_0' \cup \tilde\Delta_1$ is another 3-sphere in S^4. It follows from the previous paragraph that $\tilde\Sigma_1$ is isotopic rel $\tilde\Delta_0'$ to $\tilde\Sigma_0$ and so $\tilde\Delta$ is isotopic to $\tilde\Delta_0$ rel $\partial\tilde\Delta_0$ and hence is unknotted.

Now we show that ii) implies i). Let Σ_1 be a 3-sphere in S^4. We can assume that it coincides with the unknotted 3-sphere Σ_0, constructed above, near the 3-disc Δ_0'. Let Δ be the closed 3-disc in Σ_1 complementary to Δ_0'. By hypothesis, Δ becomes unknotted in a n-fold branched cover of S^4 branched over $\partial\Delta_0$, given by $p : S^4 \to S^4$. In this cover let $\tilde\Delta$ be a preimage of Δ, $\tilde\Delta_0$ be the preimage of Δ_0 that coincides with $\tilde\Delta$ near their common boundary, and $E_1, E_2, \cdots, E_n$ be the preimages of Δ_0' cyclically ordered about $\partial\Delta_0$ with $\tilde\Delta_0 \cup \tilde\Delta$ lying in the region W bounded by $E_n \cup E_1$ that contains no other E_i's. If $\tilde\Delta$ is isotopic to $\tilde\Delta_0$ via an isotopy supported in W, then by composing with p we obtain an isotopy between Δ and Δ_0 supported away from Δ_0' and hence one between Σ_1 and Σ_0. Since the isotopy from $\tilde\Delta$ to $\tilde\Delta_0$ is supported away from $\partial\Delta_0$ it follows that when lifted to the infinite cyclic branched cover the support of the isotopy hits only finitely many preimages of $\tilde\Delta_0'$. Thus the original n could have been chosen so that the support of the isotopy is disjoint from some E_i, $i \neq 1, n$. Since the region between E_1 and E_i (resp. E_i and E_n) is a relative product, the isotopy can be modified to be supported in W.

The equivalence of ii) and iii) follows from Theorem 9.7. $\square$

Remark 9.12. An unpublished consequence of [Ma2] due to Barry Mazur and rediscovered in conversations between the second author and Toby Colding is the following. If the Schönflies conjecture is false, then there exists an $f \in \mathrm{Diff}_0(S^1 \times S^3)$ such that $f(x_0 \times S^3)$ is isotopically non standard even after lifting to any finite sheeted covering of $S^1 \times S^3$.

Definition 9.13. We say that the 3-ball $\Delta \subset S^1 \times B^3$ is *virtually unknotted* if it becomes unknotted after lifting to some finite cover of $S^1 \times B^3$.

Proposition 9.14. *If $f : S^1 \times B^3 \to S^1 \times B^3$ is the result of finitely many pairwise disjoint barbell implantations each possibly raised to some power in $\mathbb{Z}$, then for $n \in \mathbb{N}$ sufficiently large the lift f_n to the n-fold cyclic cover is isotopic to* id *modulo* $\mathrm{Diff}(B^4 \operatorname{fix} \partial)$.

Proof. For n sufficiently large f_n is homotopic to the composition of finitely many maps each supported in a 4-ball. ☐

Corollary 9.15. *Every knotted 3-ball Δ arising from finitely many barbell implantations is virtually unknotted.* ☐

Proposition 9.16. *For each $k \in \mathbb{N}$ there exist a barbell $\mathcal{B}_k \subset S^1 \times B^3$ whose preimage in the k-fold cover is the disjoint union of k copies of $\mathcal{B}(\delta_4)$ and hence $\Delta_{\mathcal{B}_k}$ lifts to a knotted 3-ball in the k-fold cover.*

Proof. One readily constructs $\mathcal{B}_k$. The implantation of the preimage is $(\beta_{\delta_4})^k$. Since $W_3((\beta_{\delta_4})^k) = kW_3(\delta_4) \neq 0$ by Theorem 8.3, the result follows. ☐

10. More Applications

In the next proposition we list some consequences of Proposition 3.7 and Lemma 3.9. We list the consequences in dimension four, although as we see in the proof, all these statements have high-dimensional analogues.

Theorem 10.1.
(1) *$\pi_0 \mathrm{Emb}(B^2, S^2 \times B^2) \simeq \mathbb{Z}$, and this is an isomorphism under the concatenation operation. The group $\pi_1 \mathrm{Emb}(B^2, S^2 \times B^2)$ is free abelian group of rank two. More generally,*

$$\pi_k \mathrm{Emb}(B^2, S^2 \times B^2) \simeq \pi_{k+1} \mathrm{Emb}(B^1, B^4) \times \pi_k \Omega^2 S^2.$$

(2) *$\pi_0 \mathrm{Emb}(B^3, S^1 \times B^3)$ is an abelian group with the concatenation operation. Moreover it contains an infinitely generated free subgroup.*
(3) *$\mathrm{Emb}_u(B^2, B^4)$ is connected, with $\pi_1 \mathrm{Emb}_u(B^2, B^4)$ containing an infinitely generated free subgroup.*
(4) *$\mathrm{Emb}_u(S^2, S^4)$ is connected, with $\pi_1 \mathrm{Emb}_u(S^2, S^4)$ containing an infinitely generated free subgroup.*

Proof. **(1)** As we have seen, when $n \geq 4$,

$$\mathrm{Emb}(B^1, B^n) \times \Omega S^{n-2} \simeq B\, \mathrm{Emb}(B^2, S^{n-2} \times B^2).$$

The first non-trivial homotopy group of $\mathrm{Emb}(B^1, B^n)$ is known to be $\pi_{2n-6} \mathrm{Emb}(B^1, B^n) \simeq \mathbb{Z}$, generated by the *Haefliger trefoil* [Bu2]. In [Bu2] the space $\mathrm{Emb}(B^1, B^n)$ is denoted $\mathcal{K}_{n,1}$. The first non-trivial homotopy group of ΩS^{n-2} is $\pi_{n-3} \Omega S^{n-2} \equiv \pi_{n-2} S^{n-2} \simeq \mathbb{Z}$.

(2) Our technique for showing the (families) of diffeomorphisms of $S^1 \times B^n$ are non-trivial factors through the fibration $\mathrm{Diff}(S^1 \times B^n \text{ fix } \partial) \to \mathrm{Emb}(B^n, S^1 \times B^n)$.

(3) See Corollary 3.11, when $n \geq 4$ we have $\mathrm{Emb}_u(B^{n-2}, B^n) \simeq B\,\mathrm{Emb}(B^{n-1}, S^1 \times B^{n-1})$.

(4) There is a homotopy-equivalence [Bu2]

$$\mathrm{Emb}(S^j, S^n) \simeq SO_{n+1} \times_{SO_{n-j}} \mathrm{Emb}(B^j, B^n).$$

The simplest way to think of this is to consider elements of $\mathrm{Emb}(B^j, B^n)$ as smooth embeddings $\mathbb{R}^j \to \mathbb{R}^n$ that restricts to the standard inclusion $x \to (x, 0)$ outside of the ball B_j. One can conjugate such embeddings via a stereographic projection map, converting the embeddings $\mathbb{R}^j \to \mathbb{R}^n$ to embeddings $S^j \to S^n$ that are standard on a hemisphere. One can then post-compose such an embedding with an isometry of S^n. We are in the fortunate circumstance where the homotopy-fiber of the map $SO_{n+1} \times \mathrm{Emb}(B^j, B^n) \to \mathrm{Emb}(S^j, S^n)$ can be identified, and it is the orbits of the SO_{n-j}-action, acting diagonally on the product.

Thus $\mathrm{Emb}_u(S^2, S^4)$ is a bundle over SO_5/SO_2 with fiber $\mathrm{Emb}_u(B^2, B^4)$. $\pi_2 SO_5/SO_2 \simeq \mathbb{Z}$ and this group maps isomorphically to the subgroup of index two in $\pi_1 SO_2$, which maps to zero in $\pi_1 \mathrm{Emb}_u(B^2, B^4)$, so our map $\pi_1 \mathrm{Emb}_u(B^2, B^4) \to \pi_1 \mathrm{Emb}_u(S^2, S^4)$ is injective. □

Remark 10.2. The first sentence of Conclusion (1) is Theorem 10.4 of [Ga1]. The proof here is different, generalizable and arguably more direct.

Allen Hatcher's proof of the Smale Conjecture [Ha1] together with his and Ivanov's work on spaces of incompressible surfaces [HI] has as the consequence that the component of the unknot in the embedding space $\mathrm{Emb}(S^1, S^3)$ has the homotopy type of the subspace of great circles, i.e. the unit tangent bundle $UTS^3 \simeq S^3 \times S^2$. From the perspective of the homotopy-equivalence $\mathrm{Emb}(S^1, S^3) \simeq SO_4 \times_{SO_2} \mathrm{Emb}(B^1, B^3)$ this is equivalent to saying the unknot component of $\mathrm{Emb}(B^1, B^3)$ is contractible, $\mathrm{Emb}_u(B^1, B^3) \simeq \{*\}$.

In dimension four we do not know the full homotopy type of $\mathrm{Diff}(S^4)$, although there is the recent progress of Watanabe [Wa1] where he shows the rational homotopy groups of $\mathrm{Diff}(S^4)$ do not agree with those of O_5. In this regard, this paper asserts the the analogy to Hatcher and Ivanov's spaces of incompressible surfaces results [HI] are also false in dimension 4, in particular contrast with the theorem $\mathrm{Emb}(B^2, S^1 \times B^2) \simeq \{*\}$.

Hatcher and Wagoner [HW] (see Cor. 5.5) have computed the mapping class group of $S^1 \times B^n$ for a range of n. Specifically

$$\pi_0 \mathrm{Diff}(S^1 \times B^n \text{ fix } \partial) \simeq \Gamma^{n+1} \oplus \Gamma^{n+2} \oplus \left(\bigoplus_{\infty} \mathbb{Z}_2 \right)$$

provided $n \geq 6$. The Hatcher-Wagoner diffeomorphisms survive the map $\pi_0 \mathrm{Diff}(S^1 \times B^n \text{ fix } \partial) \to \pi_0 \mathrm{Diff}(S^1 \times S^n)$, as do the diffeomorphisms we prove are isotopically non-trivial.

11. Conjectures and Questions

Conjecture 11.1. *The map $p : \pi_1 \operatorname{Emb}(S^1, S^1 \times S^3; S^1_0) \to \pi_0(\operatorname{Diff}(S^1 \times B^3 \operatorname{fix} \partial) / \operatorname{Diff}(B^4 \operatorname{fix} \partial))$ induced by isotopy extension has kernal the subgroup with $W_2 = 0$. In particular the implantations $\beta_{\theta_k}, k \geq 2$ and $\beta_{\alpha_k}, k \geq 1$ are isotopically nontrivial.*

We thank Maggie Miller for bringing to our attention the following question.

Question 11.2. *If Δ_1 is a knotted 3-ball in S^4 and T_0, T_1 are respectively obtained from Δ_0, Δ_1 by attaching a small 3-dimensional 1-handle h, then is T_1 knotted, i.e. is not isotopic rel ∂T_1 to T_0? Does Δ_1 become unknotted after finitely many such stabilizations?*

Conjecture 11.3. *For each $g \geq 0$ there exists 3-dimensional genus-g handlebodies $V_0, V_1 \subset S^4$ such that $\partial V_0 = \partial V_1$ and the set of V_0-compressible simple closed curves in ∂V_0 coincides with that of V_1, but V_1 is not isotopic to V_0 via an isotopy that fixes ∂V_1.*

Question 11.4. *i) Determine $\pi_0 \left(\operatorname{Diff}(S^1 \times B^n \textit{ fix } \partial) / \operatorname{Diff}(B^{n+1} \textit{ fix } \partial)\right)$ for $n = 4, 5$. ii) Do all elements become trivial after lifting to the 2-fold cover of $S^1 \times B^n$.*

Remark 11.5. As already noted Hatcher has determined such groups for $n \geq 6$ and all its elements become trivial passing to 2-fold covers. On the other hand by Proposition 9.16 some elements remain nontrivial when passing to such covers, when $n = 3$.

The following is a restatement of a special case of Lemma 3.9, where as explained there, the isomorphism is given by *slicing the embedding*. Here notation is as in Theorem 10.1.

Theorem 11.6. $\pi_0 \operatorname{Emb}(B^3, S^1 \times B^3) \simeq \pi_1 \operatorname{Emb}_u(B^2, B^4)$.

Problem 11.7. Use this to prove or disprove the Schönflies conjecture, e.g. to prove the conjecture show that loops in $\operatorname{Emb}_u(B^2, B^4)$ are *homotopically* trivial where at discrete times of the homotopy one can do geometric moves corresponding to passing to finite sheeted coverings. Such a move might change the homotopy class and may arise from certain *reimbeddings* of the loop.

Question 11.8. *Does there exists a barbell $\mathcal{B} \subset S^4$ whose implantation represents a nontrivial element of $\pi_0 \operatorname{Diff}_0(S^4)$?*

12. Appendix: G(p,q) is a Whitehead product

The goal of this section is to prove the following result.

Theorem 12.1. *Let I_0 denote the standard oriented properly embedded arc in $S^1 \times B^3$. Under the homomorphism $W_3 : \pi_2 \operatorname{Emb}(I, S^1 \times B^3; I_0) \to \pi_5(C_3(S^1 \times B^3, I_0))/R$, then up to sign independent of p and q, $W_3(G(p,q)) = t_1^p t_2^q [w_{13}, w_{23}]$.*

This section gives a detailed proof of the theorem. It should be noted that Kosanovic has a general technique of this sort. Specifically, the Taylor tower $L_k \to T_k \to T_{k-1}$ associated to an embedding space has its associated homotopy long exact sequence, which gives us short exact sequence for computing $\pi_i T_k$. This group is an extension of a subgroup of $\pi_i T_{k-1}$ by a quotient group of $\pi_i L_k$. Kosanovic computed an explicit map from the quotient group of $\pi_i L_k$ to the kernel of $\pi_i \operatorname{Emb}(I, M) \to \pi_i T_{k-1}$ for the smallest i with $\pi_i L_k$ non-trivial [Ko3] in the case of the embedding spaces of the form $\operatorname{Emb}(I, M)$. In our case this map gives precisely the description of the closure as a linear combination of Whitehead brackets.

Remark 12.2. While stated for $S^1 \times B^3$ the analogous statement and proof holds for any orientable 4-manifold M. In the general case J_0 is an oriented properly embedded arc in M with $\{a_1, a_2, a_3\} \subset J_0$, $a_1 < a_2 < a_3$, the basepoint for $C_3(M)$. In the statement below λ_1, λ_2 are paths from a_1, a_2 to a_3 respectively and $G(\lambda_1, \lambda_2)$ is defined exactly like $G(p, q)$ except that λ_1, λ_2 represent elements of $\pi_1(M; J_0)$.

Theorem 12.3. *Let J_0 denote an oriented properly embedded arc in the oriented 4-manifold M. Under the homomorphism $W_3 : \pi_2 \operatorname{Emb}(I, M; I_0) \to \pi_5(M, J_0)/R$, then up to sign independent of λ_1 and λ_2, $W_3(G(\lambda_1, \lambda_2)) = t_1^{\lambda_1} t_2^{\lambda_2}[w_{13}, w_{23}]$.*

Idea of Proof of Theorem 12.1: We are motivated by the mapping space model of Sinha [Si1] that was derived from the work of Goodwillie, Klein and Weiss, e.g. see [GKW]. $G(p, q)$ induces a map $C_3\langle I\rangle \times I^2 \to C_3\langle S^1 \times B^3\rangle$ which is a map of the 5-ball into $C_3\langle S^1 \times B^3\rangle$. By closing off the boundary facets we will essentially modify this map to an element of $\pi_5(C_3\langle S^1 \times B^3\rangle)$ which is rationally generated by Whitehead products and under our map, up to sign, $G(p, q)$ is taken to $t_1^p t_2^q [w_{13}, w_{23}]$. We close off the boundary facets essentially subject to the constraints of the mapping space model, thus our element of π_5 is well defined up to other elements that might be obtained by closing off with the same constraints. An element z of $\pi_5(C_3(S^1 \times B^3))$ modulo those elements is by definition $W_3(z)$.

Notation 12.4. $C_n(M)$ denotes the *configuration space* of distinct ordered n-tuples of M and $C_n\langle M\rangle$ denotes the quotient of the Fulton - McPherson compactification as defined in Definition 4.1 of [Si1], though there it is denoted by $C_n\langle [M]\rangle$. In this section $n = 3$ and M is one of I or $S^1 \times B^3$. For our purposes it suffices to consider the connected component of $C_3\langle I\rangle$ which is the simplex $\{(x, y, z) | 0 \leq x \leq y \leq z \leq 1\}$, so from now on $C_3\langle I\rangle$ will denote that simplex.

Definition 12.5. We continue to view $S^1 \times B^3$ as $D^2 \times S^1 \times [-1, 1]$ with I_0 a geodesic arc through the origin of $D^2 \times \{x_0\} \times \{0\}$. We fix an identification of $[0, 1]$ with I_0 by $1_{I_0} : I \to I_0$ and will frequently implicitly identify one with the other. In particular, we abuse notation by having $(a_1, a_2, a_3) = (1/4, 1/2, 3/4) \in I_0$ denote the *basepoint* for both $C_3(I)$ and $C_3(S^1 \times B^3)$. Let $U_1 = [j_0, j_1], U_2 = [k_0, k_1], U_3 = [g_0, g_1]$ be disjoint closed intervals in $(0, 1)$ containing a_1, a_2, a_3 in their interiors. We color U_1, U_2 and U_3 *blue, red* and *green* respectively. See Figure 60 a).

We introduced $G(p, q)$ in Figure 19 a) as the bracket (B_p, R_q). To minimize notation we drop the subscripts. In the next definition we specify B, R and their end homotopies more precisely and denote the resulting bracket by $\alpha_{t,u} \in \Omega\Omega \operatorname{Emb}(I, S^1 \times B^3; I_0)$.

Definition 12.6. We represent (B, R) by the chord diagram pair shown in Figure 60 b) and equivalently in band lasso notation in Figure 60 c). The positive chords (resp. negative chords) spin around $[g_{.12}, g_{.20}] \subset U_3$ (resp. $[g_{.80}, g_{.88}] \subset U_3$); the corresponding

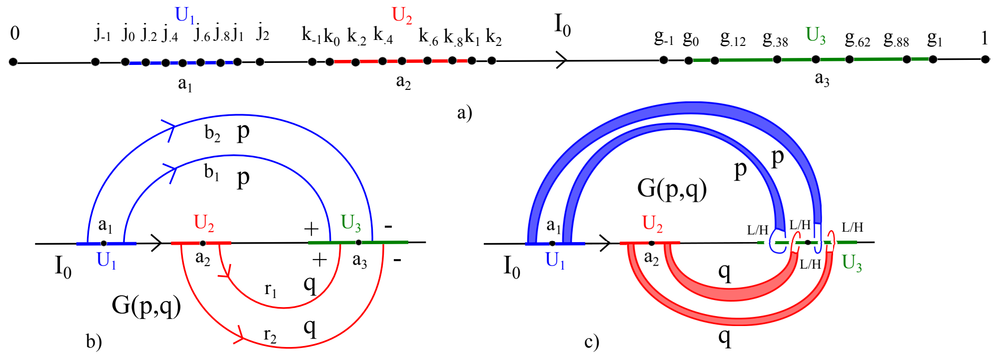


FIGURE 60.

lasso discs intersect U_3 at $g_{.15}$ and $g_{.18}$ (resp. $g_{.82}$ and $g_{.85}$). The blue (resp. red) chords are called b_1, b_2 (resp. r_1, r_2) and the spinnings about these chords are respectively supported in the domain in $[j_{.6}, j_{.8}]$ and $[j_{.2}, j_{.4}]$ (resp. $[k_{.6}, k_{.8}]$ and $[k_{.2}, k_{.4}]$). The end homotopies for B and R are the *undo homotopies* ν_b, ν_r of Definition 12.8. We represent the resulting bracket (B, R) by $\alpha_{t,u} \in \Omega\Omega\,\mathrm{Emb}(I, S^1 \times B^3; I_0)$.

We define the *left shifted* (resp. right shifted) mappings B^L, R^L (resp. B^R, R^R) as in Figure 61. Here the ends of b_2 and r_2 (resp. b_1, r_1) have been isotoped so that they spin about U_3 to the left (resp. right) of a_3 close to the subinterval $[g_{.12}, g_{.38}]$ (resp. $[g_{.62}, g_{.88}]$) around which b_1, r_1 (resp. b_2, r_2) already spin.

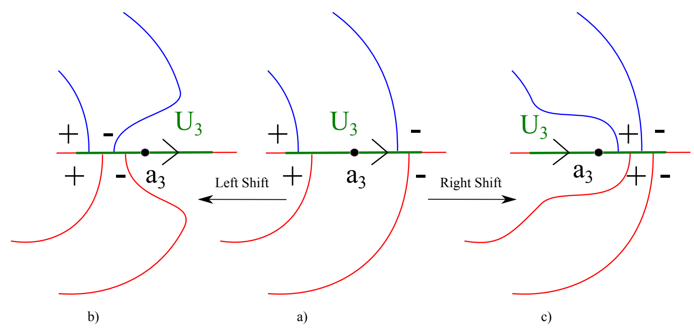


FIGURE 61.

Remark 12.7. The range support of a shift is contained in a small neighborhood of U_3.

We now describe homotopies in $\Omega\,\mathrm{Maps}(I, S^1 \times B^3; I_0)$ of B and R and their L and R shifted versions to the constant map to 1_{I_0}.

Definition 12.8. The *back track* homotopy is the homotopy in $\Omega\,\mathrm{Maps}(I, M; I_0)$ of a spinning to 1_{I_0} corresponding to withdrawing the band and lasso and the *undo* homotopy is defined in Definition 4.7. We now elaborate on these and their variants in our current context.

The *blue back track homotopy* is the homotopy β^B of B to the constant 1_{I_0} as shown in Figure 62. Each element of the homotopy is a loop of immersed intervals that are embeddings when restricted to U_1, though there may be intersections with U_3. The first part of the homotopy contracts the lasso 3-ball to the arc at the top of the band, so at that moment β_t^B is a loop that sends an arc to the top of the band and then withdraws it. In a similar manner we define the *red back track homotopy* β^R as well as the back track homtopies for B^L, B^R etc.

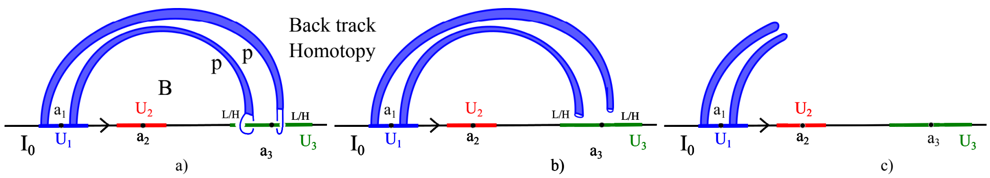


FIGURE 62.

The *blue undo* homotopy ν^B, a homotopy in $\Omega\,\mathrm{Emb}(I, S^1 \times B^3; I_0)$, is defined in Figures 63 a) - g) via continuous transformations of bands, lassos and lasso discs. Figure 63 a) shows $\nu_0^B = B$. To go from ν_0^B to $\nu_{.25}^B$ we zip up the bands to a single one from which two lassos κ_0, κ_1 simultaneously emanate as in Figure 63 b). Figure 63 c) is a detail of 34 b). Note that the original lasso discs D_0, D_1 which are also the lasso discs for κ_0 and κ_1 each intersect U_3 in one point. To go from $\nu_{.25}^B$ to $\nu_{.4}^B$ we zip up the lasso discs to obtain a single lasso disc D and lasso κ. Here D contains D_0 and D_1 as subdiscs, the zipping is disjoint from U_3 and D intersects U_3 in two points. See Figure 63 d). To obtain $\nu_{.5}^B$ we properly isotope D to be disjoint from U_3. See Figure 63 e). This isotopy induces an isotopy of the lasso sphere. The key point is that the lasso spheres remain disjoint from U_3 throughout the isotopy, since at each moment the interior of one hemisphere is in the future while the other is in the past. Finally, we withdraw the lasso and bands as in the back track homotopy. In a similar manner we define the *red undo* homotopy as well as the undo homotopies for B^L, B^R etc.

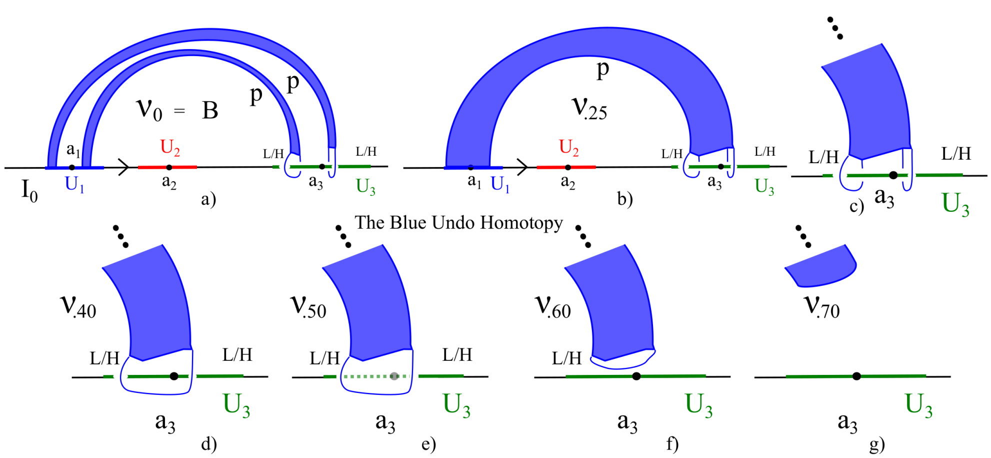


FIGURE 63.

In a natural way define the *transition homtopy* T which is a homotopy between the back track and undo homotopies. It's support is in a small neighborhood of the supports of the undo and back track homotopies. We denote by T^B (resp. T^R) the transition homotopy for B (resp. R).

Summary of Intersections: We now catalogue the intersections and self-intersections of the immersions $I \to S^1 \times B^3$ that arise from evaluating at parameter points the homotopies $\beta^B, \beta^R, \nu^B, \nu^B, T^B, T^R$, as well as their L and R shifted versions.

1) In the domain B is supported in U_1 as are all the homotopies of B. All homotopies of B are homotopies of loops of embeddings when restricted to U_1. R is supported in U_2 as are all the homotopies of R. All homotopies of R are homotopies of loops of embeddings when restricted to U_2.

Call a double point of a given map or between two maps of $I \to S^1 \times B^3$ to be of U_i/U_j *type* if it involves the image of a point of U_i intersecting the image of one from U_j.

2) Double points between $\nu^B(e,f)$ and $\nu^R(e',f')$ are only of type U_1/U_2, similarly for their shifted versions. ν^B and ν^R and their shifted versions are homotopies of loops of embeddings.

3) Double points of a $\beta^B(e,f)$ are only of type U_1/U_3, similarly for its shifted versions. There are no U_1/U_2 double points between a $\beta^b(e,f)$ and a $\beta^R(e',f')$. Double points of a $\beta^R(e,f)$ are only of type U_2/U_3, similarly for its shifted versions

4) Double points involving transitional homotopies or their shifted versions are only of type $U_1/U_2, U_1/U_3$ or U_2/U_3.

5) Any U_3 double point involving an L shifted map (resp. R shifted map) occurs in $[g_{.12}, g_{.38}]$ (resp. $[g_{.62}, g_{.88}]$.

Definition 12.9. Define two open covers $\{U_L, U_R\}, \{U_\beta, U_\nu\}$ of $\partial C_3\langle I\rangle$ called the *standard decompositions* as indicated in Figure 64 a).

$U_L := \{(\omega_1,\omega_2,\omega_3) \in \partial C_3\langle I\rangle | \omega_3 > g_{-1}\}$

$U_R := \{(\omega_1,\omega_2,\omega_3) \in \partial C_3\langle I\rangle | \omega_3 < g_0\}$

$U_\beta := \{(\omega_1,\omega_2,\omega_3) \in \partial C_3\langle I\rangle | \omega_3 < k_2\} \cup (j_{-1}, j_2) \times (k_{-1}, k_2) \times 1$

$U_\nu := \{(\omega_1,\omega_2,\omega_3) \in \partial C_3\langle I\rangle | \omega_3 > k_1\} \setminus [j_0, j_1] \times [k_0, k_1] \times 1.$

For $(t,u) \in I^2$ we define two open covers of $\partial C_3\langle I\rangle \times (t,u)$ called the L-R and β-ν open covers. For the L-R cover, use the standard $\{U_L, U_R\}$ cover, independent of (t,u). Let $\epsilon > 0$ be very small and P as in Figure 63 b). For $t,u \in P \setminus N_\epsilon(\partial P)$ use the standard $\{U_\beta, U_\nu\}$ decomposition. For $(t,u) \in N_\epsilon(\partial P)$ define U_β as above and $U_\nu = \partial C_3\langle I\rangle$. For $(t,u) \notin P \cup N_\epsilon(\partial P)$, define $U_\nu = \partial C_3\langle I\rangle$ and $U_\beta = \emptyset$.

Remark 12.10. $P \setminus N_\epsilon(\partial P)$ contains all the parameters that are both blue and red and $I \times I \setminus P \cup N_\epsilon(P)$ contains the parameters corresponding to the end homtopies.

Definition 12.11. Let E^3 denote the 3-ball $C_3\langle I\rangle \cup (\partial C_3\langle I\rangle \times [0,1])$ and $E^5 := E^3 \times I^2$. Here points are denoted (ω, v, t, u) where $(t,u) \in I^2$, $\omega \in C_3\langle I\rangle$ and $v \in [0,1]$. Also, $v = 0$ unless $\omega \in \partial C_3\langle I\rangle$.

We define $\alpha : E^5 \to \text{Maps}(I, S^1 \times B^3; I_0)$. First, define $\alpha : C_3\langle I\rangle \times I^2 \to \text{Maps}(I, S^1 \times B^3; I_0)$ by $\alpha^\omega_{t,u} = \alpha_{t,u}$. Extend α to $(\partial C_3\langle I\rangle \times [0,1]) \times I^2$ as follows. For $v \in [0, 1/4]$, according to $\omega \in U_L$ or U_R, monotonically shift $\alpha_{t,u}$ to the left or right. So, if say $\omega \in U_L \setminus U_R$, then $\alpha^\omega_{.25,t,u}$ is totally shifted to the left. For $\omega \in U_L \cap U_R$ it is somewhere between the right and the left i.e. is *partially shifted*. Next, for $v \in [1/4, 1]$ homotope

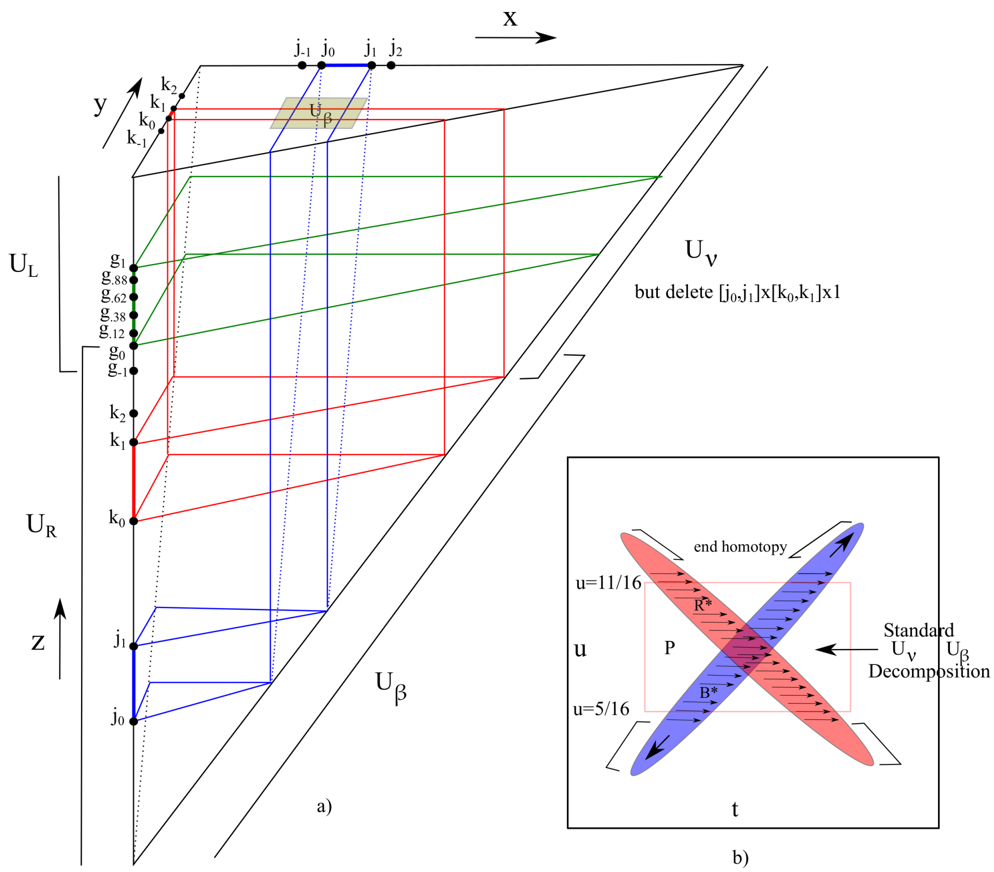


FIGURE 64.

$\alpha^\omega_{.25,t,u}$ to 1_{I_0} according to whether it is in U_β or U_ν. I.e. if $\omega \in U_\beta \setminus U_\nu$ (resp. $U_\nu \setminus U_\beta$) the back track (resp. undo) homotopy is used. In $U_\nu \cap U_\beta$ the transition homotopy is used.

Definition 12.12. For $i = 1, 2$ let $p_i : I \to U_i$ denote the retraction which fixes U_i pointwise and let p_3 the retraction to a_3. Letting $p = (p_1, p_2, p_3)$ we obtain $p : C_3\langle I\rangle \to C_3(I)$. Let $\gamma : I \to S^1 \times B^3$ be a proper immersion supported in the domain on $U_1 \cup U_2$ such that $\gamma|U_1$ and $\gamma|U_2$ are embeddings. Let $0 \le x_0 \le y_0 \le z_0 \le 1$ and $(x_1, y_1, z_1) := (p_1(x_0), p_2(y_0), a_3)$. Let x_s be the constant speed path in I from x_0 to x_1 with y_s and z_s similarly defined. We say that (x_0, y_0, z_0) is *γ-ready* if for all $s > 0, \gamma(x_s), \gamma(y_s), \gamma(z_s)$ are pairwise disjoint.

Lemma 12.13. *If γ is embedded, then any $(x, y, z) \in C_3\langle I\rangle$ is γ-ready.* $\square$

Notation 12.14. In what follows if $a, b \in I$, then $[a, b]$ denotes the closed interval, possibly a point, between a and b or b and a.

Proposition 12.15. *If $\omega \in \partial C_3\langle I\rangle$, then ω is $\alpha^\omega_{v,t,u}$ ready for all v, t, u.*

Proof. Case 1: $(t, u) \in P \setminus \text{int}(N_\epsilon(\partial P))$.

Proof of Case 1: Let $\omega = (x_0, y_0, z_0)$ and let α denote $\alpha^\omega_{v,t,u}$. If for some $s > 0$, $\alpha(x_s) = \alpha(y_s)$, then $x_s \neq y_s$ since these are unit speed paths with $x_0 \le y_0$ and $x_1 < y_1$. Therefore, either $x_s \in U_1$ and $y_s \in U_2 \cup U_3$ or $x_s \in U_2$ and $y_s \in U_3$. Similar statements hold for x_s, z_s and y_s, z_s. Thus it suffices to show that there are no U_i/U_j intersections involving two of x_s, y_s or z_s.

We first show that there is no such U_i/U_j intersection involving a z_s. Recall that $z_1 = a_3$. If $z_0 \geq a_3$, then no $z_s \in [z_0, a_3]$ is involved with a U_1/U_2 intersection. Since $\omega \in U_L \setminus U_R$ then z_s cannot be involved with a U_i/U_3 intersection. If $z_0 \in [k_1, a_3]$, then again $z_s \in [z_0, a_3]$ cannot be involved with a U_1/U_2 intersection. If $a_3 \geq z_0 \geq k_2$, then $\omega \in U_\nu \setminus U_\beta$, so $z_s \in [z_0, a_3]$ cannot be involved with a U_i/U_3 intersection. If $z_0 \in [k_1, k_2]$, then $\omega \in U_R \setminus U_L$ so $z_s \in [z_0, a_3]$ cannot be involved with a U_i/U_3 intersection. If $z_0 \leq k_1$, then $\omega \in U_\beta \setminus U_\nu$ so there are no U_1/U_2 intersections and since $\omega \in U_R \setminus U_L$, z_s cannot be involved with a U_i/U_3 intersection.

It remains to show that there is no U_i/U_j intersection involving both a y_s and an x_s. If $y_0 \geq k_1$, then $y_1 = k_1$ and hence $y_s \in [y_0, y_1]$ is not involved with a U_1/U_2 intersection. If $y_0 \geq k_2$, then $\omega \in U_\nu \setminus U_\beta$ so $y_s \in [y_0, y_1]$ is not involved with U_i/U_3 intersections. If $y_0 \leq k_2$, then so are x_0, x_1 and y_1 and hence $([y_0, y_1] \cup [x_0, x_1]) \cap U_3 = \emptyset$ so $y_s \in [y_0, y_1]$ cannot be involved with a U_i/U_3 intersection. If $y_0 \in ([0, k_0] \cup [k_1, k_2])$, then $[y_0, y_1] \cap \operatorname{int} U_2 = \emptyset$, so $y_s \in [y_0, y_1]$ cannot be involved with a U_1/U_2 intersection. If $x_0 \notin \operatorname{int}(U_1)$, then neither is x_1. Therefore if $y_0 \leq k_2$ and $x_0 \notin \operatorname{int}(U_1)$, then $y_s \in [y_0, y_1]$ cannot be involved with a U_1/U_2 intersection. If $y_0 \in [k_0, k_1]$ and $x_0 \in [j_0, j_1]$, then $y_0 = y_1, x_0 = x_1$. Being in $\partial C_3\langle I \rangle$ it follows that $z_0 = 1$ and hence $\omega \in U_\beta \setminus U_\mu$ which implies that they cannot be involved in a U_1/U_2 intersection. $\square$

Case 2: $(t, u) \notin P \setminus \operatorname{int}(N_\epsilon(\partial P))$.

Proof of Case 2 Here $\alpha | U_i = \mathrm{id}$ for $i = 1$ or 2. We will assume the case of $i = 2$, the other case being similar. Hence it suffices to show that there are no U_1/U_3 intersections. Such intersections are ruled out by the argument of Case 1. $\square$

Definition 12.16. Define $F_s : E^5 \to C_3\langle S^1 \times B^3 \rangle$ by
$$F_s(\omega, v, t, u) = \alpha^{\omega}_{v,t,u}((1-s)\omega + s(p(\omega))).$$

Definition 12.17. Define $J_{U_3} := U_3 \times [-1, 0] \subset D^2 \times S^1 \times [-1, 1]$, the *$U_3$-cohorizontal space* and for $s \in U_3, J_s := s \times [-1, 0]$, the *s-cohorizontal space*. Let f_1, f_2, f_3 be the coordinate functions for $F_1 : E^5 \to C_3(S^1 \times B^3)$. Call a point of $f_1^{-1}(J_{a_3})$ (resp. $f_2^{-1}(J_{a_3})$) a *blue cohorizontal* (resp. *red cohorizontal*) point.

Definition 12.18. As in Definition 4.21, $B, R \in \Omega \operatorname{Emb}(I, S^1 \times B^3; I_0)$ give rise to $F^B, F^R \in \Omega\Omega \operatorname{Emb}(I, S^1 \times B^3; I_0)$. Let S^B and S^R denote the parameter supports for F^B and F^R, which by convention lie in the blue and red regions of Figure 64 b). Let F^B_u denote $F^B | [0,1] \times u$. By reparametrizing $[0,1] \times [0,1]$ we will assume that for each u, length$(([0,1] \times u) \cap S^B) < \delta$, where δ is very small and S^B is contained in the δ neighborhood of the arc from $(1/8, 1/8)$ to $(7/8, 7/8)$. Similar statements hold for F^R, where the domain support of F^R is denoted S^R and is contained in the δ neighborhood of the arc from $(7/8, 1/8)$ to $(1/8, 7/8)$. Define the *blue slab* (resp. *red slab*) $= ((x, y, z), v) \in E^3$ such that $x \in U_1$ (resp. $y \in U_2$). Define $P_{z_0} = \{(x, y, z, v) \in E^3 | z = z_0\}$. Let π_E denote the projection of E^5 to E^3, $\pi_t : E^5 \to E^3 \times I \times I$ the projection to the first I factor, $\pi_v : E^5 \to E^3 \times I \times I$ the projection to the second I factor and $\pi_v : (\partial C^3 \times [0,1]) \times I^2 \to [0,1]$ the projection to the $[0,1]$ factor. Define π_{pr} to be the projection of $D^2 \times S^1 \times [-1,1] \to D^2 \times S^1 \times 0$.

Proposition 12.19. *$S_b := f_1^{-1}(J_{a_3})$ (the blue sphere) and $S_r := f_2^{-1}(J_{a_3})$ (the red sphere) are disjoint 2-spheres whose union is the standard Hopf link in B^5. If β is a path from ∂B^5 to S_b (resp. S_r), then $f_1(\beta)$ (resp. $f_2(\beta)$) represents the class $p \in \pi_1(S^1 \times B^3; I_0)$ (resp. $q \in \pi_1(S^1 \times B^3; I_0)$).*

Remark 12.20. Note that $S_b = F_1^{-1}(J_{a_3} \times (S^1 \times B^3) \times (S^1 \times B^3))$ and $S_r = F_1^{-1}((S^1 \times B^3) \times J_{a_3} \times (S^1 \times B^3))$.

Proof. Let $K_u = S_b \cap E^3 \times [0,1] \times u$ and $L_u = S_r \cap E^3 \times [0,1] \times u$. The Proposition follows from the following seven steps.

Step 1: If $(\omega, v, t, u) \in S_b \cup S_r$ where $\omega = (x, y, z)$, then either $v > 0$ or $u \notin [1/4, 3/4]$. Also either

i) $v < 1/4$, in which case $\alpha^\omega_{v,t,u}$ is a partially shifted $\alpha^\omega_{t,u}$ or

ii) $z \in [g_{-1}, g_0]$, in which case $\alpha^\omega_{v,t,u}$ is obtained from either a shifted or partially shifted $\alpha^\omega_{t,u}$ or its image under the undo homotopy.

Proof of Step 1: By construction, the lasso spheres of B and R and those of the undo homotopy project under π_{pr} to their lasso discs. The B and R lasso discs are disjoint from a_3 and their lasso discs and those of the undo homotopies are disjoint when fully shifted left or right. It follows that if $u \in [.25, .75]$, then $(S_b \cup S_r) \cap C_3\langle I\rangle \times I^2 = \emptyset$. When $z \notin [g_{-1}, g_0]$ a similar argument proves shows that $v < 1/4$. $\square$

Step 2: We can assume the following.

i) Under the left shift the b_2 (resp. r_2) lasso disc intersects a_3 when $v = .1$ (resp. $v = .2$) and under the right shift the r_1 (resp. b_1) lasso disc intersects a_3 when $v = .1$ (resp. $v = .2$).

ii) Viewing $B \in \Omega\,\mathrm{Emb}(I, S^1 \times B^3; I_0)$ as a map $[0,1] \times U_1 \to S^1 \times B^3$, then the projection of $B^{-1}(U_3)$ to $U_1 = \{j_{.3}, j_{.7}\}$. Similarly the U_2 projection of $R^{-1}(U_3)$ equals $\{k_{.3}, k_{.7}\}$.

iii) Viewing ν^B as a map $[0,1] \times [0,1] \times U_1 \to S^1 \times B^3$ where $\nu^B_0 = B$ and ν^B_1 is the constant map to id_{U_1}, then the projection of $\nu_u^{B^{-1}}(U_3)$ to U_1 consists of two, one or zero points. Here ν^B_u denotes $\nu^B | u \times [0,1] \times U_1$. In the former case the lower (resp. higher) point in U_1 is monotonically increasing (resp. decreasing) and equals $j_{.5}$ when there is exactly one point. An analogous statement holds for ν^R.

iv) The lasso discs of ν^B_u intersect U_3 in two, one or zero points. When there are two points the lower (resp. higher) point in U_3 is monotonically increasing (resp. decreasing) in u. An analogous statement holds for ν^R_u.

v) Near $\partial E_3 \times I^2$, $f_1(x, y, z, v, t, u) = p_1(x)$ and $f_2(x, y, z, v, t, u) = p_2(y)$. $\square$

Step 3: $K_u \cap P_z$ is independent of $u \in [1/4, 3/4]$ and is $\emptyset$ when $z < j_{.7}$, is one point when $z = j_{.7}$, is two points when $z \in (j_{.7}, 1)$ and an interval when $z = 1$. Furthermore,

i) If $z = j_{.7}$, then $K_u \cap P_z = (j_{.7}, j_{.7}, j_{.7}, .2)$

ii) If $z \in (j_{.7}, g_{-1})$, then $K_u \cap P_z = (j_{.7}, y_z, z, .2)$, where $y_z = \{j_{.7}, z\}$

iii) if $z \in [g_{-1}, g_0]$, then $K_u \cap P_z = (j_z, y_z, z, v_z)$, where $j_z \in [j_{.3}, j_{.7}]$ is non increasing and $y_z = \{j_z, z\}$

iv) if $z \in (g_0, 1)$, then $K_u \cap P_z = (j_{.3}, y_z, z, .1)$, where $y_z = \{j_{.3}, z\}$

v) if $z = 1$, then $K_u \cap P_1 = (j_{.3}, y_1, 1, .1)$, where $y_1 = [j_{.3}, 1]$.

An analogous statement holds for $L_u \cap P_z$. $\square$

Step 4: $f_1^{-1}(J_{a_3}) \cap f_2^{-1}(J_{a_3}) = \emptyset$.

Proof of Step 4: If $u \notin [1/4, 3/4]$ and $t \in [0,1]$, then $\alpha^{\omega}_{v,t,u} = \mathrm{id}$ when restricted to one of U_1 or U_2 and hence Step 4 follows. Now suppose that $u \in [1/4, 3/4]$. If $\omega \in C_3\langle I\rangle$, then apply Step 1 to conclude $(K_u \cup L_u) \cap (\omega, 0, t, u) = \emptyset$. If $\omega = (x_0, y_0, z_0) \in \partial C_3\langle I\rangle$, then $(\omega, v, t, u) \in K_u$ (resp. L_u) implies $x_0 \in \mathrm{int}(U_1)$ (resp. $y_0 \in \mathrm{int}(U_2)$). If $z_0 \geq g_0$, then $\omega \in U_L \setminus U_R$ and hence by Step 2, $K_u \subset \pi_v^{-1}(.1)$ while $L_u \subset \pi_v^{-1}(.2)$. If $z_0 \leq g_{-1}$, then $\omega \in U_R \setminus U_L$ and hence by Step 2, $K_u \subset \pi_v^{-1}(.2)$ while $L_u \subset \pi_v^{-1}(.1)$. If $z_0 \in [g_{-1}, g_0]$, then either $x_0 = 0$ or $x_0 = y_0$ or $y_0 = z_0$. In all three cases at least one of $x_0 \cap \mathrm{int}(U_1) = \emptyset$ or $y_0 \cap \mathrm{int}(U_2) = \emptyset$ holds and hence $K_u \cap L_u = \emptyset$. $\square$

Step 5: For $u \in [1/4, 3/4]$, $\pi_E(K_u \cup L_u)$ is independent of u and equal to a Hopf link. Also $\pi_t(K_u) \subset N_\delta(u)$ and $\pi_t(L_u) \subset N_\delta(1-u)$.

Proof of Step 5: The invariance and the Hopf link property follow by Steps 2 and 3. Figure 65 b) shows $\pi_E(K_u \cup L_u)$, a Hopf link within the blue and red slabs. If $(\omega, v, t, u) \in S_b$, then $(t, u) \in S^B$ and hence $\pi_t(\omega, v, t, u) \in N_\delta(u)$. If $(\omega, v, t, u) \in S_r$, then $(t, u) \in S^R$ and hence $\pi_t(\omega, v, t, u) \in N_\delta(1-u)$. $\square$

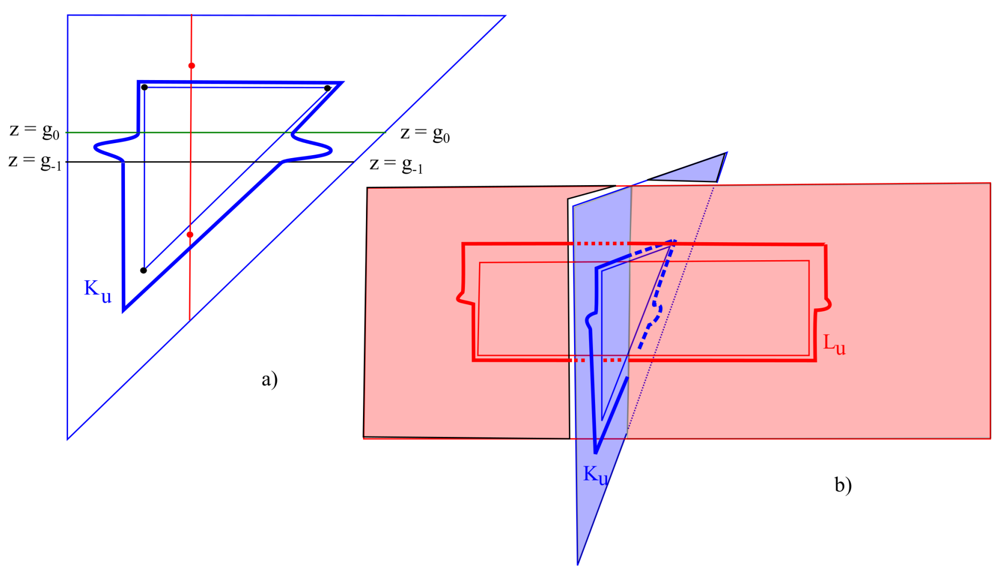


FIGURE 65.

Step 6: The surface $D_{1/4} := S_b \cap E^3 \times ([0,1] \times [0,1/4]) =$ is a disc that after proper isotopy in $E^3 \times ([1/8, 3/8] \times [0, 1/4])$ projects under π_E to a spanning disc for $K_{1/4}$. Analogous statements hold for $K_{3/4}$, $L_{1/4}$ and $L_{3/4}$.

Proof of Step 6: This is a long but routine exercise along the lines of Step 3. Here one computes the various intersections of $D_{1/4}$ with each $P_z \cap I^2$ and shows that after a small proper isotopy to eliminate arcs projecting to points, the π_E projection of $D_{1/4}$ to E_3 is embedded. $\square$

Step 7: If β is a path from ∂B^5 to S_b (resp. S_r), then $f_1(\beta)$ (resp. $f_2(\beta)$) represents the class $p \in \pi_1(S^1 \times B^3)$ (resp. $q \in \pi_1(S^1 \times B^3)$).

Proof of Step 7: Step 4 follows from the definition of $G(p,q)$. Here we view $I_0 \cup J_{a_3}$ as the basepoint. $\square$

Proof of Theorem 12.1: We will show that there is a homotopy of $F_1 = (f_1, f_2, f_3)$ to $F_1' = (f_1', f_2', f_3)$ supported away from $S_b \cup S_r$ such that throughout the homotopy $f_3 \equiv a_3$ and each of the homotopies of f_1 and f_2 are supported away from $J_{a_3} \subset S^1 \times B^3$. Furthermore, there exists disjoint regular neighborhoods $N(S_b), N(S_r)$ such that $f_1 = p_1$ off of $N(S_r)$ and $f_2 = p_2$ off of $N(S_b)$. Since each of S_b and S_r have trivial normal bundles and the restriction of f_1' and f_2' to the B^3 fibers of $N(S_b)$ and $N(S_r)$ project to degree ± 1 maps to $\partial N(a_3)$, the homotopy can be chosen so that the restriction of f_1' to a B^3 fiber is a whisker from a_1 to $\partial N(a_3)$ that goes p times around the S^1 followed by the standard quotient map of the 3-ball to $\partial N(a_3)$. The analogous statement holds for f_2' where the whisker starts at a_2 and goes q times about S^1. It follows that we have, up to sign independent of p and q, the Whitehead product $t_1^p t_2^q [w_{13}, w_{23}]$.

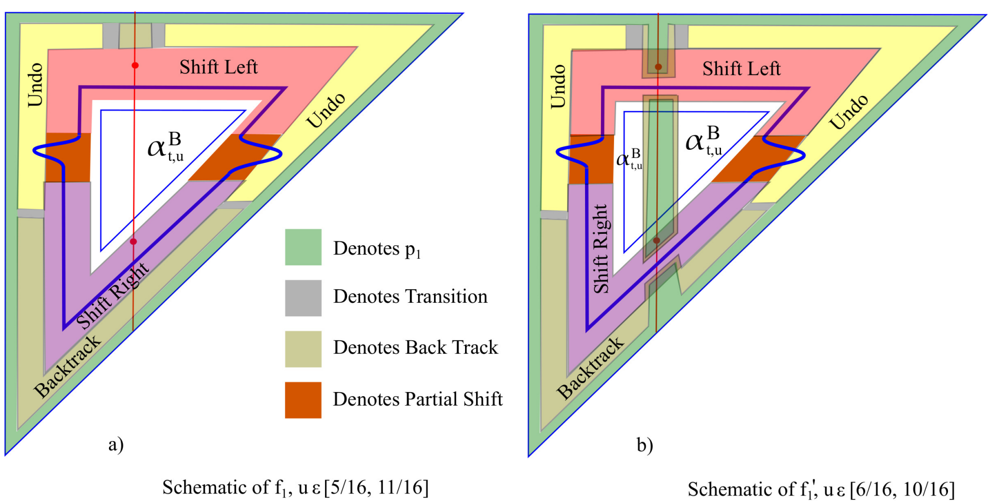


FIGURE 66.

The mapping $f_1|(\text{blue slab}) \times (t,u)$, $(t,u) \in [0,1] \times [1/4, 3/4]$ is schematically shown in Figure 66 a). The closed subset corresponding to $C_3\langle I \rangle$ is the closed region interior to the blue triangle. We denote that region by $\alpha^B_{t,u}$ since in that region $f_1(x,y,z,0,t,u) = \alpha^B_{t,u}(x,y,z)$. Since $f_1|U_1 = \mathrm{id}_{U_1}$ near the boundary of E_3, that region is denoted by p_1. In the collar, $\alpha^B_{t,u}|U_1$ is modified by the left, right, undo and backtrack homotopies as indicated in the diagram. A similar discussion holds for f_2, though in that case the diagram is of a square.

We next homotope f_1 to f_1' so that in the region $[0,1] \times [3/8, 5/8]$, f_1' is defined as in Figure 66 b). Here we modify $\alpha^B_{t,u}$ or its partially left or right shifted versions by the blue back track homotopy. Note that the homotopy is done away from where a blue lasso disc crosses a_3 under left or right shifting and hence the homotopy is supported away from J_{a_3}. In a similar manner we homotope f_2 to f_2'. Since for any (t,u) the corresponding blue bands and lasso discs are disjoint from the red ones, there is no nontrivial intersection between the images of U_1 and U_2 under the homotopy. Note that the support of f_1' is now a regular neighborhood of S_b disjoint from a regular neighborhood of S_r which is the support of f_2'. $\square$

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