ON THE TOPOLOGY OF ENDING LAMINATION SPACE

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Abstract. We show that if $S$ is a finite type orientable surface of genus $g$ and $p$ punctures where $3g + p \geq 5$, then $E\mathcal{L}(S)$ is $(n - 1)$-connected and $(n - 1)$-locally connected, where $\dim(\mathcal{PML}(S)) = 2n + 1 = 6g + 2p - 7$. Furthermore, if $g = 0$, then $E\mathcal{L}(S)$ is homeomorphic to the $p - 4$ dimensional Nobeling space.

0. Introduction

This paper is about the topology of the space $E\mathcal{L}(S)$ of ending laminations on a finite type hyperbolic surface, i.e. a complete hyperbolic surface $S$ of genus $g$ with $p$ punctures. An ending lamination is a geodesic lamination $\mathcal{L}$ in $S$ that is minimal and filling, i.e. every leaf of $\mathcal{L}$ is dense in $\mathcal{L}$ and any simple closed geodesic in $S$ nontrivally intersects $\mathcal{L}$ transversely.

Since Thurston’s seminal work on surface automorphisms in the mid 1970’s, laminations in surfaces have played central roles in low dimensional topology, hyperbolic geometry, geometric group theory and the theory of mapping class groups. From many points of view, the ending laminations are the most interesting laminations. For example, the stable and unstable laminations of a pseudo A\nosov mapping class are ending laminations [Th1] and associated to a degenerate end of a complete hyperbolic 3-manifold with finitely generated fundamental group is an ending lamination [Th4], [Bon]. Also, every ending lamination arises in this manner [BCM].

The Hausdorff metric on closed sets induces a metric topology on $E\mathcal{L}(S)$. Here, two elements $\mathcal{L}_1, \mathcal{L}_2$ in $E\mathcal{L}(S)$ are close if each point in $\mathcal{L}_1$ is close to a point of $\mathcal{L}_2$ and vice versa. In 1988, Thurston [Th2] showed that with this topology $E\mathcal{L}(S)$ is totally disconnected and in 2004 Zhu and Bonahon [ZB] showed that with respect to the Hausdorff metric, $E\mathcal{L}(S)$ has Hausdorff dimension zero.

It is the coarse Hausdorff topology that makes $E\mathcal{L}(S)$ important for applications and gives $E\mathcal{L}(S)$ a very interesting topological structure. This is the topology on $E\mathcal{L}(S)$ induced from that of $\mathcal{PML}(S)$, the space of projective measured laminations of $S$. Let $\mathcal{FPML}(S)$ denote the subspace of $\mathcal{PML}(S)$ consisting of those measured laminations whose underlying lamination is an ending lamination. (Fact: every ending lamination fully supports a measure.) Then $E\mathcal{L}(S)$ is a quotient of $\mathcal{FPML}(S)$ and is topologized accordingly. Alternatively, a sequence $\mathcal{L}_1, \mathcal{L}_2, \cdots$ converges to $\mathcal{L}$ in the coarse Hausdorff topology if after passing to subsequence, the sequence converges in the Hausdorff topology to a diagonal extension of $\mathcal{L}$, i.e. a lamination obtained by adding finitely many leaves, [Ha]. From now on $E\mathcal{L}(S)$ will have the coarse Hausdorff topology.

In 1999, Erica Klarreich [K] showed that $E\mathcal{L}(S)$ is the Gromov boundary of $\mathcal{C}(S)$, the curve complex of $S$. (See also [H1].) As a consequence of many results

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1Partially supported by NSF grants DMS-0504110 and DMS-1006553.
Version 0.918, May 20, 2011.
in hyperbolic geometry e.g. [Ber], [Th3], [M], [BCM], [Ag], [CG]; Leininger and Schleimer [LS] showed that the space of doubly degenerate hyperbolic structures on $S \times \mathbb{R}$ is homeomorphic to $\mathcal{E}L(S) \times \mathcal{E}L(S) \setminus \Delta$, where $\Delta$ is the diagonal. For other applications see [RS] and §18.

If $S$ is the thrice punctured sphere, then $\mathcal{E}L(S) = \emptyset$. If $S$ is the 4-punctured sphere or once-punctured torus, then $\mathcal{E}L(S) = \mathcal{FPML}(S) = \mathbb{R} \setminus \mathbb{Q}$. In 2000 Peter Storm conjectured that if $S$ is not one of these exceptional surfaces, then $\mathcal{E}L(S)$ is connected. Various partial results on the connectivity and local connectivity of $\mathcal{E}L(S)$ were obtained by Leininger, Mj and Schleimer, [LS], [LMS].

Using completely different methods, essentially by bare hands, we showed [G1] that if $S$ is neither the 3 or 4-punctured sphere nor the once-punctured torus, then $\mathcal{E}L(S)$ is path connected, locally path connected, cyclic and has no cut points. We also asked whether ending lamination spaces of sufficiently complicated surfaces were $n$-connected.

Here are our two main results.

**Theorem 0.1.** Let $S$ be a finite type hyperbolic surface of genus-$g$ and $p$-punctures. Then $\mathcal{E}L(S)$ is $(n - 1)$-connected and $(n - 1)$-locally connected, where $2n + 1 = \dim(\mathcal{PML}(S)) = 6g + 2p - 7$.

**Theorem 0.2.** Let $S$ be a $(4+n)$-punctured sphere. Then $\mathcal{E}L(S)$ is homeomorphic to the $n$-dimensional Nobeling space.

**Remark 0.3.** Let $T$ denote the compact surface of genus-$g$ with $p$ open discs removed and $S$ the $p$-punctured genus-$g$ surface. It is well known that there is a natural homeomorphism between $\mathcal{E}L(S)$ and $\mathcal{E}L(T)$. In particular, the topology of $\mathcal{E}L(S)$ is independent of the hyperbolic metric and hence all topological results about $\mathcal{E}L(S)$ are applicable to $\mathcal{E}L(T)$. Thus the main results of this paper are purely topological and applicable to compact orientable surfaces.

The $m$-dimensional Nobeling space $\mathbb{R}^{2m+1}_m$ is the space of points in $\mathbb{R}^{2m+1}$ with at most $m$ rational coordinates. In 1930 G. Nobeling [N] showed that the $m$-dimensional Nobeling space is a universal space for $m$-dimensional separable metric spaces, i.e. any $m$-dimensional separable metric space embeds in $\mathbb{R}^{2m+1}_m$. This extended the work of his mentor K. Menger [Me] who in 1926 defined the $m$-dimensional Menger spaces $M^{2m+1}_m$, showed that the Menger curve is universal for 1-dimensional compact metric spaces and suggested that $M^{2m+1}_m$ is universal for $m$-dimensional compact metric spaces. In 1984 Mladen Bestvina [Be] showed that any map of a compact $\leq m$-dimensional space into $M^{2m+1}_m$ can be approximated by an embedding and in 2006 Nagorko [Na] proved that any map of a $\leq m$-dimensional complete separable metric space into $\mathbb{R}^{2m+1}_m$ is approximable by a closed embedding.

A recent result of Ageev [Av], Levin [Le] and Nagorko [Na] gave a positive proof of a major long standing conjecture characterizing the $m$-dimensional Nobeling space. Nagorko [HP] recast this result to show that the $m$-dimensional Nobeling space is one that satisfies a series of topological properties that are discussed in detail in §7, e.g. the space is $(m - 1)$-connected, $(m - 1)$-locally connected and satisfies the locally finite $m$-discs property. To prove Theorem 16.1 we will show that $\mathcal{E}L(S)$ satisfies these conditions for $m = n$. 


Using [G1], Sebastian Hensel and Piotr Przytycki [HP] earlier showed that if $S$ is either the 5-punctured sphere or twice-punctured torus, then $\mathcal{EL}(S)$ is homeomorphic to the one dimensional Nobeling space. They also boldly conjectured that all ending lamination spaces are homeomorphic to Nobeling spaces.

Independently, in 2005 Bestvina and Bromberg [BB] asked whether ending lamination spaces are Nobeling spaces. They were motivated by the fact that Menger spaces frequently arise as boundaries of locally compact Gromov hyperbolic spaces, Klarreich’s theorem and the fact that the curve complex is not locally finite.

Theorem 0.2 gives a positive proof of the Hensel - Przytycki conjecture for punctured spheres. In section §19 we offer various conjectures on the topology of ending lamination spaces including three duality conjectures relating the curve complex with ending lamination space.

The methods of this paper are essentially also by bare hands. There are two main difficulties that must be overcome to generalize the methods of [G1] to prove Theorem 0.1. First of all it is problematic getting started. To prove path connectivity, given $\mu_0, \mu_1 \in \mathcal{EL}(S)$ we first chose $\lambda_0, \lambda_1 \in \mathcal{PM}(S)$ such that $\phi(\lambda_i) = \mu_i$ where $\phi$ is the forgetful map. The connectivity of $\mathcal{PM}(S)$ implies there exists a path $\mu : [0, 1] \to \mathcal{PM}(S)$ such that $\mu(0) = \mu_0$ and $\mu(1) = \mu_1$. In [G1] we found an appropriate sequence of generic such paths, projected them into lamination space, and took an appropriate limit which was a path in $\mathcal{EL}(S)$ between $\mu_0$ and $\mu_1$. To prove simple connectivity, say for $\mathcal{EL}(S)$ where $S$ is the surface of genus 2, the first step is already problematic, for there is a simple closed curve $\gamma$ in $\mathcal{EL}(S)$ whose preimage in $\mathcal{PM}(S)$ does not contain a loop projecting to $\gamma$. (See Theorem 18.8.) In the general case, the preimage is a Cech-like loop and that turns out to be good enough. The second issue is that points along a generic path in $\mathcal{PM}(S)$ project to laminations that are almost filling almost minimal, a property that was essential in order to take limits in [G1]. In the general case, the analogous laminations are not close to being either almost filling or almost minimal. To deal with this we introduce the idea of markers that is a technical device that enables us to take limits of laminations with the desired controlled properties.

This paper basically accomplishes two things. It shows that if $k \leq n$, then any generic PL map $f : B^k \to \mathcal{PM}(S)$ can be $\epsilon$-approximated by a map $g : B^k \to \mathcal{EL}(S)$ and conversely any map $g : B^k \to \mathcal{EL}(S)$ can be $\epsilon$-approximated by a map $f : B^k \to \mathcal{PM}(S)$. Here $\dim(\mathcal{PM}(S)) = 2n + 1$. See section §13 for the precise statements.

In §1 we provide some basic information and facts about ending lamination space, point out an omission in §7 [G1] and prove the following. See [LMS] and [LS] for earlier partial results.

**Theorem 0.4.** If $S$ is a finite type hyperbolic surface that is not the 3 or 4-holed sphere or 1-holed torus, then $\mathcal{PM}(S)$ is connected.

In §2 we show that if $g : S^k \to \mathcal{EL}(S)$ is continuous, then there exists a continuous map $F : B^{k+1} \to \mathcal{PM}(S)$ that extends $g$. Here $\mathcal{PM}(S)$ is the disjoint union of $\mathcal{PM}(S)$ and $\mathcal{EL}(S)$ appropriately topologized. In §3 we introduce markers. In §4 we give more facts relating the topologies of $\mathcal{EL}(S)$ and $\mathcal{PM}(S)$. In §5 we give a criterion for a sequence $f_1, f_2, \cdots$ of maps of $B^{k+1}$ into $\mathcal{PM}(S)$ that restrict to $g : S^k \to \mathcal{EL}(S)$ to converge to a continuous map $G : B^{k+1} \to \mathcal{EL}(S)$, extending $g$. The core technical work of this paper is carried out in §6-10. In §11-12 we prove Theorem 0.1. In §13 we isolate out our $\mathcal{PM}(S)$ and $\mathcal{EL}(S)$ approximation
theorems. In §14 we develop a theory of good cellulation sequences of $\mathcal{PML}(S)$, which may be of independent interest. In §15 we give various upper and lower estimates of $\dim(\mathcal{EL}(S))$ and prove that $\dim(S_{0,n+p}) = n$ and that $\pi_n(S_{0,n+p}) \neq 0$. In §16 we state Nagorko’s recharacterization of Nobeling spaces. In §17 we prove that $S_{0,n+p}$ satisfies the locally finite $n$-discs property. In §18 various applications are given. In §19 we offer some problems and conjectures.

Acknowledgements 0.5. The author learned about ending lamination spaces from hearing many beautiful lectures by Saul Schleimer and Chris Leininger. The author is grateful to Andrzej Nagorko for informative communications about Nobeling spaces and to Piotr Przytycki for many discussions and interest in this work. He thanks Mladen Bestvina, Alexander Dranishnikov and Ross Geoghegan for very helpful conversations and communications.

1. Basic definitions and facts

In what follows, $S$ or $S_{g,p}$ will denote a complete hyperbolic surface of genus $g$ and $p$ punctures. We will assume that the reader is familiar with the basics of Thurston’s theory of curves and laminations on surfaces, e.g. $\mathcal{L}(S)$ the space of geodesic laminations with topology induced from the Hausdorff topology on closed sets, $\mathcal{ML}(S)$ the space of measured geodesic laminations endowed with the weak* topology, $\mathcal{PML}(S)$ projective measured lamination space, as well as the standard definitions and properties of of train tracks. For example, see [PH], [Ha], [M], [Th1], or [FLP]. All laminations in this paper will be compactly supported. See [G1] for various ways to view and measure distance between laminations as well as for standard notation. For example, $d_{PT}(S)(x,y)$ denotes the distance between the geodesic laminations $x$ and $y$ as measured in the projective unit tangent bundle. Unless said otherwise, distance between elements of $\mathcal{L}$ are computed via the Hausdorff topology on closed sets in $PT(S)$. Sections §1 and §2 (through Remark 2.5) of [G1] are also described there.

Notation 1.1. We denote by $p: \mathcal{ML}(S) \setminus 0 \to \mathcal{PML}(S)$, the canonical projection and $\phi: \mathcal{PML}(S) \to \mathcal{L}(S)$ and $\hat{\phi}: \mathcal{ML}(S) \to \mathcal{L}(S)$, the forgetful maps. If $\tau$ is a train track, then $V(\tau)$ will denote the cell of measures supported on $\tau$ and $P(\tau)$ the polyhedron $\hat{p}(V(\tau) \setminus 0)$.

Definition 1.2. Let $\mathcal{EL}(S)$ denote the set of ending laminations on $S$, i.e. the set of geodesic laminations that are filling and minimal. A lamination $\mathcal{L} \in \mathcal{L}(S)$ is minimal if every leaf is dense and filling if the metric closure (with respect to the induced path metric) of $S \setminus \mathcal{L}$ supports no simple closed geodesic.

The following is well known.

Lemma 1.3. If $x \in \mathcal{PML}(S)$, then $\phi(x)$ is the disjoint union of minimal laminations. In particular $\phi(x) \in \mathcal{EL}(S)$ if and only if $\phi(x)$ is filling.

Proof. Since the measure on $x$ has full support, no non compact leaf $L$ is proper, i.e. non compact leaves limit on themselves. It follows that $\phi(x)$ decomposes into a disjoint union of minimal laminations. If $\phi(x)$ is filling, then there is only one such component. \[\square\]
Notation 1.4. Let $\mathcal{FPML}(S)$ denote the subspace of $\mathcal{PML}(S)$ consisting of filling laminations and $\mathcal{UPML}(S)$ denote the subspace of unfilling laminations. Thus $\mathcal{PML}(S)$ is the disjoint union of $\mathcal{FPML}(S)$ and $\mathcal{UPML}(S)$.

Definition 1.5. Topologize $\mathcal{EL}(S)$ by giving it the quotient topology induced from the surjective map $\phi : \mathcal{FPML}(S) \to \mathcal{EL}(S)$ where $\mathcal{FPML}(S)$ has the subspace topology induced from $\mathcal{PML}(S)$. After [HI] we call this the coarse Hausdorff topology. Hamenstadt observed that this topology is a slight coarsening of the Hausdorff topology on $\mathcal{EL}(S)$; a sequence $L_1, L_2 \cdots$ in $\mathcal{EL}(S)$ limits to $L \in \mathcal{EL}(S)$ if and only if any convergent subsequence in the Hausdorff topology converges to a diagonal extension $L'$ of $L$. A diagonal extension of $L$ is a lamination obtained by adding finitely many leaves.

Remark 1.6. It is well known that $\mathcal{EL}(S)$ is separable and complete. Separability follows from the fact that $\mathcal{PML}(S)$ is a sphere and $\mathcal{FPML}(S)$ is dense in $\mathcal{PML}(S)$. (E.g. $\mathcal{FPML}(S)$ is the complement of countably many codimension-1 PL-cells in $\mathcal{PML}(S)$.) Masur and Minsky [MM] showed that the curve complex $\mathcal{C}(S)$ is Gromov-hyperbolic and Klareich [K] (see also [HI]) showed that the Gromov boundary of $\mathcal{C}(S)$ is homeomorphic to $\mathcal{EL}(S)$ with the coarse Hausdorff topology. Being the boundary of a Gromov hyperbolic space, $\mathcal{EL}(S)$ is metrizable. Bonk and Schramm showed that with appropriate constants in the Gromov product, the induced Gromov metric is complete [BS]. See also [HP].

The following are characterizations of continuous maps in $\mathcal{EL}(S)$ analogous to Lemmas 1.13-1.15 of [G1]. Here $X$ is a metric space.

Lemma 1.7. A function $f : X \to \mathcal{EL}(S)$ is continuous if and only if for each $t \in X$ and each sequence $\{t_i\}$ converging to $t$, $f(t)$ is the coarse Hausdorff limit of the sequence $f(t_1), f(t_2), \cdots$. \hfill $\square$

Lemma 1.8. A function $f : X \to \mathcal{EL}(S)$ is continuous if and only if for each $\epsilon > 0$ and $t \in X$ there exists a $\delta > 0$ such that $d_X(s, t) < \delta$ implies that the maximal angle of intersection between leaves of $f(s)$ and leaves of $f(t)$ is $< \epsilon$. \hfill $\square$

Lemma 1.9. A function $f : X \to \mathcal{EL}(S)$ is continuous if and only if for each $\epsilon > 0$ and $t \in X$ there exists a $\delta > 0$ such that $d_X(s, t) < \delta$ implies that $d_{\mathcal{PT}(S)}(f'(t), (f'(s)) < \epsilon$, where $f'(s)$ (resp $f'(t)$) is any diagonal extension of $f(s)$ (resp. $f(t)$). \hfill $\square$

The forgetful map $\phi : \mathcal{PML}(S) \to \mathcal{L}(S)$ is discontinuous, for any simple closed curve viewed as a point in $\mathcal{PML}(S)$ is the limit of filling laminations and any Hausdorff limit of a sequence of filling laminations is filling.

Definition 1.10. Let $X_1, X_2, \cdots$ be a sequence of subsets of the topological space $Y$. We say that the subsets $\{X_i\}$ super converges to $X$ if for each $x \in X$, there exists $x_i \in X_i$ so that $\lim_{i \to \infty} x_i = x$. In this case we say $X$ is a sublimit of $\{X_i\}$.

We will repeatedly use the following result that first appears in [Th4]. See Proposition 3.2 [G1] for a proof.

Proposition 1.11. If the projective measured laminations $\lambda_1, \lambda_2, \cdots$ converge to $\lambda \in \mathcal{PML}(S)$, then $\phi(\lambda_1), \phi(\lambda_2), \cdots$ super converges to $\phi(\lambda)$ as subsets of $\mathcal{PT}(S)$.

The following consequence of super convergence, Proposition 1.11 is repeatedly used in this paper and in [G1].
Lemma 1.12. If \( z_1, z_2, \cdots \) is a convergent sequence in \( \mathcal{E}L(S) \) limiting to \( z_\infty \) and \( x_1, x_2, \cdots \) is a sequence in \( \mathcal{P}ML(S) \) such that for all \( i \), \( \phi(x_i) = z_i \), then any convergent subsequence of the \( x_i \)'s converges to a point of \( \phi^{-1}(z_\infty) \). \( \square \)

Proof. After passing to subsequence it suffices to consider the case that \( x_1, x_2, \cdots \) converges to \( x_\infty \) in \( \mathcal{P}ML(S) \) and that \( z_1, z_2, \cdots \) converges to \( \mathcal{L} \subset \mathcal{L}(S) \) with respect to the Hausdorff topology. By super convergence, Proposition 1.11 \( \phi(x_\infty) \) is a sublamination of \( \mathcal{L} \) and by definition of coarse Hausdorff topology \( \mathcal{L} \) is a diagonal extension of \( z_\infty \). Since \( z_\infty \) is minimal and each leaf of \( \mathcal{L} \setminus z_\infty \) is non compact and proper it follows that \( \phi(x_\infty) = z_\infty \). \( \square \)

Corollary 1.13. If \( L \subset \mathcal{E}L(S) \) is compact, then \( \phi^{-1}(L) \subset \mathcal{P}ML(S) \) is compact. \( \square \)

Corollary 1.14. \( \phi : \mathcal{F}\mathcal{P}ML(S) \rightarrow \mathcal{E}L(S) \) is a closed map. \( \square \)

Lemma 1.15. If \( \mu \in \mathcal{E}L(S) \), \( x_1, x_2, \cdots \rightarrow x \) is a convergent sequence in \( \mathcal{P}ML(S) \) and \( \lim_{i \rightarrow \infty} d_{P\mathcal{T}(S)}(\phi(x_i), \mu') = 0 \) for some diagonal extension \( \mu' \) of \( \mu \), then \( x \in \phi^{-1}(\mu) \).

Proof. After passing to subsequence and super convergence \( \phi(x_1), \phi(x_2), \cdots \) converges to \( \mathcal{L} \in \mathcal{L}(S) \) with respect to the Hausdorff topology where \( \phi(x) \subset \mathcal{L} \). If \( \phi(x) \neq \mu \), then \( d_{P\mathcal{T}(S)}(\mathcal{L}, \mu') > 0 \) and hence \( d_{P\mathcal{T}(S)}(\phi(x_i), \mu') \) is uniformly bounded away from 0 for every diagonal extension \( \mu' \) of \( \mu \). \( \square \)

Lemma 1.16. Let \( \tau \) be a train track and \( \mu \in \mathcal{E}L(S) \). Either \( \tau \) carries \( \mu \) or \( \inf\{d_{P\mathcal{T}(S)}(\phi(t), \mu) | t \in P(\tau) \} > 0 \).

Proof. If \( \inf\{d_{P\mathcal{T}(S)}(\phi(t), \mu) | t \in P(\tau) \} = 0 \), then by compactness of \( P(\tau) \) and the previous lemma, there exists \( x \in P(\tau) \) such that \( \phi(x) = \mu \) and hence \( \tau \) carries \( \mu \). \( \square \)

The following is well known, e.g. it can be deduced from Proposition 1.9 [G1].

Lemma 1.17. If \( z \in \mathcal{E}L(S) \), then \( \phi^{-1}(z) = \sigma_z \) is a compact convex cell, i.e. if \( \tau \) is any train track that carries \( z \), then \( \sigma_z = p(V) \), where \( V \subset V(\tau) \) is the open cone on a compact convex cell in \( V(\tau) \). \( \square \)

Theorem 1.18. If \( S \) is a finite type hyperbolic surface that is not the 3 or 4-holed sphere or 1-holed torus, then \( \mathcal{F}\mathcal{P}ML(S) \) is connected.

Proof. If \( \mathcal{F}\mathcal{P}ML(S) \) is disconnected, then it is the disjoint union of non empty closed sets \( A \) and \( B \). By the previous lemma, if \( \mathcal{L} \in \mathcal{E}L(S) \), then \( \phi^{-1}(\mathcal{L}) \) is connected. It follows that \( \phi(A) \cap \phi(B) = \emptyset \) and hence by Lemma 1.14, \( \mathcal{E}L(S) \) is the disjoint union of the non empty closed sets \( \phi(A), \phi(B) \) and hence \( \mathcal{E}L(S) \) is disconnected. This contradicts [G1]. \( \square \)

Remark 1.19. In the exceptional cases, \( \mathcal{F}\mathcal{P}ML(S_{0,3}) = \emptyset \), \( \mathcal{F}\mathcal{P}ML(S_{0,4}) = \mathcal{F}\mathcal{P}ML(S_{1,1}) = \mathbb{R} \setminus \mathbb{Q} \).

Definition 1.20. The curve complex \( \mathcal{C}(S) \) introduced by Harvey [Ha] is the simplicial complex with vertices the set of simple closed geodesics and \( (v_0, \cdots, v_p) \) span a simplex if the \( v_i \)'s are pairwise disjoint.

There is a natural injective continuous map \( \hat{I} : \mathcal{C}(S) \rightarrow \mathcal{M}L(S) \). If \( v \) is a vertex, then \( I(v) \) is the measured lamination supported on \( v \) with transverse measure.
1/\text{length}(v)$. Extend \( \hat{I} \) linearly on simplices. Define \( I : \mathcal{C}(S) \to \mathcal{PML}(S) \) by \( I = p \circ \hat{I} \).

**Remark 1.21.** The map \( I \) is not a homeomorphism onto its image. If \( \mathcal{C}(S)_{\text{sub}} \) denotes the topology on \( \mathcal{C}(S) \) obtained by pulling back the subspace topology on \( I(\mathcal{C}(S)) \), then \( \mathcal{C}(S)_{\text{sub}} \) is coarser than \( \mathcal{C}(S) \). Indeed, if \( C \) is a vertex, then there exists a sequence \( C_0, C_1, \cdots \) of vertices that converges in \( \mathcal{C}(S)_{\text{sub}} \) to \( C \), but does not have any limit points in \( \mathcal{C}(S) \). See §19 for some conjectures relating the topology of \( \mathcal{C}(S) \) and \( \mathcal{C}(S)_{\text{sub}} \).

**Correction 1.22.** In §7 of [G1] the author states without proof Corollary 7.4 which asserts that if \( S \) is not one of the three exceptional surfaces, then \( \mathcal{E}(S) \) has no cut points. It is not a corollary of the statement of Theorem 7.1 because as the figure-8 shows, cyclic does not imply no cut points. It is not difficult to prove Corollary 7.4 by extending the proof of Theorem 7.1 to show that given \( x \neq y \in \mathcal{E}(S) \), there exists a simple closed curve in \( \mathcal{E}(S) \) passing through both \( x \) and \( y \). Alternatively, no cut points immediately follows from the locally finite 1-discs property, Proposition 17.8 and Remark 16.3.

**Notation 1.23.** If \( X \) is a space, then \( |X| \) denotes the number of components of \( X \). If \( X \) and \( Y \) are sets, then \( X \setminus Y \) is \( X \) with the points of \( Y \) deleted, but if \( X, Y \in \mathcal{L}(S) \), then \( X \setminus Y \) denotes the union of leaves of \( X \) that are not in \( Y \). Thus by Lemma 1.3 if \( X = \phi(x), Y = \phi(y) \) with \( x, y \in \mathcal{PML}(S) \), then \( X \setminus Y \) is the union of those minimal sublaminations of \( X \) which are not sublaminations of \( Y \).

## 2. Extending Maps of Spheres into \( \mathcal{E}(S) \) to Maps of Balls into \( \mathcal{PML}(S) \)

**Definition 2.1.** Let \( \mathcal{PML}(S) \) (resp. \( \mathcal{MLEL}(S) \)) denote the disjoint union \( \mathcal{PML}(S) \cup \mathcal{E}(S) \) (resp. \( \mathcal{MLEL}(S) \cup \mathcal{E}(S) \)). Define a topology on \( \mathcal{PML}(S) \) (resp. \( \mathcal{MLEL}(S) \)) as follows. A basis consists of all sets of the form \( U \cup V \) where \( \phi^{-1}(V) \subset U \) (resp. \( \hat{\phi}^{-1}(V) \subset U \)) where \( U \) is open in \( \mathcal{PML}(S) \) (resp. \( \mathcal{MLEL}(S) \)) and \( V \) (possibly \( \emptyset \)) is open in \( \mathcal{E}(S) \). We will call this the PMLEL topology (resp. MLEL topology).

**Lemma 2.2.** The PMLEL (resp. MLEL) topology has the following properties.

i) \( \mathcal{PML}(S) \) (resp. \( \mathcal{MLEL}(S) \)) is non Hausdorff. In fact \( x \) and \( y \) cannot be separated if and only if \( x \in \mathcal{E}(S) \) and \( y \in \phi^{-1}(x) \) (resp. \( y \in \hat{\phi}^{-1}(x) \)), or vice versa.

ii) \( \mathcal{PML}(S) \) (resp. \( \mathcal{MLEL}(S) \)) is an open subspace.

iii) \( \mathcal{E}(S) \) is a closed subspace.

iv) If \( U \subset \mathcal{PML}(S) \) is a neighborhood of \( \phi^{-1}(x) \) where \( x \in \mathcal{E}(S) \), then there exists an open set \( V \subset \mathcal{E}(S) \), such that \( x \in V \) and \( \phi^{-1}(V) \subset U \), i.e. \( U \cup V \) is open in \( \mathcal{PML}(S) \).

v) A sequence \( x_1, x_2, \cdots \) in \( \mathcal{PML}(S) \) converges to \( x \in \mathcal{E}(S) \) if and only if every limit point of the sequence lies in \( \phi^{-1}(x) \). A sequence \( x_1, x_2, \cdots \) in \( \mathcal{MLEL}(S) \) bounded away from both 0 and \( \infty \) converges to \( x \in \mathcal{E}(S) \) if and only if every limit point of the sequence lies in \( \hat{\phi}^{-1}(x) \).

**Proof.** Parts i)-iii) and v) are immediate. Part iv) follows from Lemma 1.12.  \( \Box \)
Definition 2.3. Let $V$ be the underlying space of a finite simplicial complex. A generic PL map $f : V \to \mathcal{PML}(S)$ is a PL map transverse to each $B_{a_1} \cap \cdots \cap B_{a_r} \cap \Delta B_{b_1} \cap \cdots \cap \Delta B_{b_s}$, where $a_1, \ldots, a_r, b_1, \ldots, b_s$ are simple closed geodesics. (Notation as in [G1].) More generally $f : V \to \mathcal{PML}(S)$ is called a generic PL map if $f^{-1}(\mathcal{EL}(S)) = W$ is a subcomplex of $V$ and $f|\!(V \setminus W)$ is a generic PL map.

Lemma 2.4. Let $L = I(L_0)$ where $L_0$ is a finite $q$-subcomplex of $\mathcal{C}(S)$. If $f : V \to \mathcal{PML}(S)$ is a generic PL map where $\dim(V) = p$ and $p + q \leq 2n$, then $f(V) \cap L = \emptyset$. If either $p + q \leq 2n - 1$ or $p \leq n$, then for every simple closed geodesic $C$, $f(V) \cap Z = \emptyset$, where $Z = \phi^{-1}(C) \ast (\partial B_C \cap L)$.

Proof. By genericity and the dimension hypothesis, the first conclusion is immediate. For the second, $L \cap \partial B_C$ is at most $\min(q, n-1)$ dimensional, hence any simplex in the cone $(L \cap \partial B_C) \ast C$ is at most $\min(q+1, n)$ dimensional. Thus the result again follows by genericity. \hfill \Box

Remark 2.5. In this paper, $V$ will be either a $B_k$ or $S_k \times I$ and $X = \partial B_k$ or $S_k \times 1$.

Notation 2.6. Fix once and for all a map $\psi : \mathcal{EL}(S) \to \mathcal{PML}(S)$ such that $\phi \circ \psi = \text{id}_{\mathcal{EL}(S)}$. If $i : \mathcal{PML}(S) \to \mathcal{ML}(S)$ is the map sending a projective measured geodesic lamination to the corresponding measured geodesic lamination of length 1, then define $\hat{\psi} : \mathcal{EL}(S) \to \mathcal{ML}(S)$ by $\hat{\psi} = i \circ \psi$. Let $p : \mathcal{ML}(S) \setminus \emptyset \to \mathcal{PML}(S)$ denote the standard projection map and let $\hat{\phi} = \phi \circ p$.

While $\psi$ is very discontinuous, it is continuous enough to carry out the following which is the main result of this section.

Proposition 2.7. Let $g : S^{k-1} \to \mathcal{EL}(S)$ be continuous where $k \leq \dim(\mathcal{PML}(S)) = 2n+1$. Then there exists a generic PL map $F : B^k \to \mathcal{PML}(S)$ (resp. $\hat{F} : B^k \to \mathcal{ML}(S)$) such that $F|_{S^{k-1}} = g$ (resp. $\hat{F}|_{S^{k-1}} = g$) and $F(\text{int}(B^k)) \subset \mathcal{PML}(S)$ (resp. $\hat{F}(\text{int}(B^k)) \subset \mathcal{ML}(S)$).

Idea of Proof for $\mathcal{PML}(S)$: It suffices to first find a continuous extension and then perturb to a generic PL map. To obtain a continuous extension $F$, consider a sequence $K_1, K_2, \cdots$ of finer and finer triangulations of $S^{k-1}$. For each $i$, consider the map $f_i : S^{k-1} \to \mathcal{PML}(S)$ defined as follows. If $K$ is a simplex of $K_i$ with vertices $v_0, \ldots, v_m$, then define $f_i(v_j) = \psi(g(v_j))$ and extend $f_i$ linearly on $K$. Extend $f_i$ to a map of $B^k$ into $\mathcal{PML}(S)$ and extend $f_i$ and $f_{i+1}$ to a map of $F_i : S^{k-1} \times [i, i+1]$ into $\mathcal{PML}(S)$. Concatenating these maps and taking a limit yields the desired continuous map $F : B^k \cup S^{k-1} \times [1, \infty] \to \mathcal{PML}(S)$ where $\partial B^k$ is identified with $S^{k-1} \times 1, G|_{S^{k-1} \times [i, i+1]} = F_i$, and $H|S^n \times \infty = g$.

Remark 2.8. The key technical issue is making precise the phrase “extend $f_i$ linearly on $\sigma$”.

Before we prove the Proposition we establish some notation and then prove a series of lemmas.

Notation 2.9. Let $\Delta$ be the cellulation on $\mathcal{PML}(S)$ whose cells are the various $P(\tau_i)$’s, where the $\tau_i$’s are the standard train tracks to some fixed parametrized pants decomposition of $S$. If $\sigma = P(\tau_i)$ is a cell of $\Delta$ and $Y \subset \sigma$, then define the convex hull of $Y$ to be $p(C(p^{-1}(Y)))$, where $C(Z)$ is the convex hull of $Z$ in $V(\tau_i)$.
If \( x \in \mathcal{EL}(S) \), then let \( \tau_x \) denote the unique standard train track that fully carries \( x \) and \( \sigma_x \), the cell \( P(\tau_x) \).

If \( \mathcal{L} \in \mathcal{EL}(S) \), then \( \mathcal{L}' \) will denote a diagonal extension of \( \mathcal{L} \), i.e. a lamination obtained by adding finitely many non compact leaves.

**Remark 2.10.** It follows from Lemma 1.17 that \( \phi^{-1}(x) \subset \operatorname{int}(\sigma_x) \) and is closed and convex.

**Lemma 2.11.** Let \( x_1, x_2, \cdots \to x \) be a convergent sequence in \( \mathcal{EL}(S) \). If \( y \in \mathcal{PML}(S) \) is a limit point of \( \{\psi(x_i)\} \), then \( \phi(y) = x \).

*Proof.* This follows from Lemma 1.12. \( \square \)

**Definition 2.12.** Fix \( \epsilon_1 < 1/1000(\min\{d(\sigma, \sigma')|\sigma, \sigma' \text{ disjoint cells of } \Delta\}) \). For each cell \( \sigma \) of \( \Delta \) define a retraction \( r_\sigma : N(\sigma, \epsilon_1) \to \sigma \). For every \( \delta < \epsilon_1 \) define a discontinuous map \( \pi_\delta : \mathcal{PML}(S) \to \mathcal{PML}(S) \) as follows. Informally, \( \pi_\delta \) retracts a closed neighborhood of \( \delta^0 \) to \( \delta^0 \), then after deleting this neighborhood retracts a closed neighborhood of \( \Delta^1 \) to \( \Delta^1 \), then after deleting this neighborhood retracts a closed neighborhood of \( \Delta^2 \) to \( \Delta^2 \) and so on. As \( \delta \to 0 \), the neighborhoods are required to get smaller. More formally, let \( \delta(0) = \delta \). If \( \sigma \) is a 0-cell of \( \Delta \), then define \( \pi_\delta|\{N(\sigma, \delta(0)) = r_{\epsilon_1}|N(\sigma, \delta(0)) \} \) now choose \( \delta(1) \leq \delta \) such that if \( \sigma, \sigma' \) are distinct 1-cells, then \( N(\sigma, \delta(1)) \cap N(\sigma', \delta(1)) \subset N(\Delta^0, \delta) \). If \( \sigma \) is a 1-cell, then define \( \pi_\delta|N(\sigma, \delta(1)) \cap N(\Delta^0, \delta(0)) = r_{\epsilon_1}|N(\sigma, \delta(1)) \cap N(\Delta^0, \delta(0)) \). Having defined \( \pi_\delta \) on \( N(\Delta^0, \delta(0)) \cup \cdots \cup N(\Delta^k, \delta(k)) \), extend \( \pi_\delta \) in a similar way over \( N(\Delta^{k+1}, \delta(k+1)) \) using a sufficiently small \( \delta(k+1) \). We require that if \( \delta < \delta' \), then for all \( i, \delta(i) < \delta'(i) \).

**Remark 2.13.** Let \( x \in \sigma \) a cell of \( \Delta \). If \( \delta_1 < \delta_2 \leq \epsilon_1 \) and \( \sigma_1, \sigma_2 \) are the lowest dimensional cells of \( \Delta \) respectively containing \( \pi_\delta_1(x), \pi_\delta_2(x) \), then \( \sigma_2 \) is a face of \( \sigma_1 \) which is a face of \( \sigma \).

**Lemma 2.14.** Let \( f_i : B^k \to \mathcal{EL}(S) \) be a sequence of maps such that \( f_i(B^k) \to x \in \mathcal{EL}(S) \) in the Hausdorff topology on closed sets in \( \mathcal{EL}(S) \).

i) Then for \( i \) sufficiently large \( \psi(f_i(B^k)) \subset \operatorname{st}(\sigma_x) \), the open star of \( \sigma_x \) in \( \Delta \).

ii) Given \( \epsilon > 0 \) and \( \delta < \epsilon_1 \), then for \( i \) sufficiently large, if \( t \in C_i \), where \( C_i \) denotes the convex hull of \( r_{\sigma_x}(\psi(f_i(B^k))) \) in \( \sigma_x \), then \( x \in d_{\Delta}(\phi(t), \epsilon) \).

*Proof.* If i) is false, then after passing to subsequence, for each \( i \) there exists \( a_i \in B^k \) such that \( \psi(f_i(a_i)) \notin \operatorname{st}(\sigma_x) \). This contradicts Lemma 2.11 which implies that any limit point of \( \psi(f_i(a_i)) \) lies in \( \phi^{-1}(x) \subset \sigma_x \).

This argument actually shows that if \( U \) is any neighborhood of \( \phi^{-1}(x) \), then \( \psi(f_i(B^k)) \subset U \) for \( i \) sufficiently large and hence \( r_{\sigma_x}(\psi(f_i(B^k))) \subset U \cap \sigma_x \) for is sufficiently large. Since \( \phi^{-1}(x) \) is convex it follows that \( C_i \subset U \) for \( i \) sufficiently large. By super convergence there exists a neighborhood \( V \) of \( \phi^{-1}(x) \) such that if \( y \in V \), then \( x \in d_{\Delta}(\phi(y), \epsilon) \). Now choose \( U \) so that the \( r_{\sigma_x}(U) \subset \subset V \). \( \square \)

**Lemma 2.15.** If \( \epsilon > 0 \) and \( x_1, x_2, \cdots \to x \in \mathcal{EL}(S) \), then there exists \( \delta > 0 \) such that for \( i \) sufficiently large \( x \subset N_{\Delta}(\phi(\pi_\delta(\psi(x_i)))), \epsilon \).

*Proof.* Let \( U \) be a neighborhood of \( \phi^{-1}(x) \) such that \( \lambda \in U \) implies \( x \subset N_{\Delta}(\phi(\lambda), \epsilon) \). Next choose \( \delta > 0 \) so that \( \pi_\delta(N(\phi^{-1}(x), \delta)) \subset U \). Now apply Lemma 2.11 to show that for \( i \) sufficiently large the conclusion holds. \( \square \)
Lemma 2.16. Let \( \epsilon > 0, g_i : B^k \to \mathcal{E}\mathcal{L}(S) \) and \( g_i(B^k) \to x \). There exists \( N \in \mathbb{N}, \delta > 0 \), such that if \( \delta_1, \delta_2 \leq \delta, i \geq N \) and \( \sigma \) a simplex of \( \Delta \), then if \( C_i \) is the convex hull of \( \sigma \cap (\pi_{\delta_1}(\psi(g_i(B^k)))) \cup \pi_{\delta_2}(\psi(g_i(B^k))) \cup (\psi(g_i(B^k))) \) and \( t \in C_i \), then \( x \subset N_{PT(S)}(\phi(t), \epsilon) \).

Proof. Let \( V \) be a neighborhood of \( \phi^{-1}(x) \) such that if \( y \in V \), then \( x \in N_{PT(S)}(\phi(y), \epsilon) \). Let \( V_1 \subset V \) be a neighborhood of \( \phi^{-1}(x) \) such that for each cell \( \sigma \subset \Delta \) containing \( \sigma \) as a face, the convex hull of \( V_1 \cap \sigma \subset V \). Also assume that \( V_1 \cap \kappa = \emptyset \) if \( \kappa \) is a cell of \( \Delta \) disjoint from \( \sigma \). Choose \( \delta < \epsilon_1 \) so that \( d(\kappa, V_1) > 2\delta \) for all cells \( \kappa \) disjoint from \( \sigma \). Choose \( N \) so that if \( i \geq N \) and \( \delta' \leq \delta \), then \( \pi_{\delta'}(g_i(B^k)) \subset V_1 \) and \( g_i(B^k) \subset V_1 \). Therefore, if \( \delta_1, \delta_2 < \delta \) and \( i \geq N \) then \( C_i \) lies in \( V \). \( \square \)

We now address how to approximate continuous maps into \( \mathcal{E}\mathcal{L}(S) \) by PL maps into \( \mathcal{PML}(S) \).

Definition 2.17. Let \( \sigma \) be a cell of \( \Delta \). Let \( \kappa \) be a \( p \)-simplex and \( H_0 : \kappa^0 \to \sigma \), where \( \kappa^0 \) are the vertices of \( \kappa \). Define the induced map \( H : \kappa \to \sigma \), such that \( H|_{\kappa^0} = H_0 \) as follows. Let \( \hat{H} \) be the linear map of \( \kappa \) into \( \hat{\sigma} = \pi_{\delta^{-1}}(\sigma) \) such that \( \hat{H}|_{\kappa^0} = i \circ H_0 \). In a similar manner, if \( K \) is a simplicial complex and \( h|K^0 \to \mathcal{PML}(S) \) is such that for each simplex \( \kappa \), \( h(\kappa^0) \subset \sigma \), for some cell \( \sigma \) of \( \Delta \), then \( h \) extends to a map \( H : K \to \mathcal{PML}(S) \) also called the induced map. Since the linear structure on a face of a cell of \( \mathcal{ML}(S) \) is the restriction of the linear structure of the cell, \( H \) is well defined.

Lemma 2.18. Let \( g : K_1 \to \mathcal{E}\mathcal{L}(S) \) be continuous where \( K_1 \) is a finite simplicial complex. Let \( K_1, K_2, \ldots \) be such that mesh \( (K_i) \to 0 \) and each \( K_{i+1} \) is a subdivision of \( K_i \). For every \( \delta < \epsilon_1 \) there exists an \( i(\delta) \in \mathbb{N} \), monotonically increasing as \( \delta \to 0 \), such that if \( \delta' \leq \delta, i = i(\delta) \) and \( \kappa \) is a simplex of \( K_i \), then \( \pi_{\delta'}(\psi(g(\kappa))) \cup \pi_{\delta}(\psi(g(\kappa))) \) contained in a cell of \( \Delta \).

Given \( \epsilon > 0 \) there exist \( \delta(\epsilon) > 0 \) and \( N(\epsilon) \in \mathbb{N} \) such that if \( i = N(\epsilon) \), \( \kappa \) is a simplex of \( K_1 \), \( \sigma \) a simplex of \( \Delta \), \( \delta_1, \delta_2 \leq \delta(\epsilon) \) and \( C \) is the convex hull of \( \sigma \cap (\pi_{\delta_1}(\psi(g(\kappa)))) \cup \pi_{\delta_2}(\psi(g(\kappa))) \cup (\psi(g(\kappa))) \), then given \( z_1, z_2 \in C \) we have \( d_{PT(S)}(g(z_1), \phi(z_2)) < \epsilon \).

Fix \( \epsilon > 0 \). If \( \delta \) is sufficiently small, \( i > i(\delta) \) and \( H_i : K_i \to \mathcal{PML}(S) \) is the induced map arising from \( \pi_{\delta'} \psi \circ g(K_0) \), then for each \( z \in K_i, d_{PT(S)}(\phi(H_i(z)), g(z)) < \epsilon \).

Proof. Fix \( 0 < \delta < \epsilon_1 \). For each \( x \in \mathcal{E}\mathcal{L}(S) \), there exists a neighborhood \( V_x \) of \( \phi^{-1}(x) \) such that \( \pi_{\delta}(V_x) \subset \sigma_x \). By Lemma 2.2 iv) there exists a neighborhood \( U_x \) of \( x \) such that \( \psi(U_x) \subset V_x \). By compactness there exist \( U_{x_1}, \ldots, U_{x_n} \) that cover \( g(K) \). There exists \( i(\delta) > 0 \) such that if \( i \geq i(\delta) \) and \( \kappa \) is a simplex of \( K_i \), then \( g(\kappa) \subset U_x \) for some \( j \) and so \( \pi_{\delta}(\psi(g(\kappa))) \subset \sigma_{x_j} \). Since each \( \phi^{-1}(x) \) lies in the interior of a cell, it follows that \( \sigma_{x_j} \) is the lowest dimensional cell of \( \Delta \) that contains any point of \( \pi_{\delta}(\psi(g(\kappa))) \). Now let \( \delta' < \delta \). As above, \( \pi_{\delta'}(\psi(g(\kappa))) \subset \sigma_y \) for some \( y \in g(K_1) \) where \( \sigma_y \) is a minimal dimensional cell. It follows from Remark 2.13 that \( \sigma_{x_j} \) is a face of \( \sigma_y \).

The proof of the second conclusion follows from that of the first and the proof of Lemma 2.16.

If the third conclusion of the lemma is false, then after passing to subsequence there exist \( (\kappa_1, \omega_1, i_1, t_1), (\kappa_2, \omega_2, i_2, t_2), \ldots \) such that for all \( i, \kappa_{i+1} \) is a codimension-0 subsimplex of \( \kappa_i \) which is a simplex of \( K_{i+1} \).
Proof of Proposition 2.7: Let $K_1, K_2, \ldots$ be subdivisions of $S^{k-1}$ such that mesh$(K_i) \to 0$ and each $K_{i+1}$ is a subdivision of $K_i$. Pick $\delta_j < \delta(1/j)$ and $n_j > \max\{i(\delta_j), N(1/j)\}$ such that for $i \geq n_j$ and $\delta \leq \delta_j$ the induced map $H_k : K_i \to \mathcal{PML}(S)$ arising from $\pi \circ \psi \circ g|K_0^i$ satisfies the third conclusion of Lemma 2.18. Assume that $n_1 < n_2 < \cdots$. Replace the original $\{K_i\}$ sequence by the subsequence $\{K_{n_i}\}$.

With this new sequence, let $f_j : K_j \to \mathcal{PML}(S)$ be the induced map arising from $\pi_{\delta_j} \circ \psi \circ g|K_0^i$.

Apply Lemma 2.18 to find, after passing to a subsequence of the $K_i$’s, a sequence of maps $f_j : K_j \to \mathcal{PML}(S)$ so that each $f_j$ satisfies the conclusions of those results for $\epsilon = 1/j$. Let $\delta_j$ denote the $\delta$ used to define $f_j$. Note that with this new sequence of $K_i$’s, each $K_j$ satisfies the first two conclusions of Lemma 2.18 using $\delta_j$.

Define a triangulation $T$ on $S^{k-1} \times [0, \infty)$ by first letting $T|S^{k-1} \times j = K_j$ and then extending in a standard way to each $S^{k-1} \times [j, j + 1]$ so that if $\zeta$ is a simplex of $T|S^{k-1} \times [j, j + 1]$, then $\zeta^0 \subset (\kappa_x \times j) \cup (\kappa_1 \times (j + 1))$ where $\kappa$ is a simplex of $K_j$ and $\kappa_1 \subset \kappa$ is a simplex of $K_{j+1}$. By Lemma 2.18, $\pi_{\delta_j} \circ (\psi(g(\kappa^0))) \cup \pi_{\delta_{j+1}}(\psi(g(\kappa_1^0)))$ lie in the same cell of $\Delta$ so the induced maps on $T|S^{k-1} \times [j, j + 1]$ extend to one called $f_{j+1}$ on $T|S^{k-1} \times [j, j + 1]$.

Since $k \leq \dim \mathcal{PML}(S), f_1$ extends to a map $f_1'$ of $B^k$ into $\mathcal{PML}(S)$. Define $F : B^k \cup S^{k-1} \times [1, \infty)$ to $\mathcal{PML}(S)$ so that $F|B^k = f_1'$ and $F|S^{k-1} \times \infty = g$. It remains to show that $F$ is continuous at each $(z, \infty) \in S^{k-1} \times \infty$. Let $(z_1, t_1), (z_2, t_2), \cdots \to (z, \infty)$. By passing to subsequence we can assume that $\phi(F(z_i, t_i)) \to L \in \mathcal{L}(S)$ where convergence is in the Hausdorff topology. If $L$ is not a diagonal extension of $g(z)$, then $L$ is transverse to $g(z)$ and hence $d_{\mathcal{PML}(S)}(F(z_i, t_i), g(z)) > \epsilon$ for $i$ sufficiently large and some $\epsilon > 0$. By passing to subsequence we can assume that $z_i \in \kappa_i$, where $\kappa_i$ is a simplex in $K_i$, where $\kappa_1 \supset \kappa_2 \supset \cdots$ is a nested sequence of simplices and $\cap \kappa_i = z$. Let $n_i$ denote the greatest integer in $t_i$. Apply the second conclusion of Lemma 2.18 to $\kappa_i, \delta_{n_i}, \delta_{n_i+1}$ to conclude that $\lim_{i \to \infty} d_{\mathcal{PML}(S)}(z, \phi(F(z_i, t_i))) = 0$, a contradiction.

Remark 2.19. If there exists a train track $\tau$ such that each $z \in g(S^{k-1})$ is carried by $\tau$ and $V \subset P(\tau)$ is the convex hull of $\phi^{-1}(g(S^{k-1}))$, then there exists a continuous extension $F : B^k \to \mathcal{PML}(S)$ such that $F(\text{int}(B^k)) \subset V$. Indeed, since $V$ is convex we can dispense with the use of the $\pi_k$’s and directly construct the maps $f_i, f_{i+1}, f_1'$ to have values within $V$.

The following local version is needed to prove local $(k-1)$-connectivity of $\mathcal{E}(S)$, when $k \leq n$ and $\dim(\mathcal{PML}(S)) = 2n + 1$.

Proposition 2.20. If $z \in \mathcal{E}(S)$, then for every neighborhood $U$ of $\phi^{-1}(z)$ there exists a neighborhood $V$ of $\phi^{-1}(z)$ such that if $g : S^{k-1} \to \mathcal{E}(S)$ is continuous and
ψ−1(g(Sk−1)) ⊂ V, then there exists a generic PL map \( F : B^k \to \mathcal{PML}(S) \) such that \( F|\text{int}(B^k) \subset U \) and \( F|S^{k-1} = g \).

**Proof.** There exists a parametrized pair of pants decomposition of \( S \) such that \( z \) is fully carried by a maximal standard train track \( \tau \). Thus \( \phi^{-1}(z) \) is a closed convex set in \( \text{int}(V(\tau)) - 0 \). If \( U = p^{-1}(U) \), then \( \hat{U} \cap \text{int}(V(\tau)) \) is a neighborhood of \( \phi^{-1}(z) \), since \( \tau \) is maximal. Let \( \hat{V} \subset \text{int}(V(\tau)) \) be a convex neighborhood of \( \phi^{-1}(x) \) saturated by open rays through the origin such that \( \hat{V} \subset \hat{U} \). Let \( V = p(\hat{V}) \). Then \( V \) is a convex neighborhood of \( \phi^{-1}(x) \) with \( V \subset U \). By Remark 2.19 if \( \psi(g(S^{k-1})) \subset V \), then there exists a continuous map \( F : B^k \to \mathcal{PML}(S) \) such that \( F|S^{k-1} = g \) and \( F|\text{int}(B^k) \subset V \). Now replace \( F \) by a generic perturbation. \( \square \)

### 3. Markers

In this section we introduce the idea of a *marker* which is a technical device for controlling geodesic laminations in a hyperbolic surface. In the next section, using markers, we will show that under appropriate circumstances a sequence of maps \( f_i : B^k \to \mathcal{PML}(S) \), \( i = 1, 2, \cdots \) extending a given continuous map \( g : S^{k-1} \to \mathcal{E}(S) \) converges to an extension \( f_\infty : B^k \to \mathcal{E}(S) \). As always, \( S \) will denote a finite type surface with a fixed complete hyperbolic metric.

**Definition 3.1.** Let \( \alpha_0, \alpha_1 \) be open embedded geodesic arcs in \( S \). A *path* from \( \alpha_0 \) to \( \alpha_1 \) is a continuous map \( f : [0, 1] \to S \) such that for \( i = 0, 1, f(i) \subset \alpha_i \). Two paths are *path homotopic* if they are homotopic through paths from \( \alpha_0 \) to \( \alpha_1 \). Given two path homotopic paths \( f, g \) from \( \alpha_0 \) to \( \alpha_1 \), a lift \( \hat{\alpha}_0 \) of \( \alpha_0 \) to \( \mathbb{H}^2 \) determines unique lifts \( \hat{f}, \hat{g}, \hat{\alpha}_1 \) respectively of \( f, g, \alpha_1 \) so that \( \hat{f}, \hat{g} \) are homotopic paths from \( \hat{\alpha}_0 \) to \( \hat{\alpha}_1 \).

Define \( d_H(f, g) = d_H(\hat{f}(I), \hat{g}(I)) \), where \( d_H \) denotes Hausdorff distance measured in \( \text{PT}(\hat{S}) \). Note that this is well defined independent of the lift of \( \alpha_0 \).

**Definition 3.2.** A *marker* \( \mathcal{M} \) for the hyperbolic surface \( S \) consists of two embedded (though not necessarily pairwise disjoint) open geodesic arcs \( \alpha_0, \alpha_1 \) called *posts* and a path homotopy class \([\alpha]\) from \( \alpha_0 \) to \( \alpha_1 \). A representative \( \beta \) of \([\alpha]\) is said to *span* \( \mathcal{M} \). The marker \( \mathcal{M} \) is an \( \epsilon \)-marker if whenever \( \beta \) and \( \beta' \) are geodesics in \( S \) spanning \( \mathcal{M} \), then \( d_H(\beta, \beta') < \epsilon \) and \( \text{length}(\beta) \geq 1 \).

Let \( C \) be a simple closed geodesic in \( S \). A **C-marker** is a marker \( \mathcal{M} \) such that if \( \beta \) is a geodesic arc spanning \( \mathcal{M} \), then \( \beta \) is transverse to \( C \) and \( |\beta \cap C| > 4g + p + 1 \) where \( g = \text{genus}(S) \) and \( p \) is the number of punctures.

In a similar manner we define the notion of *closed* \( \epsilon \) or *closed C-marker*. Here the posts are closed geodesic arcs. In this case the requirement \( d_H(\beta, \beta') < \epsilon \) is replaced by \( d_H(\beta, \beta') \leq \epsilon \). If \( \mathcal{M} \) is an \( \epsilon \) or C-marker, then \( \mathcal{M} \) will denote the corresponding closed \( \epsilon \) or C-marker.

**Definition 3.3.** We say that the geodesic \( L \) hits the marker \( \mathcal{M} \) if there exists \( \geq 3 \) distinct embedded arcs in \( L \) that span \( \mathcal{M} \). We allow for the possibility that distinct arcs have non trivial overlap. We say that the geodesic lamination \( \mathcal{L} \) *hits* the marker \( \mathcal{M} \) if there exists a leaf \( L \) of \( \mathcal{L} \) that hits \( \mathcal{M} \). If \( b_1, \cdots, b_m \) are simple closed geodesics, then we say that \( \mathcal{M} \) is \( \mathcal{L} \)-free of \( \{b_1, \cdots, b_m\} \) if some leaf \( L \notin \{b_1, \cdots, b_m\} \) of \( \mathcal{L} \) hits \( \mathcal{M} \).

**Lemma 3.4.** Let \( S \) be a finite type hyperbolic surface with a fixed hyperbolic metric. Given \( \epsilon > 0 \) there exists \( N(\epsilon) \in \mathbb{N} \) such that if \( \beta \) is an embedded geodesic arc,
length(β) ≤ 2 and L ∈ \mathcal{L}(S) is such that |\mathcal{L} ∩ β| > N(ε), then there exists an ε-marker \mathcal{M} hit by L with posts α₀, α₁ ⊂ β.

Proof. L has at most 6|χ(S)| boundary leaves. Thus some leaf of \mathcal{L} hits β at least |\mathcal{L} ∩ β|/(6|χ(S)|) times. Since length(β) is uniformly bounded, if |\mathcal{L} ∩ β| is sufficiently large, then three distinct segments of some leaf must have endpoints in β, be nearly parallel and have length ≥ 2. Now restrict to appropriate small arcs of β to create α₀ and α₁ and let [α] be the class represented by the three segments. □

Lemma 3.5. If L ∈ \mathcal{L}(S) has a non compact leaf L, then for every ε > 0 there exists an ε-marker hit by L ∈ \mathcal{L}.

Corollary 3.6. If L ∈ \mathcal{E}L(S), then for every ε > 0 there exists an ε-marker hit by L.

The next lemma states that hitting a marker is an open condition.

Lemma 3.7. If \mathcal{L} ⊂ \mathcal{L}(S) hits the marker \mathcal{M}, then there exists a δ > 0 such that if L′ ∈ \mathcal{L}(S) and L ⊂ N_{PT(S)}(L′, δ), then L′ hits \mathcal{M}.

By super convergence we have

Corollary 3.8. If x ∈ \mathcal{P}ML(S) is such that φ(x) hits the marker \mathcal{M}, then there exists an open set U containing x such that y ∈ U implies that φ(y) hits \mathcal{M}.

Lemma 3.9. Let C ⊂ S a simple closed geodesic. There exists a k > 0 such that if L ∈ \mathcal{L}(S) and |C ∩ L| > k, then there exists a C-marker that is hit by L.

Proof. An elementary topological argument shows that if k is sufficiently large, then there exists a leaf L containing 5 distinct, though possibly overlapping embedded subarcs u₁, · · · , u₅ with endpoints in C which represent the same path homotopy class rel C such that the following holds. Each arc uᵢ intersects C more than 4g + p + 1 times and fixing a preimage \tilde{C} of C to \tilde{S}, these arcs have lifts to arcs \tilde{u}₁, · · · , \tilde{u}₅ in \tilde{S} starting at \tilde{C} and ending at the same preimage \tilde{C}' . After reordering we can assume that \tilde{u}₁ and \tilde{u}₅ are outermost.

Let \tilde{α}_0 and \tilde{α}_₁ be the maximal closed arcs respectively in \tilde{C} and \tilde{C}' with endpoints in \cup \tilde{u}_i and for i = 0, 1 let \tilde{α}_i = π(\tilde{α}_i), where π is the universal covering map.

This gives rise to a C-marker \mathcal{M} with posts α₀, α₁ where αᵢ = int(\tilde{α}_i) and u₂, u₃, u₄ represent the path homotopy class. Note that if the geodesic arc β spans \mathcal{M}, then β lifts to \tilde{β} with endpoints in \tilde{α}_0 and \tilde{α}_₁. Being a geodesic it lies in the geodesic rectangle formed by \tilde{α}_0, \tilde{α}_₁, \tilde{u}_₁, \tilde{u}_₅. Thus it intersects C more than 4g+p+1 times. □

In the rest of this section V will denote the underlying space of a finite simplicial complex. In application, V = Bᵏ or Sᵏ × I.

Definition 3.10. A marker family \mathcal{J} of V is a finite collection (\mathcal{M}_₁, W₁), · · · , (\mathcal{M}_m, W_m) where each \mathcal{M}_i is a marker and each W_j is a compact subset of V. Let f : V → \mathcal{P}ML\mathcal{E}L(S). We say that f hits the marker family \mathcal{J} if for each 1 ≤ i ≤ m and t ∈ Wᵢ, φ(f(t)) hits \mathcal{M}_i. Let C = {b₁, · · · , bₙ} be a set of simple closed geodesics. We say that \mathcal{J} is f-free of C if for each 1 ≤ i ≤ m and t ∈ Wᵢ, \mathcal{M}_i is φ(f(t))-free of C. More generally, if U ⊂ V, then we say that f hits \mathcal{J} along U (resp. \mathcal{J} is f-free of C along U) if for each 1 ≤ i ≤ m and t ∈ U ∩ Wᵢ, φ(f(t)) hits \mathcal{M}_i (resp.
\( \mathcal{M}_t \) is \( \phi(f(t)) \)-free of \( C \). We say that the homotopy \( F : V \times I \to \mathcal{PML}(S) \) is \( \mathcal{J} \)-marker preserving if for each \( t \in I \), \( F|V \times t \) hits \( \mathcal{J} \).

Note that if \( \mathcal{J} \) is \( F \)-free of \( C \), then \( F \) is in particular a \( \mathcal{J} \)-marker preserving homotopy.

An \( \epsilon \)-marker cover (resp. \( C \)-marker cover) of \( V \) is a marker family \((\mathcal{M}_1, W_1), \ldots, (\mathcal{M}_m, W_m)\) where each \( \mathcal{M}_t \) is an \( \epsilon \)-marker (resp. \( C \)-marker) and the interior of the \( W_i \)’s form an open cover of \( V \).

The next lemma gives us conditions for constructing \( \epsilon \) and \( C \)-marker families.

**Lemma 3.11.** Let \( S \) be a finite type hyperbolic surface such that \( \dim(\mathcal{PML}(S)) = 2n + 1 \). Let \( B \) be a finite simplicial complex.

i) If \( \epsilon > 0 \) and \( f : V \to \mathcal{PML}(S) \) is a generic PL map such that \( \dim(V) \leq n \), then there exists an \( \epsilon \)-marker family \( \mathcal{E} \) hit by \( f \).

ii) Given the simple closed geodesic \( C \), there exists \( N(C) \in \mathbb{N} \) such that if \( f : V \to \mathcal{PML}(S) \) is such that for all \( t \in V \), \( |\phi(f(t)) \cap C| \geq N(C) \), then there exists a \( C \)-marker family \( \mathcal{PML}(S) \) hit by \( f \).

Proof of i). Since \( k \leq n \) and \( f \) is generic, for each \( t \in V \), \( A(\phi(t)) \neq \emptyset \), where \( A(\phi(t)) \) is the arational sublamination of \( \phi(t) \). By Lemma 3.4 for each \( t \) there exists an \( \epsilon \)-marker \( \mathcal{M}_t \) and compact set \( W_t \) such that \( t \in \text{int}(W_t) \) and for each \( s \in W_t \), \( \phi(f(s)) \) hits \( \mathcal{M}_t \). The result follows by compactness of \( V \).

Proof of ii). Given \( C \), choose \( N(C) \) as in Lemma 3.9. Thus for each \( t \in V \) there exists a \( C \)-marker \( \mathcal{M}_t \) and compact set \( U_t \) such that \( t \in \text{int}(U_t) \) and for each \( s \in U_t \), \( \phi(f(s)) \) hits \( \mathcal{M}_t \). The result follows by compactness of \( V \).

4. CONVERGENCE LEMMAS

This section establishes various criteria to conclude that a sequence of ending laminations converges to a particular ending lamination or to show that two ending laminations are close in \( \mathcal{E}(S) \). We also show that markers give neighborhood bases of elements \( \mathcal{L} \in \mathcal{E}(S) \) and sets in \( \mathcal{PML}(S) \) of the form \( \phi^{-1}(\mathcal{L}) \), where \( \mathcal{L} \in \mathcal{E}(S) \).

**Lemma 4.1.** Let \( \mu \in \mathcal{E}(S) \) and

\[ W_{\epsilon}(\mu) = \{ \mathcal{L} \in \mathcal{E}(S) | d_{PT(S)}(\mathcal{L}, \mu') < \epsilon, \mu' \text{is a diagonal extension of } \mu \} \]

Then \( W(\mu) = \{ W_{\epsilon}(\mu) | \epsilon > 0 \} \) is a neighborhood basis of \( \mu \in \mathcal{E}(S) \).

Proof. By definition of coarse Hausdorff topology, \( W_{\epsilon}(\mu) \) is open in \( \mathcal{E}(S) \). Therefore if the lemma is false, then there exists a sequence \( \mathcal{L}_1, \mathcal{L}_2, \ldots \) such that \( \mathcal{L}_i \in W_{1/i}(\mu) \) all \( i \) and a \( c > 0 \) such that for all \( i, \mathcal{L}_i \notin N_{PT(S)}(\mu', c) \) for all diagonal extensions \( \mu' \) of \( \mu \). After passing to subsequence we can assume that \( \{ \mathcal{L}_i \} \to \mathcal{L}_\infty \) with respect to the Hausdorff topology. If \( \mathcal{L}_\infty \) is not a diagonal extension of \( \mu \), then \( \mathcal{L}_\infty \) is transverse to each diagonal extension of \( \mu \) and hence there exists an \( \epsilon > 0 \) such that \( d_{PT(S)}(\mu', \mathcal{L}_i) > \epsilon \) for all \( i \) sufficiently large and every diagonal extension of \( \mu \), a contradiction.

**Lemma 4.2.** Let \( \mu \in \mathcal{E}(S) \), \( \mu' \) a diagonal extension and \( x_1, x_2, \ldots \in \mathcal{PML}(S) \) such that \( \lim_{t \to \infty} d_{PT(S)}(\phi(x_t), \mu') = 0 \), then after passing to subsequence \( x_i \to x_\infty \in \phi^{-1}(\mu) \).
Proof. After passing to subsequence we can assume that \( x_i \to x_\infty \in \mathcal{PML}(S) \). If \( \phi(x_\infty) \neq \mu \), then \( \phi(x_\infty) \) intersects \( \mu \) transversely. Let \( p \in \phi(x_\infty) \cap \mu \) and \( L \) the leaf of \( \mu \) containing \( p \). Then \( \phi(x_\infty) \) intersects \( L \) at \( p \) at some angle \( \theta > 0 \). By super convergence, for \( i \) sufficiently large \( \phi(x_i) \) intersects \( \mu \) at \( p_i \in L \) at angle \( \theta_i \), where \( p_i \) is very close to \( p \) (distance measured intrinsically in \( L \)) and \( \theta_i \) is very close to \( \theta \). Since every leaf of \( \mu' \) is dense in \( \mu \), it follows that there exists \( N > 0 \) such that if \( J \) is a segment of a leaf of \( \mu' \) at least length \( N \) and \( i \) is sufficiently large, then \( J \cap \phi(x_i) \neq \emptyset \) with angle of intersection at some point at least \( \theta/2 \). Thus \( d_{PT(S)}(\phi(x_i),\mu') \) must be uniformly bounded below, else some \( \phi(x_i) \) would have a transverse self intersection.

\[ \square \]

Lemma 4.3. If \( K \subset \mathcal{PML}(S) \) and \( L \subset S(L) \) are compact and \( K \cap \phi^{-1}(L) = \emptyset \), then there exists \( \delta > 0 \) such that if \( d_{PT(S)}(\phi(x),\mu) < \delta \) where \( x \in \mathcal{PML}(S) \) and \( \mu \in L \), then \( x \notin K \).

Proof. Otherwise there exists sequences \( x_1, x_2, \ldots \to x_\infty \), \( \mu_1, \mu_2, \ldots \to \mu_\infty \) such that the \( x_i \)'s \( \in K \) and the \( \mu_i \)'s \( \in L \) and \( \lim_{i \to \infty} d_{PT(S)}(\phi(x_i),\mu_i) = 0 \). After passing to subsequence we can assume that the \( \mu_i \)'s converge to a diagonal extension of \( \mu_\infty \). This contradicts Lemma 4.2. \[ \square \]

Lemma 4.4. If \( L \subset S(L) \) is compact and \( U \subset \mathcal{PML}(S) \) is open such that \( \phi^{-1}(L) \subset U \), then there exists a neighborhood \( V \) of \( L \) such that \( \phi^{-1}(V) \subset U \).

Proof. Let \( W_\epsilon(L) = \{ \mathcal{L} \in S(L) | d_{PT(S)}(\mathcal{L},\mu') < \epsilon \} \) is a diagonal extension of \( \mu \) \( \in L \). Then \( W_\epsilon(L) \) is open and for \( \epsilon \) sufficiently small \( \phi^{-1}(W_\epsilon(L)) \subset U \) otherwise taking \( K = \mathcal{PML}(S) \setminus U \) one obtains a contradiction to the previous lemma. \[ \square \]

Lemma 4.5. Let \( \mu \in S(L) \) and \( M_1, M_2, \ldots \) a sequence of markers such that for every \( i \in \mathbb{Z} \), \( M_i \) is a \( 1/i \)-marker hit by \( \mu \). If \( U_i = \{ \mathcal{L} \in S(L) | \mathcal{L} \text{ hits } M_i \} \), then \( U = \{ U_i \} \) is a neighborhood basis of \( \mu \) in \( S(L) \).

Proof. By definition of \( 1/i \)-marker, if \( \mathcal{L} \) hits \( M_i \), then \( d_{PT(S)}(\mathcal{L},\mu) < 1/i \). Therefore, for all \( i \), \( U_i \subset W_{1/i} \). Since each \( U_i \) is open in \( S(L) \) the result follows. \[ \square \]

Lemma 4.6. Let \( \mu \in S(L) \). For each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( \{ \mathcal{L}^1, \mathcal{L}^2, \ldots, \mathcal{L}^k, z \} \subset S(L) \), \( d_{PT(S)}(\mathcal{L}^k, z) < \delta \), \( d_{PT(S)}(\mathcal{L}^i, \mathcal{L}^{i+1}) < \delta \) for \( 1 \leq i \leq k-1 \) and \( d_{PT(S)}(\mu', \mathcal{L}^1) < \delta \) for some diagonal extension \( \mu' \) of \( \mu \), then \( d_{PT(S)}(\mu, z) < \epsilon \).

Proof. We give the proof for \( k = 1 \), the general case being similar. If the Lemma is false, then there exists a sequence \( (\mathcal{L}_i, z_i, \delta_i) \) for which the lemma fails, where \( \delta_i \to 0 \). After passing to subsequence we can assume that \( \mathcal{L}_i \to \mathcal{L}_\infty \) and \( z_i \to z_\infty \) with respect to the Hausdorff topology. Since \( z_\infty \) is nowhere tangent to \( \mu \), it is transverse to \( \mu \). Since the ends of every leaf of every diagonal extension of \( \mu \) is dense in \( \mu \) and \( \mu \) is filling, there exists \( K > 0 \) such that any length \( K \) immersed segment lying in a leaf of \( z_\infty \) intersects any length \( K \) segment lying in any diagonal extension of \( \mu \) at some angle uniformly bounded away from \( 0 \). Thus a similar statement holds for each \( z_i,i \) sufficiently large, where \( K \) is replaced by \( K + 1 \). Therefore if \( \delta_i \) is sufficiently small and \( i \) is sufficiently large, then \( \mathcal{L}_i \) has length \( K + 2 \) immersed segments \( \sigma_1, \sigma_2 \) such that \( \sigma_1 \) is nearly parallel to a leaf of \( \mu' \) and \( \sigma_2 \) is nearly parallel to a leaf of \( z_i \). This implies that \( \sigma_1 \) nontrivially intersects \( \sigma_2 \) transversely, a contradiction. \[ \square \]
**Lemma 4.7.** Let \( \epsilon > 0 \). Let \( \tau_1, \tau_2, \ldots \) be a full unzipping sequence of the transversely recurrent train track \( \tau_1 \). If each \( \tau_i \) fully carries the geodesic lamination \( \mathcal{L} \), then there exists \( N > 0 \) such that if \( \mathcal{L}_1 \) is carried by \( \tau_i \), for some \( i \geq N \), then \( d_{PT(S)}(\mathcal{L}_1, \mathcal{L}) < \epsilon \).

**Proof.** This follows from the proof of Lemma 1.7.9 [PH] (see also Proposition 1.9 [G1]). That argument shows that each biinfinite train path of each \( \tau_i \) is a uniform quasi-geodesic and that given \( L > 0 \), there exists \( N > 0 \) such that any length \( L \) segment lying in a leaf of a lamination carried by \( \tau_i \), \( i \geq N \), is isotopic to a leaf of \( \mathcal{L} \) by an isotopy such that the track of a point has uniformly bounded length. \( \square \)

**Lemma 4.8.** If \( \tau \) is a train track that carries \( \mu \in \mathcal{E}\mathcal{L}(S) \), then \( \tau \) fully carries a diagonal extension of \( \mu \).

**Proof.** By analyzing the restriction of \( \tau \) to each closed complementary region of \( \mu \), it is routine to add diagonals to \( \mu \) to obtain a lamination fully carried by \( \tau \). \( \square \)

**Lemma 4.9.** Let \( \kappa \) be a transversely recurrent train track that carries \( \mu \in \mathcal{E}\mathcal{L}(S) \). Given \( \delta > 0 \) there exists \( N > 0 \) so that if \( \tau \) is obtained from \( \kappa \) by a sequence of \( \geq N \) full splittings (i.e., along all the large branches) and \( \tau \) carries both \( \mu \) and \( \mathcal{L} \in \mathcal{L}(S) \), then \( d_{PT(S)}(\mathcal{L}, \mu) < \delta \).

**Proof.** It suffices to show that \( d_{PT(S)}(\mathcal{L}, \mu') < \delta_1 \) for some diagonal extension \( \mu' \) of \( \mu \) and some \( \delta_1 > 0 \), that depends on \( \delta \) and \( \mu \). Since there are only finitely many train tracks obtained from a given finite number of full splittings of \( \kappa \) it follows that if the lemma is false, then there exist \( \tau_1, \tau_2, \ldots \) such that \( \tau_1 = \kappa, \tau_2, \ldots \) is a full splitting of \( \tau_{-1} \) and for each \( i \in \mathbb{N} \) there exists \( \mathcal{L}_i \in \mathcal{L}(S) \) carried by some splitting of \( \tau_{n_i} \), with \( d_{PT(S)}(\mathcal{L}_i, \mu) > \delta \) and \( n_i \to \infty \). Note that \( \mathcal{L}_i \) is also carried by \( \tau_{n_i} \). Since \( \mu \) has only finitely many diagonal extensions we can assume from the previous lemma that each \( \tau_i \) fully carries a fixed diagonal extension \( \mu' \) of \( \mu \).

On the other hand, there is a full unzipping sequence \( \tau'_1 = \tau_1, \tau'_2, \ldots \) with the property that each \( \tau'_i \) carries exactly the same laminations as some \( \tau_{m_i} \) and \( m_i \to \infty \). Thus by Lemma 4.7 it follows that for \( i \) sufficiently large \( d_{PT(S)}(\mathcal{L}_i, \mu') < \delta_1 \), a contradiction. \( \square \)

For once and for all fix a parametrized pants decomposition of \( S \), with the corresponding finite set of standard train tracks.

**Proposition 4.10.** Given \( \epsilon > 0, \mu \in \mathcal{E}\mathcal{L}(S) \), there exists \( N > 0, \delta > 0 \) such that if \( \tau \) is obtained from a standard train track by \( N \) full splittings and \( \tau \) carries \( \mu \), then the following holds. If \( \mathcal{L} \in \mathcal{L}(S) \) is carried by \( \tau \), \( z \in \mathcal{L}(S) \) and \( d_{PT(S)}(\mathcal{L}, z) < \delta \), then \( d_{PT(S)}(\mu, z) < \epsilon \). In particular, if \( \mathcal{L} \) is carried by \( \tau \), then \( d_{PT(S)}(\mu, \mathcal{L}) < \epsilon \).

**Proof.** Apply Lemmas 4.9 and 4.6. \( \square \)

5. A CRITERION FOR CONSTRUCTING CONTINUOUS MAPS OF COMPACT MANIFOLDS INTO \( \mathcal{E}\mathcal{L}(S) \)

This section is a generalization of the corresponding one of [G1] where a criterion was established for constructing continuous paths in \( \mathcal{E}\mathcal{L}(S) \). Our main result is much more general and is technically much simpler to verify. It will give a criterion for extending a continuous map \( g : S^{k-1} \to \mathcal{E}\mathcal{L}(S) \) to a continuous map \( \mathcal{L} : B^k \to \mathcal{E}\mathcal{L}(S) \), though it is stated in a somewhat more general form.
Recall that our compact surface $S$ is endowed with a fixed hyperbolic metric. Let $\{C_i\}_{i \in \mathbb{N}}$ denote the set of simple closed geodesics in $S$.

**Notation 5.1.** If $U_j$ is a finite open cover of a compact set $V$, then its elements will be denoted by $U_j(1), \cdots U_j(k_j)$.

**Proposition 5.2.** Let $V$ be the underlying space of a finite simplicial complex and $W$ the subspace of a subcomplex. Let $g : W \to \mathcal{E}L(S)$ and for $i \in \mathbb{N}$ let $f_i : V \to \mathcal{PML}E \mathcal{L}(S)$ be continuous extensions of $g$. Let $\mathcal{L}_m(t)$ denote $\phi(f_m(t))$. Let $\epsilon_1, \epsilon_2, \cdots$ be such that for all $i$, $\epsilon_i/2 > \epsilon_{i+1} > 0$. Let $U_1, U_2, \cdots$ be a sequence of finite open covers of $V$. Suppose that each $U_j(k)$ is assigned both an $\epsilon_j$-marker $\alpha_j(k)$ and a $C_j$-marker $\beta_j(k)$. Assume that the following two conditions hold.

**(sublimit)** For each $t \in U_j(k)$ and $m \geq j$, $\mathcal{L}_m(t)$ hits $\alpha_j(k)$.

**(filling)** For each $t \in U_j(k)$ and $m \geq j$, $\mathcal{L}_m(t)$ hits $\beta_j(k)$.

Then there exists a continuous map $L : V \to \mathcal{E}L(S)$ extending $g$ so that for $t \in V$, $L(t)$ is the coarse Hausdorff limit of $\{\mathcal{L}_m(t)\}_{m \in \mathbb{N}}$.

**Proof.** Fix $t$. We first construct a minimal and filling $L(t)$. After passing to subsequence we can assume that the sequence $\mathcal{L}_m(t)$ converges in the Hausdorff topology to a lamination $\mathcal{L}'(t)$. If $t \in U_i(j)$, then the filling and sublimit conditions imply that if $k > i$, then some arcs $\gamma_i(k), \sigma_i(k)$ in leaves of $\mathcal{L}_k(t)$ respectively span the $\epsilon_i$ and $C_i$-markers $\alpha_i(j)$ and $\beta_i(j)$. This implies that arcs in $\mathcal{L}'(t)$ span the corresponding closed markers and hence, $\mathcal{L}'(t)$ intersects each $C_i$ transversely and hence $\mathcal{L}'(t)$ contains no closed leaves. Thus spanning arcs in $\mathcal{L}'(t)$ are embedded (as opposed to wrapping around a closed geodesic) and hence $|\mathcal{L}'(t) \cap C_i| > 4g + p + 1$ for all $i$. Let $L(t)$ be a minimal sublamination of $\mathcal{L}'(t)$. If $L(t)$ is not filling, then there exists a simple closed geodesic $C_i$ disjoint from $L(t)$ that can be isotoped into any neighborhood of $L(t)$ in $S$. An elementary topological argument shows that $|C \cap \mathcal{L}'(t)| \leq 4g + p + 1$, contradicting the filling condition.

We next show that $L(t)$ is independent of subsequence. Let $\mathcal{L}'_0(t) \in \mathcal{E}L(S)$ be a lamination that is the Hausdorff limit of the subsequence $\{\mathcal{L}_k(t)\}$ and $\mathcal{L}_0(t)$ the sublamination of $\mathcal{L}'_0(t)$ in $\mathcal{E}L(S)$. By the sublimit condition each of $\mathcal{L}'(t), \mathcal{L}'_0(t)$ have arcs that span the same set $\{\alpha_i\}$ of closed markers, where $\alpha_i$ is an $\epsilon_i$-marker with associated open set $U_i \subset V$, where $t \in U_i$. Since $\epsilon_i \to 0$, the lengths of the initial posts $\{\alpha_{i_0}\}$ go to 0. Thus after passing to a subsequence of the initial posts, $\{\alpha_{i_0}\} \to x \in S$. Now let $v_{i\cdot}$ be the unit tangent vector to the initial point of some spanning arc of $\alpha_{i\cdot}$. After passing to another subsequence, $v_{i\cdot} \to v$ a unit tangent vector to $x$. The sublimit condition implies that $v$ is tangent to a leaf of both $\mathcal{L}'(t)$ and $\mathcal{L}'_0(t)$ and hence $\mathcal{L}'(t)$ and $\mathcal{L}'_0(t)$ have a leaf in common. It follows that $L(t) = \mathcal{L}_0(t)$.

We apply Lemma 1.9 to show that $f$ is continuous at $t$. Let $v$ and $\{\alpha_{i\cdot}\}_{i \in \mathbb{N}}$ be as in the previous paragraph, where $\{\alpha_{i\cdot}\}_{i \in \mathbb{N}}$ is the final subsequence produced in that paragraph. Fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for $i \geq N$, $d_{PT(S)}(v_{i\cdot}, v) \leq \epsilon$ where $v_{i\cdot}$ is any unit tangent vector to the initial point of a spanning arc of $\alpha_{i\cdot}$. Therefore if $m \geq N_j$ and $s \in U_{N_j}$, then $d_{PT(S)}(\mathcal{L}_m(s), v) \leq \epsilon$. Since this is true for all $m \geq N_j$ it follows that for all $s \in U_{N_j}$, $d_{PT(S)}(\mathcal{L}'(s), \mathcal{L}'(t)) \leq \epsilon$. \hfill $\square$
6. Pushing off of $B_C$

Given a generic PL map $f : B^k \to PMLC(L)(S)$ and a simple closed geodesic $C$, this section will describe homotopies of $f$ such that if $f_1$ is a resulting map, then $f_1^{-1}(BC) = \emptyset$. The map $f_1$ is said to be obtained from $f$ by pushing off of $C$. Various technical properties associated with such push off’s will be obtained. The concept of relatively pushing $f$ off of $C$ will be introduced and analogous technical results will be established. In subsequent sections we will produce a sequence $f_1, f_2, \cdots$ satisfying the hypothesis of Proposition 5.2, where $f_{i+1}$ is obtained by relatively pushing $f_i$ off of a finite set of geodesics, one at a time.

**Remark 6.1.** Recall the convention that $n$ is chosen so that $\dim(PMLC(S)) = 2n + 1$. Let $C$ be a simple closed geodesic. Let $\lambda_C$ denote the projective measure lamination with support $C$. As in [G1], we denote by $C_C$ the PL 2n-ball consisting of those projective measured laminations that have intersection number 0 with $\lambda_C$. Recall that $B_C$ is the cone of the PL $(2n-1)$-sphere $\delta B_C$ to $\lambda_C$, where $\delta B_C$ consists of those points of $B_C$ which do not have $C$ as a leaf. Furthermore, if $x \in B_C \setminus \lambda_C$, then $x = p((1-t)\lambda + t\lambda_C)$, for some $\lambda \in MLC(S)$ representing a unique $\lambda \in \delta C$, some $\hat{\lambda}_C \in MLC(S)$ representing $\hat{\lambda}_C$ and some $t < 1$.

**Definition 6.2.** The ray through $x \in B_C \setminus \lambda_C$ is the set of points $r(x)$ in $PMLC(S)$ represented by measured laminations of the form \{$(1-t)\lambda + t\lambda_C | 0 \leq t \leq 1$\} where $\hat{\lambda}$ and $\hat{\lambda}_C$ are as above. If $K \subset B_C$ and $K \cap \lambda_C = \emptyset$, then define $r(K) = \cup_{x \in K} r(x)$.

**Remarks 6.3.** Note that $r(x)$ is well defined and $r(K)$ is compact if $K$ is compact.

Using the methods of [Th1], [PH] or [G1] it is routine to show that there exists a neighborhood of $B_C$ homeomorphic to $2B^{2n} \times [-1,1]$, where $2B^{2n}$ denotes the radius-2 2n-ball about the origin in $\mathbb{R}^{2n}$, such that $B_C$ is identified with $B^{2n} \times 0$, $\lambda_C$ is identified with $(0,0)$ and for each $x \in B_C \setminus \lambda_C$, $r(x)$ is identified with a ray through the origin with an endpoint on $S^{2n-1} \times 0$.

While the results in this section are stated in some generality, on first reading one should imagine that if $f : V \to PMLC(L)(S)$, then $V = B^k$ and $f^{-1}(EL(S)) = S^{k-1}$, where $k \leq 2n$.

**Definition 6.4.** Let $V$ be the underlying space of a finite simplicial complex and $W$ that of a subcomplex. If $f : V \to PMLC(L)(S)$ and $W = f^{-1}(EL(L)(S))$, then the generic PL map $f_1 : V \to PMLC(S)$ is said to be obtained from $f$ by $\delta$-pushing off of $B_C$; if there exists a homotopy $F : V \times I \to PMLC(L)(S)$, called a $(C, \delta)$ push off homotopy such that

i) $f_1^{-1}(B_C) = \emptyset$,

ii) $F(t,s) = f(t)$ if either $s = 0$ or $d_V(t, f^{-1}(B_C)) \geq \delta$ or $d_V(t, W) \leq \delta$ or $d_{\mathcal{PMLC}}(f(t), B_C) \geq \delta$ and

iii) for each $t \in V$ such that $d_{\mathcal{PMLC}}(f(t), B_C) < \delta$ there exists an $x \in f(V) \cap B_C$ such that for all $s \in [0, 1]$, $d_{\mathcal{PMLC}}(F(t, s), r(x)) < \delta$. Furthermore if $f(t, s) \in B_C$, then $F(t, s) \in r(f(t))$.

**Lemma 6.5.** If $f : V \to PMLC(L)(S)$ is a generic PL map, $\dim(V) \leq 2n$ and $C$ is a simple closed geodesic, then for all sufficiently small $\delta > 0$ there exists a $(C, \delta)$ push off homotopy of $f$.

**Proof.** It follows by super convergence and compactness that if $C$ is a simple closed geodesic, then $f^{-1}(B_C)$ is a compact set disjoint from some neighborhood
of \( W \). By genericity of \( f \), i.e. Lemma 2.4, there exists an \( \epsilon_1 > 0 \) such that 
\[
d_{PML}(f(V), A_C) \geq \epsilon_1.
\]
Consider a natural homotopy \( F : ((2B^{2n} \setminus \epsilon B^{2n}) \times [-1,1]) \to 2B^{2n} \times [-1,1] \) from the inclusion to a map whose image is disjoint from \( B^{2n} \times 0 \), which is supported in an \( \epsilon \)-neighborhood of \((B^{2n} \setminus \epsilon B^{2n}) \times 0 \) and where points in \((B^{2n} \setminus \epsilon B^{2n}) \times 0 \) are pushed radially out from the origin. Let 
\[
g : N(B_C) \to 2B^{2n} \times [-1,1]
\]
denote the parametrization given by Remark 6.3. The desired \((C, \delta)\)-homotopy is obtained by appropriately interpolating the trivial homotopy outside of a very small neighborhood of \( f \) restricted to a small neighborhood of \( f^{-1}(B_C) \) where \( \epsilon \) is sufficiently small and then doing a small perturbation to make \( f_1 \) generic. 

\[\square\]

**Lemma 6.6.** Let \( C \) be a simple closed geodesic and \( f : V \to PML(S) \) be a generic PL map with \( \dim(V) \leq 2n \). Let \( J \) be a marker family of \( V \) hit by \( f \) that is free of \( C \). If \( \delta \) is sufficiently small, then any \((C, \delta)\) homotopy \( F \) from \( f \) to \( f_1 \) is \( J \)-marker preserving, free of \( C \).

**Proof.** Since there are only finitely many markers in a marker family, it suffices to show that if \( K \subset V \) is compact and \( \phi(f(t)) \) hits the marker \( M \) free of \( C \) at all \( t \in K \), then for \( \delta \) sufficiently small \( \phi(F(t,s)) \) hits \( M \) free of \( C \) at all \( t \in K \) and \( s \in I \). This is a consequence of super convergence and compactness. Indeed, if \( x \in f(K) \cap B_C \), then there exists a leaf of \( \phi(x) \) distinct from \( C \) hits \( M \). Since all points in \( r(x) \setminus \lambda_C \) have the same underlying lamination this fact holds for all \( y \in r(x) \). By super convergence it holds at all points in a neighborhood of \( r(x) \) in \( PML(S) \). Let \( U \) be the union of these neighborhoods over all \( x \in f(K) \cap B_C \). By compactness of \( K \) and \( B_C \), there exists a \( \eta > 0 \) such that if \( y \in B_C \) and \( d_{PML}(y, f(t)) \leq \eta \) for some \( t \in K \), then \( N_{PML}(r(y), \eta) \subset U \). Any \((C, \delta)\) homotopy with \( \delta < \eta \) satisfies the conclusion of the lemma. 

\[\square\]

**Definition 6.7.** If \( x \in PML(S) \) and \( A \) is a simple closed geodesic, then define 
\[
g(x, A) \in Z_{2n} \cup \infty \text{ the geometric intersection number of } x \text{ with } A \text{ by } g(x, A) = \min\{|\phi(x) \cap A'| | A' \text{ is isotopic to } A\}. \]
If \( f : V \to PML(S) \) define the geometric intersection number of \( f \) with \( A \) by 
\[
g(f, A) = \min\{|g(\phi(f(t)), A)| | t \in V \}. \]
If \( 0 < g(f, A) < \infty \), then we say that the multi-geodesic \( J \) is a stryker curve for \( A \) if for some \( t \in V \), \( J \subset \phi(f(t)) \) and \( |J \cap A| = g(f, A) \). We call \( J \) the \( f(t) \)-stryker curve or sometimes the stryker curve at \( f(t) \).

**Remark 6.8.** Note that \(|\phi(x) \cap A'|\) is minimized when \( A = A' \) unless \( A \) is a leaf of \( \phi(x) \) in which case \( g(x, A) = 0 \).

**Lemma 6.9.** If \( f : V \to PML(S) \), \( A \) is a simple closed geodesic and \( 0 < g(f, A) < \infty \), then the set of stryker curves is finite. Also 
\[
m(f, A) = \{ t \in V \setminus \phi(f(t)) \cap A = g(f, A) \} \text{ is compact. Finally } m(f, A) \text{ is the disjoint union of the compact sets } m_{J_1}(f, A), \ldots, m_{J_n}(f, A) \text{ where } t \in m_{J_i}(f, A) \text{ implies that } J_i \text{ is the stryker curve at } f(t). \]

**Proof.** Super convergence implies that \( V \setminus m(f, A) \) is open, hence \( m(f, A) \) is compact. If the first assertion is false, then there exists \( t_1, t_2, \ldots \) converging to \( t \) such that if \( J_i \) denotes the stryker curve at \( t_i \), then the \( J_i \)'s are distinct. By compactness, \( t \in m(f, A) \), so let \( J \) be the stryker curve at \( t \). Super convergence implies that if \( s \) is sufficiently close to \( t \), then either \( s \notin m(f, A) \) or \( J \) is the stryker curve at \( s \), a contradiction. The final assertion again follows from super convergence. 

\[\square\]
Lemma 6.10. Let \( f : V \to \mathcal{PMEL}(S) \) be a generic PL map and \( A \) and \( C \) disjoint simple closed geodesics such that \( 0 < g(f, A) < \infty \). Then for \( \delta \) sufficiently small, any \((C, \delta)\) push off \( f_1 \) satisfies \( g(f_1, A) \geq g(f, A) \). If equality holds and \( J \) is a stryker curve for \( f_1 \), then \( J \) is a stryker curve for \( f \).

Proof. If \( t \in f^{-1}(BC) \), then \( |(\phi(f(t)) \cap C) \cap A| \leq |\phi(f(t)) \cap A| \geq g(f, A) \). By super convergence there exists a neighborhood \( U' \) of \( r(f(t)) \) such that \( y \in U' \) implies that \( g(\phi(y), A) \geq g(f, A) \) and hence there exists a neighborhood \( U \) of \( r(f(V) \cap BC) \) with the same property. If \( \delta \) is sufficiently small to have any \((C, \delta)\) push off homotopy supported in \( U \), then \( g(f_1, A) \geq g(f, A) \).

Now assume that equality holds. If \( t \in m(f, A) \cap f^{-1}(BC) \) and \( J \) is the stryker curve at \( f(t) \), then by super convergence there exists a neighborhood \( U' \) of \( r(f(t)) \) such that if \( x \in U' \) and \( g(x, A) = g(f, A) \), then \( J \) is the stryker curve at \( x \). Thus, there exists a neighborhood \( U \) of \( f(V) \cap BC \) such that \( x \in U \) and \( g(x, A) = g(f, A) \), then the stryker curve at \( x \) is a stryker curve of \( f \). If \( \delta \) is sufficiently small to have any \((C, \delta)\) push off of \( f \) supported in \( U \), then the second conclusion holds.

This argument proves the following sharper result.

Lemma 6.11. If \( f : V \to \mathcal{PMEL}(S) \) and \( \eta > 0 \), then there exists \( \delta > 0 \) such that if \( \delta \) is sufficiently small and \( f_1 \) is the result of a \((C, \delta)\) push off homotopy of \( f \), then \( m(f_1, A) \subset N_{\nu}(m(f, A), \eta) \) and if \( t \in m(f_1, A) \) and \( d_{\nu}(t, m_{I_{x}}(f, A)) < \eta \), then \( J_{1} \) is the stryker curve at \( f_{1}(t) \).

We need relative versions of generalizations of the above results.

Definition 6.12. Let \( V \) be the underlying space of a finite simplicial complex and \( W \) that of a finite subcomplex. Let \( f : V \to \mathcal{PMEL}(S) \) be a generic PL map with \( f^{-1}(\mathcal{EL}(S)) = W \). Let \( K \subset f^{-1}(BC) \) be closed. We say that the generic PL map \( f_{1} : V \to \mathcal{PMEL}(S) \) is obtained from \( f \) by \((C, \delta, K)\) pushing off if there exists a homotopy \( F : V \times I \to \mathcal{PMEL}(S) \) called a \((C, \delta, K)\) push off homotopy such that

i) \( f_{1}(K) \cap BC = \emptyset \),

ii) \( F(t, s) = f(t) \) if either \( s = 0 \) or \( d_{\nu}(t, K) \geq \delta \) or \( d_{\nu}(t, W) \leq \delta \) or \( d_{\mathcal{PMEL}(S)}(f(t), f(K)) \geq \delta \).

iii) for each \( t \in V \) such that \( d_{\mathcal{PMEL}(S)}(f(t), f(K)) < \delta \) there exists an \( x \in r(f(K)) \) such that for all \( s \in [0, 1] \), \( d_{\mathcal{PMEL}(S)}(F(t, s), r(x)) < \delta \). Furthermore, if \( F(t, s) \in BC \), then \( F(t, s) \in r(f(t)) \) and if \( f_{1}(t) \in \text{int}(BC) \), then \( f(t) \in \text{int}(BC) \).

Lemma 6.13. If \( f : V \to \mathcal{PMEL}(S) \) is a generic PL map, \( \text{dim}(V) \leq 2n \), \( C \) a simple closed geodesic and \( K \) a closed subset of \( f^{-1}(BC) \), then for every sufficiently small \( \delta > 0 \) there exists a \((C, \delta, K)\) push off homotopy of \( f \).

Proof. For \( \delta \) sufficiently small let \( U \subset V \setminus N_{\nu}(W, \delta) \) be open such that \( K \subset U \subset N_{\nu}(K, \delta/10) \cap f^{-1}(N_{\mathcal{PMEL}(S)}(r(f(K)), \delta/10)) \). Let \( \rho : V \to [0, 1] \) be a continuous function such that \( \rho(K) = 1 \) and \( \rho(V \setminus U) = 0 \). If \( F(t, s) \) defines a \((C, \delta)\) push off homotopy, then \( F(t, \rho(t)s) \) suitably perturbed defines a \((C, \delta, K)\) push off homotopy.

Lemma 6.14. Let \( V \) be a finite \( p \)-complex and \( f : V \to \mathcal{PMEL}(S) \) a generic PL map, \( C \) a simple closed geodesic, \( L \subset \mathcal{PMEL}(S) \) a finite \( q \)-subcomplex of \( \mathcal{EL}(S) \) and \( K \) a closed subset of \( f^{-1}(BC) \). If \( p+q \leq 2n-1 \) or \( p \leq n \), then for every sufficiently small \( \delta > 0 \) any \((C, \delta, K)\) push off homotopy of \( f \) is supported away from \( L \).
Proof. Let $Z = (\partial B_C \cap L) * C$. By Lemma 2.4, $f(V) \cap (Z \cup L) = \emptyset$ and hence $r(f(K)) \cap (Z \cup L) = \emptyset$. Thus the conclusion of the lemma holds provided $\delta < d_{\mathcal{PMEL}}(S)(r(f(K)), Z \cup L)$. \hfill \square

We have the following relative version of Lemma 6.6.

**Lemma 6.15.** Let $b_1,b_2,\cdots,b_r,C$ be simple closed geodesics, $f : V \to \mathcal{PMEL}(S)$ a generic PL map and $K \subset f^{-1}(B_C)$ be compact. Let $\mathcal{F}$ be a marker family that is $f$-free of $\{b_1,\cdots,b_r,C\}$. If $\delta$ is sufficiently small, then $\mathcal{F}$ is $F$-free of $\{b_1,\cdots,b_r,C\}$ for any $(C,\delta,K)$ push off homotopy $F$ of $f$. \hfill \square

**Definition 6.16.** Let $f : V \to \mathcal{PMEL}(S)$. Let $Y$ be a compact subset of $V$ and $A$ a simple closed geodesic. Define $g(f,A;Y) = \min\{g(f(t),A)|t \in Y\}$, the $Y$-geometric intersection number of $f$ and $A$. If $0 < g(f,A;Y) < \infty$, define the $Y$-stryker curves for $f$ and $A$ to be those multi-geodesics $\sigma$ such that for some $t \in Y$, $\sigma \subset \phi(f(t))$ and $|\sigma \cap A| = g(f,A;Y)$. Define $m(f,A;Y) = \{t \in Y|g(f(t),A) = g(f,A;Y)\}$ and if $\sigma$ is a $Y$-stryker curve then define $m_\sigma(f,A;Y) = \{t \in Y|\sigma \subset \phi(f(t))$ and $|\sigma \cap A| = g(f,A;Y)\}$. We say that the set $B$ of simple closed geodesics solely hits the marker $M$ at $t$ if each leaf of the lamination $\phi(f(t))$ that hits $M$ lies in $B$. If $Z \subset V$ then let $S(f,M,B,Z)$ denote the set of points in $Z$ where $f$ solely hits $M$.

The proof of Lemma 6.9 holds for in the relative case.

**Lemma 6.17.** If $f : V \to \mathcal{PMEL}(S)$, $A$ is a simple closed geodesic, $Y \subset V$ is compact and $0 < g(f,A;Y)$, then the set of $Y$-stryker curves is finite. Also $m(f,A;Y)$ is compact and is the disjoint union of compact sets $m_{J_1}(f,A;Y), \cdots, m_{J_m}(f,A;Y)$ where $t \in m_{J_i}(f,A;Y)$ implies that $J_i$ is a $Y$-stryker curve at $f(t)$. \hfill \square

By super convergence we have the following result.

**Lemma 6.18.** If $f : V \to \mathcal{PMEL}(S)$ is a generic PL map, $B$ is a finite set of simple closed geodesics, $Z \subset V$ is compact, then $S(f,M,B,Z)$ is compact. \hfill \square

We have the following analogy of Lemma 6.10.

**Lemma 6.19.** Let $f : V \to \mathcal{PMEL}(S)$, be a generic PL map, $\eta > 0$ and $A$ and $C$ disjoint simple closed geodesics. Let $K$ be a closed subset of $f^{-1}(B_C)$ and $Y \subset V$ be compact. If $0 < g(f,A;Y) < \infty$, then there exists a neighborhood $U$ of $Y$ such that for $\delta$ sufficiently small, any $(C,\delta,K)$ push off $f_1$ satisfies $g(f_1,A;U) \geq g(f,A;Y)$. If equality holds, then $m(f_1,A;U) \subset N_V(m(f,A;Y),\eta)$ and if $t \in m(f_1,A;U)$ and $d_V(t,m_{J_i}(f,A;Y)) < \eta$, then $J_i$ is the $U$-stryker curve to $A$ at $f_1(t)$. \hfill \square

7. Marker tags

**Definition 7.1.** Let $A$ be a simple closed multi-geodesic in $S$. We say that $\tau$ is a tag for $A$, if $\tau$ is a compact embedded geodesic curve (with $\partial \tau$ possibly empty) transverse to $A$ such that $\partial \tau \subset A$ and $\text{int}(\tau) \cap A \neq \emptyset$.

Let $M$ be a marker hit by the simple closed multi-geodesic $A$. Then $r \geq 3$ distinct subarcs of $A$ span $M$, where $r \in \mathbb{N}$ is maximal. These arcs run from $\alpha_0$ to $\alpha_1$, the posts of $M$. Suppose that the initial points of these arcs intersect $\alpha_0$ at $c_1,\cdots,c_r$. Let $\tau$ be the maximal subarc of $\alpha_0$ with endpoints in $\{c_1,\cdots,c_r\}$. Such a tag is called a marker tag.
Given $f : V \to \mathcal{PML}(S)$ that hits the marker $M$ we may need to find a new $f$ that hits the marker $M$ free of a particular multi-geodesic. Tags are introduce
to measure progress in that effort. We will find a sequence of push off homotopies
whose resulting maps intersect a given tag more and more so that we can ultimately
invoke the following result.

**Lemma 7.2.** Let $f : V \to \mathcal{PML}(S)$, $M$ a marker, $A$ a simple closed multi-
geodesic that hits $M$ and $\tau$ the corresponding marker tag. Let $b_1, \ldots, b_r$ be simple
closed geodesics such that for all $t \in f^{-1}(B_A)$, $|(f(t)) \setminus (\bigcup_{i=1}^r b_i \cup A)| \cap \tau \geq 3(3g - 3 + p)$, then $M$ is $f$-free of $\{A, b_1, \ldots, b_r\}$ along $f^{-1}(B_A)$.

**Proof.** If $f(t) \in B_A$, then any leaf $L$ of $\phi(f(t))$ distinct from $A$ with $L \cap \tau \geq m$
has at least $m$ distinct subarcs that span $M$. If $L$ is a non compact leaf of $\phi(f(t))$
and $L \cap \tau \neq \emptyset$, then $|L \cap \tau| = \infty$, since $L$ is non proper. If only closed geodesics of
$\phi(f(t))$ intersect $\tau$, then since $\phi(f(t))$ can have at most $3g - 3 + p$ such geodesics,
one of them $L$ distinct from $\{b_1, \ldots, b_r, A\}$ must satisfy $|L \cap \tau| \geq 3$. \hfill \Box

**Definition 7.3.** Let $\tau$ be a tag for the multi-geodesic $A$, $f : V \to \mathcal{PML}(S)$
a generic PL map and $Y$ a compact subset of $f^{-1}(B_A)$. Define $g(f, \tau, Y) = \min\{|(\phi(f(t)) \setminus A) \cap \tau| : t \in Y\}$ the geometric intersection number of $f$ with $\tau$ along $Y$.

If $0 < g(f, \tau, Y) < \infty$, then define the multi-geodesic $J$ to be a $Y$-stryker curve
for $\tau$ if $J \subset \phi(f(t))$, $J \cap A = \emptyset$ and $|J \cap \tau| = g(f, \tau, Y)$.

The proof of Lemma 6.9 readily generalizes to the following result.

**Lemma 7.4.** If $f : V \to \mathcal{PML}(S)$ is a generic PL map, $\tau$ is a tag for the
simple closed geodesic $A$ and $Y$ is closed in $f^{-1}(B_A)$, then the set of $Y$-stryker
curves is finite. Also the set $m(f, \tau, Y) = \{t \in Y | g(f, \tau, Y) = |(\phi(f(t)) \setminus A) \cap \tau|\}$
is compact and and canonically partitions as the disjoint union of the compact
sets $m_{J_1}(f, \tau, Y), \ldots, m_{J_k}(f, \tau, Y)$ where $J_i$ is the $Y$-stryker curve to $\tau$ at all $t \in m_{J_i}(f, \tau, Y)$. \hfill \Box

Similarly, Lemma 6.19 generalizes to the following result.

**Lemma 7.5.** Let $f : V \to \mathcal{PML}(S)$ be a generic PL map, $\tau$ a tag for the
simple closed geodesic $A$, $Y$ a closed subset of $f^{-1}(B_A)$ and $0 < g(f, \tau, Y) < \infty$.
Let $C$ be a simple closed geodesic such that $C \cap (A \cup \tau) = \emptyset$ and $K$ a closed subset
of $f^{-1}(B_C)$. If $\eta > 0$, then there exists a neighborhood $U$ of $Y$ such that for $\delta$
sufficiently small, any $(C, \delta, K)$ push off $f_1$ satisfies $g(f_1, \tau, Y) \geq g(f, \tau, Y)$, where
$Y_1 = f_1^{-1}(B_A) \cap U$. If equality holds, then $m(f_1, \tau, Y_1) \subset N_\delta(m(f, \tau, Y), \eta)$ and if$t \in m(f_1, \tau, Y_1)$ and $d_\delta(t, m_{J_1}(f_1, \tau, Y)) < \eta$, then $J_i$ is the $Y_1$ stryker curve to $\tau$
at $f_1(t)$. In particular if $J$ is a $Y_1$-stryker curve for $f_1$ and $\tau$, then $J$ is a $Y$-stryker
curve for $f$ and $\tau$. \hfill \Box

8. Marker Cascades

This very technical section begins to address the following issue. To invoke
Proposition 5.2 we need to find a sequence $f_1, f_2, \ldots$ satisfying the sublimit and
filling conditions, in particular satisfying the property that $f_i^{-1}(B_{C_i}) = \emptyset$ for $i \leq j$.
We cannot just create $f_i$ from $f_{i-1}$ by pushing off of $C_i$, because $C_i$ may be needed
to hit previously constructed markers. To make $C_i$ free of these markers we may
need to relatively push off of other curves. We may not be able to push off of those.
curves because they in turn are needed to hit markers. In subsequent sections we shall see that finiteness of \( S \), genericity of \( f \) and the \( k \leq n \) condition will force this process to terminate. Thus before we push off of \( C \), we will do a sequence of relative push offs of other curves.

We introduce the notion of marker cascade to keep track of progress. Given \( f : V \to \mathcal{MLEL}(S) \), markers \( \mathcal{M}_1, \cdots, \mathcal{M}_m \) and pairwise disjoint simple closed curves \( a_1, \cdots, a_v \), a marker cascade is a (complicated) measure of how far \( \mathcal{M}_1, \cdots, \mathcal{M}_m \) are from being free of \( a_1, \cdots, a_v \). At the end of this section we will show that under appropriate circumstances relative pushing preserves freedom as measured by a marker cascade. The next section shows that judicious relative pushing increases the level of freedom. See Proposition 9.1.

**Definition 8.1.** Let \( V \) be the underlying space of a finite simplicial complex. Associated to \( f : V \to \mathcal{MLEL}(S) \) a generic PL map, \( J = (\mathcal{M}_1, W_1), \cdots, (\mathcal{M}_m, W_m) \) a marker family hit by \( f \), \( \mathcal{M}_1 < \cdots < \mathcal{M}_m \) the ordering induced from this enumeration and \( a_1, \cdots, a_v \) a sequence of pairwise disjoint simple closed geodesics we define a marker cascade \( C \) which is a \( v+1 \)-tuple \((\mathcal{A}_1, \cdots, \mathcal{A}_v, \mathcal{P})\) where each \( \mathcal{A}_i \) is a 3-tuple \((\mathcal{A}_i(i), \mathcal{A}_i(ii), \mathcal{A}_i(iii))\) that is defined below and \( \mathcal{P} \) is a finite set of \( v \)-tuples defined in Definition 8.5. To start with \( \mathcal{A}_i \) is organized as follows.

- \( \mathcal{A}_i(i) \) is either a marker \( \mathcal{M}_{j_i} \) or \( \infty \).
- \( \mathcal{A}_i(ii) \in \mathbb{Z}_{\geq 0} \cup \infty \) is the geometric intersection number of \( f \) with the tag \( \tau_i \) associated to \( a_i \) and \( \mathcal{M}_{j_i} \) along the compact set \( m_i(C) \subset V \), unless \( \mathcal{A}_i(ii) = \infty \) in which case \( \mathcal{A}_i(iii) = \infty \).
- \( \mathcal{A}_i(iii) \) is the set of stryker curves for \( \tau_i \) along the compact set \( m_i(C) \subset V \) unless \( \mathcal{A}_i(ii) = \infty \) in which case \( \mathcal{A}_i(iii) = \infty \).

We define the \( \mathcal{A}_i \)'s and the auxiliary \( m_i(C) \)'s as follows.

- \( \mathcal{A}_1(i) \) is defined to be the maximal marker \( \mathcal{M}_{j_1} \) such that \( \mathcal{M}_1 \) is \( f \)-free of \( a_1 \) along \( W_i \) for all \( i < 1 \). If \( a_1 \) is free of \( J \), then define \( \mathcal{A}_1(i) = \infty \).

If \( \mathcal{M}_{j_1} \) exists, then define \( \tau_1 \) to be the marker tag associated to \( a_1 \) and \( \mathcal{M}_{j_1} \) and define \( m_1(C) = \{ t \in S(f, a_1, \mathcal{M}_{j_1}, W_1)]g(f, \tau_1, S(f, a_1, \mathcal{M}_{j_1}, W_1)) = \{(\phi(f(t)) \setminus a_1) \cap \tau_1|\} \).

- \( \mathcal{A}_1(ii) = g(f, \tau_1, m_1(C)) \) if \( m_1(C) \neq \emptyset \) or \( \mathcal{A}_1(ii) = \infty \) otherwise.
- \( \mathcal{A}_1(iii) \) is defined to be Striker_1 the set of \( m_1(C) \)-stryker curves for \( \tau_1 \), unless \( \mathcal{A}_1(ii) = \infty \) in which case \( \mathcal{A}_1(iii) = \infty \).

Having defined \( \mathcal{A}_i, i < u \), then \( \mathcal{A}_u \) is defined as follows. (The reader is encouraged to first read Remark 8.2). To start with define \( B_u^1, \cdots, B_u^q \) where \( B_u^r = \{ b_u^1, \cdots, b_u^r \} \), where \( b_u^r = a_u \) and for \( q < u, b_u^r = a_q \) if \( r < q \) and \( b_u^r = \emptyset \) otherwise.

- \( \mathcal{A}_u(i) \) is defined to be either the maximal marker \( \mathcal{M}_{u_j} \), such that \( r < u_j \) implies that \( \mathcal{M}_r \) is free of \( B_u^r \) along \( m_{u-1}(C) \cap W_r \) or \( \mathcal{A}_u(i) = \infty \) if for all \( r \leq m, \mathcal{M}_r \) is free of \( B_u^r \) along \( m_{u-1}(C) \cap W_r \).
Remark 8.2. In words $\mathcal{A}_u(i) = M_{u_i}$ is the maximal marker such that all lower markers are free of $\{a_1, \cdots, a_i\}$, where applicable. Where applicable means two things. First, the only relevant points are those of $m_{u-1}(C)$. Second if say $M_1, M_2, M_3$ are free of $a_1$ but $M_4$ is not and along $m_1(C)$, $M_1, M_2, M_3$ are free of $\{a_1, a_2\}$ and $M_4$ is free of $a_2$ but $M_6$ is not free of $a_2$, then $M_2 = M_6$. In particular, $a_1$ is irrelevant when considering $M_{p}$, for $p \geq 4$. In this case $B^4_1 = B^4_2 = B^4_3 = \{a_1, a_2\}$ and for $r > 3$, $B^r_2 = \{a_2\}$. Note that if $M_1$ is free of $\{a_1, a_2\}$ along $m_1(C)$ but $M_2$ is not, then $M_{2_2} = M_2$.

Definition 8.1 continued If $M_{u_j}$ exists, then define $\tau_u$ to be the marker tag arising from $a_u$ and $M_{u_j}$. Let $S_u = S(f, B^u_{u_j}, M_{u_j}, W_{u_j}) \cap m_{u-1}(C)$. Define $m_u(C) = \{t \in S_u | g(f, \tau_u, S_u) = |\phi(f_t) \setminus a_u) \cap \tau_u|\}$.

$\mathcal{A}_u(i)$ is defined to be either $g(f, \tau_u, m_u(C))$ or $\infty$ if $m_u(C) = \emptyset$.

$\mathcal{A}_u(iii)$ is defined to be the set Stryker $\tau_u$ which is either the set of $m_u(C)$-stryker curves for $\tau_u$ if $m_u(C) \neq \emptyset$ or $\infty$ otherwise.

We say that the cascade $C$ is finished if $m_v(C) = \emptyset$ and active otherwise. We say that the cascade is based on $\{a_1, \cdots, a_v\}$ and has length $v$. For $r \leq v$, then the length-$r$ cascade based on $\{a_1, \cdots, a_i\}$ is called the length-$r$ subcascade and denoted $C_r$. Note that $C_r$ and $C$ have the same values of $\mathcal{A}_1, \cdots, \mathcal{A}_r$.

Notation 8.3. The data corresponding to a cascade depends on $f$. When the function must be explicitly stated, we will use notation such as $C(f), m_i(C, f), \mathcal{A}_p(f)$ or $\mathcal{A}_i(i, f)$.

We record for later use the following result.

Lemma 8.4. Let $J$ be a marker family hit by the generic PL map $f : V \to \mathcal{PML}L(S)$. If $C$ is an active cascade based on $a_1, \cdots, a_v$, then for every $t \in m_v(C)$, each $a_i$ is a leaf of $\phi(f(t))$.

Proof. By definition $m_1(C) \subset S(f, a_1, M_{1_1}, W_{1_j})$, hence $a_1$ is a leaf of $\phi(f(t))$ at all $t \in m_1(C)$.

Now assume the lemma is true for all subcascades of length $< u$. Let $t \in m_u(C)$. Let $A = B^u_{u_j}$. By definition, $M_{u_j}$ is not $f$-free of $A$ along $m_{u-1}(C)$, but $M_{u_j}$ is $f$-free of $A \setminus a_u$ along $m_{u-1}(C)$. Since $m_u(C) \subset S(f, A, W_{u_j}) \cap m_{u-1}(C)$ it follows that $a_u$ is a leaf of $\phi(f(t))$.\hfill $\square$

Definition 8.5. Let $C$ be an active cascade. To each $t \in m_v(C)$ corresponds a $v$-tuple $(p_1, \cdots, p_v)$ where $p_j$ is the (possibly empty) stryker multi-geodesic for $\tau_j$ at $t$. Such a $(p_1, \cdots, p_v)$ is called a packet. There are only finitely many packets, by the finiteness of stryker curves. Thus $m_v(C)$ canonically decomposes into a disjoint union of closed sets $S_1, \cdots, S_r$ such that each point in a given $S_j$ has the same packet. Let $P = \{P_1, \cdots P_r\}$ denote the set of packets, the last entry in the definition of $C$. We will use the notation $P(f)$, when needed to clarify the function on which this information is based.

Definition 8.6. In what follows all cascades use the same set of markers and simple closed geodesics, however the function $f : V \to \mathcal{PML}L(S)$ will vary. We put an equivalence relation on this set of cascades and then partially order the classes. We say that $C(g)$ is equivalent to $C(f)$ if $P(f) = P(g)$ and for all $r$, $\mathcal{A}_r(f) = \mathcal{A}_r(g)$. We
lexicographically partial order the equivalence classes by comparing the \(v\)-tuples \(A_i(C(f)), \ldots, A_i(C(f)), P(f)\) using the rule that \(A_r(i, f) \leq A_r(i, g)\) if \(M_r(f) \leq M_r(g)\), with \(\infty\) being considered the maximal value and \(A_r(ii, f) \leq A_r(ii, g)\) if their values satisfy that inequality and \(A_r(iii, f) \leq A_r(iii, g)\) if Stryker, \(r \subset\) Stryker, \(f\). Finally, \(P(f) \leq P(g)\) if \(P(g) \subset P(f)\).

**Remark 8.7.** More or less, \(C(f) < C(g)\) means that the markers’s are freer with respect to the function \(g\) than with respect to \(f\). In particular, this inequality holds if \(M_1\) and \(M_2\) are \(g\)-free of \(a_1\), but only \(M_1\) and \(f\)-free of \(a_1\). If \(M_1\) is both \(g\)-free and \(f\)-free of \(a_1\) and \(M_2\) is neither \(f\)-free nor \(g\)-free of \(a_1\), then \(A_i(iii)\) is a measure of how close \(M_2\) is from being free of \(a_1\). The bigger the number, the closer to freedom as motivated by Lemma 7.2. If this number is the same with respect to both \(f\) and \(g\), then \(A_i(\cdot)\) measures how much work is needed to raise the number. More stryker multi-geodesics with respect to \(f\), than \(g\), means more needs to be done to \(f\), so again \(C(f) < C(g)\). If \(A_i(f) = A_i(g)\), then \(A_2\) is used to determine the ordering. Finally, if all the \(A_i\)’s are equal, then \(f\) having more packets means that more sets of \(V\) need to be cleaned up to finish the cascade.

**Proposition 8.8.** Let \(V\) be the underlying space of a finite simplicial complex. Let \(f : V \rightarrow P, MLE(L(S))\) a generic PL map, \(\dim(V) \leq n\) and \(J = (M_1, W_1), \ldots, (M_m, W_m)\) a marker family hit by \(f\). Let \(C\) be an active cascade based on \(\{a_1, \ldots, a_v\}\), \(C\) a simple closed geodesic disjoint from the \(a_i\)’s and the \(\tau_i\)’s and \(K \subset m_C(C) \cap f^{-1}(B_C)\) compact. For \(1 \leq i \leq m\), let \(B_{u_i} = \{a_u < u_j\} \cup \{C\}\). Assume that for \(1 \leq i \leq m\), \(M_i\) is \(f\)-free of \(B_{u_i}\) along \(K \cap W_i\). If \(\delta\) is sufficiently small and \(f_1\) is obtained from \(f\) by a \((C, \delta, K)f\) push off homotopy, then \([C(f)] \leq [C(f_1)]\) and the homotopy from \(f\) to \(f_1\) is \(J\)-marker preserving.

**Proof.** Since \(J\) is \(f\)-free of \(C\) along \(K\), it follows by Lemma 6.15 that for \(\delta\) sufficiently small, any \((C, \delta, K)\) homotopy is \(J\) marker preserving. It remains to show that if \(\delta\) is sufficiently small and \(f_1\) is the resulting map, then \([C(f)] \leq [C(f_1)]\).

From \(C(f)\) we conclude that for all \(u \in \{1, \ldots, v\}\) and \(q < u_j, M_q\) is \(f\)-free of \(B_u^q\) along \(m_u(C) \cap W_q\). By hypothesis each \(M_q\) is also \(f\)-free of \(B_u^q\) along \(W_q \cap K\). Note that \(B_u^q \subset B_{u_i}\) when \(q < u_j\). By Lemma 6.15, if \(\delta\) is sufficiently small, then there exists a neighborhood \(V\) of \(K\) such that any \((C, \delta, K)\) homotopy is supported in \(V\) and each \(M_i\) is \(f\)-free of \(B_{u_i}\) along \(W_i \cap V\). It follows that with respect to the lexicographical ordering \((A_i(i, f), \ldots, A_v(i, f)) \leq (A_1(i, f_1), \ldots, A_v(i, f_1))\).

A similar argument using Lemma 6.19 shows that if \(\delta\) is sufficiently small and \(A_i(i, f) = A_i(i, f_1)\) for \(1 \leq i \leq u\), then \(A_i(i, f) = A_i(i, f_1)\) for \(1 \leq i \leq u\).

By Lemma 7.4 there exists \(\eta > 0\) such that if \(d_y(t_1, t_2) < 2\eta\) and \(t_1, t_2 \in m_u(C)\) for some \(u\), then \(f(t_1), f(t_2)\) have the same stryker curve to \(\tau_u\). \(\delta\) be sufficiently small to satisfy Lemma 7.5 with this \(\eta\) in addition to the previously required conditions. That lemma implies that if \(A_i(i, f) = A_i(i, f_1)\) and \(A_i(i, f) = A_i(i, f_1)\) for all \(1 \leq u\), then \(\text{Stryker}_i(f_1) \subset \text{Stryker}_i(f)\) for all \(i \leq u\).

Finally, Lemma 7.5 with this choice of \(\eta\) also implies that if \((A_1(f), \ldots, A_v(f)) = (A_1(f_1), \ldots, A_v(f_1))\), then \(P(f_1) \subset P(f)\). It follows that \([C(f)] \leq [C(f_1)]\). \(\square\)

**Remark 8.9.** Note that \([C(f)] \leq [C(f_1)]\) holds with respect to the lexicographical ordering, but may not hold entry-wise, since there may be no direct comparison between later entries once earlier ones differ. For example, say \(2 = A_1(i, f) < A_1(i, f_1) = 3\), then showing \(A_2(i, f) = 3\) involves verifying that \(M_2\) is \(f\)-free of \(\{a_2\}\) while showing \(A_2(i, f_1) = 3\) involves verifying that \(M_2\) is \(f_1\)-free of \(\{a_1, a_2\}\).
9. Finishing Cascades

The main result of this section is the following.

**Proposition 9.1.** Let $V$ be the underlying space of a finite simplicial complex $: V \to \mathcal{PML}(S)$ be a generic PL map such that $k = \dim(V) \leq n$ where $\dim(\mathcal{PML}(S)) = 2n + 1$. Let $J$ be a marker family hit by $h$ and $C$ be an active cascade. Then there exists a marker preserving homotopy of $h$ to $h'$ such that $C$ is finished with respect to $h'$ and $[C(h)] < [C(h')]$. The homotopy is a concatenation of relative push offs.

If $L \subset \mathcal{PML}(S)$ is a finite subcomplex of $C(S)$, then the homotopy can be chosen to be disjoint from $L$.

**Lemma 9.2.** Let $f_i : V \to \mathcal{PML}(S)$, $i \in \mathbb{N}$ be generic PL maps, $J$ be a marker family and let $\{C(f_i)\}$ be active cascades based on the same set of simple closed geodesics. Any sequence $[C(f_1)] \leq [C(f_2)] \leq \cdots$ has only finitely many terms that are strict inequalities.

**Proof.** There are only finitely many possible values for $A_u(i, f_j)$.

It follows from Lemma 7.2 that each $A_u(ii, f_j)$ is uniformly bounded above.

Indeed, there are only finitely many $a_i$’s and only finitely many markers in $J$, thus only finitely many marker tags arising from them. If $\tau_u$ is such a tag, then for $i \in \mathbb{N}$, $g(f_i, \tau, m_u(C(f_i))) \leq |(\cup_{j=1}^u a_j) \cap \tau_u| + 3(3g - 3 + 2p)$.

The finiteness of stryker curves (Lemma 4.5) shows that the number of possibilities for both $P(f_j)$ and each $A_u(iii, f_j)$ is bounded. □

**Proof of Proposition 9.1.** It suffices to show that given any active cascade $C(h_1)$, there exists a $\mathcal{J}$ marker preserving homotopy from $h_1$ to $h_2$, that is a concatenation of relative push offs, such that $[C(h_1)] < [C(h_2)]$. For if $C(h_2)$ is not finished, then we can similarly produce an $h_3$ with $[C(h_2)] < [C(h_3)]$. Finally we obtain a finished $C(h_3)$ else we contradict Lemma 9.2.

We retain the convention that $\dim(\mathcal{PML}(S)) = 2n + 1$. We will assume that $k = n$ leaving the easier $k < n$ case to the reader. The proof is by downward induction on $L = \text{length}(C)$. Suppose that $C(h)$ is an active cascade based on $\{a_1, \cdots, a_L\}$. Since the $a_i$’s are pairwise disjoint, it follows that $L \leq n + 1$. Let $\tau_i$ denote the marker tag associated to $a_i$ and $\mathcal{M}_{a_i}(h)$. Let $R' = N(a_1 \cup \cdots \cup a_L \cup \tau_1 \cdots \cup \tau_L)$. Let $R$ be $R'$ together with all its complementary discs, annuli and pants. Let $T = S \setminus (\text{int}(R))$. Note that $T = \emptyset$ if $L \geq n$.

Let $\mathcal{A}(h(t))$ denote the arational sublamination of $\phi(h(t))$, i.e. the sublamination obtained by deleting all the compact leaves. Since $h$ is generic and $k = n$, it follows that $\mathcal{A}(h(t)) \neq \emptyset$ for all $t \in B^\circ$. The key observation is that if $\mathcal{A}(h(t)) \cap R \neq \emptyset$, then $t \notin m_L(C(h))$. To start with, Lemma 8.4 implies that $a_1, \cdots, a_L$ are leaves of $\phi(h(t))$. This implies that $\mathcal{A}(h(t)) \cap \tau_i \neq \emptyset$, where $\tau_i$ is the marker tag between $a_i$ and $\mathcal{M}_{a_i}$. By Lemma 7.2 $\phi(h(t))$ hits $\mathcal{M}_{a_i}$ and hence $t \notin m_i(C(h))$ contradicting the fact that $m_L(C(h)) \subset m_i(C(h))$. It follows that $C(h)$ is finished if $L \geq n$.

Note that if $k < n$, then this argument shows that $C$ is finished if $L \geq k$. In particular, if $k = 1$, (the path connectivity case) all cascades are finished.

Now assume that the Proposition is true for all cascades of length greater than $L$, where $L < n$. Let $C$ be an active cascade of length $L$.

Let $P = (p_1, \cdots, p_L) \in \mathcal{P}(h)$ and $S_P$ the closed subset of $m_L(C)$ consisting of points whose packet is $P$. Let $\sigma$ be the possibly empty multi-geodesic $p_1 \cup \cdots \cup p_L$.

By definition, if $t \in S_P$, then each $p_i$ is a leaf of $\phi(h(t))$ and by Lemma 8.4 each of
\(a_1, \ldots, a_L\) is a leaf of \(\phi(h(t))\). Let \(Y' = N(a_1 \cup \cdots \cup a_L \cup \tau_1 \cdots \cup \tau_L \cup \sigma)\) and \(Y'\) be \(Y\) together with all complementary discs, annuli and pants. Let \(Z = S' \setminus \text{int}(Y)\).

Note that \(Z \neq \emptyset\), else for all \(t \in S_P\), \(|\phi(h(t)) \cap \tau_i(t)\) = \(\infty\) for some \(i(t)\) which is a contradiction as before. Let \(C\) be a simple closed geodesic isotopic to some component of \(\partial Z\).

Observe that \(C\) is neither an \(a_i\) nor is \(C \subset \sigma\). Indeed, since each \(a_i\) is crossed transversely by a tag at an interior point it follows that no \(a_i\) is isotopic to a component of \(\partial Y'\) and hence \(\partial Y\). Similarly, each component of \(\text{Stryker}_i\) is transverse to \(\tau_i\) at an interior point, hence no component of \(\sigma\) is isotopic to a component of \(\partial Y\). Next observe that \(S_P \subset f^{-1}(B_C)\). Indeed, \(t \in S_P\) implies that for all \(i, \phi(h(t)) \cap \tau_i \subset a_i \cup \sigma\), hence \(C\) is either a leaf of \(\phi(h(t))\) or \(C \cap \phi(h(t)) = \emptyset\).

Extend the cascade \(C(h)\) to the length \(L + 1\) cascade \(C'(h)\) by letting \(C\) be our added \(a_{L+1}\). If \(C'(h)\) is active, then by induction, there is a marker preserving homotopy of \(h\) to \(h_1\) so that \(C'(h_1)\) is finished and \(|C'(h_1)| < |C'(h_1)|\). If \(C'(h)\) is finished, then let’s unify notation by denoting \(h\) by \(h_1\). In both cases, by restricting to the length \(L\) subspace either \(|C(h)| < |C(h)|\) or \(|C(h)| = |C(h)|\) and each \(M_i\) is \(h_1\)-free of \(B_C\) for all \(t \in m_L(C(h_1)) \cap W_i\), where \(B_C\) is as in the statement of Lemma 8.8. Since freedom is an open condition and \(S_P(h_1)\) is compact as are all the \(W_i\)’s, there exists \(U \subset V\) open such that \(m_L(C(h_1)) \subset U\) and each \(M_i\) is \(h_1\)-free of \(B_C\) for all \(t \in U \cap W_i\).

If \(|C(h)| = |C(h)|\), then invoke Lemma 8.8 by taking \(f = h_1\), keeping the original \(J\) and \(C\), using the above constructed \(C\) and setting \(K = S_P(h_1)\). Choose \(\delta\) sufficiently small to satisfy the conclusion of Lemma 8.8 and so that any \((C, \delta, K)\) push off homotopy is supported in \(U\).

To complete the proof it suffices to show that if \(\delta\) is sufficiently small and \(h_2\) is the resulting map, then \(|C(h_1)| < |C(h_2)|\). Since Lemma 8.8 implies that \(|C(h_1)| \leq |C(h_2)|\) it suffices to show that \(P \notin P(h_2)\) if for all \(i, A_i(h_1) = A_i(h_2)\). If \(P \in P(L_2)\), then let \(t \in S_P(h_2)\). As above \(h_2(t) \in B_C\). Since \(h_2\) is the result of a relative push off homotopy it follows from the last sentence of Definition 6.12 iii) that either \(\phi(h_2(t)) = \phi(h_1(t))\) or \(\phi(h_2(t)) \neq \phi(h_1(t))\) but \(C \cup \phi(h_2(t)) = \phi(h_1(t))\).

In the former case, \(t \in S_P(h_1)\) and hence both \(h_1(t), h_2(t) \in B_C\), contradicting the fact that \(h_2\) is the result of a \((C, \delta, K)\) push off homotopy. In the latter case \(t \in U \setminus S_P(h_1)\), hence there exists an \(M_i\) that is \(h_1\)-free of \(B_C\) but \(t \in U \cap W_i\), contradicting the fact that \((C, \delta, K)\) push off homotopy is supported in \(U\).

Since \(k \leq n\) Lemma 2.4 it follows that \(h(V) \cap K = \emptyset\). By Lemma 6.14, by using sufficiently small \(\delta\)’s all \((C, \delta, K)\) pushoff homotopies in the above proof could have been done to avoid \(L\).

\textbf{Corollary 9.3. Let} \(V\) be the underlying space of a finite simplicial complex. \(S\) be a finite type surface and \(f : V \to \mathcal{PMLC}(S)\) be a generic PL map and \(\dim(V) \leq n\), where \(\dim(\mathcal{PMLC}(S)) = 2n + 1\). Let \(J\) be a marker family hit by \(f\) and \(C\) a simple closed geodesic. Then \(f\) is homotopic to \(f_1\) via a marker preserving homotopy such that \(g(f, C) > 0\). If \(L \subset \mathcal{PMLC}(S)\) is a finite subcomplex of \(\mathcal{C}(S)\), then the homotopy can be chosen to be disjoint from \(L\).

\textbf{Proof.} By Lemma 6.6, if \(J\) is free of \(C\) and \(\delta\) is sufficiently small, then any \((C, \delta)\) push off homotopy is \(J\)-marker preserving. If \(f_1\) is the resulting map, then \(g(f_1, C) > 0\).
If $J$ is not free of $C$, then let $C$ be the active length-1 cascade based on $C$. By Proposition 9.1 $f$ is homotopic to $f'$ by a marker preserving homotopy such that $C(f')$ is finished and the homotopy is a concatenation of relative push off homotopies. Now argue as in the first paragraph.

\[\square\]

10. Stryker Cascades

The main result of this section is

**Proposition 10.1.** Let $V$ be the underlying space of a finite simplicial complex. Let $f : V \rightarrow \mathcal{PMLEC}(S)$ be a generic PL map such that $\dim(V) \leq n$ where $\dim(\mathcal{PMLEC}(S)) = 2n+1$. If $a_1$ is a simple closed geodesic such that $\infty > g(f, a_1) > 0$ and $J$ a marker family hit by $f$, then there exists then there exists a marker preserving homotopy of $f$ to $f_1$ such that $g(f_1, a_1) > g(f, a_1)$. The homotopy is a concatenation of relative push offs. If $L \subset \mathcal{PMLEC}(S)$ is a finite subcomplex of $C(S)$, then the homotopy can be chosen to be disjoint from $L$.

**Remarks 10.2.** The proof is very similar to that of Corollary 9.3, except that stryker cascades are used in place of marker cascades. A stryker cascade is essentially a marker cascade except that the first term is a curve $a_1$ with $g(f, a_1) > 0$.

Closely following the previous two sections, we give the definition of stryker cascade and prove various results about them.

**Definition 10.3.** Associated to $f : V \rightarrow \mathcal{PMLEC}(S)$ a generic PL map, $J = (\mathcal{M}_1, W_1), \ldots, (\mathcal{M}_m, W_m)$ a marker family hit by $f$, $\mathcal{M}_1 < \cdots < \mathcal{M}_m$ the ordering induced from this enumeration and $a_1, \ldots, a_v$ a sequence of pairwise disjoint simple closed geodesics with $g(f, a_1) > 0$ we define a stryker cascade $C$ which is a $v+1$-tuple $(A_1, \ldots, A_v, \mathcal{P})$ where $A_1$ is a 2-tuple $(A_1(i), A_1(ii))$ and for $i \geq 2$, each $A_i$ is a 3-tuple $(A_i(i), A_i(ii), A_i(iii))$ and $\mathcal{P}$ is a finite set of $v$-tuples defined in Definition 10.6.

Define $A_1 = (g(f, a_1), \text{Stryker}_1)$ where Stryker$_1$ is the set of stryker curves to $f$ and $a_1$.

Define $m_1(C) = \{t \in V | g(f, a_1) = |\phi(f(t)) \cap a_1|\}$.

Having defined $A_i, 1 \leq i < u$, then $A_u$ is defined as follows. To start with define $B_u^1, \ldots, B_u^n$ where $B_u^r = \{b_u^r, \ldots, b_u^r\}$, where $b_u^r = a_u$ and for $2 \geq q < u, b_u^r = a_q$ if $r < q_j$ and $b_u^r = \emptyset$ otherwise.

$A_u(i)$ is defined to be either the maximal marker $\mathcal{M}_{u,i}$, such that $r < u_i$ implies that $\mathcal{M}_r$ is free of $B_u^r$ along $m_{u-1}(C) \cap W_r$ or $A_u(i) = \infty$ if for all $r \leq m$, $\mathcal{M}_r$ is free of $B_u^r$ along $m_{u-1}(C) \cap W_r$.

If $\mathcal{M}_{u,i}$ exists, then define $\tau_u$ to be the marker tag arising from $a_u$ and $\mathcal{M}_{u,i}$. Let $S_u = S(f, B_u^{a_u}, \mathcal{M}_{u,i}, W_{u,i}) \cap m_{u-1}(C)$. Define $m_u(C) = \{t \in S_u | g(f, \tau_u, S_u) = |\phi(f(t)) \cap a_u| \cap \tau_u|\}$.

$A_u(ii)$ is defined to be either $g(f, \tau_u, m_u(C))$ or $\infty$ if $m_u(C) = \emptyset$.

$A_u(iii)$ is defined to be the set Stryker$_u$ which is either the set of $m_u(C)$-stryker curves for $\tau_u$ if $m_u(C) \neq \emptyset$ or $\infty$ otherwise.
We say that the cascade $C$ is finished if $m_v(C) = \emptyset$ and active otherwise. We say that the cascade is based on $\{a_1, \cdots, a_v\}$ and has length $v$. For $r \leq v$, then length-$q$ cascade based on $\{a_1, \cdots, a_r\}$ is called the length-$r$ subcascade and denoted $C_r$. Note that $C_r$ and $C$ have the same values of $A_1, \cdots, A_r$.

**Notation 10.4.** The data corresponding to a cascade depends on $f$. When the function must to be explicitly stated, we will use notation such as $C(f)$, $m_v(C, f)$, $A_p(f)$ or $A_r(ii, f)$.

We record for later use the following result whose proof is essentially identical to that of Lemma 8.4.

**Lemma 10.5.** Let $J$ be a marker family hit by the generic PL map $f : V \to PML(S)$. If $C$ is an active stryker cascade based on $a_1, \cdots, a_v$, then for every $t \in m_v(C)$ and $i \geq 2$, each $a_i$ is a leaf of $\phi(f(t))$. □

**Definition 10.6.** Let $C$ be an active stryker cascade. To each $t \in m_v(C)$ corresponds an $v$-tuple $(p_1, \cdots, p_v)$ where $p_1$ is the stryker multi-geodesic for $a_1$ at $t$ and for $i \geq 2$, $p_i$ is the (possibly empty) stryker multi-geodesic for $\tau_i$ at $t$. Such a $(p_1, \cdots, p_v)$ is called a packet. There are only finitely many packets, by the finiteness of stryker curves. Thus $m_v(C)$ canonically decomposes into a disjoint union of closed sets $S_1, \cdots, S_r$ such that each point in a given $S_j$ has the same packet. Let $P = \{P_1, \cdots, P_r\}$ denote the set of packets, the last entry in the definition of $C$.

**Definition 10.7.** Use the direct analogy of Definition 8.6 to put an equivalence relation on the set of stryker cascades based on the same ordered set of simple closed curves and to partially order the classes.

**Proposition 10.8.** Let $f : V \to PML(S)$ a generic PL map, dim$(V) \leq n$ and $J = (M_1, W_1), \cdots, (M_m, W_m)$ a marker family $J$ hit by $f$. Let $C$ be an active stryker cascade based on $\{a_1, \cdots, a_v\}$, $C$ a simple closed geodesic disjoint from the $a_i$'s and $\tau_j$'s and let $K \subset m_v \cap f^{-1}(B_C)$. For $1 \leq i \leq m$, let $B_C = \{a_{p_i} : p_i \geq 2 | i < p_j \} \cup \{C\}$. Assume that for $1 \leq i \leq m$, $M_i$ is $f$-free of $B_C$ along $K \cap W_i$. If $f_1$ is obtained from $f$ by a $(C, \delta, K)$ push off homotopy and $\delta$ is sufficiently small, then $|C(f)| \leq |C(f_1)|$ and the homotopy from $f$ to $f_1$ is $J$-marker preserving.

**Proof.** The fact $A_1(f) \leq A_1(f_1)$ follows from Lemma 6.19. The rest of the argument follows as in the proof of Proposition 8.8. □

We have the following analogue of Lemma 9.2.

**Lemma 10.9.** Let $f_i : V \to PML(S)$, $i \in \mathbb{N}$ be generic PL maps, $J$ a marker family and let $\{C_{(f_i)}\}$ be active stryker cascades based on the same set of simple closed geodesics. Any sequence $|C_{(f_1)}| \leq |C_{(f_2)}| \leq \cdots$ with $g(f_1, a_1) = g(f_2, a_1) = \cdots$ has only finitely many terms that are strict inequalities. □

**Proposition 10.10.** Let $h : V \to PML(S)$ be a generic PL map such that $k = \dim(V) \leq n$ where $\dim(PML(S)) = 2n + 1$. Let $J$ be a marker family hit by $h$ and let $C$ be an active stryker cascade with $\infty > g(h, a_1) > 0$. Then there exists a marker preserving homotopy of $h$ to $h'$ such that $|C(h)| < |C(h')|$ and either $g(h', a_1) > g(h, a_1)$ or $C(h')$ is finished. The homotopy is a concatenation of relative push offs. If $L \subset PML(S)$ is a finite subcomplex of $C(S)$, then the homotopy can be chosen to be disjoint from $L$. 

Proof. As in the proof of Proposition 9.1 it suffices to show that given any active stryker cascade \( C(h_1) \) there exists a marker preserving homotopy of \( h_1 \) to \( h_2 \), that is a concatenation of relative push offs, such that \( |C(h_1)| < |C(h_2)| \). Here we invoke Lemma 10.9 instead of Lemma 9.2.

Again we discuss the \( k=n \) case, the easier \( k < n \) cases being left to the reader. The proof by downward induction on the length of the cascade follows essentially exactly that of Proposition 9.1 until the step of proving it for length-1 cascades. So now assume that the Proposition has been proved for cascades of length \( \geq 2 \).

Let \( C(h) \) be a length-1 cascade based on \( a_1 \). Here \( P \), the set of packets, consists of the set of stryker multi-curves to \( h \) and \( a_1 \). Let \( P \) be one such multi-curve. Let \( Y' = N(a_1 \cup P) \) and \( Y \) be the union of \( Y' \) and all components of \( S \setminus Y' \) that are discs, annuli and pants. Again, genericity and the condition \( k = n \) implies that \( Y \neq S \). Let \( C \) be a simple closed geodesic isotopic to a component of \( \partial Y \). The condition \( g(h, a_1) > 0 \) implies that \( C \) is not isotopic to \( a_1 \). Since each component of \( P \) non trivially intersects \( a_1 \), \( C \) is not isotopic to any component of \( P \). Let \( C' \) denote the length-2 stryker cascade based on \( \{ a_2, C \} \). By induction there exists a marker preserving homotopy of \( h \) to \( h_1 \) that is a concatenation of relative push offs such that \( |C'(h)| < |C'(h_1)| \) and either \( g(h_1, a_1) > g(h, a_1) \) or \( C'(h_1) \) is finished. In the latter case we have either \( |C(h)| < |C(h_1)| \) or \( |C(h)| = |C(h_1)| \) and for each \( 1 \leq i \leq m, M_i \) is free of \( C \) along \( M_i \). Now argue as in the proof of Proposition 9.1 that if \( h_2 \) is the result of a \( (C, \delta, K) \) homotopy, where \( \delta \) is sufficiently small and \( K = S_P \), then either \( g(h_2, a_1) > g(h_1, a_1) \) or \( g(h_2, a_1) = g(h_1, a_1) \) and \( |C(h_1)| < |C(h_2)| \).

Proof of Proposition 10.1. Let \( C(f) \) be the length-1 stryker cascade based on \( a_1 \). By Proposition 10.10 there exists a marker preserving homotopy from \( f \) to \( f_1 \) that is the concatenation of relative push off homotopies such that \( C(f) < C(f_1) \). Therefore, either \( g(f_1, a_1) > g(f, a) \) or equality holds and \( |P(C(f_1))| < |P(C(f))| \). After finitely many such homotopies we obtain \( f' \) such that \( g(f', a_1) > g(f_1, a) \).

11. \( \mathcal{E}(S) \) is \( (n-1) \)-CONNECTED

Theorem 11.1. Let \( V \) be the underlying space of a finite simplicial complex and \( W \) that of a subcomplex. Let \( S \) be a finite type hyperbolic surface with \( \dim(P, \mathcal{M}, \mathcal{L}(S)) = 2n + 1 \). If \( f : V \to \mathcal{P, M, L}(S) \) is a generic PL map, \( f^{-1}(\mathcal{E}(S)) = W \), \( \dim(V) \leq n \) and \( f \) is a marker family hit by \( f \), then there exists a map \( g : V \to \mathcal{E}(S) \) such that \( g|W = f \) and \( g \) hits \( \mathcal{F} \). The map \( g \) is the concatenation of (possibly infinitely many) relative push offs. If \( L \subset \mathcal{P, M, L}(S) \) is a finite subcomplex of \( \mathcal{C}(S) \), then the homotopy can be chosen to be disjoint from \( L \).

Proof. Let \( C_1, C_2, \ldots \) be an enumeration of the simple closed geodesics on \( S \). It suffices to find a sequence, \( f_0, f_1, f_2, \ldots \) of extensions of \( f|W \) and sequences \( \{ \mathcal{E}_i \} \), \( \{ S_i \} \) of marker covers, such that \( \mathcal{E}_i \) is a marker covering of \( V \) by \( 1/i \) markers, \( S_i \) is a marker covering of \( V \) by \( C_i \) markers and for \( i \leq j \), \( f_j \) hits each \( \mathcal{E}_i \) and \( S_i \) family of markers. Furthermore, \( f_{i+1} \) is obtained from \( f_i \) by a finite sequence of relative push offs. Then Proposition 5.2 produces \( g \) as the limit of the \( f_i \)'s. Since \( f_{i+1} \) is obtained from \( f_i \) by concatenating finitely many push offs, concatenating all these push off homotopies produces a map \( F : V \times [0, \infty] \to \mathcal{P, M, L}(S) \) with \( F|V \times \infty = g \). The proof of Proposition 5.2 shows that \( F \) is continuous. It follows from Lemma 6.14 that the homotopy can be chosen disjoint from \( L \).
Suppose that \( f_0, f_1, \ldots, f_{j-1}, \mathcal{E}_1, \ldots, \mathcal{E}_{j-1}, \) and \( \mathcal{S}_1, \ldots, \mathcal{S}_{j-1} \) have been constructed so that \( f_p \) hits each \( \mathcal{E}_p \) and \( \mathcal{S}_p \) family of markers whenever \( p \leq q \leq j - 1 \).

We will extend the sequence by constructing \( f_j, \mathcal{E}_j \) and \( \mathcal{S}_j \) to satisfy the corresponding properties. The theorem then follows by induction. In what follows, \( \mathcal{J}_i \) denotes the marker family that is the union of all the markers in the \( \mathcal{E}_r \) and \( \mathcal{S}_s \) marker families, where \( r, s \leq i \).

**Step 1. Construction of \( f_j \) and \( \mathcal{S}_j \):**

**Proof.** Let \( N(C_j) \) be as in Lemma 3.11 ii). Given \( f_{j-1} \), obtain \( f_j^1 \) by applying Corollary 9.3 to \( f_j \) and the marker cover \( \mathcal{J}_{j-1} \). Now repeatedly apply Proposition 10.1 to obtain \( f_j^2, f_j^3, \ldots, f_j^{N(C_j)} \) with the property that \( f_j^j \) is a generic PL map obtained from \( f_{j-1} \) via a \( \mathcal{J}_{j-1} \) marker preserving homotopy such that \( g(f_j^j, C) \geq q \).

Let \( f_j = f_j^{N(C_j)} \) and apply Lemma 3.11 ii) to obtain \( \mathcal{S}_j \).

**Step 2. Construction of \( \mathcal{E}_j \):**

**Proof.** Apply Lemma 3.11 i) to the generic PL map \( f_j \) to obtain \( \mathcal{E}_j \) a marker cover by \( 1/j \) markers.

**Theorem 11.2.** If \( S \) is a finite type hyperbolic surface such that \( \dim(\mathcal{PML}(S)) = 2n + 1 \), then \( \mathcal{EL}(S) \) is \((n - 1)\)-connected.

**Proof.** Let \( k \leq n \) and \( g : S^{k-1} \to \mathcal{EL}(S) \) be continuous and \( f : B^k \to \mathcal{PML}(S) \) be a generic PL extension of \( g \), provided by Proposition 2.7. Now apply Theorem 11.1 to extend \( g \) to a map of \( B^k \) into \( \mathcal{EL}(S) \).

**12. \( \mathcal{EL}(S) \) is \((n - 1)\)-locally connected**

**Theorem 12.1.** If \( S \) is a finite type hyperbolic surface, then \( \mathcal{EL}(S) \) is \((n - 1)\)-locally connected, where \( \dim(\mathcal{PML}(S)) = 2n + 1 \).

**Proof.** We need to show that if \( L \in \mathcal{EL}(S) \), and \( U' \subset \mathcal{EL}(S) \) is an open set containing \( L \), then there exists an open set \( U \subset U' \) containing \( L \) so that if \( k \leq n \), \( g : S^{k-1} \to \mathcal{EL}(S) \) and \( g(S^{k-1}) \subset U \), then \( g \) extends to a map \( F : B^k \to \mathcal{EL}(S) \) such that \( F(B^k) \subset U' \).

Choose a pair of pants decomposition and parametrizations of the the cuffs and pants so that some complete maximal train track \( \tau \) fully carries \( L \). Thus \( V(\tau) \) is a convex subset of \( \mathcal{ML}(S) \) that contains \( \hat{\phi}^{-1}(L) \) in its interior.

Let \( \mathcal{M}_1, \mathcal{M}_2, \ldots \) be a sequence of markers such that each \( \mathcal{M}_i \) is a \( 1/i \)-marker that is that is hit by \( L \). Let \( U_i = \{ x \in \mathcal{EL}(S) | \phi(x) \text{ hits } \mathcal{M}_i \} \). Then each \( U_i \) is an open set containing \( L \) and by Lemma 4.5 there exists \( N \in \mathbb{N} \) such that if \( i \geq N \), then \( \hat{\phi}^{-1}(U_i) \subset \text{int}(V(\tau)) \cap \hat{\phi}^{-1}(U') \). Reduce the ends of each post of \( \mathcal{M}_N \) slightly to obtain \( \mathcal{M}'_N \) so that if \( U'_N = \{ x \in \mathcal{EL}(S) | \phi(x) \text{ hits } \mathcal{M}'_N \} \) and \( \hat{\phi}_N = \{ x \in \mathcal{EL}(S) | \phi(x) \text{ hits } \mathcal{M}_N \} \), then \( L \in U'_N \subset \hat{\phi}_N \subset U_N \). Let \( W \subset \hat{\phi}^{-1}(U'_N) \) be an open convex subset of \( \mathcal{ML}(S) \) that contains \( \hat{\phi}^{-1}(L) \) and is saturated by rays through the origin. By reducing \( W \) we can assume that \( \hat{W} = \text{int}(V(\tau)) \) for some train track \( \tau \) carrying \( L \). Next choose \( j \) such that \( \hat{\phi}^{-1}(U_j) \subset W \). Let \( U = U_j \) and \( \mathcal{M} = \mathcal{M}_j \).

Let \( k \leq n \) and \( g : S^{k-1} \to \mathcal{EL}(S) \) with \( g(S^{k-1}) \subset U \). Since \( \hat{W} \) is convex and contains \( \hat{\phi}^{-1}(U) \), we can apply Proposition 2.7 and Remark 2.19 to find a generic PL map \( f_0 : B^k \to \mathcal{PML}(S) \) extending \( g \) such that \( f_0(\text{int}(B^k)) \subset W \). Thus for
all \( t \in B^k, f_0(t) \) hits \( \mathcal{M} \). Theorem 11.1 produces \( f : B^k \to \mathcal{E}(S) \) extending \( g \) such that for \( t \in B^k \) and \( i \in \mathbb{N}, f_i(t) \) hits \( \mathcal{M} \). Therefore, for each \( t \in B^k, f(t) \) hits the closed marker \( M' \) and hence \( f(B^k) \subset U \subset U' \). \( \square \)

13. \( \mathcal{PML}(S) \) and \( \mathcal{E}(S) \) Approximation Lemmas

The main technical results of this paper are that under appropriate circumstances any map into \( \mathcal{E}(S) \) can be closely approximated by a map into \( \mathcal{PML}(S) \) and vice versa. In this section we isolate out these results.

We first give a mild extension of Lemma 2.18, which is about approximating a map into \( \mathcal{E}(S) \) by a map into \( \mathcal{PML}(S) \). The subsequent two results are about approximating maps into \( \mathcal{PML}(S) \) by maps into \( \mathcal{E}(S) \).

Lemma 13.1. Let \( K \) be a finite simplicial complex, \( g : K \to \mathcal{E}(S) \) and \( \epsilon > 0 \). Then there exists a generic PL map \( h : K \to \mathcal{PML}(S) \) such that for each \( t \in K \),

\[
d_{\mathcal{PML}(S)}(\phi(h(t)), g(t)) < \epsilon \quad \text{and} \quad d_{\mathcal{PML}(S)}(h(t), \phi^{-1}(g(t'))) < \epsilon \quad \text{some} \ t' \in K \text{ with} \ d_K(t, t') < \epsilon.
\]

Proof. Given \( g \), Lemma 2.18 produces mappings \( h_i : K \to \mathcal{PML}(S) \) such that

\[
d_{\mathcal{PML}(S)}(\phi(h_i(t)), g(t)) < \epsilon/i. \]

By super convergence we can assume that each \( h_i \) is a generic PL map satisfying the same conclusion.

We show that if \( i \) is sufficiently large, then the second conclusion holds for \( h = h_i \). Otherwise, there exists a sequence \( t_1, t_2, \cdots \) of points in \( K \) for which it fails respectively for \( h_1, h_2, \cdots \). Since for each \( i \), \( h_i^{-1}(\mathcal{FPML}(S)) \) is dense in \( K \), it follows from Lemma 1.12 that we can replace the \( t_i \)'s by another such sequence having

\[
d_{\mathcal{PML}(S)}(h_i(t_i), \phi^{-1}(g(t_i))) \quad \epsilon/2, \ t_i \to t_\infty, \ h_i(t_i) \to x_\infty, \ \phi(h_i(t_i)) \in \mathcal{E}(S) \quad i \in \mathbb{N}. \]

By Lemmas 4.6 and 4.1, \( \phi(h_i(t_i)) \to g(t_\infty) \) and hence by Lemma 1.12, \( \phi(x_\infty) = g(t_\infty) \). Therefore, \( \lim_{i \to \infty} d_{\mathcal{PML}(S)}(h_i(t_i), \phi^{-1}(g(t_\infty))) \leq \lim_{i \to \infty} d_{\mathcal{PML}(S)}(h_i(t_i), x_\infty) = 0 \), a contradiction. \( \square \)

Lemma 13.2. Let \( V \) be the underlying space of a finite simplicial complex. Let \( W \) be that of a finite subcomplex. \( S \) be a finite type hyperbolic surface with \( \dim(\mathcal{PML}(S)) = 2n+1 \) and let \( \epsilon > 0 \). If \( h : V \to \mathcal{PML}(S) \) is a generic PL map, \( h^{-1}(\mathcal{E}(S)) = W \) and \( \dim(V) \leq n \), then there exists \( g : V \to \mathcal{E}(S) \) such that \( g|W = h \) and for each \( t \in V \),

\[
d_{\mathcal{E}(S)}(g(t), \phi(h(t))) < \epsilon.
\]

Proof. Since \( k \leq n \), \( \mathcal{A}(\phi(h(t))) \neq \emptyset \) for every \( t \in V \), where \( \mathcal{A}(\phi(h(t))) \) denotes the arational sublamination of \( \phi(h(t)) \). Therefore, by Lemma 3.5 for each \( t \in V \), there exists an \( \epsilon \)-marker \( M_t \) that is hit by \( \phi(h(t)) \). By Lemma 3.7, \( M_t \) is hit by all \( \phi(h(s)) \) for all \( s \) sufficiently close to \( t \). By compactness of \( V \), there exists an \( \epsilon \)-marker family \( J \) hit by \( h \). By Theorem 11.1 there exists \( g : V \to \mathcal{E}(S) \) such that \( g \) hits \( J \). Since both \( h \) and \( g \) hit \( J \), the conclusion follows. \( \square \)

Lemma 13.3. \( S \) be a finite type hyperbolic surface with \( \dim(\mathcal{PML}(S)) = 2n+1 \). Let \( V \) be the underlying space of a finite simplicial complex with \( \dim(V) \leq n \). For \( i \in \mathbb{N} \), let \( \epsilon_i > 0 \) and \( h_i : V \to \mathcal{PML}(S) \) be generic PL maps. If for each \( z \in \mathcal{E}(S) \) there exists \( \delta_z > 0 \) so that for \( i \) sufficiently large \( d_{\mathcal{PML}(S)}(h_i(V), \phi^{-1}(z)) > \delta_z \), then there exists \( g_1, g_2, \cdots : V \to \mathcal{E}(S) \) such that

i) For each \( i \in \mathbb{N} \) and \( t \in V \),

\[
d_{\mathcal{E}(S)}(g_i(t), \phi(h_i(t))) < \epsilon_i.
\]

ii) For each \( z \in \mathcal{E}(S) \), there exists a neighborhood \( U_z \subset \mathcal{E}(S) \) of \( z \) such that for \( i \) sufficiently large \( g_i(V) \cap U_z = \emptyset \).
Given \( z \in \mathcal{EL}(S) \) and \( \delta_z > 0 \), it follows from Lemma 4.2 that there exists \( \epsilon(z) > 0 \) such that if \( x \in \mathcal{PM}(S) \) and \( d_{\mathcal{PM}(S)}(\phi(x), z') < \epsilon(z) \), then \( d_{\mathcal{PM}(S)}(x, \phi^{-1}(z)) < \delta_z \) and if \( y \in \mathcal{EL}(S) \) and \( d_{\mathcal{EL}(S)}(y, z') \leq \epsilon(z) \), then \( \phi^{-1}(y) \subset N_{\mathcal{PM}(S)}(\phi^{-1}(z), \delta_z) \).

Here \( z' \) is a diagonal extension of \( z \). Let \( U_z \) be a neighborhood of \( z \) such that \( y \in U_z \) implies \( y \in N_{\mathcal{PT}(S)}(z', \epsilon(z)/2) \) for some diagonal extension \( z' \) of \( z \).

Now let \( f : V \to \mathcal{PM}(S) \) be a generic PL map with \( k \leq n \). If \( 0 < \epsilon < \epsilon(z) \) and \( d_{\mathcal{PM}(S)}(f(V), \phi^{-1}(z)) > \delta_z \), then we now show that there exists \( g : V \to \mathcal{EL}(S) \) such that for every \( t \in V \), \( d_{\mathcal{PT}(S)}(g(t), \phi(f(t))) < \epsilon \) and \( g(V) \cap U_z = \emptyset \). By definition of \( \epsilon(z) \), for every \( t \in V \) and diagonal extension \( z' \) of \( z \), \( d_{\mathcal{PT}(S)}(\phi(f(t)), z') > \epsilon(z) \) and in particular \( d_{\mathcal{PT}(S)}(A(t), z') > \epsilon \), where \( A(t) \) is the arational sublamination of \( \phi(f(t)) \). By Lemma 3.4 for each \( t \in V \), there exists an \( \epsilon/2 \) marker hit by \( f(t) \).

Since hitting markers is an open condition and \( V \) is compact there exists an \( \epsilon/2 \) marker family \( \mathcal{J} \) hit by \( f \). Now apply Theorem 11.1 to obtain \( g : V \to \mathcal{EL}(S) \) such that \( g \) hits \( \mathcal{J} \). Therefore for each \( t \in V \), \( d_{\mathcal{PT}(S)}(g(t), \phi(f(t))) \leq \epsilon/2 \) and hence \( g(t) \subseteq N_{\mathcal{PT}(S)}(z', \epsilon(z)/2) \) for all diagonal extensions \( z' \) of \( z \). Therefore \( g(t) \notin U_z \).

It follows that \( g(V) \cap U_z = \emptyset \).

Since \( \mathcal{EL}(S) \) is separable and metrizable it is Lindelöf, hence there exists a countable cover of \( \mathcal{EL}(S) \) of the form \( \{ U_{n_i} \} \). By hypothesis for every \( j \in \mathbb{N} \), there exists \( n_j \in \mathbb{N} \) such that if \( i \geq n_j \) implies \( d_{\mathcal{PM}(S)}(h_i(V), \phi^{-1}(z_j)) > \delta_{z_j} \). We can assume that \( n_1 < n_2 < \cdots \). Let \( m_i = \max\{ j | n_j \leq i \} \).

For \( i \in \mathbb{N} \) apply the argument of the second paragraph with \( f = h_i \) and \( \epsilon = \min\{ \epsilon(z_1), \cdots, \epsilon(z_{n_m}), \epsilon_i \} \) to produce \( g_i : V \to \mathcal{EL}(S) \) satisfying for all \( t \in V \), \( d_{\mathcal{PT}(S)}(g_i(t), \phi(h_i(t))) < \epsilon \) and such that \( g_i(V) \cap U_{z_j} = \emptyset \) for \( j \leq m_i \), assuming that \( m_i \neq \emptyset \). It follows that \( g_1, g_2, \cdots \) satisfy the conclusion of the Lemma.

14. GOOD CELLULATIONS OF \( \mathcal{PM}(S) \)

The main result of this section, Proposition 14.6 produces a sequence of cellulations of \( \mathcal{PM}(S) \) such that each cell is the polytope of a train trace and face relations among cells correspond to carrying among train tracks. Each cellulation is a subdivision of the previous one, subdivision of cells corresponds to splitting of train tracks and every train track eventually gets split arbitrarily much.

**Definition 14.1.** If \( \tau \) is a train track, then let \( V(\tau) \) denote the set of measured laminations carried by \( \tau \) and \( P(\tau) \) denote the polyhedron that is the quotient of \( V(\tau) \setminus \emptyset \), by rescaling. A train track is **generic** if exactly three edges locally emanate from each switch. All train tracks in this section are generic. We say that \( \tau_2 \) is obtained from \( \tau_1 \) by a **single splitting** if \( \tau_2 \) is obtained by splitting without collision along a single large branch of \( \tau_1 \). Also \( \tau' \) is a full splitting of \( \tau \) if it is the result of a sequence of single splittings along each large branch of \( \tau \).

**Remark 14.2.** By elementary linear algebra if \( \tau_R \) and \( \tau_L \) are the train tracks obtained from \( \tau \) by a single splitting, then \( V(\tau_R) = V(\tau) \) or \( V(\tau_L) = V(\tau) \) or \( V(\tau_R), V(\tau_L) \) are obtained by slicing \( V(\tau) \) along a codimension-1 plane through the origin.

**Definition 14.3.** We say that a finite set \( R(\tau) \) of train tracks is **descended** from \( \tau \), if there exists a sequence of sets of train tracks \( R_1 = \{ \tau \}, R_2, \cdots, R_k = R(\tau) \) such that \( R_{i+1} \) is obtained from \( R_i \) by deleting one train track \( \tau \in R_i \) and replacing it either by the two train tracks resulting from a single splitting of \( \tau \) if \( P(\tau) \) is split, or one of the resulting tracks with polyhedron equal to \( P(\tau) \) otherwise. Finally,
if a replacement track is not recurrent, then replace it by its maximal recurrent subtrack.

**Remark 14.4.** If $R(\tau)$ is descended from $\tau$, then $P(\tau) = \cup_{\kappa \in R(\tau)} P(\kappa)$, each $P(\kappa)$ is codimension-0 in $P(\tau)$ and any two distinct $P(\kappa)$’s have pairwise disjoint interiors. Thus, any set of tracks descended from $\tau$ gives rise to a subdivision of $\tau$’s polyhedron. A consequence of Proposition 14.6 is that after subdivision, the lower dimensional faces of the subdivided $P(\tau)$ are also polyhedra of train tracks.

**Definition 14.5.** Let $\Delta$ be a cellulation of $\mathcal{PML}(S)$ and $T$ a finite set of birecurrent generic train tracks. We say that $T$ is associated to $\Delta$ if there exists a bijection between elements of $T$ and cells of $\Delta$ such that if $\sigma \in \Delta$ corresponds to $\tau_\sigma \in T$, then $P(\tau_\sigma) = \sigma$.

**Proposition 14.6.** Let $S$ be a finite type hyperbolic surface. There exists a sequence of cellulations $\Delta_0, \Delta_1, \cdots$ of $\mathcal{PML}(S)$ such that

i) Each $\Delta_{i+1}$ is a subdivision of $\Delta_i$.

ii) Each $\Delta_i$ is associated to a set $T_i$ of train tracks.

iii) If $\sigma_j$ is a face of $\sigma_k$, then $\tau_{\sigma_k}$ is carried by $\tau_{\sigma_j}$. If $\sigma_p \in \Delta_i, \sigma_q \in \Delta_j, \sigma_p \subset \sigma_q, \dim(\sigma_p) = \dim(\sigma_q)$ and $i > j$, then $\tau_{\sigma_p}$ is obtained from $\tau_{\sigma_q}$ by finitely many single splittings and possibly deleting some branches.

iv) There exists a subsequence $\Delta_{i_0} = \Delta_0, \Delta_{i_1}, \Delta_{i_2}, \cdots$ such that each complete $\tau \in T_{i_N}$ is obtained from a complete $\tau' \in T_0$ by $N$ full splittings.

**Definition 14.7.** A sequence of cellulations $\{\Delta_i\}$ satisfying the conclusions of Proposition 14.6 is called a good cellulation sequence.

**Proof.** Let $T_0$ be the set of standard train tracks associated to a parametrized pants decomposition of $S$. As detailed in [PH], $T_0$ gives rise to a cellulation $\Delta_0$ of $\mathcal{PML}(S)$ such that each $\sigma \in \Delta_0$ is the polyhedron of a track in $T_0$.

The idea of the proof is this. Suppose we have constructed $\Delta_0, \Delta_1, \cdots, \Delta_i$, and $T_0, T_1, \cdots T_i$ satisfying conditions i)-iii). Since any full splitting of a train track is the result of finitely many single splittings it suffices to construct $\Delta_{i+1}$ and $T_{i+1}$ satisfying i)-iii) such that the complete tracks of $T_{i+1}$ consist of the complete tracks of $T_i$, except that a single complete track $\tau$ of $T_i$ is replaced by $\tau_R$ and/or $\tau_L$. Let $T'_i$ denote this new set of complete tracks. The key technical Lemma 14.8 implies that if $\sigma$ is a codimension-1 cell of $\Delta_i$ with associated train track $\tau_\sigma$, then every element of some set of train tracks descended from $\tau_\sigma$ is carried by an element of $T'_i$. This implies that $\sigma$ has been subdivided in a manner compatible with the subdivision of the top dimensional cell $P(\tau)$. Actually, we must be concerned with lower dimensional cells too and new cells that result from any subdivision. Careful bookkeeping together with repeated applications of Lemma 14.8 deals with this issue.

**Lemma 14.8.** Suppose that $\kappa$ is carried by $\tau$. Let $\tau_R$ and $\tau_L$ be the train tracks obtained from a single splitting of $\tau$ along the large branch $b$. Then there exists a set $R(\kappa)$ of train tracks descended from $\kappa$ such that for each $\kappa' \in R(\kappa)$, $\kappa'$ is carried by one of $\tau_R$ or $\tau_L$.

If $P(\tau) = P(\tau_R)$ (resp. $P(\tau_L)$), then each $\kappa'$ is carried by the maximal recurrent subtrack of $\tau_R$ (resp. $\tau_L$).
Proof. Let $N(\tau)$ be a fibered neighborhood of $\tau$. Being carried by $\tau$, we can assume that $\kappa \subset \text{int}(N(\tau))$ and is transverse to the ties. Let $J \times [0,1]$ denote $\pi^{-1}(b)$ where $\pi : N(\tau) \to \tau$ is the projection contracting each tie to a point. Here $J = [0,1]$ and each $J \times t$ is a tie. Let $\partial_s(J \times i)$, $i \in \{0,1\}$ denote the singular point of $N(\tau)$ in $J \times i$. The canonical projection $h : J \times [0,1]$ to $[0,1]$ gives a height function on $J \times I$.

We can assume that distinct switches of $\kappa$ inside of $J \times I$ occur at distinct heights and no switch occurs at heights 0 or 1. Call a switch $s$ a down (resp. up) switch if two branches emanate from $s$ that lie below (resp. above) $s$. A down (resp. up) switch is a bottom (resp. top) switch if the branches emanating below (resp. above) $s$ extend to smooth arcs in $\kappa \cap J \times I$ that intersect $J \times 0 \setminus \partial_s(J \times 0)$ (resp. $J \times 1 \setminus \partial_s(J \times 1)$) in distinct components. Furthermore $h(s)$ is minimal (resp. maximal) with that property. There is at most one top switch and one bottom switch. Let $s_T$ (resp. $s_B$) denote the top (resp. bottom) switch if it exists.

We define the, possibly empty, b-core as the unique embedded arc in $\kappa \cap J \times I$ transverse to the ties with endpoints in the top and bottom switches. We also require that $h(s_B) < h(s_T)$. Uniqueness follows since $\kappa$ has no bigons.

We now show that if the b-core is empty, then $\kappa$ is carried by one of $\tau_R$ or $\tau_L$. To do this it suffices to show that after normal isotopy, $\kappa$ has no switches in $J \times I$. By normal isotopy we mean isotopy of $\kappa$ within $N(\tau)$ through train tracks that are transverse to the ties. It is routine to remove, via normal isotopy, the switches in $J \times I$ lying above $s_T$ and those lying below $s_B$. Thus all the switches can be normally isotoped out of $J \times I$ if either $s_T$ or $s_B$ do not exist or $h(s_B) > h(s_T)$.

If $s_T$ exists, then since the b-core = $\emptyset$ all smooth arcs descending from $s_T$ hit only one component of $J \times 0 \setminus \partial_s(J \times 0)$. Use this fact to first normally isotope $\kappa$ to remove from $J \times I$ all the switches lying on smooth arcs from $s_T$ to $J \times 0$ and then to isotope $s_T$ out of $J \times I$.

Now assume that the b-core exists. Let $u_1, \ldots, u_r, d_1, \ldots, d_s$ be respectively the up and down switches of $\kappa$ that lie on the core. For $i \in \{1,\ldots, s\}$, let $u(i)$ be the number of up switches in the b-core that lie above the down switch $d_i$. Define $C(\kappa) = \sum_{i=1}^{s} u(i)$. Define $C(\kappa) = 0$ if the b-core is empty. Note that if $C(\kappa) = 1$, then $r = s = 1$ and $u_1 = s_T$ and $d_1 = s_B$. Furthermore splitting $\kappa$ along the large branch between $s_T$ and $s_B$ gives rise to tracks whose b-cores are empty and hence are carried by one of $\tau_R$ or $\tau_S$.

Assume by induction that the first part of the lemma holds for all train tracks $\kappa$ with $C(\kappa) < k$. Let $\kappa$ be a track with $C(\kappa) = k$. Let $e$ be a large branch of $\kappa$ lying in $\kappa$’s b-core. The two train tracks obtained by splitting along $e$ have reduced $C$-values. Therefore, the first part of the lemma follows by induction.

Now suppose that $P(\tau) = P(\tau_R)$. It follows that $P(\tau_L) \subset P(\tau_R)$ and hence $P(\tau_L) = P(\tau_C)$, where $\tau_C$ is the train track obtained by splitting $\tau$ along $b$ with collision. Therefore if $\kappa'$ is the result of finitely many simple splittings, and is carried by $\tau_L$, then the maximal recurrent subtrack of $\kappa'$ is carried by $\tau_C$ and hence by the maximal recurrent subtrack of $\tau_R$. \hfill \Box

Remarks 14.9. i) If each $u_j$ is to the left of the b-core and each $d_j$ is to the right of the b-core, then if $\kappa_R$ and $\kappa_L$ are the tracks resulting from a single splitting along a large branch in the b-core, then $C(\kappa_R) = C(\kappa) - 1$, while $C(\kappa_L) = 0$. 

ii) Any large branch of \( \kappa \) that intersects the b-core is contained in the b-core. It follows that if \( \kappa' \) is the result of finitely many single splittings of \( \kappa \), then \( C(\kappa') \leq C(\kappa) \).

Now assume we have a sequence \((\Delta_0, T_0), \cdots, (\Delta_i, T_i)\) of cellulations and associated train tracks that satisfy i)-iii) of the proposition. Let \( \sigma \) be a cell of \( \Delta_i \) with associated train track \( \tau \). Since any full splitting of a track is the concatenation of splittings along large branches, to complete the proof of the proposition, it suffices to prove the following claim.

**Claim.** If \( \tau_R \) and \( \tau_L \) are the result of a single splitting of \( \tau \), then there exists a cellulation \( \Delta_{i+1} \) with associated train tracks \( T_{i+1} \) extending the sequence and satisfying i)-iii), so that \( \tau \) is replaced with the maximal recurrent subtracks of \( \tau_R \) and/or \( \tau_L \) and if \( \tau' \in T_i \) is such that \( \dim(P(\tau')) \geq \dim(P(\tau)) \) and \( \tau' \neq \tau \), then \( \tau' \in T_{i+1} \).

**Proof.** Proof by induction on \( \dim(\sigma) \). We will assume that each of \( P(\tau_R) \) and \( P(\tau_L) \) are proper subcells of \( P(\tau) \), for proof in the general case is similar. The claim is trivial if \( \dim(\sigma) = 0 \). Now assume that the claim is true if \( \dim(\sigma) < k \). Assuming that \( \dim(\sigma) = k \) let \( \sigma_1, \cdots, \sigma_p \), be the \((k-1)\)-dimensional faces of \( \sigma \) with corresponding train tracks \( \kappa_1, \cdots, \kappa_p \). By Lemma 14.8 there exists sets \( R(\kappa_1), \cdots, R(\kappa_p) \) descended from the \( \kappa_i \)'s such that any train track in any of these sets is carried by one of \( \tau_R \) and \( \tau_L \).

If \( R(\kappa_1) \neq \{\kappa_1\} \), then there exists a single splitting of \( \kappa_1 \) into \( \kappa_1^1 \) and \( \kappa_1^2 \) such that \( R(\kappa_1) = R(\kappa_1^1) \cup R(\kappa_1^2) \), where \( R(\kappa_1^i) \) is descended from \( \kappa_1^i \). (As usual, only one may be relevant and it might have stuff deleted.) By induction there exists a cellulation \( \Delta_1^i \) with associated \( T_1^i \) such that the sequence \((\Delta_0, T_0), \cdots, (\Delta_i, T_i), (\Delta_1^i, T_1^i)\) satisfies i)-iii) and in the passage from \( T_i \) to \( T_1^i \), \( P(\kappa_1) \) is the only polyhedron of dimension \( \geq k - 1 \) that gets subdivided and it is replaced by \( P(\kappa_1^1) \) and \( P(\kappa_1^2) \). By repeatedly applying the induction hypothesis we obtain the sequence \((\Delta_0, T_0), \cdots, (\Delta_i, T_i), (\Delta_1^i, T_1^i), \cdots, (\Delta_p^p, T_p^p)\) satisfying i)-iii) such that in the passage from \( \Delta_i \) to \( \Delta_p^p \), each \( \sigma_j \) is subdivided into the polyhedra of \( R(\kappa_j) \) and no cells of dimension \( \geq k \) are subdivided and their associated train tracks are unchanged. It follows that if \( \Delta_{i+1} \) is obtained by subdividing \( \Delta_p^p \) by replacing \( \sigma \) by \( P(\tau_R) \), \( P(\tau_L) \) and \( P(\tau_C) \) and \( T_{i+1} \) is the set of associated train tracks, then \((\Delta_1, T_1), \cdots, (\Delta_i, T_i), (\Delta_{i+1}, T_{i+1})\) satisfies i)-iii) and hence the induction step is completed.

**Lemma 14.10.** Let \( \Delta_1, \Delta_2, \cdots \) be a good cellulation sequence.

i) If \( \sigma \) is a cell of \( \Delta_i \), then \( \phi(\sigma) \cap \mathcal{E}(S) \) is closed in \( \mathcal{E}(S) \).

ii) If \( \mathcal{L} = \phi(x) \) some \( x \in \mathcal{PM}(S) \), then for each \( i \in \mathbb{N} \) there exists a unique cell \( \sigma \in \Delta_i \) such that \( \phi^{-1}(\mathcal{L}) \subset \text{int}(\sigma) \).

iii) If \( U \) is an open set of \( \mathcal{PM}(S) \) that is the union of open cells of \( \Delta_i \), then \( \phi(U) \cap \mathcal{E}(S) \) is open in \( \mathcal{E}(S) \).

iv) If \( \sigma \) is a cell of \( \Delta_i \), then \( \text{int}(\sigma) \) is open in \( \phi(\sigma) \cap \mathcal{E}(S) \).

v) If \( \mu \in \mathcal{E}(S) \) and \( \phi^{-1}(\mu) \subset \text{int}(\sigma) \), \( \sigma \) a cell of \( \Delta_i \), then there exists \( \epsilon(\mu) > 0 \) such that \( \mathcal{B}(\mu, \epsilon(\mu)) = \{x \in \mathcal{PM}(S) | d_{\mathcal{P}(S)}(\phi(x), \mu) < \epsilon(\mu)\} \subset \hat{\sigma} \).

**Proof.** Conclusion i) follows from Lemma 1.14.

Conclusion ii) follows from the fact that if \( \tau \) carries \( \mathcal{L} \), then \( \phi^{-1}(\mathcal{L}) \subset P(\tau) \).

Conclusion iii) follows from Conclusion i) and the fact that \( \phi(U) \cap \phi(\mathcal{PM}(S) \setminus U) \cap \mathcal{E}(S) = \emptyset \).
Conclusion iv) follows from Conclusion i) and the fact that $\phi(\text{int}(\sigma)) \cap \phi(\partial \sigma) \cap \mathcal{E}L(S) = \emptyset$.

Let $\kappa$ be a cell of $\Delta_i$ with associated train track $\tau$. If $\tau$ carries $\mu$, then $\kappa \cap \sigma \neq \emptyset$ and hence by ii) $\sigma$ is a face of $\kappa$ and hence $\text{int}(\kappa) \subset \hat{\sigma}$. If $\tau$ does not carry $\mu$, then by Lemma 1.16 $B(\mu, \epsilon) \cap P(\tau) = \emptyset$ for $\epsilon$ sufficiently small. Since $\Delta_i$ has finitely many cells, the result follows. \hfill \Box

We have the following PML-approximation result for good cellulation sequences.

**Proposition 14.11.** Let $\Delta_1, \Delta_2, \cdots$ be a good cellulation sequence, $K$ a finite simplicial complex, $i \in \mathbb{N}$ fixed and $g : K \to \mathcal{E}L(S)$. Then there exists $h : K \to \mathcal{PML}(S)$ a generic PL map such that for each $t \in K$, there exists $\sigma(t)$ a cell of $\Delta_i$ such that $h(t) \cup \phi^{-1}(g(t)) \subset \hat{\sigma}(t)$.

**Proof.** Given $\mu \in \mathcal{E}L(S)$, let $\sigma$ be the cell of $\Delta_i$ such that $\phi^{-1}(\mu) \subset \text{int}(\sigma)$. Let $\epsilon(\mu)$ be as in Lemma 14.10 v). By Lemmas 4.6 and 4.1 there exists a neighborhood $U_\mu \subset \mathcal{E}L(S)$ of $\mu$ and $\delta_\mu > 0$ such that $d_{\mathcal{PT}(S)}(\mathcal{L}, z) < \delta_\mu, z \in U_\mu, \mathcal{L} \in \mathcal{L}(S)$ implies that $d_{\mathcal{PT}(S)}(\mathcal{L}, \mu) < \epsilon(\mu)$.

The compactness of $g(K)$ implies that there exists a finite open cover $U_{\mu_1}, \cdots, U_{\mu_k}$ of $g(K)$. Let $\delta = \min\{\delta_{\mu_1}, \cdots, \delta_{\mu_k}\}$. Then any $\delta$ PML-approximation as in Proposition 13.1 satisfies the conclusion of the proposition. \hfill \Box

**Lemma 14.12.** If $\sigma = P(\tau)$ is a cell of $\Delta_i$, $\sigma'$ is a face of $\sigma$ and $\mu \in \phi(\sigma') \cap \mathcal{E}L(S)$, then $\tau$ fully carries a diagonal extension of $\mu$.

**Proof.** Apply Lemma 4.8 and Proposition 14.6 iii). \hfill \Box

**Lemma 14.13.** Let $\Delta_1, \Delta_2, \cdots$ be a good cellulation sequence. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $i \geq N$, $\mathcal{L}_1 \in \mathcal{E}L(S), \mathcal{L}_2 \in \mathcal{L}(S)$ and $\mathcal{L}_1, \mathcal{L}_2$ are carried by $\tau$ some $\tau \in T_i, i \geq N$, then $d_{\mathcal{PT}(S)}(\mathcal{L}_1, \mathcal{L}_2) < \epsilon$.

**Proof.** Apply Propositions 14.6 and 4.10. \hfill \Box

**Definition 14.14.** If $\sigma$ is a cell of $\Delta_i$, then define the open star $\hat{\sigma} = \cup_{\eta \in \Delta_i | \sigma \subset \eta} \text{int}(\eta)$.

**Lemma 14.15.** If $\sigma$ is a cell of $\Delta_i$, then $\hat{\sigma}$ is open in $\mathcal{PML}(S)$ and contractible. Indeed, it deformation retracts to $\text{int}(\sigma)$.

**Lemma 14.16.** If $\sigma$ is a face of $\kappa$, then $\hat{\kappa} \subset \hat{\sigma}$.

**Definition 14.17.** Define $U(\sigma) = \phi(\hat{\sigma}) \cap \mathcal{E}L(S)$.

**Lemma 14.18.** $\mathcal{B} = \{U(\sigma) | \phi(\text{int}(\sigma)) \cap \mathcal{E}L(S) \neq \emptyset$ and $\sigma \in \Delta_i$ for some $i \in \mathbb{N}\}$ is a neighborhood basis of $\mathcal{E}L(S)$.

**Proof.** By Lemma 4.1 it suffices to show that for each $\mu \in \mathcal{E}L(S)$ and $\epsilon > 0$, there exists a neighborhood $U(\sigma)$ of $\mu$ such that $z \in U(\sigma)$ implies $d_{\mathcal{PT}(S)}(z, \mu) < \epsilon$.

Choose $N$ such that the conclusion of Lemma 14.13 holds. Let $\sigma$ be the simplex of $\Delta_N$ such that $\phi^{-1}(\mu) \subset \text{int}(\sigma)$. If $z \in U(\sigma)$, then $\phi^{-1}(z) \in P(\tau)$ where $\sigma$ is a face of $\tau$. Therefore, $z$ and $\mu$ are carried by $\tau$ and hence $d_{\mathcal{PT}(S)}(\mu, z) < \epsilon$. \hfill \Box

**Lemma 14.19.** Let $\mathcal{U}$ be an open cover of $\mathcal{E}L(S)$. Then there exists a refinement $\mathcal{U}_2$ of $\mathcal{U}$ by elements of $\mathcal{B}$, maximal with respect to inclusion, such that for each $U(\sigma_2) \in \mathcal{U}_2$ there exists $U \in \mathcal{U}$ and $\delta(U(\sigma_2)) > 0$ with the following property. If $x \in \hat{\sigma}_2$ and $z \in \mathcal{E}L(S)$ are such that $d_{\mathcal{PT}(S)}(\phi(x), z) < \delta(U(\sigma_2))$, then $z \in U$. 

Proof. It suffices to show that if $\mu \in \mathcal{EL}(S)$ and $U \in \mathcal{U}$, then there exists $\sigma_2$ and $\delta(\sigma_2)$ such that the last sentence of the lemma holds. By Lemma 4.1 there exists $\delta_1 > 0$ such that if $z \in \mathcal{EL}(S)$ and $d_{P\mathcal{M}\mathcal{L}}(z, \mu) < \delta_1$, then $z \in U$. By Proposition 4.10 there exists $N_1 > 0$ and $\delta > 0$ such that if $\tau$ is obtained by fully splitting any one of the train tracks associated to the top dimensional cells of $\Delta_1$ at least $N_1$ times, $\mu$ is carried by $\tau$, $\mathcal{L} \in \mathcal{L}(S)$ is carried by $\tau$, $z \in \mathcal{EL}(S)$ with $d_{P\mathcal{M}\mathcal{L}}(z, \mathcal{L}) < \delta$, then $d_{P\mathcal{M}\mathcal{L}}(z, \mu) < \delta_1$ and hence $z \in U$. Therefore, if $\sigma$ is the cell of $\Delta_1$ such that $\text{int}(\sigma)$ contains $\phi^{-1}(\mu)$, then let $\sigma_2 = \sigma$ and $\delta(U(\sigma_2)) = \delta$. □

15. Bounds on the dimension of $\mathcal{EL}(S)$

In this section we give an upper bound for $\dim(\mathcal{EL}(S))$ for any finite type hyperbolic surface $S$. When $S$ is either a punctured sphere or torus we give lower bounds for $\dim(\mathcal{EL}(S))$. For punctured spheres these bounds coincide. We conclude that if $S$ is the $n + 4$ punctured sphere, then $\dim(\mathcal{EL}(S)) = n$.

To compute the lower bounds we will show that $\pi_n(\mathcal{EL}(S)) \neq 0$ for $S = S_{0,4+n}$ or $S_{1,1+n}$. Since $\mathcal{EL}(S)$ is $(n-1)$-connected and $(n-1)$-locally connected Lemma 15.4 applies.

To start with, $\mathcal{EL}(S)$ is a separable metric space and hence covering dimension, inductive dimension and cohomological dimension of $\mathcal{EL}(S)$ all coincide [HW]. By dimension we mean any of these equal values.

The next lemma is key to establishing the upper bound for $\dim(\mathcal{EL}(S))$.

Lemma 15.1. Let $S = S_{g,p}$, where $p > 0$ and let $\{\Delta_i\}$ be a good cellulation sequence. If $\sigma$ is a cell of $\Delta_i$ and $\phi(\sigma) \cap \mathcal{EL}(S) \neq \emptyset$, then $\dim(\sigma) \geq n + 1 - g$, where $\dim(\mathcal{P}\mathcal{M}\mathcal{L}(S)) = 2n + 1$.

If $S = S_{g,0}$ and $\sigma$ is as above, then $\dim(\sigma) \geq n + 2 - g$.

Proof. We first consider the case $p \neq 0$. By Proposition 14.6, $\sigma = P(\tau)$ for some train track $\tau$. It suffices to consider the case that $\tau$ is generic. A generic track with $e$ edges has $2e/3$ switches, hence $\dim(V(\tau)) \geq e - 2e/3 = e/3$, e.g. see page 116 [PH]. Since $\tau$ carries an element of $\mathcal{EL}(S)$, all complementary regions are discs with at most one puncture. After filling in the punctures, $\tau$ has say $f$ complementary regions all of which are discs, thus $2 - 2g = \chi(S_\tau) = 2e/3 - e + f$ and hence $\dim(V(\tau)) \geq e/3 = f - 2 + 2g$. Therefore, $\dim(P(\tau)) = \dim(V(\tau)) - 1 \geq f - 3 + 2g \geq p - 3 + 2g = n + 1 - g$, since $2n + 1 = 6g + 2p - 7$.

If $p = 0$, then $f \geq 1$ and hence the above argument shows $\dim(P(\tau)) = \dim(V(\tau)) - 1 \geq f - 3 + 2g \geq 1 - 3 + 2g = n + 2 - g$. □

Corollary 15.2. If $S = S_{g,p}$ with $p > 0$ (resp. $p = 0$), then for each $m \in \mathbb{Z}$, $\mathcal{EL}(S) = U(\sigma_1) \cup \cdots \cup U(\sigma_k)$, where $\{\sigma_1, \ldots, \sigma_k\}$ are the cells of $\Delta_m$ of dimension $\geq n + 1 - g$ (resp. $\geq n + 2 - g$). □

Proposition 15.3. Let $S = S_{g,p}$. If $p > 0$ (resp. $p = 0$), then $\dim(\mathcal{EL}(S)) \leq 4g + p - 4 = n + g$ (resp. $\dim(\mathcal{EL}(S)) \leq 4g - 5 = n + g - 1$), where $\dim(\mathcal{P}\mathcal{M}\mathcal{L}(S)) = 2n + 1$.

Proof. To minimize notation we give the proof for the $g = 0$ case. The general case follows similarly, after appropriately shifting dimensions and indices.

Let $E_i$ denote those elements of $\mu \in \mathcal{EL}(S)$ such that $\mu = \phi(x)$ for some $x$ in the $r$-skeleton $\Delta_i^r$ of some $\Delta_i$. We will show that for each $i$, $\dim(E_i) \leq \max\{i - (n + 1), -1\}$. We use the convention that $\dim(X) = -1$ if $X = \emptyset$. 

By Lemma 15.1 $\Delta_i^n \cap \mathcal{EL}(S) = \emptyset$ for all $i \in \mathbb{N}$ and $r \leq n$ and hence $\mathcal{E}_n = \emptyset$. Now suppose by induction that for all $k < m$, $\dim(\mathcal{E}_{n+k}) \leq k$. We will show that $\dim(\mathcal{E}_{n+m}) \leq m$. Now $\mathcal{E}_{n+m} = \bigcup_{i \in \mathbb{N}} \mathcal{E}_{n+i} \cap \mathcal{EL}(S)$, each $\phi(\Delta_i^{n+m}) \cap \mathcal{EL}(S)$ is closed in $\mathcal{EL}(S)$ by Lemma 1.14 and $\mathcal{EL}(S)$ is separable metric, hence to prove that $\dim(\mathcal{E}_{n+m}) \leq m$ it suffices to show by the sum theorem (Theorem III 2 [HW]) that $\dim(\phi(\Delta_i^{n+m}) \cap \mathcal{EL}(S)) \leq m$ for all $i \in \mathbb{N}$.

Let $X = \phi(\Delta_i^{n+m}) \cap \mathcal{EL}(S)$. To show that the inductive dimension of $X$ is $\leq m$ it suffices to show that if $U$ is open in $X$, $\mu \in U$, then there exists $V$ open in $X$ with $\mu \in V$ and $\partial V \subset \mathcal{E}_{n+m}$.

Let $\sigma \in \Delta_i$, some $j \geq i$, such that $\mu \in U(\sigma) \cap X \subset U$. Such a $\sigma$ exists by Lemma 14.18. Now $\hat{\sigma} = \bigcup_{u \in j} \text{int}(\sigma_u)$ where the union is over all cells in $\Delta_j$ having $\sigma$ as a face. Since $\Delta_i^{n+m} \subset \Delta_j^{n+m}$ it follows that after reindexing, $U(\sigma) \cap X \subset \text{int}(\sigma_1) \cup \cdots \cup \text{int}(\sigma_q)$, where $\sigma_1, \cdots, \sigma_q$ are those $\sigma_u$’s such that $\text{dim}(\sigma_i) \leq n + m + 1$.

By Lemma 1.14 $(\bigcup_{i=1}^q \text{int}(\sigma_i)) \cap \mathcal{EL}(S)$ is closed in $\mathcal{EL}(S)$ and hence restricts to a closed set in $X$. It follows that $\partial(U(\sigma) \cap X) \cap (\bigcup_{i=1}^q \text{int}(\partial\sigma_i)) \cap \mathcal{EL}(S) \subset \mathcal{E}_{n+m}$. □

To establish our lower bounds on $\dim(\mathcal{EL}(S))$ we will use the following basic result.

**Lemma 15.4.** Let $X$ be a separable metric space such that $X$ is $(n-1)$-connected and $(n-1)$-locally connected. If $\pi_n(X) \neq 0$, then $\dim(X) \geq n$.

**Proof.** (Dranishnikov [Dr]) Let $f : S^n \to X$ be an essential map. If $\dim(X) \leq n-1$, then by [HW] $\dim(f(S^n)) \leq \dim(X)$ and hence $\dim(f(X))$ is at most $n-1$. By Bothe [Bo] there exists a compact, metric, absolute retract $Y$ such $\dim(Y) \leq n$ and $f(S^n)$ embeds in $Y$. By Theorem 10.1 [Hu], since $Y$ is metric and $X$ is $(n-1)$-connected and $(n-1)$-locally connected, the inclusion of $f(S^n)$ into $X$ extends to a map $g : Y \to X$. Now $Y$ is contractible since it is an absolute retract. (The cone of $Y$ retracts to $Y$.) It follows that $f$ is homotopically trivial, which is a contradiction. □

We need the following controlled homotopy lemma for $\mathcal{PM}(S)$ approximations that are very close to a given map into $\mathcal{EL}(S)$.

**Lemma 15.5.** Let $K$ be a finite simplicial complex. Let $g : K \to \mathcal{EL}(S)$. Let $U \subset \mathcal{PM}(S)$ be a neighborhood of $\phi^{-1}(g(K))$. There exists $\delta > 0$ such that if $f_0, f_1 : K \to \mathcal{PM}(S)$ and for every $t \in K$ and $i \in \{0, 1\}$, $\phi(f_t(i))$, $g(t)) = \delta$, then there exists a homotopy from $f_0$ to $f_1$ supported in $U$.

**Proof.** Let $\Delta_1, \Delta_2, \cdots$ be a good cellulation sequence of $\mathcal{PM}(S)$. Given $x \in \mathcal{EL}(S)$ and $i \in \mathbb{N}$ let $\sigma_i^x$ denote the cell of $\Delta_i$ such that $\phi^{-1}(x) \subset \text{int}(\sigma_i^x)$. Such a $\sigma_i^x$ exists since each cell of $\Delta_i$ is associated to a train track. By Lemma 4.4, given $x \in \mathcal{EL}(S)$ there exists a neighborhood $W_x$ of $x$ such that $\phi^{-1}(W_x) \subset U$ and lies in a small neighborhood of the cell $\phi^{-1}(x)$, i.e. that only intersects $\sigma_x^i$ and higher dimensional cells that have $\sigma_x^i$ as a face. Therefore, $y \in W_x$ implies that $\sigma_x^i$ is a face of $\sigma_y^i$ and hence $\sigma_x^i \subset \phi^{-1}(g(K))$.

Let $U' \subset \mathcal{PM}(S)$ be a neighborhood of $\phi^{-1}(g(K))$ such that $\bar{U'} \subset U$. By Lemmas 14.18 and 4.4, for $i$ sufficiently large, $\phi^{-1}(U(\sigma_x^i)) \subset U'$. This implies that $\sigma_x^i \subset U$, since $\mathcal{PM}(S)$ is dense in $\mathcal{PM}(S)$. Let $U_i = \cup_{x \in g(K)} \sigma_x^i$. By compactness of $g(K)$ and the previous paragraph it follows that for $i$ sufficiently large $U_i \subset U$. Fix such an $i$. Let $\Sigma = \{\sigma \in \Delta_i \mid$ some $\sigma_x^i$ is a face of $\sigma\}$. 
There exists $\delta > 0$ and a function $W : g(K) \to \Sigma$ such that if $y \in PML(S), x \in g(K)$ and $d_{PT(S)}(x, \phi(y)) < \delta$, then $y \in W(x)$. Furthermore, if $d_{PT(S)}(x_1, \phi(y)) < \delta$, $\cdots$, $d_{PT(S)}(x_n, \phi(y)) < \delta$, then after reindexing the $x_i$’s, $W(x_1) \subset \cdots \subset W(x_n)$. Modulo the epsilonics (to find $W$ convergence), $W$ is defined as follows. Define $d : g(K) \to \mathbb{N}$ by $d(x) = \text{dim}(\sigma^*_x)$. Let $p$ be the minimal value of $d(g(K))$. Define $W(x) = \sigma^*_x$ if $d(x) = p$. Next define $W(x') = \sigma^*_x$ if $d_{PT(S)}(x, x')$ is very small, where $d(x) = p$. To guarantee that the second sentence holds, we require that if both $d_{PT(S)}(x_1, x')$ and $d_{PT(S)}(x_2, x')$ are very small, then $W(x_1) = W(x_2)$. If $W(x)$ has not yet been defined and $d(x) = p + 1$, then define $W(x) = \sigma^*_x$. Next define $W(x') = \sigma^*_x$ if $W(x')$ hasn’t already been defined and $d_{PT(S)}(x, x')$ is very very small, where $d(x) = p + 1$. Again we require that if $d_{PT(S)}(x_1, x')$ and $d_{PT(S)}(x_2, x')$ are very very small, then after reindexing $W(x_1) \subset W(x_2)$. Inductively, continue to define $W$ on all of $g(K)$. Since $\text{dim}(\Delta_n)$ is finite this process eventually terminates. Take $\delta$ to be the minimal value used to define smallness.

Next subdivide $K$ such that the following holds. For every simplex $\kappa$ of $K$, there exists $t \in \kappa$ such that for all $s \in \kappa$, $d_{PT(S)}(\phi(f_0(s)), g(t)) < \delta$ and $d_{PT(S)}(\phi(f_1(s)), g(t)) < \delta$. Thus by the previous paragraph, for each simplex $\kappa$ of $K$, $f_0(\kappa) \cup f_1(\kappa) \subset W(g(t))$ for some $t \in \kappa$ where $t$ satisfies the above property. Let $\sigma(\kappa)$ denote the maximal dimensional simplex of $\Sigma$ with these properties. Note that by the second sentence of the previous paragraph, $\sigma(\kappa)$ is well defined and if $\kappa'$ is a face of $\kappa$, then $\sigma(\kappa')$ is a face of $\sigma(\kappa)$.

We now construct the homotopy $F : K \times I \to PML(S)$ from $f_0$ to $f_1$. Assume that $K$ has been subdivided as in the previous paragraph. If $v$ is a vertex of $K$, then both $f_0(v), f_1(v) \in \hat{\sigma}(v)$ which is contractible by Lemma 14.15. Thus $F$ extends over $v \times I$ such that $F(v \times I) \subset \hat{\sigma}(v)$. Assume by induction that if $\eta$ is a simplex of $K$ and $\text{dim}(\eta) < m$, then $F$ has been extended over $\eta \times I$ with $F(\eta) \subset \hat{\sigma}(\eta)$. If $\kappa$ is an $m$-simplex, then the contractibility of $\hat{\sigma}(\kappa)$ enables us to extend $F$ over $\kappa \times I$, with $F(\kappa \times I) \subset \hat{\sigma}(\kappa)$. \hfill \Box

**Theorem 15.6.** If $S = S_{0,4+n}$ or $S_{1,1+n}$, then $\pi_n(\mathcal{E}(S)) \neq 0$.

**Proof.** Let $S$ denote either $S_{0,4+n}$ or $S_{1,1+n}$. In either case $\dim(PML(S)) = 2n + 1$. By Harer [Hr] (see also Ivanov [Iv], [IJ]) the curve complex $\mathcal{C}(S)$ is homotopy equivalent to a non trivial wedge of $n$-spheres. Thus, there exists an essential map $h : S^n \to \mathcal{C}(S)$. We can assume that $h(S^n)$ is a simplicial map with image a finite subcomplex $L$ of the $n$-skeleton of $\mathcal{C}(S)$ and hence $0 \neq [h_*(S^n)] \in H_n(L)$. We abuse notation by letting $L$ denote the image of $L$ under the natural embedding, given in Definition 1.20. Let $Z$ be a simplicial $n$-cycle that represents $[h_*(S^n)]$. Let $\sigma$ be a $n$-simplex of $L$ in the support of $Z$. Let $f_0 : S^n \to PML(S) \setminus L$ be a PL embedding which links $\sigma$, i.e. there exists an extension $F_0 : B^{n+1} \to PML(S)$ of $f_0$ such that $F_0$ is transverse to $L$ and intersects $L$ at a single point in $\text{int}(\sigma)$. It follows that $f_0(S^n)$ has non trivial linking number with $Z$ and hence $f_0$ is homotopically non trivial as a map into $PML(S) \setminus L$. The theorem is now a consequence of the following lemma. \hfill \Box

**Lemma 15.7.** Let $S$ be a finite type hyperbolic surface with $\dim(PML(S)) = 2n + 1$. Let $L$ be an $n$-dimensional subcomplex of $\mathcal{C}(S)$ that is naturally embedded in $PML(S)$. If $\pi_n(PML(S) \setminus L) \neq 0$, then $\pi_n(\mathcal{E}(S)) \neq 0$. Indeed, if $f : S^n \to \mathcal{C}(S)$ is a simple embedding that represents $[h_*(S^n)]$, then $f_0(S^n)$ is a non trivial $n$-cycle in $PML(S) \setminus L$. The proof of the theorem is now a consequence of the following fact.
Theorem 16.1.  

Proof of the following theorem.

\[ \text{dim}(PML) \geq n \]

Remark 15.11.

Proposition 15.8.

Proof of Step 1.  Let \( \Sigma \) be a triangulation of \( S^n \). It suffices to consider the case that \( f \) is a generic PL map. Thus \( f(\Sigma^0) \subset \text{FPMLEL}(S) \). Extend \( f|\Sigma^0 \times I \) by \( f(t,s) = f(t,0) \) where \( f(t,1) \) is viewed as an element of \( \text{EL}(S) \). Now for some \( u \leq n \), assume that \( f \) extends as desired to \( \Sigma^u \times I \cup S^n \times 0 \). If \( \sigma \) is a \((u+1)\)-simplex of \( \Sigma \), then \( f|\sigma \times 0 \cup \partial \sigma \times I \) maps into \( \text{PMLEL}(S) \) with \( f|\sigma \times 0 \cup \partial \sigma \times I \) maps into \( \text{EL}(S) \) and the rest mapping into \( \text{PMLEL}(S) \setminus L \). By Lemma 11.1 \( f \) extends to \( \sigma \times I \) such that \( f(\sigma \times 1) \subset \text{EL}(S) \) and \( f(\sigma \times 0,1) \subset \text{PMLEL}(S) \setminus L \). Step 1 now follows by induction.

Proof of Step 2. Otherwise there exists \( G : B^{n+1} \rightarrow \text{EL}(S) \) extending \( g \). Since \( L \subset \text{C}(S) \), \( L \subset \phi^{-1}(G(B^{n+1})) = \emptyset \). Let \( F : B^{n+1} \rightarrow \text{PMLEL}(S) \) be a \( \delta - \text{PMLEL}(S) \) approximation of \( G \). By Lemma 13.1 \( F|B^{n+1} \cap I = \emptyset \) for \( \delta \) sufficiently small. Now \( F|S^n \) and \( f|S^n \times s \) are respectively \( \delta, \delta' \)-\( \text{PMLEL}(S) \) approximations of \( g \), where \( \delta' \rightarrow 0 \) as \( s \rightarrow 1 \). Since \( \delta \) can be chosen arbitrarily small it follows from Lemma 15.5 that \( F \) can be chosen such that for \( s \) sufficiently large, \( f|S^n \times s \) and \( F|S^n \) are homotopic via a homotopy supported in \( \text{PMLEL}(S) \setminus L \). By concatenating this homotopy with \( F \), we conclude that \( f \) is homotopically trivial via a homotopy supported in \( \text{PMLEL}(S) \setminus L \); which is a contradiction.

Proposition 15.8. If \( S = S_{0,p} \), then \( \text{dim}(\text{EL}(S)) \geq p - 4 \). If \( S = S_{1,p} \), then \( \text{dim}(\text{EL}(S)) \geq p - 1 \). In either case if \( \text{dim}(\text{PMLEL}(S)) = 2n+1 \), then \( \text{dim}(\text{EL}(S)) \geq n \).

Proof. By Theorems 11.2 and 12.1, \( \text{EL}(S) \) is \((n-1)\)-connected and \((n-1)\)-locally connected. By Theorem 15.6, \( \pi_n(\text{EL}(S)) \neq 0 \). Therefore by Lemma 15.4, \( \text{dim}(\text{EL}(S)) \geq n \).

By Proposition 15.8 and Proposition 15.3 we obtain;

Theorem 15.9. If \( S \) is the \((4+n)\)-punctured sphere, then \( \text{dim}(\text{EL}(S)) = n \). 

Theorem 15.10. If \( S \) is the \((n+1)\)-punctured torus, then \( n + 1 \geq \text{dim}(\text{EL}(S)) \geq n \).

Remark 15.11. In a future paper we will show that if \( S = S_{g,p} \) with \( p > 0 \), then \( \text{dim}(\text{EL}(S)) \leq n + g - 1 \).

16. Nobeling spaces

The \( n \)-dimensional Nobeling space \( \mathbb{R}^{2n+1} \) is the space of points in \( \mathbb{R}^{2n+1} \) with at most \( n \) rational coordinates. The goal of the next two sections is to complete the proof of the following theorem.

Theorem 16.1. If \( S \) is the \( n \) + \( p \) punctured sphere, then \( \text{EL}(S) \) is homeomorphic to the \( n \)-dimensional Nobeling space.
By Luo [Luo], $C(S_{2,0})$ is homeomorphic to $C(S_{0,6})$ and thus by Klarreich [K] $\mathcal{EL}(S_{2,0})$ is homeomorphic to $\mathcal{EL}(S_{0,6})$.

**Corollary 16.2.** If $S$ is the closed surface of genus-2, then $\mathcal{EL}(S)$ is homeomorphic to the Nobeling surface, i.e. the 2-dimensional Nobeling space.

**Remarks 16.3.** Using [G1], Sebastian Hensel and Piotr Przytycki earlier proved that the ending lamination space of the 5-times punctured sphere is homeomorphic to the Nobeling curve. They used Luo [Luo] and Klarreich [K] to show that $\mathcal{EL}(S_{1,2})$ is also homeomorphic to the Nobeling curve.

Hensel and Przytycki boldly conjectured that if $\dim(PML(S)) = 2n + 1$, then $\mathcal{EL}(S)$ is homeomorphic to $\mathbb{R}^{2n+1}$. Theorem 16.1 gives a positive proof of their conjecture for punctured spheres.


**Remarks 16.4.** Historically, the $m$-dimensional Nobeling space was called the universal Nobeling space of dimension $m$ and a Nobeling space was one that is locally homeomorphic to the universal Nobeling space. In 2006, S. Ageev [Ag], Michael Levin [Le] and Andrzej Nagorko [Na] independently showed that any two connected Nobeling spaces of the same dimension are homeomorphic. The 0 and 1-dimensional versions of that result were respectively given in [AU] 1928 and [KLT] 1997.

These spaces were named after Georg Nobeling who showed [N] in 1930 that any $m$-dimensional separable metric space embeds in an $m$-dimensional Nobeling space. This generalized a result by Nobeling's mentor, Karl Menger, who defined [Me] in 1926 the Menger compacta and showed that any 1-dimensional compact metric space embeds in the Menger curve. A topological characterization of $m$-dimensional Menger compacta was given in [Be] by Mladen Bestvina in 1984.

The following equivalent form of the topological characterization of $m$-dimensional Nobeling spaces is due to Andrzej Nagorko.

**Theorem 16.5.** (Nagorko [HP]) A topological space $X$ is homeomorphic to the $m$-dimensional Nobeling space if and only if the following conditions hold.

i) $X$ is separable

ii) $X$ supports a complete metric

iii) $X$ is $m$-dimensional

iv) $X$ is $(m - 1)$-connected

v) $X$ is $(m - 1)$-locally connected

vi) $X$ satisfies the locally finite $m$-discs property

**Definition 16.6.** [HP] The space $X$ satisfies the locally finite $m$-discs property if for each open cover $\{U\}$ of $X$ and each sequence $f_i : B^m \to X$, there exists a sequence $g_i : B^m \to X$ such that

i) for each $x \in X$ there exists a neighborhood $U$ of $x$ such that $g_i(B^m) \cap U = \emptyset$ for $i$ sufficiently large.

ii) for each $t \in B^m$, there is a $U \in \mathcal{U}$ such that $f_i(t), g_i(t) \in U$.

**Proof of Theorem 16.1.** By Remark 1.6 $\mathcal{EL}(S)$ is separable and supports a complete metric. If $S = S_{0,n+4}$, then conditions iii)-v) of Theorem 16.5 respectively follow from Theorems 11.2, 12.1 and 15.9.
To complete the proof of Theorem 16.1 it suffices to show that $\mathcal{EL}(S_0,n+4)$ satisfies the locally finite n-discs property.

17. The locally finite n-discs property

Proposition 17.1. If $S$ is the $(n+4)$-punctured sphere, then $\mathcal{EL}(S)$ satisfies the locally finite n-discs property.

Our proof is modeled on the proof, due to Andrzej Nagorko [HP], that $\mathbb{R}_n^{2n+1}$ satisfies the locally finite n-discs property. In particular, we modify his notions of participates and attracting grid.

From now on $S$ will denote the $(n+4)$-punctured sphere. Let $\mathcal{U}$ be an open cover of $\mathcal{EL}(S)$. Let $\mathcal{U}_2$ denote the refinement of $\mathcal{U}$ produced by Lemma 14.19 and let $\Delta_1, \Delta_2, \cdots$ denote a good cellulation sequence.

Definition 17.2. We say that $\sigma \in \Delta_i$ participates in $\mathcal{U}_2$ if $U(\sigma') \in \mathcal{U}_2$ for some face $\sigma'$ of $\sigma$.

Define $A_i = \{ \sigma \in \Delta_k, k \leq i | \sigma \text{ participates in } \mathcal{U}_2 \}$ and define $\Gamma_i = \mathcal{PML}(S) \setminus \cup_{\sigma \in A_i} \delta \sigma$.

Remarks 17.3. i) By Lemma 15.1 and Definition 14.18 each cell of $A_i$ has dimension $\geq n + 1$.

ii) Note that $\Gamma_i$ is obtained from $\Gamma_{i+1}$ by attaching the cells of $A_{i+1} \setminus A_i$.

Definition 17.4. Call $\Gamma_i$ the $i$th approximate attracting grid and $\cap_{i=1}^\infty \Gamma_i$ the attracting grid.

Definition 17.5. Let $Y_i \subset \mathcal{PML}(S) \setminus \Gamma_i$ denote the dual cell complex to $A_i$. Abstractly it is a simplicial complex with vertices the elements of $A_i$ and $\{v_0, v_1, \cdots, v_k\}$ span a simplex if for all $i, v_i \subset \partial v_{i+1}$.

Remark 17.6. Since $\dim(\mathcal{PML}(S)) = 2n + 1$ and each cell of $A_i$ has dimension at least $n + 1$ it follows that $\dim(Y_i) \leq n$. This is the crucial fact underpinning the proof of Proposition 17.1.

The next result follows by standard PL topology.

Lemma 17.7. For every $a > 0$ there exists $N(\Gamma_i)$ a regular neighborhood of $\Gamma_i$ in $\mathcal{PML}(S)$ such that $N(\Gamma_i) \subset N_{\mathcal{PML}(S)}(\Gamma_i, a)$. Furthermore, there exists a homeomorphism

$$q : \partial(N(\Gamma_i)) \times [0,1) \to \mathcal{PML}(S) \setminus (\text{int}(N(\Gamma_i)) \cup Y_i)$$

such that $q|\partial N(\Gamma_i) \times 0$ is the canonical embedding and a retraction

$$p : \mathcal{PML}(S) \setminus Y_i \to N(\Gamma_i)$$

that is the identity on $N(\Gamma_i)$, quotients $[0,1)$ fibers to points and has the following additional property. If $\sigma \in A_i$ and $x \in \text{int}(\sigma) \setminus Y_i$, then $p(x) \in \text{int}(\sigma)$. \qed

Proof of Proposition 17.1. Let $f_i : B^n \to \mathcal{EL}(S), i \in \mathbb{N}$ be a sequence of continuous maps. Being compact $f_i(B^n)$ is covered by finitely many elements $U(\sigma_1), \cdots, U(\sigma_1)^-$ of $\mathcal{U}_2$ and hence $\phi^{-1}(f_i(B^n)) \subset \mathcal{PML}(S) \setminus \Gamma_{n_i}$, for some $n_i \in \mathbb{N}$.

Let $\epsilon_i = \min\{\delta(U(\sigma_j)) | \sigma_j \in A_{n_i} \}$ where $\delta(U(\sigma_j))$ is as in Lemma 14.19.

Since both $\Gamma_{n_i}$ and $\phi^{-1}(f_i(B^n))$ are compact and disjoint it follows that $d_{\mathcal{PML}(S)}(\Gamma_{n_i}, \phi^{-1}(f_i(B^n))) = \alpha_i > 0$. 


Claim. It suffices to show that for each \( i \in \mathbb{N} \) there exists \( h_i : B^n \to \mathcal{PML}(S) \), a generic PL map, such that for every \( t \in B^n \)

i) \( d_{\mathcal{PML}(S)}(\Gamma_{n_i}, h_i(t)) < 1/i \) and

ii) \( h_i(t) \cup \phi^{-1}(f_i(t)) \subset \hat{\sigma}_j \) for some \( \sigma_j \in A_{n_i} \).

Proof. To start with Condition i) implies that for every \( z \in \mathcal{E}(S) \), there exists \( \delta_z > 0 \) such that \( d_{\mathcal{PML}(S)}(h_i(B^n), \phi^{-1}(z)) > \delta_z \) for \( i \) sufficiently large. This uses the fact that for each \( z \in \mathcal{E}(S) \), \( \phi^{-1}(z) \) is compact and disjoint from \( \Gamma \) for \( i \) sufficiently large.

Now apply the \( \mathcal{E}(S) \)-approximation Lemma 13.3 to the \( \{\varepsilon_i\} \) and \( \{h_i\} \) sequences to obtain maps \( g_i : B^n \to \mathcal{E}(S) \), \( i \in \mathbb{N} \) such that

a) For every \( z \in \mathcal{E}(S) \) there exists \( U_z \) open in \( \mathcal{E}(S) \) such that \( g_i(B^n) \cap U_z = \emptyset \) for \( i \) sufficiently large, and

b) if \( t \in B^n \), then \( d_{\mathcal{P}(S)}(g_i(t), \phi(h_i(t))) < \varepsilon_i \).

Conclusion a) gives the local finiteness condition. By ii) of the Claim, if \( t \in B^n \), then \( h_i(t) \cup \phi^{-1}(f_i(t)) \) lie in the same \( \hat{\sigma}_j \). Since \( \varepsilon_i < \delta(U(\sigma_j)) \), Lemma 14.19 implies that \( g_i(t) \) lies in \( U \in \mathcal{U} \) for some \( U(\sigma_j) \subset U \). To apply that lemma, let \( x = h_i(t) \) and \( z = g_i(t) \). Since \( f_i(t) \in U(\sigma_j) \) the approximation condition holds too. \( \square \)

Fix \( i > 0 \). Let \( a_i > 0 \) and \( \varepsilon_i \) be as above. Let \( a = \min\{1/2i, a_i/2\} \). Using this \( a \), let \( p : \mathcal{PML}(S) \setminus Y_{n_i} \to N(\Gamma_{n_i}) \) be the retraction given by Lemma 17.7.

By the PML-approximation Lemma 14.11, there exists \( h'_i : B^n \to \mathcal{PML}(S) \) such that for each \( t \in B^n \), \( d_{\mathcal{PML}(S)}(h'_i(t), \phi^{-1}(f_i(t))) \subset \hat{\sigma}_j \) for some cell \( \sigma \) of \( \Delta_{n_i} \). Since \( \dim(B^n) + \dim(Y_{n_i}) \leq 2n < 2n + 1 \) we can assume that \( h'_i(B^n) \cap Y_{n_i} = \emptyset \) and this property still holds. Let \( h_i = p \circ h'_i \) perturbed slightly to be a generic PL map.

By construction \( p(h'_i(B^n)) \subset p(\mathcal{PML}(S) \setminus Y_{n_i}) \subset N(\Gamma_{n_i}) \subset N_{\mathcal{PML}(S)}(\Gamma_{n_i}, 1/2i) \). Thus Condition i) of the Claim holds for \( h_i \), if it is obtained by a sufficiently small perturbation of \( p \circ h'_i \).

By Lemma 17.7 and Lemma 14.10 ii) \( p \circ h'_i \) and \( \phi^{-1}(z) \) lie in the same \( \hat{\sigma}_j \). Thus Condition ii) holds for \( p \circ h'_i \). Since each \( \hat{\sigma}_j \) is open, this condition holds for any sufficiently small perturbation of \( p \circ h'_i \) and so it holds for \( h_i \). This completes the proof of Proposition 17.1 and hence Theorem 16.1. \( \square \)

After appropriately modifying dimensions, the proof of Proposition 17.1 generalizes to a proof of the following.

**Proposition 17.8.** Let \( S = S_{g,p} \). Then \( \mathcal{E}(S) \) satisfies the locally finite k-discs property for all \( k \leq m \), where \( m = n - g \), if \( p \neq 0 \) and \( m = n - (g - 1) \), if \( p = 0 \). \( \square \)

18. Applications

By Klarreich [K] (see also [H1]), the Gromov boundary \( \partial C(S) \) of the curve complex \( C(S) \) is homeomorphic to \( \mathcal{E}(S) \). We therefore obtain the following results.

**Theorem 18.1.** Let \( C(S) \) be the curve complex of the surface \( S \) of genus \( g \) and \( p \) punctures. Then \( \partial C(S) \) is \((n - 1)\)-connected and \((n - 1)\)-locally connected. If \( g = 0 \), then \( \partial C(S) \) is homeomorphic to the \( n \)-dimensional Nobeling space. Here \( n = 3g + p - 4 \). Also \( \partial C(S_{2,0}) = \mathbb{R}^2_n \) and \( \partial C(S_{1,2}) = \mathbb{R}^3_n \).

**Remark 18.2.** The cases of \( S_{0,5} \) and \( S_{1,2} \) were first proved in [HP].
Theorem 18.3. Let $C(S)$ be the curve complex of the $p+n$ punctured sphere, then the asymptotic dimension of $C(S)$ is equal to $n + 1$.

Proof. As $C(S)$ is a Gromov-hyperbolic space [MM], its asymptotic dimension is equal to $1 + \dim(\partial C(S))$ by Buyalo and Lebedeva [BL]. Now apply Theorem 15.9.

Let $S$ be a finite type hyperbolic surface. Let $DD(S)$ denote the space of doubly degenerate marked hyperbolic structures on $S \times \mathbb{R}$. These are the complete hyperbolic structures with limit set all of $S_g^2$ whose parabolic locus corresponds to the cusps of $S$. It is topologized with the algebraic topology. See §6 [LS] for more details. As a consequence of many major results in hyperbolic 3-manifold geometry, Leininger and Schleimer proved the following.

Theorem 18.4. [LS] $DD(S)$ is homeomorphic to $\mathcal{E}L(S) \times \mathcal{E}L(S) \setminus \Delta$, where $\Delta$ is the diagonal.

Corollary 18.5. If $S$ is the $(n+p)$-punctured sphere, then $DD(S)$ is homeomorphic to $\mathbb{R}^{2n+1}_n \times \mathbb{R}^{2n+1}_n \setminus \Delta$. In particular $DD(S)$ is $(n-1)$-connected and $(n-1)$-locally connected.

The subspace of marked hyperbolic structures on $S \times \mathbb{R}$ which have the geometrically finite structure $Y$ on the $-\infty$ relative end is known as the Bers slice $B_Y$. As in [LS], let $\partial_0 B_Y(S)$ denote the subspace of the Bers slice whose hyperbolic structures on the $\infty$ relative end are degenerate.

Theorem 18.6. [LS] $\partial_0 B_Y(S)$ is homeomorphic to $\mathcal{E}L(S)$.

Corollary 18.7. If $S$ is a hyperbolic surface of genus-$g$ and $p$-punctures, then $\partial_0 B_Y(S)$ is $(n-1)$-connected and $(n-1)$-locally connected. If $g = 0$, then $\partial_0 B_Y(S)$ is homeomorphic to the $n$-dimensional Noveling space. Here $n = 3g + p - 4$.

Theorem 18.8. If $S$ is a $p$-punctured sphere, $p \geq 5$, $S_{2,0}$ or $S_{1,2}$, then there exists a simple closed curve $\alpha$ in $\mathcal{E}L(S)$ such that $\phi^{-1}(\alpha)$ contains no simple closed curve that projects to $\alpha$ under $\phi$.

Proof. By the proof of Theorem 9.1 [G1] there exists a 1-simplex $\sigma \subset \mathcal{PML}(S)$ such that $\phi(\sigma) = z \in \mathcal{E}L(S)$ and $\sigma = \lim_{i \to -\infty} \eta_i$ where $\eta_i$ is a 1-simplex in $\mathcal{PML}(S)$ consisting of those projective measured laminations supported on two disjoint simple closed geodesics. By Theorem 9.1 [G1] each $\eta_i$ is a limit of 1-simplices $\kappa_i$ with $\phi(\kappa_i) \in \mathcal{E}L(S)$. Thus $\sigma = \lim_{i \to -\infty} \sigma_i$ where $\phi(\sigma_i) = z_i \in \mathcal{E}L(S)$ and $z, z_1, z_2, \ldots$ are distinct. Let $\partial \sigma = x \cup y$ and $\partial \sigma_i = x_i \cup y_i$. We can assume that $x_i \to x$ and $y_i \to y$. Let $\beta_i$ (resp. $\gamma_i$) be a generic PL path from $x_i$ to $x_{i+1}$ (resp. $y_i$ to $y_{i+1}$). We can assume that $\lim(\beta_i) = x$, $\lim(\gamma_i) = y$ and using [Ma], [Ve] or [Ke] that for each $i \in \mathbb{N}$ there exist $p_i \in \beta_i$ and $q_i \in \gamma_i$ such that $\phi(p_i)$ and $\phi(q_i)$ are uniquely ergodic. Furthermore, the various $\phi(p_i)$'s and $\phi(q_i)$'s are distinct. The $\sigma_i$'s can be oriented so that the concatenation $\alpha_1 \ast \alpha_2 \ast \beta_1 \ast \alpha_3 \ast \cdots$ is a well defined (topologist sine curve like) path defined on $[−2, 0]$ that limits on $\sigma$. Appropriately applying Lemma 13.3 we obtain a path $g : [−2, 0] \to \mathcal{E}L(S)$ such that $g(−2) = z_1$ and for $i \in \mathbb{N}$, $g(−1/(2i − 1)) = \phi(p_{2i−1})$ and $g(−1/(2i)) = \phi(q_{2i})$ and $g(0) = z$. Since $\mathcal{E}L(S)$ is a Noveling space, we readily find an embedded path $f : [−2, 0] \to \mathcal{E}L(S)$ with the same properties that additionally is the concatenation of countably many subpaths limiting to $z$ such that on each subpath all but one coordinate of $\mathbb{R}^{2n+1}$
is constant. Observe that $f$ extends to an embedding $F : S^1 \to \mathcal{EL}(S)$, where $S^1$ is the natural quotient of $[-2, 2]$. By construction, there exists no continuous map $h : S^1 \to \mathcal{PML}(S)$ such that $\phi(h(t)) = F(t)$, thus $\phi^{-1}(F(S^1))$ contains no simple closed curve that projects to $F(S^1)$. \qed

19. Problems and Conjectures

To start with we restate the following long-standing classical question.

**Problem 19.1.** Find a topological characterization of the Nobeling type space $\mathbb{R}^p_q$, the space of points in $\mathbb{R}^p$ with at most $q$ rational coordinates.

**Remarks 19.2.** i) The fundamental work of Ageev [Av], Levin [Le] and Nagorko [Na] solve this problem for $p = 2m + 1$ and $q = m$.

ii) Of particular interest are the spaces of the form $\mathbb{R}^{2n+1}_m$, where $m \geq n$.

iii) In analogy to Theorem 16.5, does it suffice that $X$ be separable, complete metric, $q$-dimensional, $(q-1)$-connected, $(q-1)$-locally connected, and satisfy the locally finite $(p-q-1)$-discs property?

We offer the following very speculative;

**Conjecture 19.3.** Let $S$ be a $p$-punctured surface of genus-$g$. Then $\mathcal{EL}(S)$ is homeomorphic to $\mathbb{R}^{2n+1}_{m+k}$, where $n = 3g + p - 4$ and $k = 0$ if $g = 0$, $k = g - 1$ if both $p \neq 0$ and $q \neq 0$, and $k = g - 2$ if $p = 0$.

The value of $k$ in this conjecture is motivated by the following duality conjecture.

**Conjecture 19.4.** (Duality Conjecture) If $S$ is the $p$-punctured surface of genus-$g$, then

1) $\dim(\mathcal{EL}) + h\dim(\mathcal{C}(S)) + 1 = \dim(\mathcal{PML}(S))$, where $h(\dim(\mathcal{C}(S)))$ is the homological dimension of the curve complex.

If $\dim(\mathcal{PML}(S)) = 2n + 1$, then

2) for every $m$, there is a natural injection $H^m_\ast(\mathcal{EL}(S)) \to H^{2n-m}(\mathcal{C}(S))$ and

3) for every $m$, there is a natural injection of $H_m(\mathcal{C}(S)) \to \hat{H}^{2n-m}(\mathcal{EL}(S))$.

**Remark 19.5.** Harer [Hr] computed the homological dimension of $\mathcal{C}(S)$, hence duality conjecture 1) is equivalent to the conjecture that $\dim(\mathcal{EL}(S)) = n$ if $g = 0$; $\dim(\mathcal{EL}(S)) = n + (g - 1)$ if $p \neq 0$ and $q \neq 0$; and $\dim(\mathcal{EL}(S)) = n + (g - 2)$ if $p = 0$.

**Remarks 19.6.** In the above conjectures, homology is Steenrod and cohomology is Čech. Both coincide with the corresponding singular theory for CW-complexes. When $m = 0$ or $2n - m = 0$, the above is reduced homology or cohomology.

Except for a single dimension all the maps and groups should be trivial.

The conjecture is motivated by the observation that $\mathcal{EL}(S)$ and $\mathcal{C}(S)$ almost live in $\mathcal{PML}(S) = S^{2n+1}$ as complementary objects and hence Alexander - Sitnikov duality [Sit] applies. Indeed, since $\mathcal{PML}(S)$ is the disjoint union of $\mathcal{FPML}(S)$ and $\mathcal{UPML}(S)$ the following holds by Alexander - Sitnikov duality.

**Theorem 19.7.** If $S$ is a finite type surface and $\dim(\mathcal{PML}(S)) = 2n + 1$, then

1) $H^m_\ast(\mathcal{FPML}(S))$ is isomorphic to $\hat{H}^{2n-m}(\mathcal{UPML}(S))$ and
2) $H^2_n(U\mathcal{P}\mathcal{M}\mathcal{L}(S))$ is isomorphic to $\check{H}^{2n-m}(F\mathcal{P}\mathcal{M}\mathcal{L}(S))$. □

Remarks 19.8. Again we use reduced homology and cohomology for $m = 0$ or $2n - m = 0$.

Now $U\mathcal{P}\mathcal{M}\mathcal{L}(S)$ has a CW-structure $U\mathcal{P}\mathcal{M}\mathcal{L}(S)_{cw}$ and there is an inclusion $\mathcal{C}(S) \to U\mathcal{P}\mathcal{M}\mathcal{L}(S)_{cw}$ which induces a homotopy equivalence [G2]. However, the topology on $U\mathcal{P}\mathcal{M}\mathcal{L}(S)_{cw}$ is finer than that of $U\mathcal{P}\mathcal{M}\mathcal{L}(S)$ for essentially the same reason that the topology on $\mathcal{C}(S)$ is finer than that of $\mathcal{C}(S)_{sub}$. See Remark 1.21. There is the analogy that $\mathcal{C}(S)$ is to $\mathcal{C}(S)_{sub}$ as the infinite wedge of circles is to the Hawaiian earings. Note that the inclusion of the former into the latter induces proper injections of $H_1$ and $H^1$. For that reason the maps in Conjecture 19.4 are injections rather than isomorphisms.

The map $\phi : F\mathcal{P}\mathcal{M}\mathcal{L}(S) \to \mathcal{E}\mathcal{L}(S)$ is a closed map (Corollary 1.14) and point inverses are cells (Lemma 1.17); hence by [Sp] or [Sk] we obtain the following result.

**Theorem 19.9.** Let $S$ be a finite type surface. Then for all $m$

1) $\phi_* : H^m(F\mathcal{P}\mathcal{M}\mathcal{L}(S)) \to H^m(\mathcal{E}\mathcal{L}(S))$ is an isomorphism and

2) $\phi^* : \check{H}^m(\mathcal{E}\mathcal{L}(S)) \to H^m(F\mathcal{P}\mathcal{M}\mathcal{L}(S))$ is an isomorphism.

**Corollary 19.10.** Let $S$ be a finite type surface and $\dim(\mathcal{P}\mathcal{M}\mathcal{L}(S)) = 2n + 1$, then for all $m$ there are natural isomorphisms

1) $\check{H}^m(\mathcal{E}\mathcal{L}(S)) \cong H^{2n+m}_m(U\mathcal{P}\mathcal{M}\mathcal{L}(S))$ and

2) $H^{2n-m}_m(\mathcal{E}\mathcal{L}(S)) \cong \check{H}^{2n-m}(U\mathcal{P}\mathcal{M}\mathcal{L}(S))$.

**Remark 19.11.** It follows from Corollary 19.10 and Remarks 19.8 that to prove parts 2) and 3) of the duality conjecture it suffices to show that the bijective map $I : U\mathcal{P}\mathcal{M}\mathcal{L}(S)_{cw} \to U\mathcal{P}\mathcal{M}\mathcal{L}(S)$ induces inclusions in Steenrod homology and Cech cohomology. That will also be a big step towards proving 1), since it implies that $\dim(\mathcal{E}\mathcal{L}(S))$ is at least as large as the conjectured value.

**Conjecture 19.12.** Let $S$ be a finite type surface. The bijective maps $I : \mathcal{C}(S) \to \mathcal{C}(S)_{sub}$ and $I : U\mathcal{P}\mathcal{M}\mathcal{L}(S)_{cw} \to U\mathcal{P}\mathcal{M}\mathcal{L}(S)$ induce inclusions in Steenrod homology and Cech cohomology.

**Question 19.13.** Let $S$ be a finite type surface. Is $\mathcal{E}\mathcal{L}(S)$ $m$-connected (resp. $m$-locally connected) if and only if $F\mathcal{P}\mathcal{M}\mathcal{L}(S)$ is $m$-connected (resp. $m$-locally connected).

**Remark 19.14.** It follows from Theorem 1.18 that $\mathcal{E}\mathcal{L}(S)$ is connected if and only if $F\mathcal{P}\mathcal{M}\mathcal{L}(S)$ is connected.

**Definition 19.15.** Let $X$ be a regular cell complex. We say that $X$ is super symmetric if $X/\text{Isom}(X)$ is a finite complex.

**Example 19.16.** Examples of such spaces are the curve complex and arc complex of a finite type surface and the disc complex of a handlebody.

**Question 19.17.** What spaces arise as the boundaries of super symmetric Gromov-hyperbolic regular cell complexes? Under what conditions does the $m$-dimensional Nobeling space arise? More generally, under what conditions does a Nobeling type space arise.
Remark 19.18. This is a well known question, when the cell complex is a Cayley graph of a finitely generated group. Indeed, Kapovich and Kleiner give an essentially complete answer when $\partial G$ is 1-dimensional [KK]. The Gromov boundaries of locally compact Gromov hyperbolic spaces are locally compact, thus the point of this question is to bring attention to non locally finite complexes. Note that Gromov and Champetier [Ch] assert that the generic finitely presented Gromov-hyperbolic group has the Menger curve as its boundary.

Question 19.19. Let $x \in \mathcal{E}(S)$ be non uniquely ergodic. Does $\phi^{-1}(x)$ arise as in the construction of Theorem 9.1 [G2] or as limits of cells of the form $B_{a_1} \cap \cdots \cap B_{a_k}$ where $a_1, \cdots, a_k$ are pairwise disjoint simple closed curves?
References


[Dr] A. Dranishnikov, personal communication.


