CONTINUOUS SOLUTIONS OF LINEAR EQUATIONS

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Contents

1. Introduction

Consider a system of linear equations $A \cdot \mathbf{y} = \mathbf{b}$ where the entries of

$$A = \big(a_{ij}(x_1, \dots, x_n)\big) \quad \text{and of} \quad \mathbf{b} = \big(b_i(x_1, \dots, x_n)\big)$$

are themselves continuous functions on $\mathbb{R}^n$. Our aim is to decide whether the system $A \cdot \mathbf{y} = \mathbf{b}$ has a solution $\mathbf{y} = \big(y_j(x_1, \dots, x_n)\big)$, where the $y_j(x_1, \dots, x_n)$ are also continuous functions on $\mathbb{R}^n$.

More generally, if the a_{ij} and the b_i have some regularity property, can we chose the y_j to have the same (or some weaker) regularity properties?

There are two cases when the answer is rather straightforward. If A is invertible over a dense open subset $U \subset \mathbb{R}^n$, then $\mathbf{y} = A^{-1}\mathbf{b}$ holds over U. Thus there is a continuous solution iff $A^{-1}\mathbf{b}$ extends continuously to $\mathbb{R}^n$. The case when $\operatorname{rank} A$ is constant on $\mathbb{R}^n$ can also be treated by standard linear algebra.

By contrast, if the system is underdetermined and $\operatorname{rank} A$ varies, the problem seems quite subtle. In fact, the hardest case appears to be when there is only one equation in many unknowns. It can be restated as follows.

Question 1. Let $f_1, \dots, f_r$ be continuous functions on $\mathbb{R}^n$. Which continuous functions ϕ can be written in the form

$$\phi = \textstyle\sum_i \phi_i f_i \tag{1.1}$$

where the ϕ_i are continuous functions? Moreover, if ϕ and the f_i have some regularity properties, can we chose the ϕ_i to have the same (or some weaker) regularity properties?

If the f_i have no common zero, then a partition of unity argument shows that every $\phi \in C^0(\mathbb{R}^n)$ can be written this way and the ϕ_i have the same regularity properties (e.g., being Hölder, Lipschitz or C^m) as ϕ and the f_i. By Cartan's Theorem B, if ϕ and the f_i are real analytic then the ϕ_i can also be chosen real analytic.

None of these hold if the common zero set $Z := (f_1 = \dots = f_r = 0)$ is not empty. Even if ϕ and the f_i are polynomials, the best one can say is that the ϕ_i can be chosen to be Hölder continuous; see (30.1). Thus the interesting aspects happen near the common zero set Z.

The C^∞-version of Question 1 was studied extensively (see e.g. [Mal67, Tou72]) and it played a rôle in the work of Ehrenpreis (see [Ehr70]). The continuous version studied here is closer in spirit to the following question for L^∞_{loc}:

Which functions can be written in the form $\sum_i \psi_i f_i$ where $\psi_i \in L^\infty_{loc}$?

The answer to the latter variant turns out to be rather simple. If ϕ is such then $\phi / \sum_i |f_i| \in L^\infty_{loc}$. Conversely, if this holds then

$$\phi = \sum_i \phi_i f_i \quad \text{where} \quad \phi_i := \frac{\phi}{\sum_j |f_j|} \cdot \frac{\bar f_i}{|f_i|} \in L^\infty_{loc}.$$

Equivalently, the obvious formulas

$$\sum_i |f_i| = \sum_i \frac{\bar f_i}{|f_i|} f_i \quad \text{and} \quad \phi = \frac{\phi}{\sum_i |f_i|} \sum_i |f_i| \tag{1.2}$$

show that $L^\infty_{loc}(\mathbb{R}^n) \cdot (f_1, \dots, f_r)$ is the principal ideal generated by $\sum_i |f_i|$. For many purposes it is even better to write ϕ as

$$\phi = \sum_i \psi_i f_i \quad \text{where} \quad \psi_i := \frac{\phi \bar f_i}{\sum_j |f_j|^2} \in L^\infty_{loc}. \tag{1.3}$$

Note that if ϕ is continuous (resp. differentiable) then the ψ_i given in (1.3) are continuous (resp. differentiable) outside the common zero set Z; again indicating the special role of Z.

The above formulas also show that the discontinuity of the ψ_i along Z can be removed for certain functions.

Lemma 2. *For a continuous function ϕ the following are equivalent.*

(1) $\phi = \sum_i \phi_i f_i$ *where the ϕ_i are continuous functions such that* $\lim_{x \to z} \phi_i = 0$ *for every i and every $z \in Z$.*
(2) $\lim_{x \to z} \frac{\phi}{\sum_i |f_i|} = 0$ *for every $z \in Z$.* □

Similar conditions do not answer Question 1. First, if the ψ_i defined in (1.3) are continuous, then $\phi = \sum_i \psi_i f_i$ is continuous, but frequently one can write $\phi = \sum_i \phi_i f_i$ with ϕ_i continuous yet the formula (1.3) defines discontinuous functions

ψ_i. This happens already in very simple examples, like $f_1 = x, f_2 = y$. For $\phi = x$ (1.3) gives

$$x = \frac{x^2}{x^2+y^2} \cdot x + \frac{xy}{x^2+y^2} \cdot y$$

whose coefficients are discontinuous at the origin.

An even worse example is given by $f_1 = x^2, f_2 = y^2$ and $\phi = xy$. Here ϕ can not be written as $\phi = \phi_1 f_1 + \phi_2 f_2$ but every inequality that is satisfied by x^2 and y^2 is also satisfied by $\phi = xy$. We believe that there is no universal test or formula as above that answers Question 1. At least it is clear that $C^0(\mathbb{R}^n) \cdot (x, y)$ is not a principal ideal in $C^0(\mathbb{R}^n)$.

Nonetheless, these examples and the concept of axis closure defined by [Bre06] suggest several simple necessary conditions. These turn out to be equivalent to each other, but they do not settle Question 1.

The algebraic version of Question 1 was posed by H. Brenner, which led him to the notion of the *continuous closure* of ideals [Bre06]. We learned about it from a lecture of M. Hochster. It seems to us that the continuous version is the more basic variant. In turn, the methods of the continuous case can be used to settle several of the algebraic problems [Kol10].

3 (Pointwise tests)**.** For a continuous function ϕ and for a point $p \in \mathbb{R}^n$ the following are equivalent.

1. For every sequence $\{x_j\}$ converging to p there are $\psi_{ij} \in \mathbb{C}$ such that $\lim_{j\to\infty} \psi_{ij}$ exists for every i and $\phi(x_j) = \sum_i \psi_{ij} f_i(x_j)$ for every j.
2. We can write $\phi = \sum_i \psi_i^{(p)} f_i$ where the $\psi_i^{(p)}(x)$ are continuous at p.
3. We can write $\phi = \phi^{(p)} + \sum_i c_i^{(p)} f_i$ where $c_i^{(p)} \in \mathbb{C}$ and $\lim_{x\to p} \frac{\phi^{(p)}}{\sum_i |f_i|} = 0$.

If $\phi = \sum_i \phi_i f_i$ where the ϕ_i are continuous functions, then we obtain the $\psi_{ij}, \psi_i^{(p)}$ by restriction and $\phi = \bigl(\sum_i (\phi_i - \phi_i(p)) f_i\bigr) + \sum_i \phi_i(p) f_i$ shows that ϕ satisfies the third test. Conversely, if ϕ satisfies (3) then $\phi_p := \phi - \sum_i c_i^{(p)} f_i$ is continuous and $\lim_{x\to p} \frac{\phi_p}{\sum_i |f_i|} = 0$. By Lemma 2 we can write

$$\phi = \sum_i \psi_i^{(p)} f_i \quad \text{where} \quad \psi_i^{(p)} := c_i^{(p)} + \frac{\phi_p \bar{f}_i}{\sum_j |f_j|^2}$$

and the $\psi_i^{(p)}(x)$ are continuous at p. Thus (2) and (3) are equivalent. One can see their equivalence with (1) directly, but for us it is more natural to obtain it by showing that they are all equivalent to the Finite set test to be introduced in (26).

If the common zero set $Z := (f_1 = \cdots = f_r = 0)$ consists of a single point p, then the $\psi_i^{(p)}(x)$ constructed above are continuous everywhere. More generally, if Z is a finite set of points then these tests give necessary and sufficient conditions for Question 1. However, the following example of Hochster shows that the pointwise test for every p does not give a sufficient condition in general.

3.4 Example. [Hoc10] Take $\{f_1, f_2, f_3\} := \{x^2, y^2, xyz^2\}$ and $\phi := xyz$. Pick a point $p = (a, b, c) \in \mathbb{R}^3$. If $c \neq 0$ then we can write

$$xyz = \tfrac{1}{c} xyz^2 + \tfrac{1}{c}(c - z)xyz \quad \text{and} \quad \lim_{(x,y,z)\to(a,b,c)} \frac{(c-z)xyz}{|x^2| + |y^2| + |xyz^2|} = 0,$$

thus (3.3) holds. Note that if $a=b=0$, then $\frac{1}{c}xyz^2$ is the only possible constant coefficient term that works. As $c\to 0$, the coefficient $\frac{1}{c}$ is not continuous, thus xyz can not be written as $xyz=\phi_1x^2+\phi_2y^2+\phi_3xyz^2$ where the ϕ_i are continuous. Nonetheless, if $c=0$ then

$$\lim_{(x,y,z)\to(a,b,0)} \frac{xyz}{|x^2|+|y^2|+|xyz^2|}=0.$$

shows that (3.3) is satisfied (with all $c_i^{(a,b,0)}=0$).

One problem is that the coefficients $c_i^{(p)}$ are not continuous functions of p. In general, they are not even functions of p since a representation as in (3.2) or (3.3) is not unique. Still, this suggests a possibility of reducing Question 1 to a similar problem on the lower dimensional set $Z=(f_1=\cdots=f_r=0)$.

We present two methods to answer Question 1.

The first method starts with $f_1,\dots,f_r$ and ϕ and decides if $\phi=\sum_i\phi_if_i$ is solvable or not. The union of the graphs of all discontinuous solutions $(\phi_1,\dots,\phi_r)$ is a subset $\mathcal{H}\subset\mathbb{R}^n\times\mathbb{R}^r$. Then we use the tests (3.1–3) repeatedly to get smaller and smaller subsets of $\mathcal{H}$. After $2r+1$ steps, this process stabilizes. This follows [Fef06, Lem.2.2]. It was adapted from a lemma in [BMP03], which in turn was adapted from a lemma in [Gla58]. At the end we use Michael's theorem [Mic56] to get a necessary and sufficient criterion. The dependence on ϕ is somewhat delicate.

The second method considers the case when the f_i are polynomials (or real analytic functions). The method relies on the observation that formulas like (1.2–1.3) give a continuous solution to $\phi=\sum_i\phi_if_i$; albeit not on $\mathbb{R}^n$ but on some real algebraic variety mapping to $\mathbb{R}^n$. Following this idea, we transform the original Question 1 on $\mathbb{R}^n$ to a similar problem on a real algebraic variety Y for which the solvability on any finite subset is equivalent to continuous solvability.

The algebraic method also shows that if ϕ is Hölder continuous (resp. semi-algebraic and continuous) and the equation (1.1) has a continuous solution then there is also a solution where the ϕ_i are Hölder continuous (resp. semialgebraic and continuous) (29). By contrast, if can happen that ϕ is a continuous rational function on $\mathbb{R}^3$, the equation (1.1) has a continuous semialgebraic solution but has no continuous rational solutions [Kol11].

Both of the methods work for any linear system of equations $A\cdot\mathbf{y}=\mathbf{b}$.

2. The Glaeser–Michael method

Fix positive integers n, r and let Q be a compact metric space.

4 (Singular affine bundles)**.** By a singular affine bundle (or *bundle* for short), we mean a family $\mathcal{H}=(H_x)_{x\in Q}$ of affine subspaces $H_x\subseteq\mathbb{R}^r$, parametrized by the points $x\in Q$. The affine subspaces H_x are the *fibers* of the bundle $\mathcal{H}$. (Here, we allow the empty set $\emptyset$ and the whole space $\mathbb{R}^r$ as affine subspaces of $\mathbb{R}^r$.) A *section* of a bundle $\mathcal{H}=(H_x)_{x\in Q}$ is a continuous map $f:Q\to\mathbb{R}^r$ such that $f(x)\in H_x$ for each $x\in Q$. We ask:

(2.1) How can we tell whether a given bundle of $\mathcal{H}$ has a section?

For instance, let $f_1,\dots,f_r$ and φ be given real-valued functions on Q. For $x\in Q$, we take

$$H_x=\{(\lambda_1,\dots,\lambda_r)\in\mathbb{R}^r:\lambda_1f_1(x)+\cdots+\lambda_rf_r(x)=\varphi(x)\}. \tag{2.2}$$

Then a section $(\phi_1, \dots, \phi_r)$ of the bundle (2.2) is precisely an r-tuple of continuous functions solving the equation

$$\phi_1 f_1 + \cdots + \phi_r f_r = \varphi \text{ on } Q. \tag{2.3}$$

To answer question (2.1), we introduce the notion of "Glaeser refinement". (Compare with [Gla58], [BMP03], [Fef06].) Let $\mathcal{H} = (H_x)_{x \in Q}$ be a bundle. Then the *Glaeser refinement* of $\mathcal{H}$ is the bundle $\mathcal{H}' = (H'_x)_{x \in Q}$, where, for each $x \in Q$,

$$H'_x = \{\lambda \in H_x : \operatorname{dist}(\lambda, H_y) \to 0 \text{ as } y \to x \quad (y \in Q)\}. \tag{2.4}$$

One checks easily that

$$\mathcal{H}' \text{ is a } \textit{subbundle} \text{ of } \mathcal{H}, \text{ i.e., } H'_x \subseteq H_x \text{ for each } x \in Q \tag{2.5}$$

and

$$\text{the bundles } \mathcal{H} \text{ and } \mathcal{H}' \text{ have the same sections.} \tag{2.6}$$

Starting from a given bundle $\mathcal{H}$, and iterating the above construction, we obtain a sequence of bundles $\mathcal{H}^0, \mathcal{H}^1, \mathcal{H}^2, \cdots$, where $\mathcal{H}^0 = \mathcal{H}$, and $\mathcal{H}^{i+1}$ is the Glaeser refinement of $\mathcal{H}^i$ for each i. In particular, $\mathcal{H}^{i+1}$ is a subbundle of $\mathcal{H}^i$, and all the bundles $\mathcal{H}^i$ have the same sections.

We will prove the following results.

Lemma 5 (Stabilization Lemma)**.** $\mathcal{H}^{2r+1} = \mathcal{H}^{2r+2} = \cdots$

Lemma 6 (Existence of Sections)**.** *Let $\mathcal{H} = (H_x)_{x \in Q}$ be a bundle. Suppose that $\mathcal{H}$ is its own Glaeser refinement, and suppose each fiber H_x is non-empty. Then $\mathcal{H}$ has a section.*

The above results allow us to answer question (2.1). Let $\mathcal{H}$ be a bundle, let $\mathcal{H}^0, \mathcal{H}^1, \mathcal{H}^2, \cdots$ be its iterated Glaeser refinements, and let $\mathcal{H}^{2r+1} = (\widetilde{H}_x)_{x \in Q}$. Then $\mathcal{H}$ has a section if and only if each fiber $\widetilde{H}_x$ is non-empty.

The bundle (2.2) provides an interesting example. One checks that its Glaeser refinement is given by $\mathcal{H}^1 = (H^1_x)_{x \in Q}$, where

$$H^1_x = \Big\{(\lambda_1, \dots, \lambda_r) \in \mathbb{R}^r : \big|\textstyle\sum_1^r \lambda_i f_i(y) - \varphi(y)\big| = o\big(\sum_1^r |f_i(y)|\big) \text{ as } y \to x\Big\}.$$

Thus, the necessary condition (3) for the existence of continuous solutions of (2.3) asserts precisely that the fibers H^1_x are all non-empty.

In Hochster's example (3.4), the equation (2.3) has no continuous solutions, because the second Glaeser refinement $\mathcal{H}^2 = (H^2_x)_{x \in Q}$ has an empty fiber, namely H^2_0.

We present self-contained proofs of (5) and (6), for the reader's convenience. A terse discussion would simply note that the proof of [Fef06, Lem.2.2] also yields (5), and that one can easily prove (6) using Michael's theorem [Mic56], [BL00].

7 (Proof of the Stabilization Lemma)**.** Let $\mathcal{H}^0, \mathcal{H}^1, \mathcal{H}^2, \cdots$ be the iterated Glaeser refinements of $\mathcal{H}$; and let $\mathcal{H}^i = (H^i_x)_{x \in Q}$ for each i.

We must show that $H^\ell_x = H^{2r+1}_x$ for all $x \in Q$, $\ell \geq 2r+1$. If $H^{2r+1}_x = \emptyset$, then the desired result is obvious.

For non-empty H^{2r+1}_x, it follows at once from the following.

Claim 7.1$_k$. Let $x \in Q$. If $\dim H^{2k+1}_x \geq r - k$, then $H^\ell_x = H^{2k+1}_x$ for all $\ell \geq 2k+1$.

We prove (7.1_k) for all $k \geq 0$, by induction on k. In the case $k = 0$, (7.1_k) asserts that

$$\text{If } H_x^1 = \mathbb{R}^r, \text{ then } H_x^\ell = \mathbb{R}^r \text{ for all } \ell \geq 1. \tag{2.7}$$

By definition of Glaeser refinement, we have

$$\dim H_x^{\ell+1} \leq \liminf_{y \to x} \dim H_y^\ell. \tag{2.8}$$

Hence, if $H_x^1 = \mathbb{R}^r$, then $H_y^0 = \mathbb{R}^r$ for all y in a neighborhood of x. Consequently, $H_y^\ell = \mathbb{R}^r$ for all y in a neighborhood of x, and for all $\ell \geq 0$. This proves (7.1_k) in the base case $k = 0$. For the induction step, we fix k and assume (7.1_k) for all $x \in Q$. We will prove (7.1_{k+1}). We must show that

$$\text{If } \dim H_x^{2k+3} \geq r - k - 1, \text{ then } H_x^\ell = H_x^{2k+3} \text{ for all } \ell \geq 2k+3. \tag{2.9}$$

If $\dim H_x^{2k+1} \geq r - k$, then (2.9) follows at once from (7.1_k). Hence, in proving (2.9), we may assume that $\dim H_x^{2k+1} \leq r - k - 1$. Thus,

$$\dim H_x^{2k+1} = \dim H_x^{2k+2} = \dim H_x^{2k+3} = r - k - 1. \tag{2.10}$$

We now show that

$$H_y^{2k+2} = H_1^{2k+1} \text{ for all } y \text{ near enough to } x. \tag{2.11}$$

If fact, suppose that (2.11) fails, i.e., suppose that

$$\dim H_y^{2k+2} \leq \dim H_y^{2k+1} - 1 \text{ for y arbitrarily close to } x. \tag{2.12}$$

For y as in (2.12), our inductive assumption (7.1_k) shows that $\dim H_y^{2k+1} \leq r-k-1$. Therefore, for y arbitrarily near x, we have

$$\dim H_y^{2k+2} \leq \dim H_y^{2k+1} - 1 \leq r - k - 2.$$

Another application of (2.8) now yields $\dim H_x^{2k+3} \leq r-k-2$, contradicting (2.10). Thus, (2.11) cannot fail.

From (2.11) we see easily that $H_y^\ell = H_y^{2k+3}$ for all y near enough to x, and for all $\ell \geq 2k+3$.

This completes the inductive step (2.9), and proves the Stabilization Lemma. □

8 (Proof of Existence of Sections)**.** We give the standard proof of Michael's theorem in the relevant special case. We start with a few definitions. If $H \subset \mathbb{R}^r$ is an affine subspace and $v \in \mathbb{R}^r$ is a vector, then $H - v$ denotes the translate $\{w - v : w \in H\}$. If $\mathcal{H} = (H_x)_{x \in Q}$ is a bundle, and if $f : Q \to \mathbb{R}^r$ is a continuous map, then $\mathcal{H} - f$ denotes the bundle $(H_x - f(x))_{x \in Q}$. Note that if $\mathcal{H}$ is its own Glaeser refinement and has non-empty fibers, then the same is true of $\mathcal{H} - f$.

Let $\mathcal{H} = (H_x)_{x \in Q}$ be any bundle with non-empty fibers. We define the norm $\|\mathcal{H}\| := \sup_{x \in Q} \operatorname{dist}(0, H_x)$. Thus, $\|\mathcal{H}\|$ is a non-negative real number or $+\infty$.

Now suppose that $\mathcal{H} = (H_x)_{x \in Q}$ is a bundle with non-empty fibers, and suppose that $\mathcal{H}$ is its own Glaeser refinement.

Proposition 9. $\|\mathcal{H}\| < +\infty$.

Proof. Given $x \in Q$, we can pick $w_x \in H_x$ since H_x is non-empty. Also, $\operatorname{dist}(w_x, H_y) \to 0$ as $y \to x$ ($y \in Q$), since $\mathcal{H}$ is its own Glaeser refinement. Hence, there exists an open ball B_x centered at x, such that $\operatorname{dist}(w_x, H_y) \leq 1$ for all $y \in Q \cap B_x$. If follows that $\mathrm{dist}(0, H_y) \le |w_x| + 1$ for all $y \in Q \cap B_x$. We can cover the compact space Q by finitely many of the open balls B_x ($x \in Q$); say,

$$Q \subset B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_N}.$$

Since $\mathrm{dist}(0, H_y) \le |w_{x_i}| + 1$ for all $y \in Q \cap B_{x_i}$, it follows that

$$\mathrm{dist}(0, H_y) \le \max\{|w_{x_i}| + 1 : i = 1, 2, \ldots, N\} \text{ for all } y \in Q.$$

Thus $\|\mathcal{H}\| < +\infty$. □

Proposition 10. *Given $\varepsilon > 0$, there exists a continuous map $g : Q \to \mathbb{R}^r$ such that*

$$\mathrm{dist}(g(y), H_y) \le \varepsilon \textit{ for all } y \in Q,$$

and

$$|g(y)| \le \|\mathcal{H}\| + \varepsilon \textit{ for all } y \in Q.$$

Proof. Given $x \in Q$, we can find $w_x \in H_x$ such that $|w_x| \le \|\mathcal{H}\| + \varepsilon$. We know that $\mathrm{dist}(w_x, H_y) \to 0$ as $y \to x$ ($y \in Q$), since $\mathcal{H}$ is its own Glaeser refinement. Hence, there exists an open ball $B(x, 2r_x)$ centered at x, such that

$$\mathrm{dist}(w_x, H_y) < \varepsilon \text{ for all } y \in Q \cap B(x, 2r_x).$$

The compact space Q may be covered by finitely many of the open balls $B(x, r_x)$ ($x \in Q$); say

$$Q \subset B(x_1, r_{x_1}) \cup \cdots \cup B(x_N, r_{x_N}).$$

For each $i = 1, \ldots, N$, we introduce a non-negative continuous function $\widetilde{\varphi}_i$ on $\mathbb{R}^n$, supported in $B(x_i, 2r_{x_i})$ and equal to one on $B(x_i, r_{x_i})$. We then define $\varphi_i(x) = \widetilde{\varphi}_i(x)/(\widetilde{\varphi}_1(x) + \cdots + \widetilde{\varphi}_N(x))$ for $i = 1, \ldots, N$ and $x \in Q$. (This makes sense, thanks for (8).)

The φ_i form a partition of unity on Q:

- Each φ_i is a non-negative continuous function on Q, equal to zero outside $Q \cap B(x_i, 2r_{x_i})$; and
- $\sum_{i=1}^N \varphi_i = 1$ on Q.

We define

$$g(y) = \sum_{i=1}^N w_{x_i} \varphi_i(y) \text{ for } y \in Q.$$

Thus, g is a continuous map from Q into $\mathbb{R}^r$. Moreover, (8) shows that $\mathrm{dist}(w_{x_i}, H_y) \le \varepsilon$ whenever $\varphi_i(y) \ne 0$. Therefore,

$$\begin{aligned} \mathrm{dist}(g(y), H_y) &\le \sum_{i=1}^N \mathrm{dist}(w_{x_i}, H_y)\varphi_i(y) \\ &\le \varepsilon \sum_{i=1}^N \varphi_i(y) = \varepsilon \text{ for all } y \in Q. \end{aligned}$$

Also, for each $y \in Q$ we have

$$|g(y)| \le \sum_{i=1}^N |w_{x_i}| \varphi_i(y) \le \sum_{i=1}^N (\|\mathcal{H}\| + \varepsilon)\varphi_i(y) = \|\mathcal{H}\| + \varepsilon.$$

The proof of Proposition 10 is complete. □

Corollary 11. *Let $\mathcal{H}$ be a bundle with non-empty fibers, equal to its own Glaeser refinement. Then there exists a continuous map $g : Q \to \mathbb{R}^r$, such that $\|\mathcal{H} - g\| \le \frac{1}{2}\|\mathcal{H}\|$, and $|g(y)| \le 2\|\mathcal{H}\|$ for all $y \in Q$.*

Proof. If $\|\mathcal{H}\| > 0$, then we can just take $\varepsilon = \frac{1}{2}\|\mathcal{H}\|$ in Proposition 10. If instead $\|\mathcal{H}\| = 0$, then we can just take $g = 0$. □

Now we can prove the existence of sections. Let $\mathcal{H} = (H_x)_{x\in Q}$ be a bundle. Suppose the H_x are all non-empty, and assume that $\mathcal{H}$ is its own Glaeser refinement. By induction on $i = 0, 1, 2, \dots$, we define continuous maps $f_i, g_i : Q \to \mathbb{R}^r$. We start with $f_0 = g_0 = 0$. Given f_i and g_i, we apply Corollary 11 to the bundle $\mathcal{H} - f_i$, to produce a continuous map $g_{i+1} : Q \to \mathbb{R}^r$, such that $\|(\mathcal{H} - f_i) - g_{i+1}\| \le \frac{1}{2}\|\mathcal{H} - f_i\|$, and $|g_{i+1}(y)| \le 2\|\mathcal{H} - f_i\|$ for all $y \in Q$.

We then define $f_{i+1} = f_i + g_{i+1}$. This completes our inductive definition of the f_i and g_i. Note that $f_0 = 0$, $\|\mathcal{H} - f_{i+1}\| \le \frac{1}{2}\|\mathcal{H} - f_i\|$ for each i, and $|f_{i+1}(y) - f_i(y)| \le 2\|\mathcal{H} - f_i\|$ for each $y \in Q$, $i \ge 0$. Therefore, $\|\mathcal{H} - f_i\| \le 2^{-i}\|\mathcal{H}\|$ for each i, and $|f_{i+1}(y) - f_i(y)| \le 2^{1-i}\|\mathcal{H}\|$ for each $y \in Q$, $i \ge 0$. In particular, the f_i converge uniformly on Q to a continuous map $f : Q \to \mathbb{R}^r$, and $\|\mathcal{H} - f_i\| \to 0$ as $i \to \infty$.

Now, for any $y \in Q$, we have

$$\begin{aligned}\operatorname{dist}(f(y), H_y) &= \lim_{i\to\infty} \operatorname{dist}(f_i(y), H_y)\\ &= \lim_{i\to\infty} \operatorname{dist}\big(0, H_y - f_i(y)\big) \le \liminf_{i\to\infty} \|\mathcal{H} - f_i\| = 0.\end{aligned}$$

Thus, $f(y) \in H_y$ for each $y \in Q$. Since also $f : Q \to \mathbb{R}^r$ is a continuous map, we see that f is a section of $\mathcal{H}$. This completes the proof of existence of sections. □

12 (Further problems and remarks)**.** We return to the equation

$$\phi_1 f_1 + \cdots + \phi_r f_r = \varphi \ \text{ on } \ \mathbb{R}^n, \tag{2.13}$$

where $f_1, \dots, f_r$ are given polynomials.

Let X be a function space, such as $C^m_{\mathrm{loc}}(\mathbb{R}^n)$ or $C^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$ $(0 < \alpha \le 1)$. It would be interesting to know how to decide whether the equation (2.13) admits a solution $\phi_1, \dots, \phi_r \in X$. Some related examples are given in (30). If φ is real-analytic, and if (2.13) admits a continuous solution, then we can take the continuous functions ϕ_i to be real-analytic outside the common zeros of the f_i. To see this we invoke the following

Theorem 13 (Approximation Theorem, see [Nar68])**.** *Let $\phi, \sigma : \Omega \to \mathbb{R}$ be continuous functions on an open set $\Omega \subset \mathbb{R}^n$, and suppose $\sigma > 0$ on Ω. Then there exists a real-analytic function $\tilde\phi : \Omega \to \mathbb{R}$ such that $|\tilde\phi(x) - \phi(x)| \le \sigma(x)$ for all $x \in \Omega$.*

Once we know the Approximation Theorem, we can easily correct a continuous solution $\phi_1, \dots, \phi_r$ of (2.13) so that the functions ϕ_i are real-analytic outside the common zeros of $f_1, \dots, f_r$. We take $\Omega = \{x \in \mathbb{R}^n : f_i(x) \ne 0 \text{ for some } i\}$, and set $\sigma(x) = \sum_i (f_i(x))^2$ for $x \in \Omega$.

We obtain real-analytic functions $\tilde\phi_i$ on Ω such that $|\tilde\phi_i - \phi_i| \le \sigma$ on Ω. Setting $h = \sum_i \tilde\phi_i f_i - \varphi = \sum_i (\tilde\phi_i - \phi_i) f_i$ on Ω and then defining

$$\left\{\begin{array}{l} \phi_i^{\#} = \tilde\phi_i - \frac{h f_i}{f_1^2 + \cdots + f_r^2} \ \text{ on } \ \Omega \\ \phi_i^{\#} = \phi_i \ \text{ on } \ \mathbb{R}^n \setminus \Omega \end{array}\right\},$$

we see that $\sum_i \phi_i^{\#} f_i = \varphi$, with $\phi_i^{\#}$ continuous on $\mathbb{R}^n$ and real-analytic on Ω.

3. Computation of the solutions

In this section, we show how to compute a continuous solution $(\phi_1, \ldots, \phi_r)$ of the equation

$$\phi_1 f_1 + \ldots + \phi_r f_r = \phi, \tag{3.1}$$

assuming such a solution exists. We start with an example, then spend several sections explaining how to compute Glaeser refinements and sections of bundles, and finally return to (1) in the general case.

For our example, we pick Hochster's equation

$$\phi_1 x^2 + \phi_2\ y^2 + \phi_3\ xyz^2 = \phi \quad \text{on} \quad Q = [-1,1]^3, \tag{3.2}$$

where ϕ is a given, continuous, real-valued function on Q. Our goal here is to compute a continuous solution of (3.2), assuming such a solution exists.

Suppose ϕ_1, ϕ_2, ϕ_3 satisfy (3.2). Then, for every positive integer ν, we have

$$\phi_1\big(\tfrac{1}{\nu}, 0, z\big) \cdot \tfrac{1}{\nu^2} = \phi\big(\tfrac{1}{\nu}, 0, z\big),$$

$$\phi_2\big(0,\ \tfrac{1}{\nu},\ z\big) \cdot \tfrac{1}{\nu^2} = \phi\big(0,\ \tfrac{1}{\nu},\ z\big), \text{ and}$$

$$\phi_1\big(\tfrac{1}{\nu}, \tfrac{1}{\nu}, z\big) \cdot \tfrac{1}{\nu^2} + \phi_2\big(\tfrac{1}{\nu}, \tfrac{1}{\nu}, z\big) \cdot\ \tfrac{1}{\nu^2} + \phi_3\big(\tfrac{1}{\nu},\ \tfrac{1}{\nu},\ z\big) \cdot\ \tfrac{z^2}{\nu^2} = \phi\big(\tfrac{1}{\nu}, \tfrac{1}{\nu}, z\big)$$

for all $z \in [-1,1]$. Hence, it is natural to define

$$\xi_1(z) \quad = \quad \lim_{\nu\to\infty} \nu^2 \cdot \phi\big(\tfrac{1}{\nu}, 0, z\big), \tag{3.3}$$

$$\xi_2(z) \quad = \quad \lim_{\nu\to\infty} \nu^2 \cdot \phi\big(0, \tfrac{1}{\nu}, z\big) \quad \text{and} \tag{3.4}$$

$$\xi_3(z) \quad = \quad \lim_{\nu\to\infty} \nu^2 \cdot \phi\big(\tfrac{1}{\nu}, \tfrac{1}{\nu}, z\big) \qquad \text{for } z \in [-1,1]. \tag{3.5}$$

If (3.2) has a continuous solution $\overrightarrow{\phi} = (\phi_1, \phi_2, \phi_3)$, then the limits (3.3) exist, and our solution $\overrightarrow{\phi}$ satisfies

$$\phi_1(0,0,z) = \xi_1(z), \quad \phi_2(0,0,z) = \xi_2(z), \quad \text{and} \tag{3.6}$$

$$\phi_1(0,0,z) + \phi_2(0,0,z) + z^2\phi_3(0,0,z) = \xi_3(z) \tag{3.7}$$

for $z \in [-1,1]$, so that

$$\phi_3(0,0,z) = z^{-2}\ \cdot\ [\xi_3(z) - \xi_1(z) - \xi_2(z)] \quad \text{for } z\ \in [-1,1] \smallsetminus \{0\}. \tag{3.8}$$

To recover $\phi_3(0,0,0)$, we just pass to the limit in (3.8). Let us define

$$\xi = \lim_{\nu\to\infty} \nu^2 \cdot \xi_3\big(\tfrac{1}{\nu}\big) - \xi_1\big(\tfrac{1}{\nu}\big) - \xi_2\big(\tfrac{1}{\nu}\big). \tag{3.9}$$

If (3.2) has a continuous solution $\overrightarrow{\phi}$, then the limit (3.9) exists, and we have

$$\phi_3(0,0,0) = \xi. \tag{3.10}$$

Thus, $\overrightarrow{\phi}(0,0,z)(z\ \in\ [-1,1])$ can be computed from the given function ϕ. Note that $\phi_3(0,0,0)$ arises from ϕ by taking an iterated limit.

Since we assumed that $\overrightarrow{\phi}$ is continuous, we have in particular

(3.11) The functions $\phi_i(0,0,z)$ $(i = 1,2,3)$ are continuous on $[-1,1]$.

From now on, we regard $\overrightarrow{\phi}(0,0,z) = (\phi_1(0,0,z), \phi_2(0,0,z), \phi_3(0,0,z))$ as known.

Let us now define

$$\overrightarrow{\phi}^{\#}(x,y,z) = \overrightarrow{\phi}(x,y,z) - \overrightarrow{\phi}(0,0,z) = (\phi_1^{\#}(x,y,z), \phi_2^{\#}(x,y,z), \phi_3^{\#}(x,y,z)) \tag{3.12}$$

and

(3.13) $\phi^{\#}(x,y,z) = \phi(x,y,z) - [\phi_1(0,0,z)\cdot x^2 + \phi_2(0,0,z)\cdot y^2 + \phi_3(0,0,z)\cdot xyz^2]$

on Q. Then, since $\overrightarrow{\phi}$ is a continuous solution of (3.2), we see that

(3.14) $\phi^{\#}$ and all the $\phi_i^{\#}$ are continuous functions on Q;

(3.15) $\phi_i^{\#}(0,0,z) = 0$ for all $z \in [-1,1]$, $i = 1,2,3$; and

(3.16) $\phi_1^{\#}(x,y,z)\cdot x^2 + \phi_2^{\#}(x,y,z)\cdot y^2 + \phi_3^{\#}(x,y,z)\cdot xyz^2 = \phi^{\#}(x,y,z)$ on Q.

We don't know the functions $\phi_i^{\#}(i = 1,2,3)$, but $\phi^{\#}$ may be computed from the given function ϕ in (3.2), since we have already computed $\phi_i(0,0,z)(i = 1,2,3)$. (See (3.13).)

We now define $\overrightarrow{\Phi}^{\#}(x,y,z) = (\Phi_1^{\#}(x,y,z), \Phi_2^{\#}(x,y,z), \Phi_3^{\#}(x,y,z))$ to be the shortest vector $(v_1,v_2,v_3) \in \mathbb{R}^3$ such that

(3.17) $$v_1\cdot x^2 + v_2\cdot y^2 + v_3\cdot xyz^2 = \phi^{\#}(x,y,z).$$

Thus,

(3.18) $\Phi_1^{\#}(x,y,z)\cdot x^2 + \Phi_2^{\#}(x,y,z)\cdot y^2 + \Phi_3^{\#}(x,y,z)\cdot xyz^2 = \phi^{\#}(x,y,z)$ on Q.

Unless $x = y = 0$, we have

(3.19) $$\begin{aligned} \Phi_1^{\#}(x,y,z) &= \tfrac{x^2}{x^4+y^4+x^2y^2z^4}\cdot\phi^{\#}(x,y,z),\\ \Phi_2^{\#}(x,y,z) &= \tfrac{y^2}{x^4+y^4+x^2y^2z^4}\cdot\phi^{\#}(x,y,z),\\ \Phi_3^{\#}(x,y,z) &= \tfrac{xyz^2}{x^4+y^4+x^2y^2z^4}\cdot\phi^{\#}(x,y,z) \end{aligned}$$

(3.20) If $x = y = 0$, then $\Phi_i^{\#}(x,y,z) = 0$ for $i = 1,2,3$.

Since $\phi^{\#}$ may be computed from ϕ, the functions $\Phi_i^{\#}$ $(i = 1,2,3)$ may also be computed from ϕ.

Recall that $\overrightarrow{\phi}^{\#} = (\phi_1^{\#}, \phi_2^{\#}, \phi_3^{\#})$ satisfies (3.16). Since $\overrightarrow{\Phi}(x,y,z)$ was defined as the shortest vector satisfying (3.17), we learn that

(3.21) $$\left| \overrightarrow{\Phi}^{\#}(x,y,z) \right| \le \left| \overrightarrow{\phi}^{\#}(x,y,z) \right| \quad \text{for all } (x,y,z) \in Q.$$

Since also $\overrightarrow{\phi}^{\#}$ satisfies (3.14) and (3.15), it follows that

(3.22) $\Phi_i^{\#}(x,y,z) \to 0$ as $(x,y,z) \to (0,0,z')$, for each $i = 1,2,3$.

Here, $z' \in [-1,1]$ is arbitrary.

We will now check that

(3.23) $\Phi_1^{\#}, \Phi_2^{\#}, \Phi_3^{\#}$ are continuous functions on Q.

Indeed, the $\Phi_i^{\#}$ are continuous at each $(x,y,z) \in Q$ such that $(x,y) \neq (0,0)$, as we see at once from (3.14) and (3.19). On the other hand, (3.20) and (3.22) tell us that the $\Phi_i^{\#}$ are continuous at each $(x,y,z) \in Q$ such that $(x,y) = (0,0)$. Thus, (3.23) holds.

Next, we set

(3.24) $\Phi_i(x,y,z) = \Phi_i^{\#}(x,y,z) + \phi_i(0,0,z)$ for $(x,y,z) \in Q$, $i = 1,2,3$.

Since $\Phi_i^{\#}(x,y,z)$ and $\phi_i(0,0,z)$ can be computed from ϕ, the same is true of $\Phi_i(x,y,z)$.

Also, (3.11) and (3.23) imply

$$\Phi_1, \Phi_2, \Phi_3 \text{ are continuous functions on } Q. \tag{3.25}$$

From (3.13), (3.18) and (3.24), we have

$$\Phi_1(x,y,z) \cdot x^2 + \Phi_2(x,y,z) \cdot y^2 + \Phi_3(x,y,z) \cdot xyz^2 = \phi(x,y,z) \text{ on } Q. \tag{3.26}$$

Note also that the Φ_i satisfy the estimate

$$\max_{x \epsilon Q,\quad i=1,2,3} \left| \, \Phi_i(x) \, \right| \leq \; C \;\; \max_{x \epsilon Q,\quad i=1,2,3} \left| \, \phi_i(x) \, \right| \tag{3.27}$$

for an absolute constant C, as follows from (3.13), (3.21) and (3.24).

Let us summarize the above discussion of equation (3.2). Given a function $\phi : Q \to \mathbb{R}$, we proceed as follows.

Step 1: We compute the limits (3.3), (3.4), (3.5) for each $z \in [-1, 1]$, to obtain the functions $\xi_i(z) \quad (i = 1, 2, 3)$.
Step 2: We compute the limit (3.9), to obtain the number ξ.
Step 3: We read off the functions $\phi_i(0, 0, z) \qquad (i = 1, 2, 3)$ from (3.6), (3.7), (3.8), (3.10).
Step 4: We compute the function $\phi^{\#}(x, y, z)$ from (3.13).
Step 5: We compute the functions $\Phi_i^{\#}(x, y, z) \qquad (i = 1, 2, 3)$ from (3.19) $\cdots$ (3.20).
Step 6: We read off the functions $\Phi_i(x, y, z) \qquad (i = 1, 2, 3)$ from (3.24).

If, for our given ϕ, equation (3.2) has a continuous solution (ϕ_1, ϕ_2, ϕ_3), then the limits exist in Steps 1 and 2, and the above procedure produces continuous functions Φ_1, Φ_2, Φ_3 that solve equation (3.2) and satisfy estimate (3.27).
If instead the equation (3.2) has no continuous solutions, then we cannot guarantee that the limits in Steps 1 and 2 exist. It may happen that those limits exist, but the functions Φ_1, Φ_2, Φ_3 produced by our procedure are discontinuous.

This concludes our discussion of example (3.2). We devote the next several sections to making calculations with bundles. We show how to pass from a given bundle to its iterated Glaeser refinements by means of formulas involving iterated limits. After recalling the construction of "Whitney cubes" (which will be used below), we then provide additional formulas to compute a section of a given Glaeser stable bundle with non-empty fibers. These results together allow us to compute a section of any given bundle for which a section exists. Finally, we apply our results on bundles, to provide a discussion of equation (3.1) in the general case, analogous to the discussion given above for example (3.2).

3.1. **Computation of the Glaeser refinement.**

We use the standard inner product on $\mathbb{R}^r$. We define a *homogeneous bundle* to be a family $\mathcal{H}^0 = (H_x^0)_{x \in Q}$ of vector subspaces $H_x^0 \subset \mathbb{R}^r$, indexed by the points x of a closed cube $Q \subset \mathbb{R}^n$. We allow $\{0\}$ and $\mathbb{R}^r$, but not the empty set, as vector subspaces of $\mathbb{R}^r$. Note that the fibers of a homogeneous bundle are vector subspaces of $\mathbb{R}^r$, while the fibers of a bundle are (possibly empty) affine subspaces of $\mathbb{R}^r$.

Any bundle $\mathcal{H}$ with non-empty fibers may be written uniquely in the form

$$\mathcal{H} = (H_x)_{x \in Q} = (v(x) + H_x^0)_{x \in Q}, \tag{3.28}$$

where $\mathcal{H}^0 = (H_x^0)_{x \in Q}$ is a homogeneous bundle, and $v(x) \perp \; H_x^0$ for each $x \in Q$.

Let $\widetilde{\mathcal{H}}$ be the Glaeser refinement of $\mathcal{H}$, and suppose $\widetilde{\mathcal{H}}$ has non-empty fibers. Just as $\mathcal{H}$ may be written in the form (3.28), we can express $\widetilde{\mathcal{H}}$ uniquely in the form

$$\widetilde{\mathcal{H}} = (\widetilde{v}(x) + \widetilde{H}^0_x)_{x\in Q}, \tag{3.29}$$

where $\widetilde{\mathcal{H}}^0 = (\widetilde{H}^0_x)_{x\in Q}$ is a homogeneous bundle, and $\widetilde{v}(x) \perp \widetilde{H}^0_x$ for each $x \in Q$.

One checks easily that $\widetilde{\mathcal{H}}^0$ is the Glaeser refinement of $\mathcal{H}^0$. The goal of this section is to understand how the vectors $\widetilde{v}(x)(x \in Q)$ depend on the vectors $v(y)(y \in Q)$ for fixed $\mathcal{H}^0$.

To do so, we introduce the sets

$$E = \{(x,\lambda) \in Q \times \mathbb{R}^r : \lambda \perp H^0_x\}, \text{ and} \tag{3.30}$$

$$\Lambda(x) = \{\widetilde{\lambda} \in \mathbb{R}^r : (x,\widetilde{\lambda}) \text{ belongs to the closure of } E\} \text{ for } x \in Q.. \tag{3.31}$$

The following is immediate from the definitions (3.30), (3.31).

Claim 14. Given $\widetilde{\lambda} \in \Lambda(x)$, there exist points $y^\nu \in Q$ and vectors $\lambda^\nu \in \mathbb{R}^r (\nu \geqslant 1)$, such that $y^\nu \to x$ and $\lambda^\nu \to \widetilde{\lambda}$ as $\nu \to \infty$, and $\lambda^\nu \perp H^0_{y^\nu}$ for each ν. □

Note that E and $\Lambda(x)$ depend on $\mathcal{H}^0$, but not on the vectors $v(y),\ y \in Q$. The basic properties of $\Lambda(x)$ are given by the following result.

Lemma 15. *Let $x \in Q$. Then*

1. *Each $\widetilde{\lambda} \in \Lambda(x)$ is perpendicular to $\widetilde{H}^0_x$.*
2. *Given any vector $\widetilde{v} \in \mathbb{R}^r$ not belonging to $\widetilde{H}^0_x$, there exists a vector $\lambda \in \Lambda(x)$ such that $\lambda \cdot \widetilde{v} \neq 0$.*
3. *The vector space $(\widetilde{H}^0_x)^\perp \subset \mathbb{R}^r$ has a basis $\widetilde{\lambda}_1(x), \ldots, \widetilde{\lambda}_s(x)$ consisting entirely of vectors $\widetilde{\lambda}_i(x) \in \Lambda(x)$.*

Proof: To check (1), let $\widetilde{\lambda} \in \Lambda(x)$ and let $\widetilde{v} \in \widetilde{H}^0_x$. We must show that $\widetilde{\lambda} \cdot \widetilde{v} = 0$. Let $y^\nu \in Q$ and $\lambda^\nu \in \mathbb{R}^r$ $(\nu \geqslant 1)$ be as in (3.9). Since $\widetilde{v} \in \widetilde{H}^0_x$ and $(\widetilde{H}^0_y)_{y\in Q}$ is the Glaeser refinement of $(H^0_y)_{y\in Q}$, we know that distance $(\widetilde{v}, H^0_y) \to 0$ as $y \to x$. In particular, distance $(\widetilde{v}, H^0_{y^\nu}) \to 0$ as $\nu \to \infty$. Hence, there exist $v^\nu \in H^0_{y^\nu}$ $(v \geqslant 1)$ such that $v^\nu \to \widetilde{v}$ as $\nu \to \infty$. Since $v^\nu \in H^0_{y^\nu}$ and $\lambda^\nu \perp H^0_{y^\nu}$, we have $\lambda^\nu \cdot v^\nu = 0$ for each ν. Since $\lambda^\nu \to \widetilde{\lambda}$ and $v^\nu \to \widetilde{v}$ as $\nu \to \infty$, it follows that $\widetilde{\lambda} \cdot \widetilde{v} = 0$, proving (1).

To check (2), suppose $\widetilde{v} \in \mathbb{R}^r$ does not belong to $\widetilde{H}^0_x$. Since $(\widetilde{H}^0_y)_{y\in Q}$ is the Glaeser refinement of $(H^0_y)_{y\in Q}$, we know that distance $(\widetilde{v}, H^0_y)$ does not tend to zero as $y \in Q$ tends to x. Hence there exist $\epsilon > 0$ and a sequence of points $y^\nu \in Q$ $(\nu \geqslant 1)$, such that

$$y^\nu \to x \text{ as } \nu \to \infty, \quad \text{but} \quad \operatorname{dist}(\widetilde{v}, H^0_{y^\nu}) \geqslant \epsilon \text{ for each } \nu. \tag{3.32}$$

Thanks to (3.14), there exist unit vectors $\lambda^\nu \in \mathbb{R}^r$ $(\nu \geqslant 1)$, such that

$$\lambda^\nu \perp H^0_{y^\nu} \quad \text{and} \quad \lambda^\nu \cdot \widetilde{v} \geqslant \epsilon \quad \text{for each } \nu. \tag{3.33}$$

Passing to a subsequence, we may assume that the vectors λ^ν tend to a limit $\widetilde{\lambda} \in \mathbb{R}^r$as $\nu \to \infty$.

Comparing (3.33) to (3.30), we see that $(y^\nu, \lambda^\nu) \in E$ for each ν. Since $y^\nu \to x$ and $\lambda^\nu \to \widetilde{\lambda}$ as $\nu \to \infty$, the point $(x, \widetilde{\lambda})$ belongs to the closure of E, hence $\widetilde{\lambda} \in \Lambda(x)$. Also, $\widetilde{\lambda} \cdot \widetilde{v} = \lim_{\nu\to\infty} \lambda^\nu \cdot \widetilde{v} \geqslant \epsilon$ by (3.16); in particular, $\widetilde{\lambda} \cdot \widetilde{v} \neq 0$. The proof of (2) is complete. Finally, to check (3), we note that

$$\bigcap_{\widetilde{\lambda}\in\Lambda(x)} (\widetilde{\lambda}^\perp) = \widetilde{H}^0_x, \quad \text{thanks to (3.10) and (3.11)} .$$

Assertion (3) now follows from linear algebra. The proof of Lemma 15 is complete. □

Let $\widetilde{\lambda}_1(x), \cdots, \widetilde{\lambda}_s(x)$ be the basis for $(\widetilde{H}_x^0)^\perp$ given by (3), and let $\widetilde{\lambda}_{s+1}(x), \cdots, \widetilde{\lambda}_r(x)$ be a basis for $\widetilde{H}_x^0$. Thus

$$\widetilde{\lambda}_1(x), \cdots, \widetilde{\lambda_r}(x) \text{ form a basis for } \mathbb{R}^r. \tag{3.34}$$

For $1 \leq i \leq s$, the vector $\widetilde{\lambda}_i(x)$ belongs to $\Lambda(x)$. Hence, by (14), there exist vectors $\lambda_i^\nu(x) \in \mathbb{R}^r$ and points $y_i^\nu(x) \in Q$ $(\nu \geqslant 1)$, such that

$$y_i^\nu(x) \to x \text{ as } \nu \to \infty, \tag{3.35}$$

$$\lambda_i^\nu(x) \to \widetilde{\lambda}_i(x) \text{ as } \nu \to \infty, \text{ and} \tag{3.36}$$

$$\lambda_i^\nu(x) \perp H^0_{y_i^\nu(x)} \text{ for each } \nu. \tag{3.37}$$

For $s+1 \leq i \leq r$, we take $y_i^\nu(x) = x$ and $\lambda_i^\nu(x) = 0$ $(\nu \geqslant 1)$. Thus, (3.19) holds also for $s+1 \leq i \leq r$, although (3.36) holds only for $1 \leqslant i \leqslant s$.

We now return to the problem of computing $\widetilde{v}(x)(x \in Q)$ for the bundles given by (3.28) and (3.29). The answer is as follows.

Lemma 16. *Given $x \in Q$, we have $\widetilde{\lambda}_i(x) \cdot \widetilde{v}(x) = \lim_{\nu\to\infty} \lambda_i^\nu(x) \cdot v(y_i^\nu(x))$ for $i = 1, \cdots, r$. In particular, the limit in (16) exists.*

Remarks: Since $\widetilde{\lambda}_1(x), \cdots, \widetilde{\lambda}_r(x)$ form a basis for $\mathbb{R}^r$, (16) completely specifies the vector $\widetilde{v}(x)$. Note that the points $y_i^\nu(x)$ and the vectors $\widetilde{\lambda}_i(x)$, $\lambda_i^\nu(x)$ depend only on $\mathcal{H}^0$, not on the vectors $v(y)$ $(y \in Q)$.

Proof: First suppose that $1 \leq i \leq s$. Since $\widetilde{v}(x)$ belongs to the fiber $\widetilde{v}(x) + \widetilde{H}_x^0$ of the Glaeser refinement of $(v(y) + H_y^0)_{y \in Q}$, we know that $\text{dist}(\widetilde{v}(x), v(y) + H_y^0) \to 0$ as $y \to x$ $(y \in Q)$. In particular, $\text{dist}(\widetilde{v}(x), v(y_i^\nu(x)) + H^0_{y_i^\nu(x)}) \to 0$ as $\nu \to \infty$. Hence, there exist vectors $w_i^\nu(x) \in H^0_{y_i^\nu(x)}$ such that $v(y_i^\nu(x)) + w_i^\nu(x) \to \widetilde{v}(x)$ as $\nu \to \infty$. Since also $\lambda_i^\nu(x) \to \widetilde{\lambda}_i(x)$ as $\nu \to \infty$, it follows that $\widetilde{\lambda}_i(x) \cdot \widetilde{v}(x) = \lim_{v\to\infty} \lambda_i^\nu(x) \cdot [v(y_i^\nu(x)) + w_i^\nu(x)]$. However, since $w_i^\nu(x) \in H^0_{y_i^\nu(x)}$ and $\lambda_i^\nu(x) \perp H^0_{y_i^\nu(x)}$, we have $\lambda_i^\nu(x) \cdot w_i^\nu(x) = 0$ for each ν.

Therefore, $\widetilde{\lambda}_i^\nu(x) \cdot \widetilde{v}(x) = \lim_{\nu\to\infty} \lambda_i^\nu(x) \cdot v(y_i^\nu(x))$, i.e. (3.20) holds for $1 \leq i \leq s$.

On the other hand, suppose $s+1 \leq i \leq r$. Then since $\widetilde{\lambda}_i(x) \in \widetilde{H}_x^0$ and $\widetilde{v}(x) \perp \widetilde{H}_x^0$, we have $\widetilde{\lambda}_i(x) \cdot \widetilde{v}(x) = 0$. Also, in this case we defined $\lambda_i^\nu(x) = 0$, hence $\lambda_i^\nu(x) \cdot v(y_i^\nu(x)) = 0$ for each ν. Therefore, $\widetilde{\lambda}_i(x) \cdot \widetilde{v}(x) = 0 = \lim_{\nu\to\infty} \lambda_i^\nu(x) \cdot v(y_i^\nu(x))$, so that (16) holds also for $s+1 \leq i \leq r$. The proof of Lemma 16 is complete. □

3.2. **Computation of iterated Glaeser refinements.**

In this section, we apply the results of the preceding section to study iterated Glaeser refinements. Let $\mathcal{H} = (v(x) + H_x^0)_{x \in Q}$ be a bundle, given in the form (3.28). We assume that $\mathcal{H}$ has a section. Therefore, $\mathcal{H}$ and all its iterated Glaeser refinements have non-empty fibers. For $\ell \geq 0$, we write the ℓ^{th} iterated Glaeser refinement in the form

$$\mathcal{H}^{(\ell)} = (v^\ell(x) + H_x^{0,\ell})_{x \in Q}, \tag{3.38}$$

where $\mathcal{H}^{0,\ell} = (H_x^{0,\ell})_{x \in Q}$ is a homogeneous bundle, and $v^\ell(x) \perp H_x^{0,\ell}$ for each $x \in Q$. (Again, we use the standard inner product on $\mathbb{R}^r$.) In particular, $\mathcal{H}^{(0)} = \mathcal{H}$, and

$$\mathcal{H}^{0,0} = (H_x^0)_{x \in Q}, \text{ with } H_x^0 \text{ as in (3.1).} \tag{3.39}$$

One checks easily that $\mathcal{H}^{0,\ell}$ is the ℓ^{th} iterated Glaeser refinement of $\mathcal{H}^{0,0}$. Our goal here is to give formulas computing $v^\ell(x)$ in terms of the $v(y)(y \in Q)$ in (3.1).

We proceed by induction on ℓ. For $\ell = 0$, we have

$$v^0(x) = v(x) \text{ for all } x \in Q. \tag{3.40}$$

For $\ell \geq 1$, we apply the results of the preceding section, to pass from $(v^{\ell-1}(x))_{x\in Q}$ to $(v^\ell(x))_{x\in Q}$.

Claim 17. We obtain points $y_i^{\ell,\nu}(x) \in Q \quad (\nu \geq 1, \quad 1 \leq i \leq r, \quad x \in Q)$; and vectors $\widetilde{\lambda}_i^\ell(x) \in \mathbb{R}^r \quad (1 \leq i \leq r, \quad x \in Q), \widetilde{\lambda}_i^{\ell,\nu}(x) \quad (1 \leq i \leq r, \quad \nu \geq 1, \quad x \in Q)$ with the following properties.

(1) The above points and vectors depend only on $\mathcal{H}^{0,0}$, not on the family of vectors $(v(x))_{x\in Q}$,
(2) $\widetilde{\lambda}_1^\ell(x), \cdots, \widetilde{\lambda}_r^\ell(x)$ form a basis of $\mathbb{R}^r$, for each $\ell \geq 1, x \in Q$.
(3) $y_i^{\ell,\nu}(x) \to x$ as $\nu \to \infty$ for each $\ell \geq 1, \quad 1 \leq i \leq r, \quad x \in Q$.
(4) $[\widetilde{\lambda}_i^\ell(x) \cdot v^\ell(x)] = \lim_{\nu\to\infty} [\widetilde{\lambda}_i^{\ell,\nu}(x) \cdot v^{\ell-1}(y_i^{\ell,\nu}(x))]$ for each $\ell \geq 1,\ 1 \leq i \leq r,\ x \in Q$.

The last formula computes the $v^\ell(x) \ \ (x \in Q)$ in terms of the $v^{\ell-1}(y) \ (y \in Q)$ for $\ell \geq 1$, completing our induction on ℓ.

Note that we have defined the basis vectors $\widetilde{\lambda}_1^\ell(x), \cdots, \widetilde{\lambda}_r^\ell(x)$ only for $\ell \geq 1$. For $\ell = 0$, it is convenient to use the standard basis vectors for $\mathbb{R}^r$, i.e., we define

$$\widetilde{\lambda}_i^0(x) = (0, 0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^r, \text{with the 1 in the } i^{th} \text{ slot.} \tag{3.41}$$

It is convenient also to set

$$\xi_i^\ell(x) = \widetilde{\lambda}_i^\ell(x) \cdot v^\ell(x) \text{ for } x \in Q,\ \ell \geq 0, \quad 1 \leq i \leq r, \tag{3.42}$$

and to expand $\widetilde{\lambda}_i^{\ell,\nu}(x) \in \mathbb{R}^r$ in terms of the basis $\widetilde{\lambda}_1^{\ell-1}(y), \cdots, \widetilde{\lambda}_r^{\ell-1}(y) \quad$ for $y = y_i^{\ell,\nu}(x)$. Thus, for suitable coefficients $\beta_{ij}^{\ell,\nu}(x) \in \mathbb{R} \ \ (\ell \geq 1,\ \nu \geq 1, \ \ 1 \leq i \leq r, \ \ 1 \leq j \leq r, \ \ x \in Q)$ we have

$$\widetilde{\lambda}_i^{\ell,\nu}(x) = \sum_{ij}^{r} \beta_{ij}^{\ell,\nu}(x) \cdot \widetilde{\lambda}_j^{\ell-1}\big(y_i^{\ell,\nu}(x)\big) \ \text{ for } x \in Q,\ \ell \geq 1, \nu \geq 1, \ \ 1 \leq i \leq r. \tag{3.43}$$

Note that the coefficients $\beta_{ij}^{\ell,\nu}(x)$ depend only on $\mathcal{H}^{0,0}$, not on the vectors $v(y)(y \in Q)$.

Putting (3.42) and (3.43) into (17.4), we obtain a recurrence relation for the $\xi_i^\ell(x)$:

$$\xi_i^\ell(x) = \lim_{\nu\to\infty} \sum_{j=1}^{r} \beta_{ij}^{\ell,\nu}(x) \ \cdot \ \xi_j^{\ell-1}\big(y_i^{\ell,\nu}(x)\big) \quad \text{for} \quad \ell \geq 1,\ 1 \leq i \leq r,\ x \in Q. \tag{3.44}$$

For $\ell = 0$, (3.40), (3.41) and (3.42) give

$$\xi_i^0(x) = [i^{th} \text{ component of } v(x)]. \tag{3.45}$$

Since $\beta_{ij}^{\ell,\nu}(x)$ and $y_i^{\ell,\nu}(x)$ are independent of the vectors $v(y)(y \in Q)$, our formulas (3.44), (3.18) express each $\xi_i^\ell(x)$ as an iterated limit in terms of the vectors $v(y)(y \in Q)$. In particular, the $\xi_i^\ell(x)$ depend linearly on the $v(y) \ (y \in Q)$.

We are particularly interested in the case $\ell = 2r + 1$, since the bundle $\mathcal{H}^{2r+1}$ is Glaeser stable, as we proved in section X.

Since $\widetilde{\lambda}_1^{2r+1}(x), \cdots, \widetilde{\lambda}_r^{2r+1}(x)$ form a basis of $\mathbb{R}^r$ for each $x \in Q$, there exist vectors $w_1(x), \cdots, w_r(x) \in \mathbb{R}^r$ for each $x \in Q$, such that

$$v = \sum_{i=1}^{r} \widetilde{\lambda}_i^{2r+1}(x) \cdot v w_i(x) \text{ for any vector } v \in \mathbb{R}^r, \text{ and for any } x \in Q. \tag{3.46}$$

Note that the vectors $w_1(x), \cdots, w_r(x) \in \mathbb{R}^r$ depend only on $\mathcal{H}^{0,0}$, not on the vectors $v(y)(y \in Q)$.

Taking $v = v^{2r+1}(x)$ in (3.46), and recalling (3.42), we see that

$$v^{2r+1}(x) = \sum_{i=1}^{r} \xi_i^{2r+1}(x) w_i(x) \qquad \text{for each } x \in Q. \tag{3.47}$$

Thus, we determine the $\xi_i^\ell(x)$ by the recursion (3.44), (3.45), and then compute $v^{2r+1}(x)$ from formula (3.47). Since also $(H_x^{0,2r+1})_{x\in Q}$ is simply the $(2r+1)^{rst}$

Glaeser refinement of $\mathcal{H}^{0,0}$, we have succeeded in computing the Glaeser stable bundle $(v^{2r+1}(x) + H_x^{0,2r+1})_{x\in Q}$ in terms of the initial bundle as in (3.28).

Our next task is to give a formula for a section of a Glaeser stable bundle. To carry this out, we will use "Whitney cubes", a standard construction which we explain below.

3.3. **Whitney cubes.**

In this section, for the reader's convenience, we review "Whitney cubes" (see [Mal67, Ste70, Whi34]). We will work with closed cubes $Q \subset \mathbb{R}^n$ whose sides are parallel to the coordinate axes. We write $\mathrm{ctr}(x)$ and δ_Q to denote the center and side length of Q, respectively; and we write Q^* to denote the cube with center $\mathrm{ctr}(Q)$ and side length 3δ.

To "bisect" Q is to write it as a union of 2^n subcubes, each with side length $\frac{1}{2}\delta_Q$, in the obvious way; we call those 2^n subcubes the "children" of Q.

Fix a cube Q^o. The "dyadic cubes" are the cube Q^o, the children of Q^o, the children of the children of Q^o, and so forth. Each dyadic Q is a subcube of Q^o. If Q is a dyadic cube other than Q^o, then Q is a child of one and only one dyadic cube, which we call Q^+. Note that $Q^+ \subset Q^*$.

Now let E_1 be a non-empty closed subset of Q^o. A dyadic cube $Q \neq Q^o$ will be called a "Whitney cube" if it satisfies

$$\mathrm{dist}(Q^*, E_1) \geq \delta_Q, \text{ and} \tag{3.48}$$

$$\mathrm{dist}((Q^+)^*, E_1) < \delta_{Q^+}. \tag{3.49}$$

The next result gives a few basic properties of Whitney cubes. In this section, we write c, C, C', etc. to denote constants depending only on the dimension n. These symbols need not denote the same constant in different occurrences.

Lemma 18. *For each Whitney cube Q, we have*

(1) $\delta_Q \leq \mathrm{dist}(Q^*, E_1) \leq C\delta_Q$; *in particular,*
(2) $Q^* \cap E_1 = \phi$.
(3) *The union of all Whitney cubes is $Q^o \setminus E_1$.*
(4) *Any given $y \in Q^o \setminus E_1$ has a neighborhood that meets Q^* for at most C distinct Whitney cubes Q.*

Proof: Estimates (1) follow at once from (1) and (2); and (4) is immediate from (3).

To check (3), we note first that each Whitney cube Q is contained in $Q^o \smallsetminus E_1$, thanks to (2) and our earlier remark that every dyadic cube is contained in Q^o. Conversely, let $x \in Q^o \smallsetminus E_1$ be given. Any small enough dyadic cube $\widehat{Q}$ containing x will satisfy (3.48). Fix such a $\widehat{Q}$. There are only finitely many dyadic cubes Q containing x with side length greater than or equal to $\delta_{\widehat{Q}}$. Hence, there exists a dyadic cube $Q \ni x$ satisfying (3.48), whose side length is at least as large as that of any other dyadic cube $Q' \ni x$ satisfying (3.48) . We know that $Q \neq Q^o$, since (3.48) fails for Q^o. Hence, Q has a dyadic parent Q^+. We know that (3.48) fails for Q^+, since the side length of Q^+ is greater than that of Q. It follows that Q satisfies (3.49). Thus $Q \ni x$ is a Whitney cube, completing the proof of (3).

We turn our attention to (4). Let $y \in Q^o \smallsetminus E_1$. We set $r = 10^{-3}$ distance (y, E_1), and we prove that there are at most C distinct Whitney cubes Q for which Q^* meets the ball $B(x, r)$.

Indeed, let Q be such a Whitney cube. Then there exists $z \in B(y, r) \cap Q^*$. By (3.55), we have

$$\delta_Q \leq \operatorname{dist}(z, E_1) \leq C\delta_Q. \tag{3.50}$$

Since $z \in B(y, r)$, we know that $|\operatorname{dist}(z, E_1) - \operatorname{dist}(y, E_1) \quad | \leq 10^{-3} \operatorname{dist}(y, E_1)$. Hence

$$(1 - 10^{-3}) \operatorname{dist}(y, E_1) \leq \operatorname{dist}(z, E_1) \leq (1 + 10^{-3}) \operatorname{dist}(y, E_1). \tag{3.51}$$

From (3.50), (3.51) we learn that

$$c \operatorname{dist}(y, E_1) \leq \delta_Q \leq C \operatorname{dist}(y, E_1). \tag{3.52}$$

Since $z \in B(y, r) \cap Q^*$, we know also that

$$\operatorname{dist}(y, Q^*) \leq \operatorname{dist}(y, E_1). \tag{3.53}$$

For fixed y, there are at most C distinct dyadic cubes that satisfy (3.52), (3.53).

Thus, (3.6) holds and Lemma 18 is proven. □

The next result provides a partition of unity adapted to the geometry of the Whitney cubes.

Lemma 19. *There exists a collection of real-valued functions θ_Q on Q^o, indexed by the Whitney cubes Q, satisfying the following conditions.*

1. *Each θ_Q is a non-negative continuous function on Q^o.*
2. *For each Whitney cube Q, the function θ_Q is zero on $Q^o \smallsetminus Q^*$.*
3. *$\sum_Q \theta_Q = 1$ on $Q^o \smallsetminus E_1$.*

Proof: Let $\widetilde{\theta}(x)$ be a non-negative, continuous function on $\mathbb{R}^n$, such that $\widetilde{\theta}(x) = 1$ for $x = (x_1, \cdots, x_n)$ with $\max \{|x_1|, \cdots, |x_n|\} \leq \frac{1}{2}$ and $\widetilde{\theta}(x) = 0$ for $x = (x_1, \cdots, x_n)$ with $\max \{|x_1|, \cdots, |x_n|\} \geq 1$.

For each Whitney cube Q, define $\widetilde{\theta}_Q(x) = \widetilde{\theta}\big(\frac{x - \operatorname{ctr}(Q)}{\delta_Q}\big)$, for $x \in \mathbb{R}^n$. Thus, $\widetilde{\theta}_Q$ is a non-negative continuous function on $\mathbb{R}^n$, equal to 1 on Q, and equal to 0 outside Q^*. It follows easily, thanks to (3) and (4), that $\sum\limits_{Q'} \widetilde{\theta}_{Q'}$ is a non-negative continuous function on $Q^o \smallsetminus E_1$, greater than or equal to one at every point of $Q^o \smallsetminus E_1$.

Consequently, the functions θ_Q, defined by $\theta_Q(x) = \widetilde{\theta}_Q(x) \big/ \sum\limits_{Q'} \widetilde{\theta}_{Q'}(x)$ for $x \in Q^o \smallsetminus E_1$, $\theta_Q(x) = 0$ for $x \in E_1$, are easily seen to satisfy (1), (2), (3). □

Additional basic properties of Whitney cubes, and sharper versions of Lemma 19 may be found in [Mal67, Ste70, Whi34].

The partition of unity $\{\theta_Q\}$ on $Q^o \smallsetminus E_1$ is called the "Whitney partition of unity".

3.4. **The Glaeser–stable case.** In this section, we suppose we are given a Glaeser-stable bundle with non-empty fibers, written in the form

$$\mathcal{H} = (v(x) + H_x^0)_{x \in Q}, \tag{3.54}$$

where $\mathcal{H}^0 = (H_x^0)_{x \in Q}$ is a homogeneous bundle, and

$$v(x) \perp H_x^0 \quad \text{for each } x \in Q. \tag{3.55}$$

(As before, we use the standard inner product on $\mathbb{R}^r$.) Our goal here is to give a formula for a section F of the bundle $\mathcal{H}$. We will take

$$F(x) = \sum_{y \in S(x)} A(x,y)v(y) \in \mathbb{R}^r \text{ for each } x \in Q, \text{ where} \tag{3.56}$$

$$S(x) \subset Q \text{ is a finite set for each } x \in Q, \text{ and} \tag{3.57}$$

$$A(x,y) : \mathbb{R}^r \to \mathbb{R}^r \text{ is a linear map, for each } x \in Q, y \in S(x). \tag{3.58}$$

Here the sets $S(x)$ and the linear maps $A(x,y)$ are determined by $\mathcal{H}^0$; they do not depend on the family of vectors $(v(x))_{x \in Q}$.

We will establish the following result.

Theorem 20. *We can pick the $S(x)$ and $A(x,y)$ so that (3.57), (3.58) hold, and the function $F : Q \to \mathbb{R}^r$, defined by (3.56), is a section of the bundle $\mathcal{H}$. Moreover, that section satisfies*

1. $\max_{x \in Q} |F(x)| \le C \sup_{x \in Q} |v(x)|$, *where C depends only on n and r.*
2. *Furthermore, each of the sets $S(x)$ contains at most d points, where d depends only on n and r.*

Note: Since $v(x)$ is the shortest vector in $v(x) + H_x^0$ by (3.55), it follows that $\sup_{x \in Q} |v(x)| = \sup_{x \in Q}$ distance $(0, v(x) + H_x^0) = \| \mathcal{H} \| < \infty$; see our earlier discussion of Michael's Theorem.

Proof: Roughly speaking, the idea of our proof is as follows. We partition Q into finitely many "strata", among which we single out the "lowest stratum" E_1. For $x \in E_1$, we simply set $F(x) = v(x)$. To define F on $Q \smallsetminus E_1$, we cover $Q \smallsetminus E_1$ by Whitney cubes Q_ν. Each Q_ν^* fails to meet E_1, by definition, and therefore has fewer strata than Q. Hence, by induction on the number of strata, we can produce a formula for a section F_ν of the bundle $\mathcal{H}$ restricted to Q_ν^*. Patching together the F_ν by using the Whitney partition of unity, we define our section F on $Q \smallsetminus E_1$, and complete the proof of Theorem 20.

Let us begin our proof. For $k = 0, 1, \cdots, r$, the k^{th} "stratum" of $\mathcal{H}$ is defined by

$$E(k) = \{x \in Q : \dim\ H_x^0 = k\}. \tag{3.59}$$

The "number of strata" of $\mathcal{H}$ is defined as the number of non-empty $E(k)$; this number is at least 1 and at most $r + 1$. We write E_1 to denote the stratum $E(k_{\min})$, where $k_{\min}$ is the least k such that $E(k)$ is non-empty. We call E_1 the "lowest stratum".

We will prove Theorem 20 by induction on the number of strata, allowing the constants C and d on (1), (2), to depend on the number of strata, as well as on

n and r. Since the number of strata is at most $r+1$, such an induction will yield Theorem 20 as stated.

Thus, we fix a positive integer Λ, and assume the inductive hypothesis:

(H1) Theorem 20 holds, with constants $C_{\Lambda-1}, d_{\Lambda-1}$ in (3.8), (3.9), whenever the number of strata is less than Λ.

We will then prove Theorem 20, with constants C_Λ, d_Λ in (1), (2), whenever the number of strata is equal to Λ. Here, C_Λ and d_Λ are determined by $C_{\Lambda-1}, d_{\Lambda-1}, n$ and r. To do so, we start with (3.54), (3.3), and assume that

(H2) The number of strata of $\mathcal{H}$ is equal to Λ.

We must produce sets $S(x)$ and linear maps $A(x,y)$ satisfying (3.57) $\cdots$ (2), with constants C_Λ, d_Λ depending only on $C_{\Lambda-1}, d_{\Lambda-1}, n, r$. This will complete our induction, and establish Theorem 20.

For the rest of the proof of Theorem 20, we write c, C, C', etc. to denote constants determined by $C_{\Lambda-1}, d_{\Lambda-1}, n, r$. These symbols need not denote the same constant in different occurrences.

The following useful remark is a simple consequence of our assumption that the bundle (3.54) is Glaeser stable. Let $x \in E(k)$, and let

$$v_1, \cdots, v_{k+1} \;\in\; v(x) + H_x^0 \tag{3.60}$$

be the vertices of a non-degenerate affine k-simplex in $\mathbb{R}^r$. Given $\epsilon > 0$ there exists $\delta > 0$ such that for any $y \in Q \cap B(x,\delta)$, there exist $v_1', \cdots, v_{k+1}' \in v(y) + H_y^0$ satisfying $|v_i' - v_i| < \epsilon$ for each i. Here, as usual, $B(x,\delta)$ denotes the ball of radius δ about x.

Taking ϵ small enough in (3.60), we conclude that $v_1', \cdots, v_{k+1}' \in v(y) + H_y^0$ are the vertices of a non-degenerate affine k-simplex in $\mathbb{R}^r$. Therefore, (3.60) yields at once that if $x \in E(k)$, then $\dim H_y^0 \geq k$ for all $y \in Q$ sufficiently close to x. In particular, the lowest stratum E_1 is a non-empty closed subset of Q. Also, for each $k = 0, 1, 2, \cdots, r$, (3.60) shows that the map

$$x \mapsto v(x) + H_x^0 \tag{3.61}$$

is continuous from $E(k)$ to the space of all affine k-dimensional subspaces of $\mathbb{R}^r$.

Since each H_x^0 is a vector subspace of $\mathbb{R}^r$, we learn from (3.55) and (3.61) that the map $x \mapsto v(x)$ is continuous on each $E(k)$. In particular,

$$x \mapsto v(x) \text{ is continuous on } E_1. \tag{3.62}$$

Next, we introduce the Whitney cubes $\{Q_\nu\}$ and the Whitney partition of unity $\{\theta_\nu\}$ for the closed set $E_1 \subset Q$. From the previous section, we have the following results. We write δ_ν for the side length of the Whitney cube Q_ν. Note that

$$\delta_\nu \le \mathrm{dist}(Q_\nu^*, E_1) \le C\delta_\nu \text{ for each } \nu. \tag{3.63}$$

$$Q_\nu^* \cap E_1 = \phi \quad \text{for each } \nu. \tag{3.64}$$

$$\bigcup_\nu Q_\nu = Q \setminus E_1. \tag{3.65}$$

(3.66) Any given $y \in Q \setminus E_1$ has a neighborhood that meets Q_ν^* for at most C distinct Q_ν.

(3.67) Each θ_ν is a non-negative continuous function on Q, vanishing outside $Q \cap Q_\nu^*$.

$$\sum_\nu \theta_\nu(x) = 1 \quad \text{if } x \in Q \setminus E_1, \quad 0 \text{ if } x \in E_1. \tag{3.68}$$

Thanks to (3.19), we can pick points $x_\nu \in E_1$ such that

$$\mathrm{dist}(x_\nu, Q_\nu^*) \le C\delta_\nu. \tag{3.69}$$

We next prove a continuity property of the fibers $v(x) + H_x^0$.

Lemma 21. *Given $x \in E_1$ and $\epsilon > 0$, there exists $\delta > 0$ for which the following holds. Let Q_ν be a Whitney cube such that distance $(x, Q_\nu^*) < \delta$. Then*

1. $|v(x) - v(x_\nu)| < \epsilon$, *and*
2. $\mathrm{dist}(v(x), v(y) + H_y^0) < \epsilon$ *for all* $y \in Q_\nu^* \cap Q$.

Proof: Fix $x \in E_1$ and $\epsilon > 0$. Let $\delta > 0$ be a small enough number, to be picked later. Let Q_ν be a Whitney cube such that

$$\mathrm{dist}(x, Q_\nu^*) < \delta. \tag{3.70}$$

Then, by (3.19), we have

$$\delta_\nu \le \mathrm{dist}(E_1, Q_\nu^*) \le \mathrm{dist}(x, Q_\nu^*) < \delta, \tag{3.71}$$

hence, (3.69) and (3.70) yield the estimates

$$|x - x_\nu| \le \mathrm{dist}(x, Q_\nu^*) + \text{ diameter } (Q_\nu^*) + \mathrm{dist}(Q_\nu^*, x_\nu) \le \delta + C\delta_\nu \le C'\delta. \tag{3.72}$$

Since x and x_ν belong to E_1, (3.72) implies (1) thanks to (3.62), provided we take δ small enough. Also, for any $y \in Q_\nu^* \cap Q$, we learn from (3.70), (3.71) that
$|y - x| \le \text{ diameter } (Q_\nu^*) + \mathrm{dist}(x, Q_\nu^*) < C\delta_\nu + C\delta \le C'\delta$.

Since the bundle $(v(z) + H_z^0)_{z\in Q}$ is Glaeser stable, it follows that (3.26) holds, provided we take δ small enough.

We now pick $\delta > 0$ small enough that the above arguments go through. Then (3.25) and (3.26) hold. The proof of Lemma 21 is complete. □

We return to the proof of Theorem 20. For each Whitney cube Q_ν, we prepare to apply our inductive hypothesis (H1) to the family of affine subspaces

$$\mathcal{H}_\nu = (v(y) - v(x_\nu) + H_y^0)_{y \in Q_\nu^* \cap Q}. \tag{3.73}$$

Since $Q_\nu^* \cap Q$ is a closed rectangular box, but not necessarily a cube, it may happen that (3.73) fails to be a bundle. The cure is simply to fix an affine map $\rho_\nu : \mathbb{R}^n \to \mathbb{R}^n$, such that $\rho_\nu(Q^o) = Q_\nu^* \cap Q$, where Q^o denotes the unit cube.

The family of affine spaces

$$\check{\mathcal{H}}_\nu = \big(v(\rho_\nu \check{y}) - v(x_\nu) + H^0_{\rho_\nu \check{y}}\big)_{\check{y} \in Q^o} \quad \text{is then a bundle.} \tag{3.74}$$

We write (3.73) in the form

$$\mathcal{H}_\nu = \big(v_\nu(y) + H^0_y\big)_{y\in Q^*_\nu \cap Q}, \text{ where} \tag{3.75}$$

$$v_\nu(y) \perp H^0_y \text{ for each } y \in Q^*_\nu \cap Q. \tag{3.76}$$

The vector $v_\nu(y)$ is given by

$$v_\nu(y) = \Pi_y v(y) - \Pi_y v(x_\nu) \text{ for } y \in Q^*_\nu \cap Q, \text{ where} \tag{3.77}$$

Π_y denotes the orthogonal projection from $\mathbb{R}^r$ onto the orthocomplement of H^0_y.

Passing to the bundle $\check{\mathcal{H}}_\nu$, we find that

$$\check{\mathcal{H}}_\nu = (\check{v}_\nu(\check{y}) + H^0_{\rho_\nu \check{y}})_{\check{y}\in Q^o}, \text{ with} \tag{3.78}$$

$$\check{v}_\nu(\check{y}) \perp H^0_{\rho_\nu \check{y}} \text{ for each } \check{y} \in Q^o. \tag{3.79}$$

Here, $\check{v}_\nu(\check{y})$ is given by

$$\check{v}_\nu(\check{y}) = v_\nu(\rho_\nu \check{y}). \tag{3.80}$$

It is easy to check that $\check{\mathcal{H}}_\nu$ is a Glaeser stable bundle with non-empty fibers. Moreover, from (3.12) and (21), we see that the function $y \mapsto \dim H^0_y$ takes at most $\Lambda - 1$ values as y ranges over $Q^*_\nu \cap Q$. Therefore, the bundle $\check{\mathcal{H}}_\nu$ has at most $\Lambda - 1$ strata.

Thus, our inductive hypothesis (3.11) applies to the bundle $\check{\mathcal{H}}_\nu$. Consequently, we obtain the following results for the family of affine spaces $\mathcal{H}_\nu$.

We obtain sets

$$S_\nu(x) \subset Q^*_\nu \cap Q \quad \text{ for each } x \in Q^*_\nu \cap Q, \tag{3.81}$$

and linear maps

$$A_\nu(x, y) \; : \; \mathbb{R}^r \to \mathbb{R}^r \text{ for each } x \in Q^*_\nu \cap Q, y \in S_\nu(x). \tag{3.82}$$

$$\text{The sets } S_\nu(x) \text{ each contain at most } C \text{ points.} \tag{3.83}$$

$$\text{The } S_\nu(x) \text{ and } A_\nu(x, y) \text{ are determined by } (H^0_z)_{z\in Q^*_\nu \cap Q}. \tag{3.84}$$

Moreover, setting

$$F_\nu(x) = \sum_{y\in S_\nu(x)} A_\nu(x, y) v_\nu(y) \quad \text{for} \quad x \in Q^*_\nu \cap Q, \tag{3.85}$$

we find that

$$F_\nu \text{ is continuous on } Q^*_\nu \cap Q, \tag{3.86}$$

$$F_\nu(x) \in v_\nu(x) + H^0_x = v(x) - v(x_\nu) + H^0_x \quad \text{for each} \quad x \in Q^*_\nu \cap Q, \tag{3.87}$$

and

$$\max_{x\in Q^*_\nu \cap Q} \; \big| \, F_\nu(x) \, \big| \le C \quad \sup_{y\in Q^*_\nu \cap Q} \big| \, v_\nu(y) \, \big| \, . \tag{3.88}$$

Let us estimate the right-hand side of (3.88). For any Q_ν, formula (3.77) shows that

$$\sup_{y\in Q^*_\nu \cap Q} \big| \, v_\nu(y) \, \big| \le 2 \; \sup_{y\in Q} \big| \, v(y) \, \big| \, . \tag{3.89}$$

Moreover, let $x \in E_1$, $\epsilon > 0$ be given, and let δ be as in Lemma 21. Given any Q_ν such that distance $(x, Q^*_\nu) < \delta$, and given any $y \in Q^*_\nu \cap Q$, Lemma 21 tells us that

$\mid v(x) - v(x_\nu) \mid < \epsilon$ and distance $(v(x), v(y) + H_y^0) < \epsilon$.
Consequently,

$$\text{dist}(0, v(y) - v(x_\nu) + H_y^0) < 2\epsilon \text{ and } \mid v(x) - v(x_\nu) \mid < \epsilon. \tag{3.90}$$

From (3.73), (3.75), (3.76), we see that $v_\nu(y)$ is the shortest vector in $v(y) - v(x_\nu) + H_y^0$. Hence, (3.90) yields the estimate $\mid v_\nu(y) \mid < 2\epsilon$.

Therefore, we obtain the following result. Let $x \in E_1$ and $\epsilon > 0$ be given. Let δ be as in Lemma 21. Then, for any Q_ν such that distance $(x, Q_\nu^*) < \delta$, we have

$$\sup_{y \in Q_\nu^* \cap Q} \mid v_\nu(y) \mid \leq 2\epsilon, \quad \text{and} \quad \mid v(x) - v(x_\nu) \mid < \epsilon. \tag{3.91}$$

From (3.88), (3.89), (3.91), we see that

$$\max_{x \in Q_\nu^* \cap Q} \mid F_\nu(x) \mid \leq C \sup_{y \in Q} \mid v(y) \mid \tag{3.92}$$

for each ν, and that the following holds. Let $x \in E_1$ and $\epsilon > 0$ be given. Let δ be be as in Lemma 21, and let $y \in Q_\nu^* \cap Q \cap B(x, \delta)$. Then

$$\mid F_\nu(y) \mid \leq C\epsilon, \text{ and } \mid v(x) - v(x_\nu) \mid < \epsilon. \tag{3.93}$$

We now define a map $F : Q \to \mathbb{R}^r$, by setting

$$F(x) = v(x) \text{ for } x \in E_1, \text{ and} \tag{3.94}$$

$$F(x) = \sum_\nu \theta_\nu(x) \cdot F_\nu(x) + v(x_\nu) \text{ for } x \in Q \smallsetminus E_1. \tag{3.95}$$

Note that (3.95) makes sense, because the sum contains finitely many non-zero terms, and because $\theta_\nu = 0$ outside the set where F_ν is defined.

We will show that F is given in terms of the $(v(y))_{y \in Q}$ by a formula of the form (3.56), and that conditions (3.57) $\cdots$ (2) are satisfied. As we noted just after (H2), this will complete our induction on Λ, and establish Theorem 20.

First, we check that our $F(x)$ is given by (3.56), for suitable $S(x), A(x, y)$. We proceed by cases. If $x \in E_1$, then already (3.94) has the form (3.56), with

$$S(x) = \{x\} \text{ and } A(x, y) = \text{ identity.} \tag{3.96}$$

Suppose $x \in Q \smallsetminus E_1$. Then $F(x)$ is defined by (3.95).

Thanks to (3.67), we may restrict the sum in (3.95) to those ν such that $x \in Q_\nu^*$. For each such ν, we substitute (3.77) into (3.85), and then substitute the resulting formula for $F_\nu(x)$ into (3.95). We find that

$$F(x) = \sum_{Q_\nu^* \ni x} \theta_\nu(x) \cdot v(x_\nu) + \sum_{y \in S_\nu(x)} A_\nu(x, y) \cdot \big(\Pi_y v(y) - \Pi_y v(x_\nu)\big) \tag{3.97}$$

which is a formula of the form (3.4).

Thus, in all cases, F is given by a formula (3.4). Moreover, examining (3.96) and (3.97) (and recalling (3.81) $\cdots$ (3.84) as well as (3.20)), we see that (3.5), (3.6), (3.7) hold, and that in our formula (3.4) for F, each $S(x)$ contains at most C points. Thus (3.9) holds, with a suitable d_Λ in place of d.

It remains to prove (3.8), and to show that our F is a section of the bundle $\mathcal{H}$. Thus, we must establish the following.

$$F : Q \to \mathbb{R}^r \text{ is continuous.} \tag{3.98}$$

$$F(x) \in \; v(x) + H_x^0 \text{ for each } x \in Q. \tag{3.99}$$

$$\left|F(x)\right| \leq C \sup_{y\in Q} \left|v(y)\right| \text{ for each } x \in Q. \tag{3.100}$$

The proof of Theorem 20 is reduced to proving (3.98), (3.99), (3.100).

Let us prove (3.98). Fix $x \in Q$; we show that F is continuous at x. If $x \notin E_1$, then (3.66), (3.67), (3.86) and (3.95) easily imply that F is continuous at x.

On the other hand, suppose $x \in E_1$. To show that F is continuous at x, we must prove that

$$\lim_{y\to x, y\in E_1} v(y) = v(x) \quad \text{and that} \tag{3.101}$$

$$\lim_{y\to x, y\in Q\smallsetminus E_1} \sum_\nu \theta_\nu(y)F_\nu(y) + v(x_\nu) = v(x). \tag{3.102}$$

We obtain (3.101) as an immediate consequence of (3.62). To prove (3.102), we bring in (3.93). Let $\epsilon > 0$, and let $\delta > 0$ arise from ϵ, x as in (3.93). Let $y \in Q \smallsetminus E_1$; and suppose $\left|y - x\right| < \delta$. For each ν such that $y \in Q_\nu^*$, (3.93) gives

$$\left|\theta_\nu(y) \;\cdot\; [F_\nu(y) + v(x_\nu) - v(x)]\right| \leq C\epsilon\theta_\nu(y). \tag{3.103}$$

For each ν such that $y \notin Q_\nu^*$, (3.103) holds trivially, since $\theta_\nu(y) = 0$. Thus, (3.103) holds for all ν. Summing on ν, and recalling (3.68), we conclude that
$\left|\sum_\nu \theta_\nu(y) \;\cdot\; F_\nu(y) + v(x_\nu) - v(x)\right| \leq C\epsilon.$

This holds for any $y \in Q \smallsetminus E_1$ such that $\left|y - x\right| < \delta$. The proof of (3.102) is complete. Thus, (3.98) is now proven.

To prove (3.99), we again proceed by cases. If $x \in E_1$, then (3.99) holds trivially, by (3.94). On the other hand, suppose $x \in Q \smallsetminus E_1$. Then (3.87) gives $[F_\nu(x) + v(x_\nu)] \in \; v(x) + H_x^0$ for each ν such that $Q_\nu^* \ni x$.

Since also $\theta_\nu(x) = 0$ for $x \notin Q_\nu^*$, and since $\sum_\nu \theta_\nu(x) = 1$, it follows that
$\sum_\nu \theta_\nu(x) \;\cdot\; [F_\nu(x) + v(x_\nu)] \;\in\; v(x) + H_x^0$, i.e.,
$F(x) \in \; v(x) + H_x^0$. Thus, (3.99) holds in all cases.

Finally, we check (3.100). For $x \in E_1$, (3.100) is trivial from the definition (3.94). On the other hand, suppose $x \in Q \smallsetminus E_1$. For each ν such that $Q_\nu^* \ni x$, (3.92) gives

$$\left|\theta_\nu(x) \;\cdot\; [F_\nu(x) + v(x_\nu)]\right| \leq C\theta_\nu(x) \cdot \sup_{y\in Q} \left|v(y)\right|. \tag{3.104}$$

Estimate (3.104) also holds trivially for $x \notin Q_\nu^*$, since then $\theta_\nu(x) = 0$. Thus, (3.104) holds for all ν. Summing on ν, we find that

$$\left|F(x)\right| \leq \sum_\nu \left|\theta_\nu(x) \cdot [F_\nu(x) + v(x_\nu)]\right| \;\leq\; C \sup_{y\in Q} \left|v(y)\right| \cdot \sum_\nu \theta_\nu(x) = C \sup_{y\in Q} \left|v(y)\right|,$$

thanks to (3.68) and (3.95).

Thus (3.100) holds in all cases. The proof of Theorem 20 is complete. □

Let $\widetilde{F}$ be any section of the bundle $\mathcal{H}$ in Theorem 20. For each $x \in Q$, we have $|v(x)| \leq |\widetilde{F}(x)|$, since $\widetilde{F}(x) \in v(x) + H_x^0$ and $v(x) \perp H_x^0$. Therefore, the section F

produced by Theorem 20 satisfies the estimate $\max_{x\in Q}|F(x)| \le C \cdot \max_{x\in Q}|\widetilde{F}(x)|$, where C depends only on n, r.

3.5. **Computing the section of a bundle.**

Here, we combine our results from the last few sections. Let

$$\mathcal{H} = (v(x) + H_x^0)_{x\in Q} \text{ be a bundle, where} \tag{3.105}$$

$$\mathcal{H}^0 = (H_x^0)_{x\in Q} \text{ is a homogeneous bundle, and} \tag{3.106}$$

$$v(x) \perp H_x^0 \quad \text{for each } x \in Q. \tag{3.107}$$

Suppose $\mathcal{H}$ has a section. Then the iterated Glaeser refinements of $\mathcal{H}$ have non-empty fibers, and may therefore be written as

$$\mathcal{H}^\ell = (v^\ell(x) + H_x^{0,\ell})_{x\in Q} \text{ where} \tag{3.108}$$

$$\mathcal{H}^{0,\ell} = (H_x^{0,\ell})_{x\in Q} \text{ is a homogeneous bundle, and} \tag{3.109}$$

$$v^\ell(x) \perp H_x^{0,\ell} \quad \text{for each } x \in Q. \tag{3.110}$$

Let $\xi_i^\ell(x) \in \mathbb{R}$, $y_i^{\ell,\nu}(x) \in Q$, $\beta_{ij}^{\ell,\nu}(x) \in \mathbb{R}$, $w_i(x) \in \mathbb{R}^r$ be as in section 3.2. Thus,

$$\xi_i^0(x) = i^{th} \text{ component of } v(x), \text{ for } x \in Q; \tag{3.111}$$

$$\xi_i^\ell(x) = \lim_{\nu\to\infty} \sum_{j=1}^r \beta_{ij}^{\ell,\nu}(x)\ \xi_j^{\ell-1}\big(y_i^{\ell,\nu}(x)\big) \tag{3.112}$$

for $x \in Q, 1 \le \ell \le 2r+1, \quad 1 \le i \le r$, and

$$v^{2r+1}(x) = \sum_{i=1}^r \xi_i^{2r+1}(x) w_i(x) \quad \text{for } x \in Q. \tag{3.113}$$

Recall that $\beta_{ij}^{\ell,\nu}(x), y_i^{\ell,\nu}(x)$ and $w_i(x)$ are determined by the homogeneous bundle $\mathcal{H}^0$, independently of the vectors $(v(z))_{z\in Q}$. The bundle $\mathcal{H}^{2r+1} = (v^{2r+1}(x) + H_x^{0,2r+1})_{x\in Q}$ is Glaeser stable, with non-empty fibers. Hence, the results of section 3.4 apply to $\mathcal{H}^{2r+1}$. Thus, we obtain a section of $\mathcal{H}^{2r+1}$ of the form

$$F(x) = \sum_{y\in S(x)} A(x,y) v^{2r+1}(y) \qquad (\text{all } x \in Q), \tag{3.114}$$

where $S(x) \subset Q$ and $\#(S(x)) \le d$ for each $x \in Q$; and $A(x,y) : \mathbb{R}^r \to \mathbb{R}^r$ is a linear map, for each $x \in Q, y \in S(x)$. Our section F satisfies the estimate

$$\max_{x\in Q} \big|F(x)\big| \le C \max_{x\in Q} \big|\widetilde{F}(x)\big|, \text{ for any section } \widetilde{F} \text{ of } \mathcal{H}^{2r+1}. \tag{3.115}$$

Here, d and C depend only on n and r; and the $S(x)$ and $A(x,y)$ are determined by $\mathcal{H}^{0,2r+1}$, independently of the vectors $v^{2r+1}(z)$ $(z \in Q)$.

Recall that the bundles $\mathcal{H}$ and $\mathcal{H}^{2r+1}$have the same sections. Therefore, substituting (3.113) into (3.114), and setting

$$A_i(x,y) = A(x,y) w_i(y) \ \in \mathbb{R}^r \quad \text{for } x \in Q,\ y \in S(x),\ i = 1, \cdots, r, \tag{3.116}$$

we find that

$$F(x) = \sum_{y\in S(x)} \sum_1^r \ \xi_i^{2r+1}(y) A_i(x,y) \text{ for all } x \in Q. \tag{3.117}$$

Moreover, F is a section of $\mathcal{H}$, and

$$\max_{x\in Q} \left|F(x)\right| \le C \max_{x\in Q} \left|\widetilde{F}(x)\right| \text{ for any section } \widetilde{F} \text{ of } \mathcal{H}. \tag{3.118}$$

Furthermore The $A_i(x,y)$ are determined by $\mathcal{H}^0$, independently of the family of vectors $(v(z))_{z\in Q}$.

Thus, we can compute a section of $\mathcal{H}$ by starting with (3.111), then computing the $\xi_i^\ell(x)$ using the recursion (3.112), and finally applying (3.117) once we know the $\xi_i^{2r+1}(x)$. In particular, we guarantee that the limits in (3.112) exist. Here, of course, we make essential use of our assumption that $\mathcal{H}$ has a section.

3.6. **Computing a continuous solution of linear equations.**

We apply the results of the preceding section, to find continuous solutions of

$$\phi_1 f_1 + \cdots + \phi_r f_r = \phi \quad \text{on } Q. \tag{3.119}$$

Such a solution $(\phi_1, \cdots, \phi_r)$ is a section of the bundle

$$\mathcal{H} = (H_x)_{x\in Q}, \text{ where} \tag{3.120}$$

$$H_x = \{v = (v_1, \cdots, v_r) \in \mathbb{R}^r : \ v_1 f_1(x) + \cdots + v_r f_r(x) = \phi(x)\}. \tag{3.121}$$

We write $\mathcal{H}$ in the form

$$\mathcal{H} = (v(x) + H_x^0)_{x\in Q}, \text{ where} \tag{3.122}$$

$$H_x^0 = \{v = (v_1, \cdots, v_r) \in \mathbb{R}^r : v_1 f_1(x) + \cdots + v_r f_r(x) = 0\}, \text{ and} \tag{3.123}$$

$$v(x) = \phi(x) \cdot (\widetilde{\xi}_1(x), \cdots, \widetilde{\xi}_r(x)); \text{ here,} \tag{3.124}$$

$$\widetilde{\xi}_i(x) = \begin{cases} 0 & \text{if } \ f_1(x) = f_2(x) = \cdots = f_r(x) = 0 \\ f_i(x)/\big(f_1^2(x) + \cdots + f_r^2(x)\big) & \text{otherwise.} \end{cases} \tag{3.125}$$

Note that

$$v(x) \perp H_x^0 \quad \text{for each } x \in Q. \tag{3.126}$$

Specializing the discussion in the preceding section to the bundle (3.108) $\cdots$ (3.112), we obtain the following objects:

- coefficients $\beta_{ij}^{\ell,\nu}(x) \in \mathbb{R}$, for $x \in Q,\ 1 \le \ell \le 2r+1,\ \nu \ge 1, 1 \le i,j \le r$;
- points $y_i^{\ell,\nu}(x) \in Q$, for $x \in Q,\ 1 \le \ell \le 2r+1,\ \nu \ge 1,\ 1 \le i \le r$;
- finite sets $S(x) \subset Q$, for $x \in Q$; and
- vectors $A_i(x,y) \in \mathbb{R}^r$, for $x \in Q, y \in S(x),\ 1 \le i \le r$.

These objects depend only on the functions $f_1, \cdots, f_r$.

We write $A_{ij}(x,y)$ to denote the i^{th} component of the vector $A_j(x,y)$.

To attempt to solve equation (3.119), we use the following

Procedure 22. First, compute $\xi_i^\ell(x) \in \mathbb{R}$, for all $x \in Q,\ 0 \le \ell \le 2r+1,\ 1 \le i \le r$, by the recursion:

$$\xi_i^0(x) = \widetilde{\xi}_i(x) \cdot \phi(x) \quad \text{for } 1 \le i \le r; \text{ and} \tag{3.127}$$

$$\xi_i^\ell(x) = \lim_{\nu\to\infty} \textstyle\sum_{j=1}^r \beta_{ij}^{\ell,\nu}(x) \cdot \xi_j^{\ell-1}(y_i^{\ell,\nu}(x)) \tag{3.128}$$

for $1 \le i \le r, 1 \le \ell \le 2r+1$.

Then define functions $\Phi_1, \cdots, \Phi_r : Q \to \mathbb{R}$, by setting

$$\Phi_i(x) = \sum_{y\in S(x)} \sum_{j=1}^{r} A_{ij}(x,y)\cdot \xi_j^{2r+1}(y) \quad \text{for} \qquad x \in Q, \quad 1 \le i \le r \tag{3.129}$$

If, for some $x \in Q$ and $i = 1, \cdots, r$, the limit in (3.128) fails to exist, then our procedure (22) fails. Otherwise, procedure (22) produces functions $\Phi_1, \cdots, \Phi_r : Q \to \mathbb{R}$. These functions may or may not be continuous.

The next result follows at once from the discussion in the preceding section. It tells us that, if equation (3.105) has a continuous solution, then procedure (22) produces an essentially optimal continuous solution of (3.105).

Theorem 23. (1) *The objects $\widetilde{\xi}_i(x), \beta_{ij}^{\ell,\nu}(x), y_i^{\ell,\nu}(x), S(x)$, and $A_{ij}(x,y)$, used in procedure (22), depend only on $f_1, \cdots, f_r$, and not on the function ϕ.*

(2) *For each $x \in Q$, the set $S(x) \subset Q$ contains at most d points, where d depends only on n and r.*

(3) *Let $\phi : Q \to \mathbb{R}$, and let $\phi_1, \cdots, \phi_r : Q \to \mathbb{R}$ be continuous functions such that $\phi_1 f_1 + \cdots + \phi_r f_r = \phi$ on Q. Then procedure (22) succeeds, the resulting functions $\Phi_1, \cdots, \Phi_r : Q \to \mathbb{R}$ are continuous, and $\Phi_1 f_1 + \cdots + \Phi_r f_r = \phi$ on Q. Moreover,*

$$\max_{\substack{x\in Q\\ 1\le i\le r}} \big|\Phi_i(x)\big| \le C \cdot \max_{\substack{x\in Q\\ 1\le i\le r}} \big|\phi_i(x)\big|$$

where C depends only on n, r.

For particular functions $f_1, \cdots, f_r$, it is a tedious, routine exercise to go through the arguments in the past several sections, and compute the $\widetilde{\xi}_i(x), \beta_i^{\ell,\nu}(x), y_i^{\ell,\nu}(x), S(x)$ and $A_{ij}(x,y)$ used in our Procedure (22). We invite the reader to carry this out for the case of Hochster's equation 3.4, and to compare the resulting formulas with those given in Section 3.

So far we have dealt with a single equation (3.119) for continuous functions $\phi_1, \ldots, \phi_r$. To handle a system of equations, we simply take $f_1, \ldots, f_r$ and ϕ to be vector valued in (3.119). In place of (3.124), (3.125) and (3.127), we now define $v(x) = \big(\xi_1^0(x), \ldots, \xi_r^0(x)\big)$ to be the shortest vector in $\mathbb{R}^r$ that solves the equation $\sum_i \xi_i^0(x) f_i(x)$ for each fixed x. (If, for some x, this equation has no solution, then (3.119) has no solution.) We can easily compute the $\xi_i^0(x)$ from $f_1(x), \ldots, f_r(x)$ and $\phi(x)$ by linear algebra. Starting from the above $\xi_i^0(x)$, we can repeat the proof of Theorem 23, with trivial changes.

4. Algebraic geometry approach

The following simple example illustrates this method.

Example 24. Which functions ϕ on $\mathbb{R}^2_{xy}$ can be written in the form

$$\phi = \phi_1 x^2 + \phi_2 y^2 \tag{4.1}$$

where ϕ_1, ϕ_2 are continuous on $\mathbb{R}^2$? (We know that the Pointwise Tests (3) give an answer in this case, but the following method will generalize better.)

An obvious necessary condition is that ϕ should vanish to order 2 at the origin. This is, however, not sufficient since xy can not be written in this form.

To see what happens, we blow up the origin. The resulting real algebraic variety $p : B_0\mathbb{R}^2 \to \mathbb{R}^2$ can be covered by two charts; one given by coordinates $x_1 = x/y, y_1 = y$ the other by coordinates $x_2 = x, y_2 = y/x$. Working in the first chart, pulling back (4.1) we get the equation

$$\phi \circ p = (\phi_1 \circ p) \cdot x_1^2 y_1^2 + (\phi_2 \circ p) \cdot y_1^2. \tag{4.2}$$

The right hand side is divisible by y_1^2, so we have our first condition

(24.1) First test. Is $(\phi \circ p)/y_1^2$ continuous?

If the answer is yes, then we divide by y_1^2, set $\psi := (\phi \circ p)/y_1^2$ and try to solve

$$\psi = \psi_1 \cdot x_1^2 + \psi_2. \tag{4.3}$$

This always has a continuous solution, but we need a solution where $\psi_i = \phi_i \circ p$ for some ϕ_i. Clearly, the ψ_i have to be constant along the line $(y_1 = 0)$. This is easily seen to be the only restriction. We thus set $y_1 = 0$ and try to solve

$$\psi(x_1, 0) = r_1 x_1^2 + r_2 \quad \text{where } r_i \in \mathbb{R}. \tag{4.4}$$

The original 2 variable problem has been reduced to a 1 variable question. Solvability is easy to decide using either of the following.

(24.2.i) Second test, Wronskian form. The following determinant is identically zero

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ \psi(a,0) & \psi(b,0) & \psi(c,0) \end{vmatrix}$$

(24.2.ii) Second test, finite set form. For every $a, b, c \in \mathbb{R}$ there are $r_i := r_i(a,b,c) \in \mathbb{R}$ (possibly depending on a, b, c) such that

$$\psi(a,0) = r_1 a^2 + r_2, \quad \psi(b,0) = r_1 b^2 + r_2 \quad \text{and} \quad \psi(c,0) = r_1 c^2 + r_2.$$

(In principle we should check what happens on the second chart, but in this case it gives nothing new.)

Working on $\mathbb{R}^n$, let us now consider the general case

$$\phi = \textstyle\sum_i \phi_i f_i.$$

As in (24), we start by blowing up either the common zero set $Z = (f_1 = \cdots = f_r = 0)$, or, what is computationally easier, the ideal $(f_1, \dots, f_r)$. We get a real algebraic variety $p : Y \to \mathbb{R}^n$.

Working in various coordinate charts on Y, we get analogs of the First test (24.1) and new equations

$$\psi = \textstyle\sum_i \psi_i g_i.$$

The solvability again needs to be checked only on an $(n-1)$-dimensional real algebraic subvariety $Y_E \subset Y$. One sees, however, that the second tests (24.2.i–ii) are both equivalent to the Pointwise tests (3), thus not sufficient in general.

Instead, we focus on what kind of question we need to solve on Y_E. This leads to the following concept.

Definition 25. A *descent problem* is a compound object

$$\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$$

consisting of a proper morphism of real algebraic varieties $p : Y \to X$, an algebraic vector bundle E on X, an algebraic vector bundle F on Y and an algebraic vector bundle map $f : p^*E \to F$. (See (31) for the basic notions related to real algebraic varieties.)

Our aim is to understand the image of $f \circ p^* : C^0(X, E) \to C^0(Y, F)$.

We have the following analog of (24.2.ii).

Definition 26. Let $\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$ be a descent problem and $\phi_Y \in C^0(Y, F)$. We say that ϕ_Y satisfies the *finite set test* if for every $y_1, \dots, y_m \in Y$ there is a $\phi_X = \phi_{X,y_1,\dots,y_m} \in C^0(X, E)$ (possibly depending on $y_1, \dots, y_m$) such that

$$\phi_Y(y_i) = f \circ p^*(\phi_X)(y_i) \quad \text{for } i = 1, \dots, m.$$

Definition 27. A descent problem $\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$ is called *finitely determined* if for every $\phi_Y \in C^0(Y, F)$ the following are equivalent.

(1) $\phi_Y \in \operatorname{im}\big[f \circ p^* : C^0(X, E) \to C^0(Y, F)\big]$.
(2) ϕ_Y satisfies the finite set test.

28 (Outline of the main result)**.** Our theorem (34) gives an algorithm to decide the answer to Question 1. The precise formulation is somewhat technical to state, so here is a rough explanation of what kind of answer it gives and what we mean by an "algorithm." There are three main parts.

Part 1. First, starting with $\mathbb{R}^n$ and $f_1, \dots, f_r$ we construct a finitely determined descent problem $\mathbf{D} = \big(p : Y \to \mathbb{R}^n, f : p^*E \to F\big)$. This is purely algebraic, can be effectively carried out and independent of ϕ.

Part 2. There is a partially defined "twisted pull-back" map $p^{(*)} : C^0(\mathbb{R}^n) \dashrightarrow C^0(Y, F)$ (32) which is obtained as an iteration of three kinds of steps

(1) We compose a function by a real algebraic map.
(2) We create a vector function out of several functions or decompose a vector function into its coordinate functions.
(3) We choose local (real analytic) coordinates $\{y_i\}$ and ask if a certain function of the form $\psi_{j+1} := \psi_j \cdot \prod_i y_i^{-m_i}$ is continuous or not where $m_i \in \mathbb{Z}$.

If any of the answers is no, then the original ϕ can not be written as $\sum_i \phi_i f_i$ and we are done. If all the answers are yes, then we end up with $p^{(*)}\phi \in C^0(Y, F)$.

Part 3. We show that $\phi = \sum_i \phi_i f_i$ is solvable iff $p^{(*)}\phi \in C^0(Y, F)$ satisfies the finite set test (26).

By following the proof, one can actually write down solutions ϕ_i, but this relies on some artificial choices. The main ingredient that we need is to choose extensions of certain functions defined on closed semialgebraic subsets to the whole $\mathbb{R}^n$. In general, there does not seem to be any natural extension, and we do not know if it makes sense to ask for the "best possible" solution or not.

Negative aspects. There are two difficulties in carrying out this procedure in any given case. First, in practice, (3) of Part 2 may not be effectively doable. Second, we may need to compose ψ_{j+1} with a real algebraic map r_{j+1} such that ψ_j vanishes on the image of r_{j+1}. Thus we really need to compute limits and work with the resulting functions. This also makes it difficult to interpret our answer on $\mathbb{R}^n$ directly.

Positive aspects. On the other hand, just knowing that the answer has the above general structure already has some useful consequences.

First, the general framework works for other classes of functions; for instance the same algebraic set-up also applies in case ϕ and the ϕ_i are Hölder continuous.

Another consequence we obtain is that if $\phi = \sum_i \phi_i f_i$ is solvable and ϕ has certain additional properties, then one can also find a solution $\phi = \sum_i \psi_i f_i$ where the ψ_i also have these additional properties. We list two such examples below; see also (12). For the proof, see (50) and (37).

Corollary 29. *Fix $f_1, \dots, f_r$ and assume that $\phi = \sum_i \phi_i f_i$ is solvable. Then:*

1. *If ϕ is semialgebraic (31) then there is a solution $\phi = \sum_i \psi_i f_i$ such that the ψ_i are also semialgebraic.*
2. *Let $U \subset \mathbb{R}^n \setminus Z$ be an open set such that ϕ is C^m on U for some $m \in \{1, 2, \dots, \infty, \omega\}$. Then there is a solution $\phi = \sum_i \psi_i f_i$ such that the ψ_i are also C^m on U.*

Examples 30. The next series of examples shows several possible variants of (29) that fail.

(1) Here ϕ is a polynomial, but the ϕ_i must have very small Hölder exponents.

For $m \geq 1$, take $\phi := x^{2m} + (x^{2m-1} - y^{2m+1})^2$ and $f_1 = x^{2m+2} + y^{2m+2}$. There is only one solution,

$$\phi_1 = \frac{x^{2m} + (x^{2m-1} - y^{2m+1})^2}{x^{2m+2} + y^{2m+2}}.$$

We claim that it is Hölder with exponent $\frac{2}{2m-1}$. The exponent is achieved along the curve $x^{2m-1} - y^{2m+1} = 0$, parametrized as $\bigl(t^{(2m+1)/(2m-1)}, t\bigr)$.

(2) Here ϕ is C^n, there is a C^0 solution but no Hölder solution.

On $[-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^1$ set $f = x^n$ and $\phi = x^n / \log|x|$. Then ϕ is C^n and $\phi = \frac{1}{\log|x|} \cdot f$. Note that $\frac{1}{\log|x|}$ is continuous but not Hölder. (These can be extended to $\mathbb{R}^1$ in many ways.)

(3) Question: If ϕ is C^∞ and there is a C^0 solution, is there always a Hölder solution?

(4) Let $g(x)$ be a real analytic function. Set $f_1 := y$ and $\phi := \sin\bigl(g(x)y\bigr)$. Then $\phi_1 := \phi/y$ is also real analytic and $\phi = \phi_1 \cdot f_1$ is the only solution. Note that $|\phi(x, y)| \leq 1$ everywhere yet $\phi_1(x, 0) = g(x)$ can grow arbitrary fast.

(5) In general there is no solution $\phi = \sum_i \psi_i f_i$ such that $\operatorname{Supp} \psi_i \subset \operatorname{Supp} \phi$ for every i. As an example, take $f_1 = x^2 + x^4$, $f_2 = x^2 + y^2$ and

$$\phi(x, y) = \begin{cases} x^4 - y^2 & \text{if } y^2 \geq x^4 \text{ and} \\ 0 & \text{if } y^2 \leq x^4. \end{cases}$$

Note that $\phi = f_1 - \phi_2 f_2$ where

$$\phi_2(x, y) = \begin{cases} 1 & \text{if } y^2 \geq x^4 \text{ and} \\ \frac{x^2 + x^4}{x^2 + y^2} & \text{if } y^2 \leq x^4. \end{cases}$$

Let $\phi = \phi_1 \cdot (x^2 + x^4) + \psi_2 \cdot (x^2 + y^2)$ be any continuous solution. Setting $x = 0$ we get that $-y^2 = \psi_2(0, y) \cdot y^2$, hence $\psi_2(0, 0) = -1$. Thus $\operatorname{Supp} \psi_2$ can not be contained in $\operatorname{Supp} \phi$.

On the other hand, given any solution $\phi = \sum_i \phi_i f_i$, let χ be a function that is 1 on $\operatorname{Supp}\phi$ and 0 outside a small neighborhood of it. Then $\phi = \chi\phi = \sum(\chi\phi_i)f_i$. Thus we do have solutions whose support is close to $\operatorname{Supp}\phi$.

4.1. **Descent problems and their scions.**

31 (Basic set-up)**.** From now on, X denotes a fixed real algebraic variety. We always think of X as the real points of a complex affine algebraic variety $X_{\mathbb{C}}$ that is defined by real equations. (All our algebraic varieties are assumed reduced, that is, a function is zero iff it is zero at every point).

By a *projective variety over* X we mean the real points of a closed subvariety $Y \subset X \times \mathbb{CP}^N$. Every such Y is again the set of real points of a complex affine algebraic variety $Y_{\mathbb{C}} \subset X_{\mathbb{C}} \times \mathbb{CP}^N$ that is defined by real equations. For instance, $X \times \mathbb{RP}^N$ is contained in the affine variety which is the complement of the hypersurface $(\sum y_i^2 = 0)$ where y_i are the coordinates on $\mathbb{P}^N$.

A variety Y over X comes equipped with a morphism $p : Y \to X$ to X, given by the first projection of $X \times \mathbb{CP}^N$. Given such $p_i : Y_i \to X$, a morphism between them is a morphism of real algebraic varieties $\phi : Y_1 \to Y_2$ such that $p_1 = p_2 \circ \phi$.

Given $p_i : Y_i \to X$, their *fiber product* is

$$Y_1 \times_X Y_2 := \bigl\{(y_1, y_2) : p_1(y_1) = p_2(y_2)\bigr\} \subset Y_1 \times Y_2.$$

This comes with a natural projection $p : Y_1 \times_X Y_2 \to X$ and $p^{-1}(x) = p_1^{-1}(x) \times p_2^{-1}(x)$ for every $x \in X$. (Note, however, that even if the Y_i are smooth, their fiber product can be very singular.) If X is irreducible, we are frequently interested only in those irreducible components that dominate X; called the *dominant components.*

$\mathcal{R}(Y)$ denotes the ring of all regular functions on Y. These are locally quotients of polynomials $p(x)/q(x)$ where $q(x)$ is nowhere zero.

By an algebraic *vector bundle* on Y we mean the restriction of a complex algebraic vector bundle from $Y_{\mathbb{C}}$ to Y. All such vector bundles can be given by patching trivial bundles on a Zariski open cover $X = \cup_i U_i$ using transition functions in $\mathcal{R}(U_i \cap U_j)$. (Note that the latter condition is not quite equivalent to our definition, but this is not important for us, cf. [BCR98, Chap.12].)

Note that there are two natural topologies on a real algebraic variety Y, the Euclidean topology and the Zariski topology. The closed sets of the latter are exactly the closed subvarieties of Y. A Zariski closed (resp. open) subset of Y is also Euclidean closed (resp. open).

A closed *basic semialgebraic subset* of Y is defined by finitely many inequalities $g_i \geq 0$. Using finite intersections and complements we get all semialgebraic subsets. A function is semialgebraic iff its graph is semialgebraic. See [BCR98, Chap.2] for a detailed treatment.

We need various ways of modifying descent problems. The following definition is chosen to consist of simple and computable steps yet be broad enough for the proofs to work. (It should become clear that several variants of the definition would also work. We found the present one convenient to use.)

Definition 32 (Scions of descent problems)**.** Let $\mathbf{D} = \bigl(p : Y \to X, f : p^*E \to F\bigr)$ be a descent problem. A *scion* of $\mathbf{D}$ is any descent problem $\mathbf{D}_s = \bigl(p_s : Y_s \to X, f_s : p_s^*E \to F_s\bigr)$ that can be obtained by repeated application of the following procedures.

(1) For a proper morphism $r : Y_1 \to Y$ set
$$r^*\mathbf{D} := \big(p \circ r : Y_1 \to X, r^*f : (p\circ r)^*E \to r^*F\big).$$
As a special case, if $Z \subset X$ is a closed subvariety then the scion $\mathbf{D}_Z = \big(p_Z : Y_Z \to Z, f_Z : p_Z^*(E|_Z) \to F|_{Y_Z}\big)$ (where $Y_Z := p^{-1}(Z)$) is called the *restriction* of $\mathbf{D}$ to Z.

(2) Given Y_w, assume that there are several proper morphisms $r_i : Y_w \to Y$ such that the composites $p_w := p \circ r_i$ are all the same. Set
$$(r_1, \dots, r_m)^*\mathbf{D} := \big(p_w : Y_w \to X, \textstyle\sum_{i=1}^m r_i^* f : p_w^*E \to \sum_{i=1}^m r_i^*F\big)$$
where $\sum_{i=1}^m r_i^* f$ is the natural diagonal map.

(3) Assume that f factors as $p^*E \stackrel{q}{\to} F' \stackrel{j}{\hookrightarrow} F$ where F' is a vector bundle and $\operatorname{rank}_y j = \operatorname{rank}_y F'$ for all y in a Euclidean dense Zariski open subset $Y^0 \subset Y$. Then set
$$\mathbf{D}' := \big(p : Y \to X, f' := q : p^*E \to F'\big).$$

(The choice of Y^0 is actually a quite subtle point. Algebraic maps have constant rank over a suitable Zariski open subset and we want this open set to determine what happens with an arbitrary continuous function. This is why Y^0 is assumed Euclidean dense, not just Zariski dense. If Y is smooth, these are equivalent properties, but not if Y is singular. As an example, consider the Whitney umbrella $Y := (x^2 = y^2 z) \subset \mathbb{R}^3$. Here $Y \setminus (x = y = 0)$ is Zariski open and Zariski dense. Its Euclidean closure does not contain the "handle" $(x = y = 0,\ z < 0)$, so it is not Euclidean dense.)

Each scion remembers all of its forebears. That is, two scions are considered the "same" only if they have been constructed by an identical sequence of procedures. This is quite important since the vector bundle F_s on a scion $\mathbf{D}_s$ does depend on the whole sequence.

Every scion comes with a *structure map* $r_s : Y_s \to Y$.

If $\phi \in C^0(Y, F)$ then $r^*\phi \in C^0(Y_1, r^*F)$ and $\sum_{i=1}^m r_i^*\phi \in C^0(Y_w, \sum_{i=1}^m r_i^*F)$ are well defined. In (3) above, $j : C^0(Y, F') \to C^0(Y, F)$ is an injection, hence there is at most one $\phi' \in C^0(Y, F')$ such that $j(\phi') = \phi$. Iterating these, for any scion $\mathbf{D}_s$ of $\mathbf{D}$ with structure map $r_s : Y_s \to Y$ we get a partially defined map, called the *twisted pull-back*,
$$r_s^{(*)} : C^0(Y, F) \dashrightarrow C^0(Y_s, F_s).$$
We will need to know which functions ϕ are in the domain of a twisted pull-back map. A complete answer is given in (43).

The twisted pull-back map sits in a commutative square
$$\begin{array}{ccc} C^0(Y,F) & \stackrel{r_s^{(*)}}{\dashrightarrow} & C^0(Y_s,F_s) \\ \uparrow & & \uparrow \\ C^0(X,E) & = & C^0(X,E). \end{array}$$
If the structure map $r_s : Y_s \to Y$ is surjective, then $r^{(*)} : C^0(Y, F) \dashrightarrow C^0(Y_s, F_s)$ is injective (on its domain). In this case, understanding the image of $f \circ p^* : C^0(X, E) \to C^0(Y, F)$ is pretty much equivalent to understanding the image of $f_s \circ p_s^* : C^0(X, E) \to C^0(Y_s, F_s)$.

We are now ready to state our main result, first in the inductive form.

Proposition 33. *Let* $\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$ *be a descent problem. Then there is a scion* $\mathbf{D}_s = \big(p_s : Y_s \to X, f_s : p_s^*E \to F_s\big)$ *with surjective structure map* $r_s : Y_s \to Y$ *and a closed subvariety* $Z \subset X$ *such that* $\dim Z < \dim X$ *and for every* $\phi \in C^0(Y, F)$ *following are equivalent.*

(1) $\phi \in \operatorname{im}\big[f \circ p^* : C^0(X, E) \to C^0(Y, F)\big]$.
(2) $r_s^{(*)}\phi$ *is defined and* $r_s^{(*)}\phi \in \operatorname{im}\big[f_s \circ p_s^* : C^0(X, E) \to C^0(Y_s, F_s)\big]$,
(3) (a) $r_s^{(*)}\phi$ *satisfies the finite set test (26) and*
 (b) $\phi|_{Y_Z} \in \operatorname{im}\big[f_Z \circ p_Z^* : C^0(Z, E|_Z) \to C^0(Y_Z, F_Z)\big]$, *where the scion* $\mathbf{D}_Z = \big(p_Z : Y_Z \to Z, f_Z : p_Z^*(E|_Z) \to F_Z\big)$ *is the restriction of* $\mathbf{D}_s$ *to Z (32.1).*

We can now set $X_1 := Z$, $\mathbf{D}_1 := \mathbf{D}_Z$ apply (33) to $\mathbf{D}_1$ and get a descent problem $\mathbf{D}_2 := \big(\mathbf{D}_1\big)_Z$. Repeating this, we obtain descent problems $\mathbf{D}_i = \big(p_i : Y_i \to X, f_i : p_i^*E \to F_i\big)$ such that the dimension of $p_i(Y_i)$ drops at every step. Eventually we reach the case where $p_i(Y_i)$ consists of points. Then the finite set test (26) gives the complete answer. The disjoint union of all the Y_i can be viewed as a single scion, hence we get the following algebraic answer to Question 1.

Theorem 34. *Let* $\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$ *be a descent problem. Then it has a finitely determined scion* $\mathbf{D}_w = \big(p_w : Y_w \to X, f_w : p_w^*E \to F_w\big)$ *with surjective structure map* $r_w : Y_w \to Y$.

That is, for every $\phi \in C^0(Y, F)$ *following are equivalent.*

(1) $\phi \in \operatorname{im}\big[f \circ p^* : C^0(X, E) \to C^0(Y, F)\big]$.
(2) *The twisted pull-back* $r_w^{(*)}\phi$ *is defined and it is contained in the image of* $f_w \circ p_w^* : C^0(X, E) \to C^0(Y_w, F_w)$,
(3) *The twisted pull-back* $r_w^{(*)}\phi$ *is defined and satisfies the finite set test (26).* □

The proof of (34) works for many subclasses of continuous functions as well. Next we axiomatize the necessary properties and describe the main examples.

4.2. **Subclasses of continuous functions.**

Assumption 35. For real algebraic varieties Z we consider vector subspaces $C^*\big(Z\big) \subset C^0\big(Z\big)$ that satisfy the following properties.

(1) (Local property) If $Z = \cup_i U_i$ is an open cover of Z then $\phi \in C^*\big(Z\big)$ iff $\phi|_{U_i} \in C^*\big(U_i\big)$ for every i.
(2) ($\mathcal{R}(Z)$-module) If $\phi \in C^*\big(Z\big)$ and $h \in \mathcal{R}\big(Z\big)$ is a regular function (31) then $h \cdot \phi \in C^*\big(Z\big)$.
(3) (Pull-back) For every morphism $g : Z_1 \to Z_2$, composing with g maps $C^*\big(Z_2\big)$ to $C^*\big(Z_1\big)$.
(4) (Descent property) Let $g : Z_1 \to Z_2$ be a proper, surjective morphism, $\phi \in C^0\big(Z_2\big)$ and assume that $\phi \circ g \in C^*\big(Z_1\big)$. Then $\phi \in C^*\big(Z_2\big)$.
(5) (Extension property) Let $Z_1 \subset Z_2$ be a closed semialgebraic subset (38). Then the twisted pull-back map $C^*\big(Z_2\big) \to C^*\big(Z_1\big)$ is surjective.

Since every closed semialgebraic subset is the image of a proper morphism (38), we can unite (4) and (5) and avoid using semialgebraic subsets as follows.

(4+5) (Strong descent property) Let $g : Z_1 \to Z_2$ be a proper morphism and $\psi \in C^*\big(Z_2\big)$. Then $\psi = \phi \circ g$ for some $\phi \in C^*\big(Z_2\big)$ iff ψ is constant on every fiber of g.

The following additional condition comparing 2 classes $C_1^* \subset C_2^*$ is also of interest.

(6) (Division property) Let $h \in \mathcal{R}\big(Z\big)$ be any function whose zero set is nowhere Euclidean dense. If $\phi \in C_1^*\big(Z\big)$ and $\phi/h \in C_2^*\big(Z\big)$ then $\phi/h \in C_1^*\big(Z\big)$.

Example 36. Here are some natural examples satisfying the assumptions (35.1–5).

(1) $C^0\big(Z\big)$, the set of all continuous functions on Z.
(2) $C^h\big(Z\big)$, the set of all locally Hölder continuous functions on Z.
(3) $S^0\big(Z\big)$, the set of continuous semialgebraic functions on Z.

Moreover, the pairs $S^0 \subset C^0$ and $S^0 \subset C^h$ both satisfy (35.6). (By contrast, by (30.2), the pair $C^h \subset C^0$ does not satisfy (35.6).)

37 (Proof of (29.1))**.** More generally, consider two classes $C_1^* \subset C_2^*$ that satisfy (35.1–5) and also (35.6). Let $\mathbf{D}$ be a descent problem and $\phi \in C_1^*(Y, F)$. We claim that if $\phi = f \circ p^*(\phi_X)$ is solvable with $\phi_X \in C_2^*(X, E)$ then it also has a solution $\phi = f \circ p^*(\psi_X)$ where $\psi_X \in C_1^*(X, E)$.

To see this, let $\mathbf{D}_w$ be a scion as in (34). By our assumption, the twisted pull-back $r_w^{(*)}\phi$ is in $C_2^*\big(Y_w, F_w\big)$ and it satisfies the finite set test. For the finite set test it does not matter what type of functions we work with. Thus we need to show that $r_w^{(*)}\phi$ is in $C_1^*\big(Y_w, F_w\big)$.

In a scion construction, this holds for steps as in (32.1–2) by (35.3). The key question is (32.3). The solution given in (43) shows that it is equivalent to (35.6). □

38 (C^*-valued functions over semialgebraic sets)**.** Let $S \subset Z$ be a closed semialgebraic subset. We can think of S as the image of a proper morphism $g : W \to Z$ (cf. [BCR98, Sec.2.7]). One can define $C^*(S)$ either as the image of $C^*(Z)$ in $C^0(S)$ or as the preimage of $C^*(W)$ under the pull-back by g. By (35.4+5), these two are equivalent.

We also have the following

(1) (Closed patching condition) Let $S_i \subset Z$ be closed semialgebraic subsets. Let $\phi_i \in C^*(S_i)$ and assume that $\phi_i|_{S_i \cap S_j} = \phi_j|_{S_i \cap S_j}$ for every i, j.
Then there is a unique $\phi \in C^*\big(\cup_i S_i\big)$ such that $\phi|_{S_i} = \phi_i$ for every i.

To see this, realize each S_i as the image of some proper morphism $g_i : W_i \to Z$. Let $W := \amalg_i W_i$ be their disjoint union and $g : W \to Z$ the corresponding morphism. Define $\psi \in C^*(W)$ by the conditions $\psi|_{W_i} = \phi_i \circ g_i$.

The patching condition guarantees that ψ is constant on the fibers of g. Thus, by (35.4+5), $\psi = \phi \circ g$ for some $\phi \in C^*\big(\cup_i S_i\big)$.

These arguments also show that each $C^*(Z)$ is in fact a module over $S^0(Z)$, the ring of continuous semialgebraic functions.

Definition 39 (C^*-valued sections)**.** By Serre's theorems, every vector bundle on a complex affine variety can be written as a quotient bundle of a trivial bundle and also as a subbundle of a trivial bundle. Furthermore, every extension of vector bundles splits.

Thus, on a real algebraic variety, every algebraic vector bundle can be written as a quotient bundle (and a subbundle) of a trivial bundle and every constant rank map of vector bundles splits.

Let F be an algebraic vector bundle on Z and $Z = \cup_i U_i$ an open cover such that $F|_{U_i}$ is trivial of rank r for every i. Let

$$C^*\big(Z, F\big) \subset C^0\big(Z, F\big)$$

denote the set of those sections $\phi \in C^0\big(Z, F\big)$ such that $\phi|_{U_i} \in C^*\big(U_i\big)^r$ for every i. If C^* satisfies the properties (35.1–2), this is independent of the trivializations and the choice of the covering.

If C^* satisfies the properties (35.1–6) then their natural analogs also hold for $C^*\big(Z, F\big)$. This is clear for the properties (35.2–4) and (35.6).

In order to check the extension property (35.5) first note that we have the following.

(1) Let $f : F_1 \to F_2$ be a surjection of vector bundles. Then $f : C^*\big(Z, F_1\big) \to C^*\big(Z, F_2\big)$ is surjective.

Now let $Z_1 \subset Z_2$ be an closed subvariety and F a vector bundle on Z_2. Write it as a quotient of a trivial bundle $\mathbb{C}^N_{Z_2}$. Every section $\phi_1 \in C^*\big(Z_1, F|_{Z_1}\big)$ lifts to a section in $C^*\big(Z_1, \mathbb{C}^N_{Z_1}\big)$ which in turn extends to a section in $C^*\big(Z_2, \mathbb{C}^N_{Z_2}\big)$ by (35.6). The image of this lift in $C^*\big(Z_2, F|_{Z_2}\big)$ gives the required lifting of ϕ_1.

4.3. **Local tests and reduction steps.**

Next we consider various descent problems whose solution is unique, if it exists.

40 (Pull-back test)**.** Let $g : Z_1 \to Z_2$ be a proper surjection of real algebraic varieties. Let F be a vector bundle on Z_2 and $\phi_1 \in C^*\big(Z_1, g^*F\big)$. When can we write $\phi_1 = g^*\phi_2$ for some $\phi_2 \in C^*\big(Z_2, F\big)$?

Answer: By (35.4), such a ϕ_2 exists iff ϕ_1 is constant on every fiber of g. This can be checked as follows.

Take the fiber product $Z_3 := Z_1 \times_{Z_2} Z_1$ with projections $\pi_i : Z_3 \to Z_1$ for $i = 1, 2$. Note that $F_3 := \pi_1^* g^* F$ is naturally isomorphic to $\pi_2^* g^* F$. We see that ϕ_1 is constant on every fiber of g iff

$$\pi_1^*\phi_1 - \pi_2^*\phi_1 \in C^*\big(Z_3, F_3\big) \quad \text{is identically } 0.$$

Note that this solves descent problems $\mathbf{D} = \big(p : Y \to X, f : p^*E \cong F\big)$ where f is an isomorphism. We use two simple cases.

(1) Assume that there is a closed subset $Z \subset X$ such that p induces an isomorphism $Y \setminus p^{-1}(Z) \to X \setminus Z$ and $\phi_Y \in C^0\big(Y, p^*E\big)$ vanishes along $p^{-1}(Z)$. Then there is a $\phi_X \in C^0\big(X, E\big)$ such that $\phi_Y = p^*\phi_X$ (and ϕ_X vanishes along Z.)

(2) Assume that there is a finite group G acting on Y such that G acts transitively on every fiber of $\big(Y \setminus (\phi_Y = 0)\big) \to X$. Then there is a $\phi_X \in C^0\big(X, E\big)$ such that $\phi_Y = p^*\phi_X$.

41 (Wronskian test)**.** Let $\phi, f_1, \dots, f_r$ be functions on a set Z. Assume that the f_i are linearly independent. Then ϕ is a linear combination of the f_i (with constant coefficients) iff the determinant

$$\begin{vmatrix} f_1(\mathbf{z}_1) & \cdots & f_1(\mathbf{z}_r) & f_1(\mathbf{z}_{r+1}) \\ \vdots & & \vdots & \vdots \\ f_r(\mathbf{z}_1) & \cdots & f_r(\mathbf{z}_r) & f_r(\mathbf{z}_{r+1}) \\ \phi(\mathbf{z}_1) & \cdots & \phi(\mathbf{z}_r) & \phi(\mathbf{z}_{r+1}) \end{vmatrix}$$

is identically zero as a function on Z^{r+1}.

Proof. Since the f_i are linearly independent, there are $\mathbf{z}_1, \dots, \mathbf{z}_r \in Z$ such that the upper left $r \times r$ subdeterminant of is nonzero. Fix these $\mathbf{z}_1, \dots, \mathbf{z}_r$ and solve the linear system

$$\phi(\mathbf{z}_i) = \sum_j \lambda_j f_j(\mathbf{z}_i) \quad \text{for } i = 1, \dots, r.$$

Replace ϕ by $\psi := \phi - \sum_i \lambda_i f_i$ and let $\mathbf{z}_{r+1}$ vary. Then our determinant is

$$\begin{vmatrix} f_1(\mathbf{z}_1) & \cdots & f_1(\mathbf{z}_r) & f_1(\mathbf{z}_{r+1}) \\ \vdots & & \vdots & \vdots \\ f_r(\mathbf{z}_1) & \cdots & f_r(\mathbf{z}_r) & f_r(\mathbf{z}_{r+1}) \\ 0 & \cdots & 0 & \psi(\mathbf{z}_{r+1}) \end{vmatrix}$$

and it vanishes iff $\psi(\mathbf{z}_{r+1})$ is identically zero. That is, when $\phi \equiv \sum_j \lambda_j f_j$. □

42 (Linear combination test)**.** Let Z be a real algebraic variety, F a vector bundle on Z and $f_1, \dots, f_r$ linearly independent algebraic sections of F.

Given $\phi \in C^*\big(Z, F\big)$, when can we write $\phi = \sum_i \lambda_i f_i$ for some $\lambda_i \in \mathbb{C}$?

Answer: One can either write down a determinental criterion similar to (41) or reduce this to the Wronskian test as follows.

Consider $q : \mathbb{P}(F) \to X$, the space of 1-dimensional quotients of F. Let $u : q^*F \to Q$ be the universal quotient line bundle. Then $\phi = \sum_i \lambda_i f_i$ iff

$$u \circ q^*(\phi) = \textstyle\sum_i \lambda_i \cdot u \circ q^*(f_i).$$

The latter is enough to check on a Zariski open cover of $\mathbb{P}(F)$ where Q is trivial. Thus we recover the Wronskian test. □

43 (Membership test for sheaf injections)**.** Let Z be a real algebraic variety, E, F algebraic vector bundles and $h : E \to F$ a vector bundle map such that $\operatorname{rank} h = \operatorname{rank} E$ on a Euclidean dense Zariski open set $Z^0 \subset Z$. Given a section $\phi \in C^*\big(Z, F\big)$, when is it in the image of $h : C^*\big(Z, E\big) \to C^*\big(Z, F\big)$?

Answer: Over Z^0, there is a quotient map $q : F|_{Z^0} \to Q_{Z^0}$ where $\operatorname{rank} Q_{Z^0} = \operatorname{rank} F - \operatorname{rank} E$ and $\operatorname{im}\big(h|_{Z^0}\big) = \ker q$. Then the first lifting condition is:

(1) $q(\phi) = 0$. Note that, in the local coordinate functions of ϕ, this is a linear condition with polynomial coefficients.

By (39.3), $h|_{Z^0}$ has an algebraic splitting $s : F|_{Z^0} \to E|_{Z^0}$. Note that s is not unique on E but it is unique on the image of h. Thus the second condition says:

(2) The section $s\big(\phi|_{Z^0}\big) \in C^*\big(Z^0, E|_{Z^0}\big)$ extends to a section of $C^*\big(Z, E\big)$.

In order to make this more explicit, choose local algebraic trivializations of E and of F. Then ϕ is given by coordinate functions $(\phi_1, \dots, \phi_m)$ and s is given by a matrix (s_{ij}) where the s_{ij} are rational functions on Z that are regular on Z^0. We can bring them to common denominator and write $s_{ij} = u_{ij}/v$ where u_{ij} and v are regular on Z. Thus

$$s\big(\phi|_{Z^0}\big) = \Big(\sum_j s_{1j}\phi_j, \dots, \sum_j s_{nj}\phi_j\Big) = \frac{1}{v}\Big(\sum_j u_{1j}\phi_j, \dots, \sum_j u_{nj}\phi_j\Big).$$

Let Φ denote the vector function in the parenthesis on the right. Then $\Phi \in C^*(Z, E)$ and we are asking if $\Phi/v \in C^*(Z, E)$ or not. This is exactly one of the question considered in Part 2 of (28).

Also, if we are considering two function classes $C_1^* \subset C_2^*$, then (43.3) and the assumption (35.6) say that a function $\phi \in C_1^*(Z, F)$ is in the image of $h : C_2^*\big(Z, E\big) \to C_2^*\big(Z, F\big)$ iff it is in the image of $h : C_1^*\big(Z, E\big) \to C_1^*\big(Z, F\big)$. □

44 (Resolution of singularities)**.** Let $\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$ be a descent problem. By Hironaka's theorems (see [Kol07, Chap.3] for a relatively simple treatment) there is a resolution of singularities $r_0 : Y' \to Y$. That is, Y' is smooth and r_0 is proper and birational (that is, an isomorphism over a Zariski dense open set). Note however, that r_0 is not surjective in general. In fact, $r_0(Y')$ is precisely the Euclidean closure of the smooth locus Y^{ns}. Thus $Y \setminus r_0(Y') \subset \operatorname{Sing}(Y)$.

We resolve $\operatorname{Sing} Y$ to obtain $r_1 : Y_1' \to \operatorname{Sing}(Y)$. The resulting map $Y' \amalg Y_1' \to Y$ is surjective, except possibly along $\operatorname{Sing}(\operatorname{Sing}(Y))$. We can next resolve $\operatorname{Sing}(\operatorname{Sing}(Y))$ and so on. After at most $\dim Y$ such steps, we obtain a smooth, proper morphism $R : Y^R \to Y$ such that Y^R is smooth and R is surjective. R is an isomorphism over Y^{ns} but it can have many irreducible components that map to $\operatorname{Sing}(Y)$.

We refer to $Y' \subset Y^R$ as the *main components* of the resolution.

Proposition 45. *Let $\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$ be a descent problem. Assume that X, Y are irreducible, the generic fiber of p is irreducible, smooth and $h(x) : E(x) \to C^0\big(Y_x, F|_{Y_x}\big)$ is an injection for general $x \in p(Y)$. Then $\mathbf{D}$ has a scion $\mathbf{D}_s = \big(p_s : Y_s \to X, f_s : p_s^*E \to F_s\big)$ with surjective structure map $r_s : Y_s \to Y$ such that*

1. *Y_s is a disjoint union $Y_s^h \amalg Y_s^v$,*
2. *$\dim p_s\big(Y_s^v\big) < \dim X$ and*
3. *f_s is an isomorphism over Y_s^h.*

Proof. Set $n = \operatorname{rank} E$ and let Y_X^{n+1} be the union of the dominant components (31) of the $n+1$-fold fiber product of $Y \to X$ with coordinate projections π_i. Let $\tilde{p} : Y_X^{n+1} \to X$ be the map given by any of the $p \circ \pi_i$. Consider the diagonal map

$$\tilde{f} : \tilde{p}^*E \to \textstyle\sum_{i=1}^{n+1} \pi_i^* F$$

which is an injection over a Zariski dense Zariski open set $Y^0 \subset Y_X^{n+1}$ by assumption. By (32), these define a scion of $\mathbf{D}$ with surjective structure map.

We want to use the Local lifting test (43) to replace $\sum_{i=1}^{n+1} \pi_i^* F$ by $\tilde{p}^*E$. For this we need Y^0 to be also Euclidean dense. To achieve this, we resolve Y_X^{n+1} as in (44) to get Y_s. The main components give Y_s^h but we may have introduced some other components Y_s^v that map to $\operatorname{Sing}(Y)$. Since the general fiber of p is smooth, Y_s^v maps to a lower dimensional subvariety of X. □

Proposition 46. *Let $\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$ be a descent problem. Assume that X, Y are irreducible and the generic fiber of p is irreducible and smooth.*

Then there is a commutative diagram

$$\begin{array}{ccc} \bar{Y} & \stackrel{\tau_Y}{\to} & Y \\ \bar{p}\downarrow & & \downarrow p \\ \bar{X} & \stackrel{\tau_X}{\to} & X \end{array}$$

where τ_X, τ_Y are proper, birational and there is a quotient bundle $\tau_X^* E \twoheadrightarrow \bar{E}$ such that $\bar{p}^*\tau_X^* E \to \tau_Y^* F$ factors through $\bar{p}^*\bar{E}$ and the descent problem

$$\bar{\mathbf{D}} = \bigl(\bar{p} : \bar{Y} \to \bar{X}, \bar{f} : \bar{p}^*\bar{E} \to \bar{F} := \tau_Y^* F\bigr)$$

satisfies the assumptions of (45). That is, $\bar{f}(x) : \bar{E}(x) \to C^0\bigl(\bar{Y}_x, \tau_Y^* F|_{\bar{Y}_x}\bigr)$ is an injection for general $x \in \bar{p}(\bar{Y})$.

Moreover, if a finite group G acts on $\mathbf{D}$ then we can choose $\bar{\mathbf{D}}$ such that the G-action lifts to $\bar{\mathbf{D}}$.

(Note that, as shown by (48), the conclusions can fail if the general fibers of p are not irreducible.)

Proof. Complexify $p : Y \to X$ to get a complex proper morphism $p_{\mathbb{C}} : Y_{\mathbb{C}} \to X_{\mathbb{C}}$ and set

$$E'_{\mathbb{C}} := \operatorname{im}\bigl[E_{\mathbb{C}} \to \bigl(p_{\mathbb{C}}\bigr)_* F_{\mathbb{C}}\bigr].$$

Let $x \in p(Y)$ be a general point. Then Y_x is irreducible and the real points Y_x are Zariski dense in the complex fiber $\bigl(Y_{\mathbb{C}}\bigr)_x$. Thus $H^0\bigl(\bigl(Y_{\mathbb{C}}\bigr)_x, F_{\mathbb{C}}\bigr) = H^0\bigl(Y_x, F\bigr)$.

So far $E'_{\mathbb{C}}$ is only a coherent sheaf which is a quotient of $E_{\mathbb{C}}$. Using (47) and then (45), we obtain $\tau_X : \bar{X} \to X$ as desired. □

47. Let X be an irreducible variety $q : E \to E'$ a map of vector bundles on X. In general we can not write q as a composite of a surjection of vector bundles followed by an injection, but the following construction shows how to achieve this after modifying X.

Let $\operatorname{Gr}(d, E) \to X$ be the universal Grassmann bundle of rank d quotients of E where d is the rank of q at a general point. At a general point $x \in X$, $q(x) : E(x) \twoheadrightarrow \operatorname{im} q(x) \subset E'(x)$ is such a quotient. Thus q gives a rational map $X \dashrightarrow \operatorname{Gr}(d, E)$, defined on a Zariski dense Zariski open subset. Let $\bar{X} \subset \operatorname{Gr}(d, E)$ denote the closure of its image and $\tau_X : \bar{X} \to X$ the projection. Then τ_X is a proper birational morphism and we have a decomposition

$$\tau_X^* q : \tau_X^* E \stackrel{s}{\twoheadrightarrow} \bar{E} \stackrel{j}{\hookrightarrow} \tau_X^* E'$$

where $\bar{E}$ is a vector bundle of rank d on $\bar{X}$, s is a rank d surjection everywhere and j is a rank d injection on a Zariski dense Zariski open subset.

4.4. **Proof of the main algebraic theorem.**

In order to answer Question 1 in general, we try to create a situation where (46) applies.

First, using (44) we may assume that Y is smooth. Next take the Stein factorization $Y \to W \to X$; that is, $W \to X$ is finite and all the fibers of $Y \to W$ are connected (hence general fibers are irreducible).

After some modifications, (45) applies to $Y \to W$, thus we are reduced to comparing $C^0(W, p_W^* E)$ and $C^0(X, E)$.

This is easy if $W \to X$ is Galois, since then the sections of $p_W^* E$ that are invariant under the Galois group descend to sections of E.

If $p : W \to X$ is a finite morphism of (smooth or at least normal) varieties over $\mathbb{C}$, the usual solution would be to take the Galois closure of the field extension $\mathbb{C}(W)/\mathbb{C}(X)$ and let $W^{Gal} \to X$ be the normalization of X in it. Then the Galois group G acts on $W^{Gal} \to X$ and the action is transitive on every fiber.

This does not work for real varieties since in general, W^{Gal} has no real points. (For instance, take $X = \mathbb{R}$ and let $W \subset \mathbb{R}^2$ be any curve given by an irreducible equation of the form $y^m = f(x)$. If $m = 2$ then W/X is Galois but for $m \geq 3$ the Galois closure W^{Gal} has no real points.) Some other problems are illustrated by the next example.

Example 48. Let $W \subset \mathbb{R}^2$ be defined by $(y^5 - 5y = x)$ with $p : W \to \mathbb{R}^1_x =: X$ the projection. Set $E = \mathbb{C}^4_X$ and $F = \mathbb{C}_W$ with $f : p^*E \to F$ given by $f(\psi(x)e_i) = y^i\psi(x)|_W$.

Note that p has degree 5 as a map of (complex) Riemann surfaces, but $p^{-1}(x)$ consists of 3 points for $-1 < x < 1$ and of 1 point if $|x| > 1$. Therefore, the kernel of $f \circ p^*(x) : \mathbb{C}^4 = E(x) \to C^0\big(W_x, F|_{W_x}\big)$ has rank 1 if $-1 < x < 1$ and rank 3 if $|x| > 1$. Thus $\ker\big(f \circ p^*\big) \subset E$ is a rank 1 subbundle on the interval $-1 < x < 1$ and a rank 3 subbundle on the intervals $|x| > 1$.

These kernels depend only on some of the 5 roots of $y^5 - 5y = x$, hence they are semialgebraic subbundles but not real algebraic subbundles.

As a replacement of the Galois closure W^{Gal}, we next introduce a series of varieties $W^{(m)}_X \to X$. The $W^{(m)}_X$ are usually reducible, the symmetric group S_m acts on them, but the S_m-action is usually not transitive on every fiber. Nonetheless, all the $W^{(m)}_X$ together provide a suitable analog of the Galois closure.

Definition 49. Let $s : W \to X$ be a finite morphism of (possibly reducible) varieties and $X^0 \subset X$ the largest Zariski open subset over which p is smooth.

Consider the m-fold fiber product $W^m_X := W \times_X \cdots \times_X W$ with coordinate projections $\pi_i : W^m_X \to W$. For every $i \neq j$, let $\Delta_{ij} \subset W^m_X$ be the preimage of the diagonal $\Delta \subset W \times_X W$ under the map (π_i, π_j). Let $W^{(m)}_X \subset W^m_X$ be the union of the dominant components in the closure of $W^m_X \setminus \cup_{i\neq j}\Delta_{ij}$ with projection $s^{(m)} : W^{(m)}_X \to X$. The symmetric group S_m acts on $W^{(m)}_X$ by permuting the factors.

If $x \in X^0$ then $\big(s^{(m)}\big)^{-1}(x)$ consists of ordered m-element subsets of $s^{-1}(x)$. Thus $\big(s^{(m)}\big)^{-1}(x)$ is empty if $|s^{-1}(x)| < m$ and S_m acts transitively on $\big(s^{(m)}\big)^{-1}(x)$ if $|s^{-1}(x)| = m$. If $\big|s^{-1}(x)\big| > m$ then S_m does not act transitively on $\big(s^{(m)}\big)^{-1}(x)$. We obtain a decreasing sequence of semialgebraic subsets

$$s^{(1)}\big(W^{(1)}_X\big) \supset s^{(2)}\big(W^{(2)}_X\big) \supset \cdots .$$

Set

$$X^0_{W,m} := X^0 \cap \Big(s^{(m)}\big(W^{(m)}_X\big) \setminus s^{(m+1)}\big(W^{(m+1)}_X\big)\Big).$$

The $X^0_{W,m}$ are disjoint, $\bigcup_m X^0_{W,m}$ is a Euclidean dense semialgebraic open subset of $p(Y) \cap X^0$ and the S_m-action is transitive on the fibers of $s^{(m)}$ that lie over $X^0_{W,m}$. Thus $s^{(m)} : W^{(m)} \to X$ behaves like a Galois extension over $X^0_{W,m}$ and together the $X^0_{W,m}$ cover most of X.

Let now $p : Y \to X$ be a proper morphism of (possibly reducible) normal varieties with Stein factorization $p : Y \xrightarrow{q} W \xrightarrow{s} X$. Let Y_X^m denote the m-fold fiber product $Y \times_X \cdots \times_X Y$ with coordinate projections $\pi_i : Y_X^m \to Y$.

Let $Y_X^{(m)} \subset Y_X^m$ denote the dominant parts of the preimage of $W_X^{(m)}$ under the natural map $q^m : Y_X^m \to W_X^m$ with projection $p^{(m)} : Y_X^{(m)} \to X$. Note that, for general $x \in X$, $\bigl(p^{(m)}\bigr)^{-1}(x)$ is empty if $p^{-1}(x)$ has fewer than m irreducible components and S_m acts transitively on the irreducible components of $\bigl(p^{(m)}\bigr)^{-1}(x)$ if $p^{-1}(x)$ has exactly m irreducible components. Thus we obtain a decreasing sequence of semialgebraic subsets $p^{(1)}\bigl(Y_X^{(1)}\bigr) \supset p^{(2)}\bigl(Y_X^{(2)}\bigr) \supset \cdots$.

Let F be a vector bundle on Y. Then $\oplus_i \pi_i^* F$ is a vector bundle on Y_X^m. Its restriction to $Y_X^{(m)}$ is denoted by $F^{(m)}$.

Note that the S_m-action on $Y_X^{(m)}$ naturally lifts to an S_m-action on $F^{(m)}$. If E is a vector bundle on X and $f : p^*E \to F$ is a vector bundle map then we get an S_m-invariant vector bundle map $f^{(m)} : \bigl(p^{(m)}\bigr)^* E \to F^{(m)}$. For each m we get a scion of $\mathbf{D}$

$$\mathbf{D}^{(m)} := \bigl(p^{(m)} : Y_X^{(m)} \to X, f^{(m)} : \bigl(p^{(m)}\bigr)^* E \to F^{(m)}\bigr).$$

Below, we will use all the $\mathbf{D}^{(m)}$ together to get a scion with Galois-like properties.

50 (Proof of (33))**.** If $\mathbf{D}_s$ is a scion of $\mathbf{D}$ with surjective structure map $r_s : Y_s \to Y$, then (33.1) $\Leftrightarrow$ (33.2) by definition and (33.2) $\Rightarrow$ (33.3) holds for any choice of Z.

Assume next that we have a candidate for $\mathbf{D}_s$ and Z such that. How do we check (33.3) $\Rightarrow$ (33.2)?

Pick $\Phi_s \in C^*\bigl(Y_s, F_s\bigr)$ and assume that there is a section $\phi_Z \in C^*\bigl(Z, E|_Z\bigr)$ whose pull-back to Y_Z equals the restriction of Φ_s. By (39), we can lift ϕ_Z to a section $\phi_X \in C^*\bigl(X, E\bigr)$. Consider next

$$\Psi_s := \Phi_s - f_s\bigl(p_s^* \phi_X\bigr) \in C^*\bigl(Y_s, F_s\bigr).$$

We are done if we can write $\Psi_s = f_s \circ p_s^*(\psi_X)$ for some $\psi_X \in C^*\bigl(X, E\bigr)$.

By assumption, Ψ_s satisfies the finite set test (26) but the improvement is that Ψ_s vanishes on Y_Z. As we saw already in (2), this can make the problem much easier. We deal with this case in (51).

Note that by [Whi34], we can choose ϕ_X to be real analytic away from Z and the rest of the construction preserves differentiability properties. Thus (29.2) holds once the rest of the argument is worked out. □

Proposition 51. *Let* $\mathbf{D} = \bigl(p : Y \to X, f : p^*E \to F\bigr)$ *be a descent problem. Then there is a closed algebraic subvariety* $Z \subset X$ *with* $\dim Z < \dim X$ *and a scion* $\mathbf{D}_s = \bigl(p_s : Y_s \to X, f_s : p_s^* E \to F_s\bigr)$ *with surjective structure map* $r_s : Y_s \to Y$ *such that the following holds.*

Let $\psi_s \in C^0(Y_s, F_s)$ *be a section that vanishes on* $p_s^{-1}(Z)$ *and satisfies the finite set test (26). Then there is a* $\psi_X \in C^0(X, E)$ *such that* ψ_X *vanishes on* Z *and* $\psi_s = f_s \circ p_s^*(\psi_X)$*.*

Proof. We may harmlessly assume that $p(Y)$ is Zariski dense in X. Using (44) we may also assume that Y is smooth.

After we construct $\mathbf{D}_s$, the plan is to make sure that Z contains all of its "singular" points. In the original setting of Question 1, Z was the set where the map

$(f_1,\dots,f_r):\mathbb{C}^r\to\mathbb{C}$ has rank 0. In the general case, we need to include points over which f_s drops rank and also points over which p_s drops rank.

During the proof we gradually add more and more irreducible components to Z. To start with, we add to Z the lower dimensional irreducible components of X, the locus where X is not normal and the (Zariski closures of) the $p(Y_i)$ where $Y_i\subset Y$ is an irreducible component that does not dominate any of the maximal dimensional irreducible components of X. We can thus assume that X is irreducible and every irreducible component of Y dominates X.

Take the Stein factorization $p:Y\stackrel{q}{\to}W\stackrel{s}{\to}X$ and set $M=\deg(W/X)$. For each $1\le m\le M$, consider the following diagram

$$\begin{array}{ccccccc}
\big(\bar q^{(m)}\big)^*\bar E^{(m)} & \cong & \bar F^{(m)} & & F^{(m)} & & F\\
 & & \downarrow & & \downarrow & & \downarrow\\
\big(t_W^{(m)}\circ s_W^{(m)}\big)^*E\twoheadrightarrow \bar E^{(m)} & & \bar Y_X^{(m)} & \stackrel{t_Y^{(m)}}{\to} & Y_X^{(m)} & \stackrel{\pi_i^{(m)}}{\to} & Y\\
 & \searrow & \downarrow \bar q^{(m)} & & \downarrow q^{(m)} & & \downarrow p\\
 & & \bar W^{(m)} & \stackrel{t_W^{(m)}}{\to} & W^{(m)} & \stackrel{s^{(m)}}{\to} & X
\end{array}\qquad (51.m)$$

where $W^{(m)}$ and its column is constructed in (49) and out of this $\bar W^{(m)}$, its column and the vector bundle $\bar E^{(m)}$ are constructed in (46). Note that the symmetric group S_m acts on the whole diagram.

The $\mathbf{D}_s$ we use will be the disjoint union of the scions

$$\bar{\mathbf{D}}_s^{(m)}:=\Big(\bar p^{(m)}:\bar Y_X^{(m)}\to X,\bar f^{(m)}:\big(\bar p^{(m)}\big)^*E\to\bar F^{(m)}\Big)\quad\text{for } m=1,\dots,M.$$

By enlarging Z if necessary, we may assume that $Y_X^{(m)}\to X$ is smooth over $X\setminus Z$ and each $t_W^{(m)}$ is an isomorphism over $X\setminus Z$. Note that, for every m,

$$X_m^0:=p^{(m)}\big(Y_X^{(m)}\big)\setminus\Big(Z\cup p^{(m+1)}\big(Y_X^{(m+1)}\big)\Big)\subset X$$

is an open semialgebraic subset of $X\setminus Z$ whose boundary is in Z. Furthermore, $p(Y)\setminus Z$ is the disjoint union of the X_m^0 and the fiber Y_x has exactly m irreducible components if $x\in X_m^0$.

Let $\Psi_s\in C^0(Y_s,F_s)$ be a section that vanishes on $p_s^{-1}(Z)$. We can then uniquely write $\Psi_s=\sum_m\Psi_s^{(m)}$ such that each $\Psi_s^{(m)}$ vanishes on $Y_s\setminus p_s^{-1}\big(X_m^0\big)$. Moreover, Ψ_s satisfies the finite set test (26) iff all the $\Psi_s^{(m)}$ satisfy it.

Thus it is sufficient to prove that each $\Psi_s^{(m)}$ is the pull-back of a section $\psi_X^{(m)}\in C^*(X,E)$ that vanishes on $X\setminus X_m^0$. For each m we use the corresponding diagram (51_m).

Each $\Psi_s^{(m)}$ lifts to a section $\bar\Psi_s^{(m)}$ of $\big(\bar q^{(m)}\big)^*\bar E^{(m)}$ that satisfies the pull-back conditions for $\bar Y^{(m)}\to\bar W^{(m)}$. Thus $\bar\Psi_s^{(m)}$ is the pull-back of a section $\bar\Psi_W^{(m)}$ of $\bar E^{(m)}$. By construction, $\bar\Psi_W^{(m)}$ is S_m-invariant and it vanishes outside $\big(t^{(m)}\circ s^{(m)}\big)^{-1}\big(X_m^0\big)$. Using a splitting of $\big(s_W^{(m)}t_W^{(m)}\big)^*E\twoheadrightarrow\bar E^{(m)}$ we can think of $\bar\Psi_W^{(m)}$ as an S_m-invariant section of $\big(t_W^{(m)}\circ s_W^{(m)}\big)^*E$. By the choice of Z, $t^{(m)}$ is an isomorphism over X_m^0, hence $\bar\Psi_W^{(m)}$ descends to an S_m-invariant section $\Psi_W^{(m)}$ of $\big(s_W^{(m)}\big)^*E$ that vanishes outside $\big(s_W^{(m)}\big)^{-1}\big(X_m^0\big)$. Therefore, by (40.2), $\Psi_W^{(m)}$ descends to a section $\psi_X^{(m)}\in C^0\big(X,E\big)$ that vanishes on $X\setminus X_m^0$. □

4.5. **Semialgebraic, real and p-adic analytic cases.**

52 (Real analytic case)**.** It is natural to ask Question 1 when the f_i are real analytic functions and $\mathbb{R}^n$ is replaced by an arbitrary real analytic variety. As before, we think of X as the real points of a complex Stein space $X_{\mathbb{C}}$ that is defined by real equations. Our proofs work without changes for descent problems $\mathbf{D} = \bigl(p : Y \to X, f : p^*E \to F\bigr)$ where Y and f are *relatively algebraic* over X.

By definition, this means that Y is the set of real points of a closed (reduced but possibly reducible) complex analytic subspace of some $X_{\mathbb{C}} \times \mathbb{CP}^N$ and that f is assumed algebraic in the $\mathbb{CP}^N$-variables.

This definition may not seem the most natural, but it is exactly the setting needed to answer Question 1 if the f_i are real analytic functions on a real analytic space.

53 (Semialgebraic case)**.** It is straightforward to consider semialgebraic descent problems $\mathbf{D} = \bigl(p : Y \to X, f : p^*E \to F\bigr)$ where X, Y are semialgebraic sets, E, F are semialgebraic vector bundles and p, f are semialgebraic maps. (See [BCR98, Chap.2] for basic results and definitions.) It is not hard to go through the proofs and see that everything generalizes to the semialgebraic case.

In fact, some of the constructions could be simplified since one can break up any descent problem $\mathbf{D}$ into a union of descent problems $\mathbf{D}_i$ such that each $Y_i \to X_i$ is topologically a product over the interior of X_i. This would allow one to make some non-canonical choices to simplify the construction of the diagrams (51.m).

It may be, however, worthwhile to note that one can directly reduce the semialgebraic version to the real algebraic one as follows.

Note first that in the semialgebraic setting it is natural to replace a real algebraic descent problem $\mathbf{D} = \bigl(p : Y \to X, f : p^*E \to F\bigr)$ by its *semialgebraic reduction* $\text{sa-red}(\mathbf{D}) := \bigl(p : Y \to p(Y), f : p^*\bigl(E|_{p(Y)}\bigr) \to F\bigr)$.

We claim that for every semialgebraic descent problem $\mathbf{D}$ there is a proper surjection $r : Y_s \to Y$ such that the corresponding scion $r^*\mathbf{D}$ is semialgebraically isomorphic to the semialgebraic reduction of a real algebraic descent problem.

To see this, first, we can replace the semialgebraic X by a real algebraic variety X^a that contains it and extend E to semialgebraic vector bundle over X^a. Not all semialgebraic vector bundles are algebraic, but we can realize E as a semialgebraic subbundle of a trivial bundle $\mathbb{C}^M$. This in turn gives a semialgebraic embedding of X into $X \times \mathrm{Gr}(\operatorname{rank} E, M)$. Over the image, E is the restriction of the algebraic universal bundle on $\mathrm{Gr}(\operatorname{rank} E, M)$. Thus, up to replacing X by the Zariski closure of its image, we may assume that X and E are both algebraic. Replacing Y by the graph of p in $Y \times X$, we may assume that p is algebraic. Next write Y as the image of a real algebraic variety. We obtain a scion where now $p : X \to Y, E, F$ are all algebraic. To make f algebraic, we use that f defines a semialgebraic section of $\mathbb{P}\bigl(\mathcal{H}om_X(p^*E, F)\bigr) \to Y$. Thus, after replacing Y by the Zariski closure of its image in $\mathbb{P}\bigl(\mathcal{H}om_X(p^*E, F)\bigr) \to Y$, we obtain an algebraic scion with surjective structure map.

54 (p-adic case)**.** One can also consider Question 1 in the p-adic case and the proofs work without any changes. In fact, if we start with polynomials $f_i \in \mathbb{Q}[x_1, \dots, x_n]$ then in Theorem 34 it does not matter whether we want to work over $\mathbb{R}$ or $\mathbb{Q}_p$; we construct the same descent problems. It is only in checking the finite set test (26) that the field needs to be taken into account: if we work over $\mathbb{R}$, we need to check the condition for fibers over all real points, if we work over $\mathbb{Q}_p$, we need to check the condition for fibers over all p-adic points.

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