The Structure of Linear Extension Operators for C^m

by

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Abstract: For any subset $E \subset \mathbb{R}^n$, let $C^m(E)$ denote the Banach space of restrictions to E of functions $F \in C^m(\mathbb{R}^n)$. It is known that there exist bounded linear maps $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$ such that $Tf = f$ on E for any $f \in C^m(E)$. We show that T can be taken to have a simple form, but cannot be taken to have an even simpler form.

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§0. Statement of Results

Fix $m, n \geq 1$, let $E \subset \mathbb{R}^n$ be given, and let $C^m(E) = \{F|_E : F \in C^m(\mathbb{R}^n)\}$, with norm

$$\| f \|_{C^m(E)} = \inf\{\| F \|_{C^m(\mathbb{R}^n)}: F \in C^m(\mathbb{R}^n) \text{ and } F|_E = f\}.$$

Here, as usual, $C^m(\mathbb{R}^n)$ denotes the space of m times continuously differentiable $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, for which the norm $\| F \|_{C^m(\mathbb{R}^n)} = \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|$ is finite. A linear extension operator for $C^m(E)$ is a bounded linear map $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$, such that $Tf|_E = f$ for all $f \in C^m(E)$.

Given $E \subset \mathbb{R}^n$, there exists a linear extension operator for $C^m(E)$. See [17] for a proof, and [1,...,29] for related work going back to Whitney. In particular, Merrien [20] constructed linear extension operators for $C^m(E)$ when $E \subset \mathbb{R}^1$, and Bromberg [3] constructed linear extension operators for $C^1(E)$ when $E \subset \mathbb{R}^n$. The existence of linear extension operators for $C^m(E)$ was explicitly conjectured by Brudnyi and Shvartsman in [9].

The purpose of this paper is to examine what a linear extension operator for $C^m(E)$ might look like. For arbitrary finite E, we showed in [11] that $C^m(E)$ admits an extension operator of bounded "depth". We recall the relevant definition from [11], in a slightly weakened form.

Let $s \geq 1$ be an integer, and let $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$ be a linear map. Then we say that T has depth s if, for every $x^0 \in \mathbb{R}^n$, there exist $x_1, \ldots, x_s \in E$ and $\lambda_1, \ldots, \lambda_s \in \mathbb{R}$, such that

$$Tf(x^0) = \sum_{i=1}^{s} \lambda_i f(x_i) \quad \text{for all } f \in C^m(E).$$

From [11], we have the following result.

<u>Theorem 1:</u> *Given $m \geq 1$ and $E \subset \mathbb{R}^n$ finite, there exists an extension operator $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$ with norm at most C and depth at most s; here, C and s depend only on m and n.*

One might be tempted to believe that the hypothesis of finite E can be dropped from Theorem 1. The following result dashes this hope.

Theorem 2: *There exists a countable compact set $E \subset \mathbb{R}^2$, for which $C^1(E)$ admits no extension operator of finite depth.*

We prove this result in Section 1 below, by exhibiting an explicit E. Our set E is very close to a counterexample given by Glaeser in [18].

Despite Theorem 2, we can get a positive result by modifying the notion of "depth". We prepare the way with the following definitions.

A "one-point differential operator on $C^m(\mathbb{R}^n)$" is a linear functional on $C^m(\mathbb{R}^n)$ of the form

(1) $\mathcal{D} : F \mapsto \sum_{|\alpha| \leq m} a_\alpha \, \partial^\alpha F(x^0)$, with $x^0 \in \mathbb{R}^n$ and $a_\alpha \in \mathbb{R}\,(|\alpha| \leq m)$.

Next, let $E \subset \mathbb{R}^n$, and let $\mathcal{D}$ be as in (1). We say that $\mathcal{D}$ is a "one-point differential operator on $C^m(E)$", provided we have

(2) $\mathcal{D}F = 0$ whenever $F \in C^m(\mathbb{R}^n)$ and $F|_E = 0$.

Evidently, if (1) and (2) hold, then we obtain a linear functional on $C^m(E)$, by mapping $f \in C^m(E)$ to $\mathcal{D}F$, for any $F \in C^m(\mathbb{R}^n)$ with $F|_E = f$. Abusing notation, we denote this functional by $f \mapsto \mathcal{D}f$.

As a trivial example, suppose E is an embedded sub-manifold in $\mathbb{R}^n$. Then any tangent vector $X \in T_{x^0}E$ is a one-point differential operator on $C^1(E)$.

The paper [2] of Bierstone-Milman-Pawłucki shows how to find all possible one-point differential operators on $C^m(E)$ for an arbitrary, given $E \subset \mathbb{R}^n$. (See also [13].)

Now let $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$ be a linear map and let $s \geq 1$. Then we say that T has "breadth" s if, given any one-point differential operator $\mathcal{D}$ on $C^m(\mathbb{R}^n)$, there exist one-point differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_s$ on $C^m(E)$, such that

$$\mathcal{D}(Tf) = \sum_{i=1}^{s} \mathcal{D}_i f \text{ for all } f \in C^m(E)\,.$$

In particular, this implies that, for any $x^0 \in \mathbb{R}^n$, we can express $Tf(x^0)$ as a sum of at most s terms of the form $\mathcal{D}_i f$, where $\mathcal{D}_i$ is a one-point differential operator on $C^m(E)$.

We are ready to state our positive result.

Theorem 3: *Given $m \geq 1$ and $E \subset \mathbb{R}^n$, there exists an extension operator $T : C^m(E) \longrightarrow C^m(\mathbb{R}^n)$, with norm at most C and breadth at most s; here, C and s depend only on m and n.*

The proof of Theorem 3 is accomplished by modifying the proof of the main result in [17], as explained in Section 2 below.

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§1. Proof of Theorem 2

We exhibit the countable compact set $E \subset \mathbb{R}^2$ from Theorem 2. Let

(1) $P_{N,k} = (x_{N,k}, y_{N,k}) = (2^{-N} + 10^{-N-k}, (-1)^k \cdot 10^{-2N-k}) \in \mathbb{R}^2$ for $N, k \geq 1$; and let

(2) $P_{N,\infty} = (2^{-N}, 0) \in \mathbb{R}^2$, for $N \geq 1$.

We define

(3) $E_N = \{P_{N,\infty}\} \cup \{P_{N,k} : k \geq 1\} \subset \mathbb{R}^2$ for $N \geq 1$, and we set

(4) $E = \{(0,0)\} \cup \bigcup_{N \geq 1} E_N$.

Note that the E_N are pairwise disjoint. As promised, E is a countable compact subset of $\mathbb{R}^2$.

To show that $C^1(E)$ admits no extension operators of finite depth, we use the following three properties of E.

Lemma 1: *Let $\tilde{E} \subset E$, and suppose $\tilde{E} \cap E_N$ is finite for each $N \geq 1$. Then $\tilde{E} \subset \{(x,y) \in \mathbb{R}^2 : y = \psi(x)\}$ for some function $\psi \in C^1(\mathbb{R})$.*

Lemma 2: *Let $s \geq 1$, and let $(P_1^n, \ldots, P_s^n)$ be a sequence of s-tuples of points of E. Then there exist an integer $N_0 \geq 1$ and an increasing infinite sequence $(n_\nu)_{\nu \geq 1}$, such that $\{P_i^{n_\nu} : \nu \geq 1, 1 \leq i \leq s\} \cap E_N$ is finite for each $N \geq N_0$.*

Lemma 3.: *Let $F \in C^1(\mathbb{R}^2)$. If $F = 0$ on E, then $\nabla F(0,0) = 0$.*

Assume these three lemmas for the moment, and suppose $T : C^1(E) \longrightarrow C^1(\mathbb{R}^2)$ is an extension operator of depth s. We will derive a contradiction.

For $n \geq 1$, let

(5) $Q^n = (0, \frac{1}{n}) \in \mathbb{R}^2$.

Since T has depth s, there exist points $P_1^n, \ldots, P_s^n \in E$ and coefficients $\lambda_1^n, \ldots, \lambda_s^n \in \mathbb{R}$, such that

$$Tf(Q^n) = \sum_{i=1}^{s} \lambda_i^n f(P_i^n) \text{ for } f \in C^1(E)\,,\ n \geq 1\,.$$

In particular, for each $n \geq 1$, we have

(6) $Tf(Q^n) = 0$ whenever $f \in C^1(E)$ with $f(P_1^n) = \cdots = f(P_s^n) = 0$.

We apply Lemma 2 to the s-tuples $(P_1^n, \ldots, P_s^n)$, $n \geq 1$.

Let N_0 and $(n_\nu)_{\nu \geq 1}$ be as in Lemma 2. We define sets

(7) $\hat{E} = \{P_i^{n_\nu} : \nu \geq 1, 1 \leq i \leq s\}$,

(8) $E^\# = \{P \in \hat{E} : P \in E_N \text{ for some } N < N_0\}$, and

(9) $\tilde{E} = \hat{E} \setminus E^\#$.

The set $\tilde{E} \cap E_N$ is finite for $N \geq N_0$ (by Lemma 2), and empty for $N < N_0$ (by (8) and (9)). Hence, Lemma 1 applies, and there exists $\psi \in C^1(\mathbb{R})$ such that

(10) $y = \psi(x)$ for all $(x, y) \in \tilde{E}$.

Now let $\theta(x, y)$ be a smooth cutoff function on $\mathbb{R}^2$, equal to one in a neighborhood of the origin, and equal to zero on E_N for $N < N_0$. We define

(11) $F(x, y) = \theta(x, y) \cdot [y - \psi(x)]$ for $(x, y) \in \mathbb{R}^2$, and

(12) $f = F|_E \in C^1(E)$.

The functions F and Tf both belong to $C^1(\mathbb{R}^2)$, and are both equal to f on E. Hence, Lemma 3 gives

(13) $\nabla(Tf)(0, 0) = \nabla F(0, 0)$.

On the other hand, we can compute $\frac{\partial}{\partial y}(Tf)(0, 0)$ and $\frac{\partial}{\partial y} F(0, 0)$, and they will turn out to be unequal.

In fact, we have $F = 0$ on $\tilde{E}$ thanks to (10), (11); and $F = 0$ on $E^{\#}$, since $\theta = 0$ on E_N for $N < N_0$. (See (8), (11).) Thus, $F = 0$ on $\hat{E}$, hence $f = 0$ on $\hat{E}$, and therefore $Tf(Q^{n_\nu}) = 0$ for $\nu \geq 1$, thanks to (6) and (7).

Recalling (5), we conclude that

(14) $\frac{\partial}{\partial y}(Tf)(0, 0) = 0$.

However, since $\theta = 1$ in a neighborhood of the origin, the definition (11) yields

(15) $\frac{\partial}{\partial y} F(0, 0) = 1$.

Thus, $\frac{\partial}{\partial y}(Tf)(0, 0)$ and $\frac{\partial}{\partial y} F(0, 0)$ are distinct, as claimed.

This contradicts (13), showing that $C^1(E)$ cannot have an extension operator of depth s.

To complete the proof of Theorem 2, it remains to establish Lemmas 1,2,3. We begin with the following elementary result, which will be used in the proof of Lemma 1.

Proposition: *Given $M \geq 1$, there exists $\psi_M \in C^1(\mathbb{R})$, with*

(16) $\mathrm{supp}\, \psi_M \subset (0,1)$,

(17) $\psi_M(10^{-k}) = (-1)^k \cdot 10^{-k}$ *for* $1 \leq k \leq M$, *and*

(18) $\| \psi_M \|_{C^1(\mathbb{R})} \leq C$, *with C independent of M.*

Proof: Fix smooth functions $\theta, \tilde{\theta}$ on $\mathbb{R}$, with $\theta(x) = 0$ for $x \leq 1/2$, $\theta(x) = 1$ for $x \geq 1$, $\tilde{\theta}(x) = 1$ for $|x| \leq 1/2$, $\tilde{\theta}(x) = 0$ for $|x| \geq 2/3$. One checks easily that

$$\psi_M(x) = \theta(10^M x) \cdot \tilde{\theta}(x) \cdot x \cos(\pi \log_{10} |x|)$$

satisfies all the conditions asserted in the proposition. ∎

Proof of Lemma 1: For each $N \geq 1$, pick

(19) $M_N > \max\{k : P_{N,k} \in \tilde{E}\}$.

We can do this, since $\tilde{E} \cap E_N$ is assumed finite. Define

(20) $\psi(x) = \sum_{N \geq 1} 10^{-2N} \psi_{M_N}(10^N \cdot [x - 2^{-N}])$ for $x \in \mathbb{R}$,

with ψ_{M_N} as in the Proposition.

Each summand in (20) is a C^1 function on $\mathbb{R}$, with the N^{th} summand having C^1 norm at most $C \cdot 10^{-N}$. (This follows easily from (18).) Hence, $\psi \in C^1(\mathbb{R})$.

From (1) and (17), we have

$$10^{-2N} \psi_{M_N}(10^N \cdot [x_{N,k} - 2^{-N}]) = 10^{-2N} \psi_{M_N}(10^{-k})$$

$$= (-1)^k \cdot 10^{-2N-k} \text{ for } 1 \leq k \leq M_N .$$

Hence, (1) and (19) yield

(21) $10^{-2N}\,\psi_{M_N}(10^N \cdot [x_{N,k} - 2^{-N}]) = y_{N,k}$ whenever $P_{N,k} \in \tilde{E}$.

On the other hand, (16) implies easily that

(22) $10^{-2N'}\,\psi_{M_{N'}}(10^{N'} \cdot [x_{N,k} - 2^{-N'}]) = 0$ whenever $N' \neq N$ $(N, N', k \geq 1)$.

Putting (21), (22) into (20), we see that

(23) $\psi(x_{N,k}) = y_{N,k}$ whenever $P_{N,k} = (x_{N,k}, y_{N,k}) \in \tilde{E}$.

For $(x, y) = P_{N,\infty}$ or $(0, 0)$, we have $y = 0$, and all the summands in (20) are equal to zero, thanks to (16). Hence,

(24) $\psi(x) = y$ whenever $(x, y) = P_{N,\infty}$ or $(0, 0)$.

From (23), (24) and (3), (4), we conclude that $\psi(x) = y$ for all $(x, y) \in \tilde{E}$, since $\tilde{E} \subset E$. The proof of Lemma 1 is complete. ■

Proof of Lemma 2: For $n \geq 1$, let $\mathcal{P}^n$ be the set

(25) $\mathcal{P}^n = \{P_1^n, \ldots, P_s^n\}$.

Suppose $\mathcal{N}$ is any set of positive integers. We say that $\mathcal{N}$ is a "sink" if there are infinitely many $n \geq 1$ for which $\mathcal{P}^n$ intersects E_N for each $N \in \mathcal{N}$. The empty set is a sink. On the other hand, no sink can have more than s elements, since the E_N are pairwise disjoint and the $\mathcal{P}^n$ have cardinality at most s. Hence there exists a sink $\bar{\mathcal{N}}$ of maximal cardinality. Thus,

(26) The set $A = \{n \geq 1 : \mathcal{P}^n$ intersects E_N for each $N \in \bar{\mathcal{N}}\}$ is infinite (since $\bar{\mathcal{N}}$ is a sink)

and

(27) Given $N \geq 1$ not belonging to $\bar{\mathcal{N}}$, there are at most finitely many $n \in A$ for which $\mathcal{P}^n$ intersects E_N. (Otherwise, $\bar{\mathcal{N}} \cup \{N\}$ would be a sink, contradicting the maximal cardinality of $\bar{\mathcal{N}}$.)

In view of (26), we can write

(28) $A = \{n_1, n_2, n_3, \ldots\}$

for an infinite increasing sequence $(n_\nu)_{\nu\geq 1}$.

Since $\bar{\mathcal{N}}$ is a sink, it has at most s elements. Hence we can pick an integer $N_0 \geq 1$ such that

(29) $N_0 > N$ for all $N \in \bar{\mathcal{N}}$.

From (27), (28), (29), we learn the following:

(30) Given $N \geq N_0$, there are at most finitely many ν for which $\mathcal{P}^{n_\nu}$ intersects E_N.

From (25) and (30), we obtain the conclusion of Lemma 2. ■

Proof of Lemma 3: Let $F \in C^1(\mathbb{R}^2)$, with $F = 0$ on E. Fix $N \geq 1$, and note that $F(P_{N,\infty}) = F(P_{N,k}) = 0$ for $k \geq 1$. Consequently,

(31) $$0 = \lim_{\substack{k\to\infty \\ (k \text{ even})}} \left[\frac{F(P_{N,k}) - F(P_{N,\infty})}{10^{-N-k}}\right] = \left(\frac{\partial F}{\partial x} + 10^{-N}\frac{\partial F}{\partial y}\right)(P_{N,\infty})$$

and

(32) $$0 = \lim_{\substack{k\to\infty \\ (k \text{ odd})}} \left[\frac{F(P_{N,k}) - F(P_{N,\infty})}{10^{-N-k}}\right] = \left(\frac{\partial F}{\partial x} - 10^{-N}\frac{\partial F}{\partial y}\right)(P_{N,\infty}).$$

(See (1), ... , (4).)

From (31) and (32), we learn that $\nabla F(P_{N,\infty}) = 0$. Taking the limit as $N \to \infty$, we conclude that $\nabla F(0,0) = 0$, proving Lemma 3. ■

We have now established Lemmas 1,2,3. Since we reduced Theorem 2 to those lemmas, the proof of Theorem 2 is complete. ■

It is an amusing exercise to construct a linear extension operator for $C^1(E)$ with $E \subset \mathbb{R}^2$ given by (1),...,(4).

§2. Sketch of Proof of Theorem 3

We recall the main result of [17], then explain how to modify it to prove Theorem 3. We begin with some notation and definitions.

We write $\mathcal{R}_x$ for the ring of m-jets of smooth real-valued functions at $x \in \mathbb{R}^n$. For $F \in C^m(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we write $J_x(F)$ to denote the m-jet of F at x.

Let $E \subset \mathbb{R}^n$ be compact. For each $x \in E$, suppose we are given an m-jet $f(x) \in \mathcal{R}_x$ and an ideal $I(x)$ in $\mathcal{R}_x$. Then $(f(x) + I(x))_{x \in E}$ is called a "family of cosets". (We allow the possibilities $I(x) = \{0\}$ and $I(x) = \mathcal{R}_x$.) The family of cosets $(f(x) + I(x))_{x \in E}$ is called "Glaeser stable" if it satisfies the following condition: Given $x_0 \in E$ and $P_0 \in f(x_0) + I(x_0)$, there exists $F \in C^m(\mathbb{R}^n)$ such that $J_{x_0}(F) = P_0$, and $J_x(F) \in f(x) + I(x)$ for all $x \in E$.

More generally, suppose Ξ is a vector space, and again let $E \subset \mathbb{R}^n$ be compact. For each $x \in E$, suppose we are given a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$, and an ideal $I(x)$ in $\mathcal{R}_x$. Then we call $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ a "family of cosets depending linearly on $\xi \in \Xi$". We say that $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ is "Glaeser stable" if, for each fixed $\xi \in \Xi$, the family of cosets $(f_\xi(x) + I(x))_{x \in E}$ is Glaeser stable.

These notions arise naturally in [16,17], and we refer the reader to those papers for the motivation.

The main result of [17] is as follows.

Theorem 4: *Let Ξ be a vector space, with seminorm $|\cdot|$. Let $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ be a Glaeser stable family of cosets depending linearly on $\xi \in \Xi$.*

Assume that for each $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with $\| F \|_{C^m(\mathbb{R}^n)} \leq 1$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

Then there exists a linear map $\xi \mapsto F_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, such that

(A) *$J_x(F_\xi) \in f_\xi(x) + I(x)$ for all $x \in E$, $\xi \in \Xi$; and*

(B) *$\| F_\xi \|_{C^m(\mathbb{R}^n)} \leq C|\xi|$ for all $\xi \in \Xi$, with C depending only on m and n.*

This result easily implies the existence of extension operators for $C^m(E)$. To prove Theorem 3, we modify Theorem 4 by introducing the notion of "s-admissible" operators, which we now explain.

Let $\hat{\Xi}$ be a set of (real) linear functionals on the linear space Ξ, and let $s \geq 1$ be an integer. Then:

- A linear functional on Ξ will be called "s-admissible" (with respect to $\hat{\Xi}$) if it can be written as a linear combination of at most s elements of $\hat{\Xi}$.
- A linear map T from Ξ to a finite-dimensional vector space V is called "s-admissible" (with respect to $\hat{\Xi}$) if, for every linear functional λ on V, the linear functional $\lambda \circ T$ on Ξ is s-admissible.
- A linear map $T : \Xi \longrightarrow C^m(\mathbb{R}^n)$ will be called "s-admissible" (with respect to $\hat{\Xi}$) if, for every $x \in \mathbb{R}^n$, the map $\xi \mapsto J_x(T\xi)$ is s-admissible as a map from Ξ to $\mathcal{R}_x$.

Our modification of Theorem 4 is as follows.

Theorem 5: *Let Ξ be a vector space, with seminorm $|\cdot|$, let $\hat{\Xi}$ be a set of linear functionals on Ξ, and let $s \geq 1$ be an integer. Let $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ be a Glaeser stable family of cosets depending linearly on $\xi \in \Xi$.*

Assume that the map $\xi \mapsto f_\xi(x)$, from Ξ into $\mathcal{R}_x$, is s-admissible with respect to $\hat{\Xi}$, for each $x \in E$.

Assume also that, for each $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with

$$\| F \|_{C^m(\mathbb{R}^n)} \leq 1\,, \text{ and } J_x(F) \in f_\xi(x) + I(x) \text{ for all } x \in E\,.$$

Then there exists a linear map $\xi \mapsto F_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, such that

(A) *$J_x(F_\xi) \in f_\xi(x) + I(x)$ for all $x \in E$, $x \in \Xi$;*

(B) *$\| F_\xi \|_{C^m(\mathbb{R}^n)} \leq C|\xi|$ for all $\xi \in \Xi$, with C depending only on m and n; and*

(C) *The map* $\xi \mapsto F_\xi$ *is* s'*-admissible, with* s' *depending only on* s, m *and* n.

We indicate briefly why Theorem 5 implies Theorem 3, and then we explain how the proof of Theorem 4 in [17] may be modified to prove Theorem 5.

Reduction of Theorem 3 to Theorem 5:

To prove Theorem 3, we may assume that the set E is compact. (In fact, for a general E, we may pass without difficulty to the closure of E, and then reduce matters to the case of closed, bounded E by a partition of unity.)

For $E \subset \mathbb{R}^n$ compact, we make the following definitions.

- $\Xi = C^m(E)$.
- $|\xi| = 2 \parallel \xi \parallel_{C^m(E)}$ for $\xi \in \Xi$.
- $\widehat{\Xi}$ is the set of all one-point differential operators on $C^m(E)$.

For each $x \in E$:

- $I(x) = \{J_x(F) : F \in C^m(\mathbb{R}^n) \text{ and } F = 0 \text{ on } E\}$.
- $V(x) =$ any complementary subspace to $I(x)$ in $\mathcal{R}_x$.
- $\pi_x : \mathcal{R}_x \longrightarrow V(x)$ is the natural projection arising from the direct sum $\mathcal{R}_x = V(x) \oplus I(x)$.

Suppose $x \in E$ and $\xi \in \Xi$. Since $\xi \in \Xi$, there exists $F \in C^m(\mathbb{R}^n)$ with $F|_E = \xi$. We define $f_\xi(x) = \pi_x(J_x(F))$. This is independent of the choice of F. (In fact, suppose $F_1, F_2 \in C^m(\mathbb{R}^n)$, with $F_i|_E = \xi$. Then $F_1 - F_2 \in C^m(\mathbb{R}^n)$ and $(F_1 - F_2)|_E = 0$. Hence, $J_x(F_1 - F_2) \in I(x)$, and therefore $\pi_x(J_x(F_1) - J_x(F_2)) = 0$.)

One checks easily that the above Ξ, $|\cdot|$, $\widehat{\Xi}$, $I(x)$, $f_\xi(x)$ satisfy the hypotheses of Theorem 5, with $s = 1$. Hence, applying Theorem 5, we obtain a linear map $\xi \mapsto F_\xi$ from Ξ into $C^m(\mathbb{R}^n)$, satisfying (A), (B), (C).

From (A), we see that $\xi \mapsto F_\xi$ is an extension operator for $C^m(E)$. Conclusion (B) controls the norm of this extension operator, and conclusion (C) tells us that it has breadth s', with s' depending only on m and n. Thus, Theorem 3 is reduced to Theorem 5.

Sketch of Proof of Theorem 5. We assume that the reader is familiar with our previous papers [11,...,17]. It is a long, routine exercise to follow the proof of Theorem 4, as given in [17], and note that at each step, we preserve s'-admissibility (although s' may increase). ("Admissibility" will always be defined with respect to $\widehat{\Xi}$, given in the hypotheses of Theorem 5.) The highlights of this tedious exercise are as follows.

- For $E \subset \mathbb{R}^n$ compact, let $C^m_{\text{jet}}(E)$ be the space of families of jets $\vec{f} = (f_x)_{x \in E}$, with $f_x \in \mathcal{R}_x$ for each $x \in E$, such that there exists $F \in C^m(\mathbb{R}^n)$ satisfying

(1) $J_x(F) = f_x$ for each $x \in E$.

The norm $\| \vec{f} \|_{C^m_{\text{jet}}(E)}$ is defined as the infimum of $\| F \|_{C^m(\mathbb{R}^n)}$ over all $F \in C^m(\mathbb{R}^n)$ satisfying (1).

The proof of the standard Whitney extension theorem [19,24,25] gives an operator $T : C^m_{\text{jet}}(E) \longrightarrow C^m(\mathbb{R}^n)$, with the following properties.

(a) $\| T \| \leq C$, with C depending only on m and n.

(b) For $\vec{f} = (f_x)_{x \in E} \in C^m_{\text{jet}}(E)$, we have $J_x(T\vec{f}) = f_x$ for each $x \in E$.

(c) For each $x_0 \in \mathbb{R}^n$ there exist $x_1, \ldots, x_k \in E$ such that, as $\vec{f} = (f_x)_{x \in E}$ varies over $C^m_{\text{jet}}(E)$, the jet $J_{x_0}(T\vec{f})$ depends only on $f_{x_1}, \ldots, f_{x_k}$. Here, k depends only on m and n.

In view of (c), we have the following result.

Let $\xi \mapsto \vec{f}_\xi = (f_{x,\xi})_{x \in E}$ be a linear map from Ξ into $C^m_{\text{jet}}(E)$. Assume that $\xi \mapsto f_{x,\xi}$ is s'-admissible, for each $x \in E$.

Then the map $\xi \mapsto T\vec{f}_\xi$ is s''-admissible from Ξ into $C^m(\mathbb{R}^n)$, where T is as above, and s'' depends only on s', m, n.

- Suppose we add to the hypotheses of Lemma 3.3 in [16] the assumption that $\xi \mapsto f_\xi(x)$ is s'-admissible for each $x \in E$. (Here, $s' \geq 1$ is given.) Then the map $\xi \mapsto \tilde{f}_\xi(x_0)$ in the conclusion of that lemma may be taken to be s''-admissible, with s'' depending only on $s', m, n, k^\#$. (That's because the $\tilde{f}_\xi(x_0)$ constructed in the proof of Lemma 3.3 in [16] depends on ξ only through the $f_\xi(x)$ for $x \in \bar{S}$, where $\bar{S} \subset E$ has cardinality less than $k^\#$.) When we apply the above lemma in [17], we take $k^\#$ to depend only on m and n.

  Therefore, the property of s'-admissibility (for some s' depending only on m, n, s) is preserved when we apply Lemma 3.3 from [16].

- Whenever we applied Theorem 5 from [16] in the proof of Theorem 4, we now apply instead Theorem 8 from [16]. Note that the notion of "depth" in [16] differs from our present notion.

- For suitable $x \in E$, let $\text{proj}_x : \mathcal{R}_x \longrightarrow \mathcal{R}_x$ be the linear map defined in Section 10 of [17]. If $\xi \mapsto g_\xi(x)$ is an s'-admissible linear map from Ξ into $\mathcal{R}_x$, then also $\xi \mapsto \text{proj}_x(g_\xi(x))$ is s'-admissible. (This follows trivially from the definition of s'-admissibility.)

- Suppose $F_\xi = \sum_\nu \theta_\nu \cdot F_\xi^\nu$ for $\xi \in \Xi$; and suppose that, for each $x \in \mathbb{R}^n$, we are given a finite set $\Omega(x)$, such that

$$J_x(F_\xi) \;=\; \sum_{\nu \in \Omega(x)} J_x(\theta_\nu \cdot F_\xi^\nu) \text{ for all } \xi \in \Xi\,.$$

  If $\xi \mapsto F_\xi^\nu$ is s'-admissible for each ν, and if $\Omega(x)$ has cardinality at most k for each $x \in \mathbb{R}^n$, then $\xi \mapsto F_\xi$ is s''-admissible, with $s'' = k \cdot s'$.

Finally, we can prove Theorem 5 by following the proof of Theorem 4 in [17], and using the above observations to keep track of s'-admissibility of every operator and functional that enters the argument. We dispense with further details.

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