Interpolation by Linear Programming I

by

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Dedicated to Louis Nirenberg

Abstract: Given $m, n \geq 2$ and $\epsilon > 0$, we compute a function taking prescribed values at N given points of $\mathbb{R}^n$, and having C^m norm as small as possible up to a factor $1 + \epsilon$. Our computation reduces matters to a linear programming problem.

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Introduction

Fix $m, n \geq 2$. We study the following

INTERPOLATION PROBLEM: *Compute a function $F \in C^m(\mathbb{R}^n)$ taking prescribed values at N given points, with the C^m-norm of F as small as possible up to a given factor $1+\epsilon$.*

In [1], we showed how to reduce the computation of such an F to a linear programming problem of size $\exp(C/\epsilon) \cdot N$, where C depends only on m, n. Our goal, here and in [2], is to improve the dependence on ϵ.

To state our result precisely, we introduce some notation. Let $\mathcal{P}$ be the vector space of real polynomials of degree at most $m-1$ on $\mathbb{R}^n$. For a function $F \in C^m$ and a point $x \in \mathbb{R}^n$, we write $J_x(F)$ (the "jet" of F at x) to denote the $(m-1)^{\text{rst}}$-order Taylor polynomial of F at x. Thus, $J_x(F) \in \mathcal{P}$.

Let $E \subset \mathbb{R}^n$ be a finite set. A "Whitney field" on E is a family $\vec{P} = (P^x)_{x \in E}$ of polynomials $P^x \in \mathcal{P}$, indexed by the points of E. We write $Wh(E)$ to denote the vector space of all Whitney fields on E. If $E \subset E^+$ are finite sets, and if $\vec{P}^+ = (P_+^x)_{x \in E^+} \in Wh(E^+)$, then we define the "restriction" of $\vec{P}^+$ to E as $\vec{P}^+|_E = (P_+^x)_{x \in E}$.

Let $Q_0 \subset \mathbb{R}^n$ be a closed cube, with sidelength δ_{Q_0}, and with sides parallel to the coordinate axes. For $F \in C^m(Q_0)$, we define the C^m-seminorm of F to be $\| F \|_{C^m(Q_0)} = \sup_{x \in Q_0} \max_{|\alpha| = m} |\partial^\alpha F(x)|$.

If $F \in C^m(Q_0)$, and if $E \subset Q_0$ is finite, then we define $J_E(F) = (J_x(F))_{x \in E} \in Wh(E)$.

We write c, C, C', etc., to denote constants depending only on m, n. These symbols may denote different constants in different occurences.

To solve our INTERPOLATION PROBLEM, we first pose an easier problem, as in [1].

JET INTERPOLATION PROBLEM: *Let $\epsilon > 0$. Given a real number $M \geq 0$ and a Whitney field $\vec{P} \in Wh(E)$, decide whether there exists $F \in C^m(Q_0)$ such that $J_E(F) = \vec{P}$ and $\| F \|_{C^m(Q_0)} \leq (1 + O(\epsilon))M$. Compute such an F, if one exists.*

A variant of the case $m = 2$ of this problem has been solved in closed form; see LeGruyer [3].

To answer the JET INTERPOLATION PROBLEM for general m, n, we will construct a finite set $E^+ \subset Q_0$ and a convex polyhedron $K \subset \mathbb{R} \oplus Wh(E^+)$. The set E^+ and the polyhedron K are determined by m, n, E and ϵ. Our solution of the JET INTERPOLATION PROBLEM is as follows.

Theorem 1: *Let* $M \geq 0$, *and let* $\vec{P} \in Wh(E)$.

(A) *Suppose there exists* $F \in C^m(Q_0)$ *such that* $\| F \|_{C^m(Q_0)} \leq M$ *and* $J_E(F) = \vec{P}$.
Then there exists $\vec{P}^+ \in Wh(E^+)$ *such that* $(M, \vec{P}^+) \in K$ *and* $\vec{P}^+|_E = \vec{P}$.

(B) *Conversely, suppose there exists* $\vec{P}^+ \in Wh(E^+)$ *such that* $(M, \vec{P}^+) \in K$ *and* $\vec{P}^+|_E = \vec{P}$.
Then there exists $F \in C^m(Q_0)$ *such that* $\| F \|_{C^m(Q_0)} \leq (1+\epsilon)M$ *and* $J_E(F) = \vec{P}$.

Moreover, our methods are constructive. We exhibit E^+, and define K by an explicit family of linear constraints. The function F in conclusion (B) is defined in terms of $\vec{P}^+$ by a formula.

Thus, the JET INTERPOLATION PROBLEM reduces to a linear programming problem. To estimate the size of that linear programming problem, we bring in the distance between nearest neighbors in E.

Theorem 2: *Suppose the finite set* $E \subset Q_0$ *consists of* N *points. Let* $0 < \eta < 1/2$, *and assume that* $|x-y| \geq \eta\delta_{Q_0}$ *for any two distinct points* $x, y \in E$. *Then the set* E^+ *contains at most* $C|\ell n\, \eta|\epsilon^{-\frac{3}{2}n}N$ *points, and the convex polyhedron* $K \subset \mathbb{R} \oplus Wh(E^+)$ *is defined by at most* $C|\ell n\, \eta|\epsilon^{-\frac{3}{2}n}N$ *linear constraints.*

Returning to our original INTERPOLATION PROBLEM, we can easily read off the following result as a consequence of Theorem 1.

Theorem 3: *Let* $f : E \longrightarrow \mathbb{R}$, *and let* $M \geq 0$.

(A) *Suppose there exists* $F \in C^m(Q_0)$ *such that* $\| F \|_{C^m(Q_0)} \leq M$, *and* $F = f$ *on* E.
Then there exists $\vec{P}^+ = (P^x)_{x \in E^+} \in Wh(E^+)$, *such that* $(M, \vec{P}^+) \in K$ *and* $P^x(x) = f(x)$ *for all* $x \in E$.

(B) *Conversely, suppose there exists* $\vec{P}^+ = (P^x)_{x \in E^+} \in Wh(E^+)$, *such that* $(M, \vec{P}^+) \in K$ *and* $P^x(x) = f(x)$ *for all* $x \in E$.
Then there exists $F \in C^m(Q_0)$ *such that* $\| F \|_{C^m(Q_0)} \leq (1+\epsilon)M$, *and* $F = f$ *on* E.

Again, the function F in conclusion (B) is given in terms of $\vec{P}^+$ by an explicit formula.

In view of Theorems 2 and 3, our INTERPOLATION PROBLEM reduces to a linear programming problem of size at most $C|\ell n\, \eta|\epsilon^{-\frac{3}{2}n}N$. Recall that the corresponding linear programming problem in [1] has size $\exp\,(C/\epsilon) \cdot N$.

Thus, we have tamed the exponential behavior in ϵ, at the cost of introducing the factor $|\ell n\, \eta|$. We believe we can sharpen Theorem 2 by using a more sophisticted choice of

E^+ and K in Theorems 1 and 3. More precisely, we believe that the size of the relevant linear programming problem can be cut down to $C\epsilon^{-(\frac{3}{2}n+1)}N$. We hope to address this point in a later paper [2]. We are grateful to Arianna Valdevit for interesting conversations.

It is a pleasure to dedicate this paper to Louis Nirenberg, and to thank Gerree Pecht for expertly LaTeX-ing my manuscript.

The remainder of this paper gives the proof of Theorems 1 and 2.

Squares and Cubes

We use the norm $|(x_1, \ldots, x_n)| = \max\{|x_1|, \ldots, |x_n|\}$ on $\mathbb{R}^n$.

A *cube* will be a Cartesian product $Q = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ of closed non-degenerate intervals $I_1, I_2, \ldots, I_n$ such that $|I_1| = |I_2| = \cdots = |I_n| < \infty$. If $I_1, \ldots, I_n$ here are dyadic intervals, then we call Q a *dyadic* cube. For any cube $Q = I_1 \times \cdots \times I_n$, we write $\delta_Q := |I_1| = \cdots = |I_n|$ to denote the *sidelength* of Q.

Any given cube Q can be written in an obvious way as a union of 2^n subcubes of sidelength $\frac{1}{2}\delta_Q$. These subcubes will be called the *children* of Q; they arise by *bisecting* Q.

Similarly, a *square* in $\mathbb{R}^n$ is a Cartesian product $\bar{Q} = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$, where, for some $j\,(1 \leq j \leq n)$, the following hold.

- $I_j = \{\xi\}$ consists of a single point.
- Each $I_i (i \neq j)$ is a non-degenerate compact interval.
- All the intervals $I_i (i \neq j)$ have the same length.

We write $\delta_{\bar{Q}} := |I_i|$ (all $i \neq j$) to denote the *sidelength* of a given square $\bar{Q}$. Any square $\bar{Q}$ may be written in an obvious way as the union of 2^{n-1} subsquares of sidelength $\frac{1}{2}\delta_{\bar{Q}}$. We call these subsquares the *children* of $\bar{Q}$, and we say that they arise by *bisecting* $\bar{Q}$.

If Q is a cube, and if $A \geq 1$ is a real number, then we write AQ for the cube obtained by dilating A about its center by a factor of A.

Thus, Q and AQ have the same center, and $\delta_{AQ} = A\delta_Q$.

The boundary of a cube $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ may be written as the union of the $2n$ *faces* $\partial Q(j, \sigma)$ $(1 \leq j \leq n, \sigma = \pm 1)$, where:

- $\partial Q(j, 1) = [a_1, b_1] \times \cdots \times [a_{j-1}, b_{j-1}] \times \{b_j\} \times [a_{j+1}, b_{j+1}] \times \cdots \times [a_n, b_n]$

and

- $\partial Q(j,-1) = [a_1,b_1] \times \cdots \times [a_{j-1},b_{j-1}] \times \{a_j\} \times [a_{j+1},b_{j+1}] \times \cdots \times [a_n,b_n]$.

(We make obvious modifications for $j=1$ and for $j=n$.)

Let Q, Q' be cubes. We say that Q, Q' *abut* if $Q \cap Q'$ is a square. If Q, Q' abut, and if $\bar{Q}$ is a face of Q or Q', then we say that Q, Q' *abut along* $\bar{Q}$, provided $Q \cap Q' \subseteq \bar{Q}$.

We note two elementary properties of cubes.

- Let $\hat{Q} \subset Q_0$ be cubes. Assume that $\delta_{\hat{Q}} \leq \frac{1}{4}\delta_{Q_0}$. Then there exists a vector $\xi_{\hat{Q}} \in \mathbb{R}^n$, such that $|\xi_{\hat{Q}}| \leq 1$, and the following holds:

  Let $x \in Q_0 \cap 2\hat{Q}$, and suppose $|\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}| < 1$ for some $0 < \lambda \leq \frac{1}{4}$. Then y lies in the interior of Q_0.

- Let Q, Q' be cubes. If $Q \cap Q' \neq \phi$ and $\delta_{Q'} \leq \frac{1}{2}\delta_Q$, then $3Q' \subseteq 3Q$.

  (Recall: By our convention, Q and Q' are closed cubes.)

Any dyadic cube Q is one of the children of Q' for precisely one dyadic cube Q'. We call that Q' the *parent* of Q, and we denote it by Q^+. Note that if Q, Q_0 are dyadic cubes and $Q \underset{\neq}{\subseteq} Q_0$, then $Q^+ \subseteq Q_0$.

Calderón-Zygmund Cubes and Their Plaques

Let Q_0 be a dyadic cube in $\mathbb{R}^n$. A *Calderón-Zygmund decomposition* of Q_0 is a finite collection CZ of dyadic subcubes of Q_0, with the following properties.

- The cubes in CZ have pairwise disjoint interiors.
- The cubes in CZ cover Q_0.
- ("Good Geometry") If $Q, Q' \in CZ$ intersect, then $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$.

Note that any given point of Q_0 belongs to $2Q$ for at most C distinct $Q \in CZ$.

Let $Q \in CZ$, and let $\bar{Q} = \partial Q(j,\sigma)$ be a face of Q. We ask: Which $Q' \in CZ$ abut Q along $\bar{Q}$? There are three cases.

Case 1: If $\bar{Q} \subset \partial Q_0(j,\sigma)$, then no cube in CZ abuts Q along $\bar{Q}$.

Case 2: If $\bar{Q} \not\subset \partial Q_0(j,\sigma)$, and if there exists $Q' \in CZ$ such that $\partial Q'(j,-\sigma) \supseteq \bar{Q}$, then that Q' is the one and only cube in CZ that abuts Q along $\bar{Q}$.

Case 3: If we are not in Case 1 or Case 2, then there exist 2^{n-1} distinct cubes $Q'_1, \dots, Q'_{2^{n-1}} \in CZ$ such that $\partial Q'_i(j,-\sigma) \subset \bar{Q}$ for each i.

These cubes Q'_i are the only CZ cubes that abut Q along $\bar{Q}$.

The faces $\partial Q'_i(j,-\sigma)$ $(i = 1, \dots, 2^{n-1})$ are the 2^{n-1} children of $\bar{Q}$.

The proofs of the above assertions will be left to the reader.

Next, for each $Q \in CZ$, and for each face $\partial Q(j,\sigma)$ of Q, we define a family of squares called the family of *plaques* $\Pi(Q,j,\sigma)$. If Q and $\partial Q(j,\sigma)$ fall into Case 1 or Case 2 above, then we define $\Pi(Q,j,\sigma)$ to consist of the single square $\partial Q(j,\sigma)$. However, if Q and $\partial Q(j,\sigma)$ fall into Case 3, then we define $\Pi(Q,j,\sigma)$ to consist of the 2^{n-1} children of $\partial Q(j,\sigma)$. Thus in all cases, $\Pi(Q,j,\sigma)$ is a finite collection of squares, whose union is the face $\partial Q(j,\sigma)$.

For each $j (1 \le j \le n)$, we define $\Pi(j) = \bigcup_{\substack{Q \in CZ \\ \sigma = \pm 1}} \Pi(Q,j,\sigma)$. Also, we define $\Pi = \bigcup_{j=1}^{n} \Pi(j)$. Any $\bar{Q} \in \Pi$ will be called a *plaque*. Given a plaque $\bar{Q} \in \Pi(j)$ we ask: For which $Q \in CZ$ and $\sigma = \pm 1$ do we have $\bar{Q} \in \Pi(Q,j,\sigma)$? The answer is as follows.

Suppose $\bar{Q} \subset \partial Q_0(j,\sigma)$. Then there exists one and only one $Q \in CZ$ such that $\bar{Q} \in \Pi(Q,j,\sigma)$; and there exists no $Q' \in CZ$ such that $\bar{Q} \in \Pi(Q',j,-\sigma)$. In this case, we define $\bar{Q}^{-\sigma} :=$ the one and only $Q \in CZ$ such that $\bar{Q} \in \Pi(Q,j,\sigma)$; and we define $\bar{Q}^{\sigma} :=$ Null.

Suppose $\bar{Q} \not\subset \partial Q_0(j,\sigma)$, both for $\sigma = 1$ and for $\sigma = -1$. Then $\bar{Q} \in \Pi(Q,j,1)$ for one and only one $Q \in CZ$; and $\bar{Q} \in \Pi(Q',j,-1)$ for one and only one $Q' \in CZ$. For the above Q, Q', we then define $\bar{Q}^1 := Q'$ and $\bar{Q}^{-1} := Q$.

The proofs of the above assertions will be left to the reader. In all cases, we thus have the following. Let $Q \in CZ$, and let $\bar{Q} \in \Pi(j)$. Then, for $\sigma = \pm 1$, we have $\bar{Q} \in \Pi(Q,j,\sigma)$ if and only if $Q = \bar{Q}^{-\sigma}$. For each $\bar{Q} \in \Pi$ and $\sigma = \pm 1$, we have $\bar{Q}^{\sigma} =$ Null or $\bar{Q}^{\sigma} \in CZ$.

Integration by Parts with Plaques

Let $r \ge 1$, and let θ, P be C^r functions on a cube Q. For $j_1, \dots, j_r \in \{1, \dots, n\}$, and for $1 \le s \le r$, integration by parts gives

$$(-1)^{s-1} \int_Q (\partial_{j_s} \cdots \partial_{j_r} \theta) \cdot (\partial_{j_1} \cdots \partial_{j_{s-1}} P) dx =$$

$$(-1)^s \int_Q (\partial_{j_{s+1}} \cdots \partial_{j_r} \theta) \cdot (\partial_{j_1} \cdots \partial_{j_s} P) dx$$

$$+(-1)^{s-1} \sum_{\sigma=\pm 1} \sigma \int_{\partial Q(j_s,\sigma)} (\partial_{j_{s+1}} \cdots \partial_{j_r} \theta) \cdot (\partial_{j_1} \cdots \partial_{j_{s-1}} P)\, dx\,.$$

It follows that

$$\int_Q (\partial_{j_1} \cdots \partial_{j_r} \theta) \cdot P\, dx =$$

$$(-1)^r \int_Q \theta \cdot (\partial_{j_1} \cdots \partial_{j_r} P)\, dx + \sum_{s=1}^{r} \sum_{\sigma=\pm 1} (-1)^{s-1} \sigma \int_{\partial Q(j_s,\sigma)} (\partial_{j_{s+1}} \cdots \partial_{j_r} \theta) \cdot (\partial_{j_1} \cdots \partial_{j_{s-1}} P)\, dx$$

$$= (-1)^r \int_Q \theta \cdot (\partial_{j_1} \cdots \partial_{j_r} P)\, dx + \sum_{s=1}^{r} \sum_{\sigma=\pm 1} (-1)^{s-1} \sigma \sum_{\bar{Q} \in \Pi(Q, j_s, \sigma)} \int_{\bar{Q}} (\partial_{j_{s+1}} \cdots \partial_{j_r} \theta) \cdot (\partial_{j_1} \cdots \partial_{j_{s-1}} P)\, dx\,.$$

(Here, we assume that Q is a cube from a Calderón-Zygmund decomposition.)

Now let CZ be a Calderón-Zygmund decomposition of a dyadic cube Q_0. For each $Q \in CZ$, suppose we are given a function $P_Q \in C^r(Q)$. Let θ be a C^r function on Q_0. For each $Q \in CZ$, we apply the above formula, with P_Q in place of P. We then sum over all $Q \in CZ$. Recalling that, for a given $\bar{Q} \in \Pi(j), Q \in CZ$, we have $\bar{Q} \in \Pi(Q, j, \sigma)$ if and only if $Q = \bar{Q}^{-\sigma}$, we derive the following formula:

$$\sum_{Q \in CZ} \int_Q (\partial_{j_1} \cdots \partial_{j_r} \theta) \cdot P_Q\, dx =$$

$$(-1)^r \sum_{Q \in CZ} \int_Q \theta \cdot (\partial_{j_1} \cdots \partial_{j_r} P_Q)\, dx$$

$$+ \sum_{1 \le s \le r} (-1)^s \sum_{\bar{Q} \in \Pi(j_s)} \int_{\bar{Q}} (\partial_{j_{s+1}} \cdots \partial_{j_r} \theta) \cdot (\partial_{j_1} \cdots \partial_{j_{s-1}} [P_{\bar{Q}^1} - P_{\bar{Q}^{-1}}])\, dx,$$

where we define $P_{\bar{Q}^\sigma} := 0$ in case $\bar{Q}^\sigma = \mathsf{Null}$.

It is convenient to rewrite the above formula using multi-indices. We have

$$\sum_{Q \in CZ} \int_Q (\partial^\alpha \theta) \cdot P_Q \, dx \;=\; (-1)^{|\alpha|} \sum_{Q \in CZ} \int_Q \theta \cdot (\partial^\alpha P) \, dx$$

$$+ \sum_{\substack{\alpha' + \alpha'' + \gamma = \alpha \\ |\gamma| = 1}} \sum_{\bar{Q} \in \Pi} c(\alpha', \alpha'', \gamma, \bar{Q}) \int_{\bar{Q}} (\partial^{\alpha'} \theta) \cdot (\partial^{\alpha''} [P_{\bar{Q}^1} - P_{\bar{Q}^{-1}}]) \, dx \,,$$

where each $c(\alpha', \alpha'', \gamma, \bar{Q})$ depends only on $\alpha', \alpha'', \gamma$, and the j for which $\bar{Q} \in \Pi(j)$.

This is our basic integration-by-parts formula for plaques. It holds whenever $\theta \in C^{|\alpha|}(Q_0)$ and $P_Q \in C^{|\alpha|}(Q)$ for each $Q \in CZ$.

Two Particular Calderón-Zygmund Decompositions

Suppose we are given two real numbers $0 < \tau < \lambda \leq 1/8$, a dyadic cube Q_0 in $\mathbb{R}^n$, and a set $E \subset Q_0$. Let $N = \#(E)$.

We assume that $2 \leq N < \infty$. For a given real number $\eta > 0$, we assume that

- $|x' - x''| \geq \eta \delta_{Q_0}$ for any two distinct points $x', x'' \in E$.

We will define two Calderón-Zygmund decompositions of Q_0.

- CZ consists of the maximal dyadic subcubes $\hat{Q} \subset Q_0$ such that $\#(E \cap 3\hat{Q}) \leq 1$.
- $CZ(\tau)$ consists of the maximal dyadic subcubes $Q \subset Q_0$ such that $\#(E \cap 3\tau^{-1} Q) \leq 1$.

It is easy to check that CZ and $CZ(\tau)$ are Calderón-Zygmund decompositions of Q_0, and that $CZ(\tau)$ refines CZ. (That is, any $\hat{Q} \in CZ$ is a union of $Q \in CZ(\tau)$.) Moreover, whenever $Q \in CZ(\tau)$, $\hat{Q} \in CZ$, and $Q \subset \hat{Q}$, we then have $c\tau\delta_{\hat{Q}} \leq \delta_Q \leq C\tau\delta_{\hat{Q}}$.

For $\hat{Q} = Q_0$ or a child of Q_0, we have $Q_0 \subset 3\hat{Q}$, hence $\#(E \cap 3\hat{Q}) \geq 2$. Consequently, $\delta_{\hat{Q}} \leq \frac{1}{4}\delta_{Q_0}$ for each $\hat{Q} \in CZ$. Hence, for each $\hat{Q} \in CZ$, there exists $\xi_{\hat{Q}} \in \mathbb{R}^n$ such that $|\xi_{\hat{Q}}| \leq 1$; and if $x \in 2\hat{Q} \cap Q_0$, $y \in \mathbb{R}^n$ and $|\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}}| < 1$, then y lies in the interior of Q_0. We fix such a $\xi_{\hat{Q}}$ for each $\hat{Q} \in CZ$.

We establish a useful property of CZ and $CZ(\tau)$.

Suppose $\hat{Q} \in CZ$ and $Q \in CZ(\tau)$. Let $x \in 2\hat{Q} \cap Q_0$ and let $y \in Q$.

If $|\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}}| \leq 1$, then $c\tau\delta_{\hat{Q}} \leq \delta_Q \leq C\tau\delta_{\hat{Q}}$.

To see this, we first make a preliminary observation.

Let $\hat{Q} \in CZ$. Then $2\hat{Q} \cap Q_0$ is covered by $\hat{Q}$, together with finitely many dyadic cubes $Q' \subset Q_0$ of sidelength $\frac{1}{2}\delta_{\hat{Q}}$, that meet $\hat{Q}$. Each such Q' satisfies $3Q' \subset 3\hat{Q}$, hence $\#(E \cap 3Q') \leq \#(E \cap 3\hat{Q}) \leq 1$ since $\hat{Q} \in CZ$. Therefore, each such Q' is contained in a cube $\hat{Q}' \in CZ$ that meets $\hat{Q}$. Thus, we have established the following:

$$(*) \quad \left[\begin{array}{l} \text{Let } \hat{Q} \in CZ, \text{ and let } x \in 2\hat{Q} \cap Q_0\,. \text{ Then there exists } \hat{Q}' \in CZ \\ \text{such that } x \in \hat{Q}' \text{ and } \hat{Q}' \text{ meets } \hat{Q}\,. \end{array} \right.$$

Now let $\hat{Q} \in CZ$, $Q \in CZ(\tau)$, $x \in 2\hat{Q} \cap Q_0$, $y \in Q$; and assume that $|\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}| \leq 1$. Applying $(*)$, we find $\hat{Q}_x \ni x$ such that $\hat{Q}_x \in CZ$ and $\hat{Q}_x \cap \hat{Q} \neq \phi$. By the Good Geometry of CZ, we have $\frac{1}{2}\delta_{\hat{Q}} \leq \delta_{\hat{Q}_x} \leq 2\delta_{\hat{Q}}$. We then have $|\frac{x-y}{\lambda\delta_{\hat{Q}}}| \leq 2$ (since $|\xi_{\hat{Q}}| \leq 1$), hence $|x-y| \leq 2\lambda\delta_{\hat{Q}} \leq 4\lambda\delta_{\hat{Q}_x} \leq \frac{1}{2}\delta_{\hat{Q}_x}$ (since $\lambda \leq \frac{1}{8}$). From $x \in \hat{Q}_x$ and $|x-y| \leq \frac{1}{2}\delta_{\hat{Q}_x}$, we conclude that $y \in 2\hat{Q}_x$.

Also, $y \in Q \subset Q_0$. Therefore, $(*)$ applies to the cube $\hat{Q}_x$ and the point y. Hence, there exists $\hat{Q}'_y \in CZ$ such that $y \in \hat{Q}'_y$ and $\hat{Q}'_y \cap \hat{Q}_x \neq \phi$.

Fix a cube $\hat{Q}_y \in CZ$ that contains Q. In particular, $\hat{Q}_y$ contains y. Since $y \in \hat{Q}_y \cap \hat{Q}'_y$, we have $\frac{1}{2}\delta_{\hat{Q}'_y} \leq \delta_{\hat{Q}_y} \leq 2\delta_{\hat{Q}'_y}$. Similarly, since $\hat{Q}'_y$ meets $\hat{Q}_x$, we have $\frac{1}{2}\delta_{\hat{Q}_x} \leq \delta_{\hat{Q}'_y} \leq 2\delta_{\hat{Q}_x}$.

Since $\frac{1}{2}\delta_{\hat{Q}} \leq \delta_{\hat{Q}_x} \leq 2\delta_{\hat{Q}}$, $\frac{1}{2}\delta_{\hat{Q}_x} \leq \delta_{\hat{Q}'_y} \leq 2\delta_{\hat{Q}_x}$, and $\frac{1}{2}\delta_{\hat{Q}'_y} \leq \delta_{\hat{Q}_y} \leq 2\delta_{\hat{Q}'_y}$, it follows that $\frac{1}{8}\delta_{\hat{Q}} \leq \delta_{\hat{Q}_y} \leq 8\delta_{\hat{Q}}$. Recalling that $c\tau\delta_{\hat{Q}_y} \leq \delta_Q \leq C\tau\delta_{\hat{Q}_y}$, we conclude that $c\tau\delta_{\hat{Q}} \leq \delta_Q \leq C\tau\delta_{\hat{Q}}$, as claimed.

Let us estimate the number of cubes $\hat{Q} \in CZ$ and $Q \in CZ(\tau)$.

For each $x \in E$, let $d(x) = \min\{|x - x'| : x' \in E, x' \neq x\}$. We consider now the set X of all pairs $(\hat{Q}, x)$, where $\hat{Q} \in CZ$, $x \in E \cap 3\hat{Q}^+$ and $\#(E \cap 3\hat{Q}^+) \geq 2$. For each $\hat{Q} \in CZ$, we have $(\hat{Q}, x) \in X$ for some x. Hence, $\#(CZ) \leq \#(X)$. On the other hand, for fixed $x \in E$, any $\hat{Q}$ such that $(\hat{Q}, x) \in X$ satisfies $x \in 3\hat{Q}^+$ and $\delta_{\hat{Q}} \geq cd(x)$. There are at most $C\log(\delta_{Q_0}/d(x))$ such dyadic cubes $\hat{Q} \subseteq Q_0$ for any given x. Consequently,

$\#(X) \leq \sum_{x \in E} C\log(\delta_{Q_0}/d(x))$, and thus,

$$\#(CZ) \leq \sum_{x\in E} C\log(\delta_{Q_0}/d(x))\,.$$

Moreover, $CZ(\tau)$ refines CZ, and we know that $Q \in CZ(\tau)$, $\hat{Q} \in CZ$, $Q \subset \hat{Q}$ imply $c\tau\delta_{\hat{Q}} \leq \delta_Q \leq C\tau\delta_{\hat{Q}}$.

Consequently,

$$\#(CZ(\tau)) \leq C\tau^{-n} \cdot \#(CZ) \leq C\tau^{-n}\sum_{x\in E}\log(\delta_{Q_0}/d(x))\,.$$

Recall that $\#(E) = N$, and that $|x - x'| \geq \eta\delta_{Q_0}$ for any two distinct $x, x' \in E$. Therefore, $d(x) \geq \eta\delta_{Q_0}$ for $x \in E$, so that our previous estimates for $\#(CZ)$ and $\#(CZ(\tau))$ yield the following,

There are at most $C|\ln\eta|N$ distinct cubes $\hat{Q} \in CZ$.

There are at most $C|\ln\eta|\tau^{-n}N$ distinct cubes $Q \in CZ(\tau)$.

In the sections to follow, when we discuss plaques, we will always be referring to the plaques arising from the Calderón-Zygmund decomposition $CZ(\tau)$. We will never make use of plaques arising from the coarser decomposition CZ.

Auxiliary Functions, Cubes, Points

We retain the assumptions of the preceding section.

We fix functions $\theta \in C^{m+2}(\mathbb{R}^n)$, $\varphi \in C^m(\mathbb{R}^n)$, with the following properties: $\theta \geq 0$ on $\mathbb{R}^n$; $\mathrm{supp}\,\theta \subset\subset \{|x| < 1\}$; $|\partial^\alpha\theta| \leq C$ on $\mathbb{R}^n$ for $|\alpha| \leq m+2$; $\int_{\mathbb{R}^n}\theta(x)dx = 1$; $\partial^\alpha(\varphi - 1)(0) = 0$ for $|\alpha| \leq m-1$; $|\partial^\alpha\varphi| \leq C$ on $\mathbb{R}^n$ for $|\alpha| \leq m$; $\mathrm{supp}\,\varphi \subset \{|x| \leq 1\}$. For each $\hat{Q} \in CZ$, we fix a cutoff function $\tilde{\chi}_{\hat{Q}} \in C^{m+2}(\mathbb{R}^n)$, such that $\tilde{\chi}_{\hat{Q}} \geq 0$ on $\mathbb{R}^n$; $\mathrm{supp}\,\tilde{\chi}_{\hat{Q}} \subset 2\hat{Q}$; $\tilde{\chi}_{\hat{Q}} \geq c$ on $\hat{Q}$; $|\partial^\alpha\tilde{\chi}_{\hat{Q}}| \leq C\delta_{\hat{Q}}^{-|\alpha|}$ on $\mathbb{R}^n$, for $|\alpha| \leq m+2$.

We then define $\chi_{\hat{Q}} = \tilde{\chi}_{\hat{Q}} \Big/ \sum_{\hat{Q}'\in CZ}\tilde{\chi}_{\hat{Q}'}$ on Q_0.

Note that $\chi_{\hat{Q}}$ is defined only on Q_0. The functions $\chi_{\hat{Q}}$ have the following properties:

$$\chi_{\hat{Q}} \geq 0 \text{ on } Q_0\,;\ |\partial^\alpha\chi_{\hat{Q}}| \leq C\delta_{\hat{Q}}^{-|\alpha|} \text{ on } Q_0\,,\ \text{ for } |\alpha| \leq m+2\,;$$

$$\chi_{\hat{Q}} = 0 \text{ on } Q_0 \setminus 2\hat{Q}\,;\ \sum_{\hat{Q}\in CZ}\chi_{\hat{Q}} = 1\,.$$

Also, any given point of Q_0 lies in $\mathsf{supp}\chi_{\hat{Q}}$ for at most C distinct $\hat{Q} \in CZ$.

For each $Q \in CZ(\tau)$, let x_Q be the center of Q.

For each $\hat{Q} \in CZ$, we pick $Q_{\mathsf{rep}}(\hat{Q}) \in CZ(\tau)$ contained in $\hat{Q}$.

If $\hat{Q}$ contains a point of E, then we take $Q_{\mathsf{rep}}(\hat{Q})$ to contain a point of E. Note that $\hat{Q}$ contains at most one point of E, since $\hat{Q} \in CZ$. We define $x_{\hat{Q}} = x_{Q_{\mathsf{rep}}(\hat{Q})}$ for $\hat{Q} \in CZ$. Thus, $x_{\hat{Q}} \in \hat{Q}$. For each plaque $\bar{Q} \in \Pi$, we write $x_{\bar{Q}}$ to denote the center of $\bar{Q}$.

From Functions to Whitney Fields

We adopt the assumptions of the last two sections. We assume also that we are given a function $F \in C^m(Q_0)$ and a positive real number M such that $|\partial^\alpha F(x)| \leq M$ for $|\alpha| = m$, $x \in Q_0$.

For each $Q \in CZ(\tau)$, we define $P_Q := J_{x_Q}(F)$.

For each $\hat{Q} \in CZ$, we define $P_{\hat{Q}} := P_{Q_{\mathsf{rep}}(Q)} = J_{x_{\hat{Q}}}(F)$.

For each $x \in E$, we define $P^x := J_x(F)$.

Thus, the $P_{\hat{Q}}(\hat{Q} \in CZ)$ are simply some of the $P_Q(Q \in CZ(\tau))$. We will derive a family of linear constraints satisfied by the P_Q, P^x, M. We begin with trivial constraints. From Taylor's theorem, we have the following.

(I) Let $\bar{Q}$ be a plaque such that $\bar{Q}^1$ and $\bar{Q}^{-1}$ are both non-null. Then $|\partial^\alpha(P_{\bar{Q}^1} - P_{\bar{Q}^{-1}})(x_{\bar{Q}})| \leq CM\delta_{\bar{Q}}^{m-|\alpha|}$ for $|\alpha| \leq m-1$.

(II) Let $\hat{Q} \in CZ$, $Q \in CZ(\tau)$, and suppose there exist $x \in 2\hat{Q} \cap Q_0$, $y \in Q$ such that $|\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}| < 1$.

Then

$|\partial^\alpha(P_Q - P_{\hat{Q}})(x_Q)| \leq CM\delta_{\hat{Q}}^{m-|\alpha|}$ for $|\alpha| \leq m-1$.

(III) Let $\hat{Q}, \hat{Q}' \in CZ$. If $2\hat{Q} \cap \hat{Q}' \neq \emptyset$, then

$|\partial^\alpha(P_{\hat{Q}} - P_{\hat{Q}'})(x_{\hat{Q}})| \leq CM\delta_{\hat{Q}}^{m-|\alpha|}$ for $|\alpha| \leq m-1$.

To derive (II) from Taylor's theorem, we recall that under the assumptions of (II), we have $c\tau\delta_{\hat{Q}} \leq \delta_Q \leq C\tau\delta_{\hat{Q}}$. Since also the distance from $2\hat{Q}$ to Q is at most $C\delta_{\hat{Q}}$ in (II), it follows that $|x_Q - x_{\hat{Q}}| \leq C\delta_{\hat{Q}}$. Similarly, to derive (III) from Taylor's theorem, we recall that, under the assumptions of (III), we have $c\delta_{\hat{Q}} \le \delta_{\hat{Q}'} \le C\delta_{\hat{Q}}$. Since also $2\hat{Q} \cap \hat{Q}' \ne \phi$, it follows that $|x_{\hat{Q}} - x_{\hat{Q}'}| \le C\delta_{\hat{Q}}$.

The goal of the next few pages is to derive less trivial constraints on the P_Q, P^x and M. To do so, we first study the functions

$$G_{\hat{Q}}(x) := P_{\hat{Q}}(x) + \int_{\mathbb{R}^n} (\lambda\delta_{\hat{Q}})^{-n}\theta\left(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}\right)[F(y) - P_{\hat{Q}}(y)]dy \text{ for } x \in 2\hat{Q} \cap Q_0\,, \hat{Q} \in CZ\,;$$

and

$$G(x) = \sum_{\hat{Q} \in CZ} \chi_{\hat{Q}}(x) G_{\hat{Q}}(x) \text{ for } x \in Q_0\,.$$

Here, $G_{\hat{Q}}$ is defined on $2\hat{Q} \cap Q_0$, while G is defined on Q_0. Indeed, we have seen that, for $x \in 2\hat{Q} \cap Q_0$, the function $y \mapsto \theta(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}})$ is supported in the interior of Q_0; hence, the integral defining $G_{\hat{Q}}(x)$ makes sense. Moreover, $G(x)$ makes sense for $x \in Q_0$, since $\chi_{\hat{Q}} = 0$ on $Q_0 \setminus 2\hat{Q}$. Thus, $G_{\hat{Q}}$ and G are indeed defined where we said they are defined.

We will derive the basic properties of $G_{\hat{Q}}$ and G.

For $x \in 2\hat{Q} \cap Q_0$, $|\alpha| = m$, we have $\partial^\alpha G_{\hat{Q}}(x) = \int_{\mathbb{R}^n} \theta(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}) \cdot \partial^\alpha F(y) dy$, since m^{th} derivatives of polynomials of degree at most $(m-1)$ vanish. Hence, our assumptions on θ and F imply the estimate

$$|\partial^\alpha G_{\hat{Q}}(x)| \le M \text{ for } |\alpha| = m\,,\ x \in 2\hat{Q} \cap Q_0\,.$$

Next, suppose $|\alpha| \le m-1$, $\hat{Q} \in CZ$, $x \in 2\hat{Q} \cap Q_0$.

Fix $Q \in CZ(\tau)$ such that $x \in Q$. Then

$$\partial^\alpha G_{\hat{Q}}(x) = \partial^\alpha P_{\hat{Q}}(x) + \int_{\mathbb{R}^n} (\lambda\delta_{\hat{Q}})^{-n}\theta\left(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}\right) \cdot [\partial^\alpha F(y) - \partial^\alpha P_{\hat{Q}}(y)]dy$$

$$= \partial^\alpha P_Q(x) + \int_{\mathbb{R}^n} (\lambda\delta_{\hat{Q}})^{-n}\,\theta(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}) \cdot [\partial^\alpha F(y) - \partial^\alpha P_Q(y)]dy$$

$$+ \int_{\mathbb{R}^n} (\lambda\delta_{\hat{Q}})^{-n}\,\theta(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}) \cdot [\partial^\alpha(P_Q - P_{\hat{Q}})(y) - \partial^\alpha(P_Q - P_{\hat{Q}})(x)]dy\,.$$

We estimate the two integrals on the right-hand side.

For y such that $\theta(\frac{x-y}{\lambda\delta_{\hat{Q}}}-\xi_{\hat{Q}}) \neq 0$, we have $|\frac{x-y}{\lambda\delta_{\hat{Q}}}-\xi_{\hat{Q}}| < 1$, hence $|\frac{x-y}{\lambda\delta_{\hat{Q}}}| \leq 2$, i.e., $|x-y| \leq 2\lambda\delta_{\hat{Q}}$. Also, for such y, and for $Q \in CZ(\tau)$ containing x, we have $c\tau\delta_{\hat{Q}} \leq \delta_Q \leq C\tau\delta_{\hat{Q}}$; since $x, x_Q \in Q$, we conclude that $|x - x_Q| \leq C\tau\delta_{\hat{Q}}$, hence $|y - x_Q| \leq C\tau\delta_{\hat{Q}} + 2\lambda\delta_{\hat{Q}} \leq C\lambda\delta_{\hat{Q}}$.

Taylor's theorem therefore yields $|\partial^\alpha F(y) - \partial^\alpha P_Q(y)| \leq CM(\lambda\delta_{\hat{Q}})^{m-|\alpha|}$ for $|\alpha| \leq m-1$. Consequently, for $x \in 2\hat{Q} \cap Q_0$ and $|\alpha| \leq m-1$, we have:

$$\left| \int_{\mathbb{R}^n} (\lambda\delta_{\hat{Q}})^{-n}\, \theta\left(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}\right) [\partial^\alpha F(y) - \partial^\alpha P_Q(y)] dy \right| \leq CM(\lambda\delta_{\hat{Q}})^{m-|\alpha|} .$$

Thus, we have estimated the first integral on right-hand side of our formula for $\partial^\alpha G_{\hat{Q}}$. We turn to the second integral. For $\theta(\frac{x-y}{\lambda\delta_{\hat{Q}}}-\xi_{\hat{Q}}) \neq 0$, we have $|x-y| \leq 2\lambda\delta_{\hat{Q}}$ as before; and $y \in Q_0$. We know also that $x, x_{\hat{Q}} \in 2\hat{Q}$, hence $|x_{\hat{Q}} - x| \leq C\delta_{\hat{Q}}$, which implies that $|x_{\hat{Q}} - y| \leq C\delta_{\hat{Q}}$.

Similarly, $x, x_Q \in Q$. Since $c\tau\delta_{\hat{Q}} \leq \delta_Q \leq C\tau\delta_{\hat{Q}}$, it follows that $|x - x_Q| \leq C\tau\delta_{\hat{Q}}$. We conclude that the distance between any two of $x, y, x_{\hat{Q}}, x_Q$ is at most $C\delta_{\hat{Q}}$. In particular, Taylor's theorem tells us that $|\partial^\beta(P_Q - P_{\hat{Q}})(x_Q)| \leq CM\delta_{\hat{Q}}^{m-|\beta|}$ for $|\beta| \leq m$; hence $|\partial^\beta(P_Q - P_{\hat{Q}})| \leq CM\delta_{\hat{Q}}^{m-|\beta|}$ everywhere on the line segment joining x to y. In particular, for $|\alpha| \leq m-1$, we have $|\nabla\partial^\alpha(P_Q - P_{\hat{Q}})| \leq CM\delta_{\hat{Q}}^{m-|\alpha|-1}$ everywhere on that line segment. Hence, $|\partial^\alpha(P_Q - P_{\hat{Q}})(y) - \partial^\alpha(P_Q - P_{\hat{Q}})(x)| \leq CM\delta_{\hat{Q}}^{m-|\alpha|-1}|x-y|$ for x, y as above. Therefore,

$$\left| \int_{\mathbb{R}^n} (\lambda\delta_{\hat{Q}})^{-n}\, \theta\left(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}\right) [\partial^\alpha(P_Q - P_{\hat{Q}})(y) - \partial^\alpha(P_Q - P_{\hat{Q}})(x)] dy \right|$$

$$\leq \int_{\mathbb{R}^n} (\lambda\delta_{\hat{Q}})^{-n}\, \theta\left(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}\right) \cdot CM\delta_{\hat{Q}}^{m-|\alpha|-1}|x-y| dy \leq C'\lambda M\delta_{\hat{Q}}^{m-|\alpha|} .$$

Thus, we have estimated both integrals on the right-hand side of our formula for $\partial^\alpha G_{\hat{Q}}(x)$. We now conclude from that formula that

$$|\partial^\alpha G_{\hat{Q}}(x) - \partial^\alpha P_Q(x)| \leq C\lambda M\delta_{\hat{Q}}^{m-|\alpha|} \quad \text{for} \quad |\alpha| \leq m-1,\, x \in 2\hat{Q} \cap Q_0,\, \hat{Q} \in CZ,$$

$$Q \in CZ(\tau), Q \ni x .$$

On the other hand, since $x, x_Q \in Q$, $Q \in CZ(\tau)$, $x \in 2\hat{Q} \cap Q_0$, $\hat{Q} \in CZ$, we know also that $|x - x_Q| \leq C\delta_Q \leq C\tau\delta_{\hat{Q}}$; hence, Taylor's theorem gives $|\partial^\alpha F(x) - \partial^\alpha P_Q(x)| \leq$

$CM(\tau\delta_{\hat{Q}})^{m-|\alpha|} \le C\lambda M\delta_{\hat{Q}}^{m-|\alpha|}$ for $\alpha, x, Q, \hat{Q}$ as above. We therefore have

$$|\partial^\alpha(G_{\hat{Q}} - F)(x)| \le C\lambda M\delta_{\hat{Q}}^{m-|\alpha|} \quad \text{for } \ x \in 2\hat{Q} \cap Q_0\,,\ \hat{Q} \in CZ,\ |\alpha| \le m-1\,.$$

Thus, we have estimated $\partial^\alpha G_{\hat{Q}}$ for $|\alpha| = m$, and $\partial^\alpha(G_{\hat{Q}} - F)$ for $|\alpha| \le m-1$. We pass from the $G_{\hat{Q}}$ to G. Recall that $|\partial^\alpha \chi_{\hat{Q}}| \le C\delta_{\hat{Q}}^{-|\alpha|}$ for $|\alpha| \le m$, that $\sum_{\hat{Q}\in CZ} \chi_{\hat{Q}} = 1$ on Q_0, and that any given point of Q_0 lies in at most C of the supports of the $\chi_{\hat{Q}}$. (Also, $\mathsf{supp}\,\chi_{\hat{Q}} \subset 2\hat{Q}$.) Hence, for $|\alpha| \le m-1$, $x \in Q_0$, we have

$$|\partial^\alpha(G-F)(x)| = \left| \sum_{\substack{\alpha'+\alpha''=\alpha \\ \hat{Q}'\in CZ}} c(\alpha',\alpha'')\,\partial^{\alpha'}\chi_{\hat{Q}'}(x) \cdot \partial^{\alpha''}(G_{\hat{Q}'} - F)(x) \right|$$

$$\le \sum_{\substack{\alpha'+\alpha''=\alpha \\ \hat{Q}'\in CZ \\ 2\hat{Q}'\ni x}} C\delta_{\hat{Q}'}^{-|\alpha'|} \cdot M\lambda\delta_{\hat{Q}}^{m-|\alpha''|} \le C\lambda M\delta_{\hat{Q}}^{m-|\alpha|} \ \text{ for any } \ \hat{Q} \in CZ \ \text{ containing } \ x\,.$$

(Here, we use the fact that $\delta_{\hat{Q}'}$ and $\delta_{\hat{Q}}$ are comparable if $2\hat{Q}' \cap \hat{Q} \ne \phi$.)

Thus, for $x \in \hat{Q}$, $\hat{Q} \in CZ$, and $|\alpha| \le m-1$, we have $|\partial^\alpha(F-G)(x)| \le C\lambda M\delta_{\hat{Q}}^{m-|\alpha|}$.

We turn to the estimation of $\partial^\alpha G(x)$ for $|\alpha| = m$, $x \in Q_0$. We have

$$\partial^\alpha G(x) = \sum_{\hat{Q}\in CZ} \chi_{\hat{Q}}(x) \cdot \partial^\alpha G_{\hat{Q}}(x) + \sum_{\substack{\hat{Q}\in CZ \\ \alpha'+\alpha''=\alpha \\ \alpha'\ne 0}} c(\alpha',\alpha'')\partial^{\alpha'}\chi_{\hat{Q}}(x) \cdot \partial^{\alpha''}(G_{\hat{Q}} - F)(x)\,.$$

Since $|\partial^\alpha G_{\hat{Q}}(x)| \le M$ for $x \in \mathsf{supp}\chi_{\hat{Q}}$, and since $\sum_{\hat{Q}\in CZ} \chi_{\hat{Q}}(x) = 1$, the first sum on the right-hand side has absolute value at most M.

Regarding the second sum, we note that a given x belongs to $\mathsf{supp}\chi_{\hat{\mathsf{Q}}}$ for at most C distinct $\hat{Q}$; moreover, each summand has absolute value at most $C\delta_{\hat{Q}}^{-|\alpha'|} \cdot C\lambda M\delta_{\hat{Q}}^{m-|\alpha''|} = C'\lambda M$, since $|\alpha'| + |\alpha''| = |\alpha| = m$. Consequently,

$$|\partial^\alpha G(x)| \le (1 + C\lambda)M \ \text{ for } \ x \in Q_0\,,\ |\alpha| = m\,.$$

Thus, we have estimated $|\partial^\alpha G|$ for $|\alpha| = m$, and $|\partial^\alpha(G-F)|$ for $|\alpha| \le m-1$.

The functions $G_{\hat{Q}}$ and G depend not merely on the P_Q, but on F. To remedy this, we therefore define

$$F_{\hat{Q}}(x) := P_{\hat{Q}}(x) + \sum_{Q \in CZ(\tau)} \int_Q (\lambda \delta_{\hat{Q}})^{-n} \theta \left(\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}} \right) \cdot [P_Q(y) - P_{\hat{Q}}(y)] dy$$

$$\text{for} \quad x \in 2\hat{Q} \cap Q_0 \,,\, \hat{Q} \in CZ;$$

and

$$\tilde{F}(x) = \sum_{\hat{Q} \in CZ} \chi_{\hat{Q}}(x) \cdot F_{\hat{Q}}(x) \;\; \text{for} \;\; x \in Q_0 \,.$$

Thus, $F_{\hat{Q}}$ and $\tilde{F}$ are determined (linearly) by the P_Q. (Recall that each $P_{\hat{Q}}$ is one of the P_Q.)

Let us compare $F_{\hat{Q}}$, $\tilde{F}$ with $G_{\hat{Q}}$, G. Recalling the definitions of $F_{\hat{Q}}$, $G_{\hat{Q}}$, we see that

$$\partial^\alpha (F_{\hat{Q}} - G_{\hat{Q}})(x) = \sum_{Q \in CZ(\tau)} \int_Q (\lambda \delta_{\hat{Q}})^{-n} \partial_x^\alpha \theta \left(\frac{x-y}{\lambda_{\hat{Q}}} - \xi_Q \right) \cdot [P_Q(y) - F(y)] dy \,,$$

$$\text{for} \quad |\alpha| \le m,\, x \in 2\hat{Q} \cap Q_0,\, \hat{Q} \in CZ.$$

For $y \in Q$, Taylor's theorem gives $|P_Q(y) - F(y)| \le CM\delta_Q^m$. Also, every $Q \in CZ(\tau)$ such that $\partial_x^\alpha \theta(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}) \neq 0$ for some $y \in Q$ satisfies $c\tau\delta_{\hat{Q}} \le \delta_Q \le C\tau\delta_{\hat{Q}}$. Therefore,

$$\begin{aligned} |\partial^\alpha (F_{\hat{Q}} - G_{\hat{Q}})(x)| &\le CM\delta_{\hat{Q}}^m \tau^m \int_{\mathbb{R}^n} (\lambda\delta_{\hat{Q}})^{-n} \left| \partial_x^\alpha \theta \left(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}} \right) \right| dy \\ &\le C\tau^m \lambda^{-|\alpha|} M \delta_{\hat{Q}}^{m-|\alpha|} \le C' \left(\tfrac{\tau}{\lambda}\right)^m M (\lambda\delta_{\hat{Q}})^{m-|\alpha|} \,. \end{aligned}$$

Since $|\partial^\alpha \chi_{\hat{Q}}(x)| \le C\delta_{\hat{Q}}^{-|\alpha|}$ for $|\alpha| \le m$, and since there are at most C distinct $\hat{Q} \in CZ$ such that $2\hat{Q} \ni x$, all of which have comparable sidelengths, it follows that

$$\left| \partial^\alpha \left\{ \sum_{\hat{Q}' \in CZ} \chi_{\hat{Q}'} \cdot (F_{\hat{Q}'} - G_{\hat{Q}'})(x) \right\} \right| \le C \left(\frac{\tau}{\lambda} \right)^m M \cdot (\lambda\delta_{\hat{Q}})^{m-|\alpha|}$$

$$\text{for} \quad |\alpha| \le m,\, x \in \hat{Q},\, \hat{Q} \in CZ.$$

That is,

$$|\partial^\alpha (\tilde{F} - G)(x)| \le C \left(\frac{\tau}{\lambda} \right)^m M \cdot (\lambda\delta_{\hat{Q}})^{m-|\alpha|} \;\; \text{for} \;\; |\alpha| \le m \,,\, x \in \hat{Q} \,,\, \hat{Q} \in CZ \,.$$

Combining this estimate with our results for G, we obtain the following estimates.

$$|\partial^\alpha \tilde{F}(x)| \le \left(1 + C\lambda + C\left(\tfrac{\tau}{\lambda}\right)^m\right) M \quad \text{for} \quad |\alpha| = m\,,\ x \in Q_0\,; \quad \text{and}$$

$$|\partial^\alpha (\tilde{F} - F)(x)| \le C\lambda\, M\, \delta_{\hat{Q}}^{m-|\alpha|} \quad \text{for} \quad |\alpha| \le m-1\,,\ x \in \hat{Q},\ \hat{Q} \in CZ\,.$$

These are our basic estimates for $\tilde{F}$. We specialize these estimates to obtain linear constraints on the P_Q, P^x, M. Recall that $\tilde{F}$ is determined linearly by the P_Q. Our constraints are as follows (together with (I), (II), (III) above).

(IV) $|\partial^\alpha \tilde{F}(x)| \le \left(1 + C\lambda + C\left(\tfrac{\tau}{\lambda}\right)^m\right) M$ for $|\alpha| = m$, x any vertex of $Q, Q \in CZ(\tau)$.

(V) $|\partial^\alpha (\tilde{F} - P^x)(x)| \le C\lambda\, M\, \delta_{\hat{Q}}^{m-|\alpha|}$ for $|\alpha| \le m-1$, $x \in \hat{Q} \cap E$, $\hat{Q} \in CZ$.

(To see that (V) holds, we recall that $P^x = J_x(F)$ for $x \in E$.)

This concludes our list of constraints. In view of our estimates on the number of cubes in CZ and $CZ(\tau)$, we have the following estimate:

The number of constraints (I)$\cdots$(V) is at most $C|\ln\eta|\tau^{-n}N$, where (recall) $N = \#(E)$, and $|x - x'| \ge \eta\, \delta_{Q_0}$ for any $x, x' \in E$ distinct.

We have proven the following basic result.

Let $P^x(x \in E)$ and M be given. Suppose there exists a function $F \in C^m(Q_0)$ such that

- $|\partial^\alpha F(x)| \le M$ for $|\alpha| = m$, $x \in Q_0$; and
- $J_x(F) = P^x$ for each $x \in E$.

  Then there exist $P_Q(Q \in CZ(\tau))$ such that the $P^x(x \in E)$, $P_Q(Q \in CZ(\tau))$ and M satisfy constraints (I)$\cdots$(V).

From Whitney Fields to Functions

In this section, we drop the assumption that there exists a function F on Q_0 with the properties assumed before. Instead, we now assume that we are given polynomials $P^x(x \in E)$ and $P_Q(Q \in CZ(\tau))$, of degree at most $m-1$; and a real number $M \ge 0$. We assume that the P^x, P_Q, M satisfy constraints (I)$\cdots$(V) from the previous section. As in that section, we define

$$F_{\hat{Q}}(x) = P_{\hat{Q}}(x) + \sum_{Q \in CZ(\tau)} \int_Q (\lambda\delta_{\hat{Q}})^{-n}\, \theta\left(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}\right) [P_Q(y) - P_{\hat{Q}}(y)]dy$$

for $\quad x \in 2\hat{Q} \cap Q_0\,,\ \hat{Q} \in CZ;$ and

$$\tilde{F}(x) \;=\; \sum_{\hat{Q}\in CZ} \chi_{\hat{Q}}(x)\, F_{\hat{Q}}(x) \quad \text{for} \quad x \in Q_0\,.$$

Since each $P_{\hat{Q}}$ is simply one of the P_Q's, the functions $F_{\hat{Q}}$ and $\tilde{F}$ are well-defined, as in the preceding section.

Let $x \in 2\hat{Q} \cap Q_0$, $\hat{Q} \in CZ$. We will estimate $\partial^\alpha F_{\hat{Q}}(x)$ for $|\alpha| = m+2$, and $|\partial^\alpha (F_{\hat{Q}} - P_{\hat{Q}})(x)|$ for $|\alpha| \leq m-1$. To do so, we apply our integration-by-parts formula: For $x, \hat{Q}$ as above, and for $|\alpha| \leq m+2$, we have

$$\partial^\alpha (F_{\hat{Q}} - P_{\hat{Q}})(x) \;=\; \sum_{Q \in CZ(\tau)} \int_Q (\lambda \delta_{\hat{Q}})^{-n} \left[\partial_x^\alpha\, \theta \left(\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}} \right) \right] \cdot [P_Q(y) - P_{\hat{Q}}(y)] dy$$

$$= \sum_{Q \in CZ(\tau)} \int_Q (\lambda \delta_{\hat{Q}})^{-n} (-1)^{|\alpha|} \left[\partial_y^\alpha\, \theta \left(\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}} \right) \right] \cdot [P_Q(y) - P_{\hat{Q}}(y)] dy$$

$$= \sum_{Q \in CZ(\tau)} \int_Q (\lambda \delta_{\hat{Q}})^{-n}\, \theta \left(\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}} \right) \cdot [\partial^\alpha P_Q(y) - \partial^\alpha P_{\hat{Q}}(y)] dy$$

$$+ \sum_{\bar{Q} \in \Pi} \sum_{\substack{\alpha' + \alpha'' + \gamma = \alpha \\ |\gamma| = 1}} c(\alpha', \alpha'', \gamma, \bar{Q}) \int_{\bar{Q}} (\lambda \delta_{\hat{Q}})^{-n} \left[\partial_y^{\alpha'}\, \theta \left(\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}} \right) \right] [\partial^{\alpha''} (P_{\bar{Q}^1} - P_{\bar{Q}^{-1}})(y)] dy$$

$$\equiv \text{Term 1} + \text{Term 2},$$

where each $c(\alpha', \alpha'', \gamma, \bar{Q})$ depends only on $\alpha', \alpha'', \gamma$, and the j for which $\bar{Q} \in \Pi(j)$.

Note that $P_{\hat{Q}}$ does not appear in Term 2. Let us first estimate Term 2. If either $\bar{Q}^1$ or $\bar{Q}^{-1}$ is Null, then $\bar{Q} \subset \partial Q_0$. However, when $x \in 2\hat{Q} \cap Q_0$, $\hat{Q} \in CZ$ and $\theta(\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}}) \neq 0$, we have $y \in Q_0^{\text{interior}}$. Therefore, $\theta(\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}}) = 0$ for $y \in \bar{Q}$, whenever either $\bar{Q}^1$ or $\bar{Q}^{-1}$ is Null. In the definition of Term 2, we may therefore restrict the sum over $\bar{Q}$ to those $\bar{Q}$ for which both $\bar{Q}^1$ and $\bar{Q}^{-1}$ are non-Null. For such $\bar{Q}$, our constraints (I) apply. By (I) and Taylor's theorem, we have $|\partial^\alpha (P_{\bar{Q}^1} - P_{\bar{Q}^{-1}})(y)| \leq CM \delta_{\bar{Q}}^{m-|\alpha|}$ for all $y \in \bar{Q}$, $|\alpha| \leq m-1$.

When $|\alpha| \geq m$, the quantity $\partial^\alpha (P_{\bar{Q}^1} - P_{\bar{Q}^{-1}})(y)$ is zero, since $P_{\bar{Q}^1}$, $P_{\bar{Q}^{-1}}$ have degree at most $m-1$. Therefore, in the sum defining Term 2, we may restrict to $|\alpha''| \leq m-1$. Furthermore, suppose $\theta(\frac{x-y}{\lambda \delta_{\hat{Q}}} - \xi_{\hat{Q}}) \neq 0$ for some $y \in \bar{Q}$, $\bar{Q} \in \Pi$. By definition of plaques, and because our cubes are defined to be closed, we have $\bar{Q} \subset \partial Q \subset Q$ for some $Q \in CZ(\tau)$;

moreover, $\frac{1}{2}\delta_Q \le \delta_{\bar{Q}} \le \delta_Q$. We have $x \in 2\hat{Q} \cap Q_0$, $\hat{Q} \in CZ$, $y \in \bar{Q} \subset Q$, $Q \in CZ(\tau)$, and $|\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}| < 1$. Therefore, by one of our previous observations, $c\tau\delta_{\hat{Q}} \le \delta_Q \le C\tau\delta_{\hat{Q}}$. Consequently, $c\tau\delta_{\hat{Q}} \le \delta_{\bar{Q}} \le C\tau\delta_{\hat{Q}}$. Thus, each $\bar{Q}$ that enters into Term 2 satisfies $c\tau\delta_{\hat{Q}} \le \delta_{\bar{Q}} \le C\tau\delta_{\hat{Q}}$, and arises from a cube $Q \in CZ(\tau)$ with sidelength $c\tau\delta_{\hat{Q}} \le \delta_Q \le C\tau\delta_{\hat{Q}}$ that meets $\{y \in \mathbb{R}^n : |\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}| < 1\}$.

The number of such $Q \in CZ(\tau)$ is at most $C(\frac{\lambda}{\tau})^n$. Consequently,

|Term 2|

$$\le \sum_{\substack{\alpha'+\alpha''+\gamma=\alpha \\ |\gamma|=1 \\ |\alpha''|\le m-1}} C\left(\tfrac{\lambda}{\tau}\right)^n \cdot (\lambda\delta_{\hat{Q}})^{-n} \cdot (\lambda\delta_{\hat{Q}})^{-|\alpha'|} \cdot M(\tau\delta_{\hat{Q}})^{m-|\alpha''|} \cdot (\tau\delta_{\hat{Q}})^{n-1}$$

$C(\frac{\lambda}{\tau})^n$	$(\lambda\delta_{\hat{Q}})^{-n}$	$(\lambda\delta_{\hat{Q}})^{-\lvert\alpha'\rvert}$	$M(\tau\delta_{\hat{Q}})^{m-\lvert\alpha''\rvert}$	$(\tau\delta_{\hat{Q}})^{n-1}$
This factor counts the number of plaques $\bar{Q}$ that contribute.	This factor is present in the integral.	This factor is a bound for $\lvert\partial_y^{\alpha'}\theta(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}})\rvert$.	This factor is a bound for $\lvert\partial^{\alpha''}(P_{\bar{Q}^1} - P_{\bar{Q}^{-1}})\rvert$.	This factor is comparable to the area of $\bar{Q}$.

$$\le \sum_{\substack{|\alpha'|+|\alpha''|=|\alpha|-1 \\ |\alpha''|\le m-1}} C\, \tau^{m-|\alpha''|-1}\lambda^{-|\alpha'|}\, M\, \delta_{\hat{Q}}^{m-|\alpha'|-|\alpha''|-1}$$

$$\le C \sum_{\substack{|\alpha''|\le m-1 \\ |\alpha''|\le|\alpha|-1}} \left(\frac{\tau}{\lambda}\right)^{m-1-|\alpha''|} \lambda^{m-|\alpha|}\, M\, \delta_{\hat{Q}}^{m-|\alpha|}.$$

Therefore,

|Term 2|

$$\le C\, \left(\tfrac{\tau}{\lambda}\right)^{m-|\alpha|} M\,(\lambda\delta_{\hat{Q}})^{m-|\alpha|} = C\, M\,(\tau\delta_{\hat{Q}})^{m-|\alpha|} \quad \text{if} \quad |\alpha| \le m-1,$$

while

|Term 2|

$$\le C\,(\lambda\delta_{\hat{Q}})^{m-|\alpha|}\, M \quad \text{if} \quad |\alpha| \le m+2.$$

We turn our attention to Term 1.

Fix $x \in 2\hat{Q} \cap Q_0$, and consider any $Q \in CZ(\tau)$ that makes a non-zero contribution to Term 1. We have $|\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_Q| < 1$ for some $y \in Q$. Hence, (II) applies. Since also $c\tau\delta_{\hat{Q}} \le \delta_Q \le C\tau\delta_{\hat{Q}}$ by an earlier observation, it follows from (II) that $|\partial^\alpha(P_Q - P_{\hat{Q}})(y)| \le CM\,\delta_{\hat{Q}}^{m-|\alpha|}$ for all $y \in Q$, provided $|\alpha| \leq m-1$. If $|\alpha| \geq m$, then $\partial^\alpha(P_Q - P_{\hat{Q}}) \equiv 0$, and Term 1 is simply 0.

Assuming that $|\alpha| \leq m-1$, we now see that

|Term 1|

$$\leq \sum_{Q \in CZ(\tau)} \int_Q (\lambda\delta_{\hat{Q}})^{-n}\, \theta\left(\frac{x-y}{\lambda\delta_{\hat{Q}}} - \xi_{\hat{Q}}\right) \cdot CM\, \delta_{\hat{Q}}^{m-|\alpha|}\, dy \;=\; CM\, \delta_{\hat{Q}}^{m-|\alpha|}.$$

Combining our estimates for Term 1 and Term 2, we obtain the estimate

$$|\partial^\alpha(F_{\hat{Q}} - P_{\hat{Q}})(x)| \leq CM\delta_{\hat{Q}}^{m-|\alpha|} + CM(\lambda\delta_{\hat{Q}})^{m-|\alpha|} \quad \text{for} \quad |\alpha| \leq m+2,\ x \in 2\hat{Q} \cap Q_0\,, \hat{Q} \in CZ\,.$$

We recall that $\mathsf{supp}\, \chi_{\hat{Q}} \subset 2\hat{Q} \cap Q_0$, $|\partial^\alpha \chi_{\hat{Q}}| \leq C\, \delta_{\hat{Q}}^{-|\alpha|}$ for $|\alpha| \leq m+2$.

Also, we recall that any given $x \in Q_0$ belongs to $\mathsf{supp}\ \chi_{\hat{Q}}$ for at most C distinct $\hat{Q} \in CZ$; and that $\hat{Q} \cap 2\hat{Q}' \neq \phi$ implies $c\delta_{\hat{Q}} \leq \delta_{\hat{Q}'} \leq C\delta_{\hat{Q}}$ for $\hat{Q}, \hat{Q}' \in CZ$.

For $|\alpha| = m+2$, we now estimate $\partial^\alpha \tilde{F}(x)$ at a given $x \in Q_0$. Fix $\hat{Q}' \in CZ$ containing x. We have

$$\partial^\alpha \tilde{F}(x) = \partial^\alpha(\tilde{F} - P_{\hat{Q}'})(x) = \partial^\alpha \sum_{\hat{Q} \in CZ} \chi_{\hat{Q}} \cdot (F_{\hat{Q}} - P_{\hat{Q}'})(x)$$

$$= \sum_{\alpha'+\alpha''=\alpha} \sum_{\hat{Q} \in CZ} c(\alpha', \alpha'')\, [\partial^{\alpha'} \chi_{\hat{Q}}(x)] \;\cdot [\partial^{\alpha''}(F_{\hat{Q}} - P_{\hat{Q}'})(x)]\,.$$

Let $\hat{Q} \in CZ$ make a non-zero contribution to the right-hand side.

We have $|\partial^{\alpha''}(F_{\hat{Q}} - P_{\hat{Q}'})(x)| \leq |\partial^{\alpha''}(F_{\hat{Q}} - P_{\hat{Q}})(x)| + |\partial^{\alpha''}(P_{\hat{Q}} - P_{\hat{Q}'})(x)|$

$$\leq CM\, \delta_{\hat{Q}}^{m-|\alpha''|} + CM(\lambda\delta_{\hat{Q}})^{m-|\alpha''|} + |\partial^{\alpha''}(P_{\hat{Q}} - P_{\hat{Q}'})(x)|\,.$$

Since $x \in \hat{Q}' \cap 2\hat{Q}$ with $\hat{Q}, \hat{Q}' \in CZ$, constraint (III) applies.

Since $x \in 2\hat{Q}$, it follows from (III) that $|\partial^{\alpha''}(P_{\hat{Q}} - P_{\hat{Q}'})(x)| \leq CM\delta_{\hat{Q}}^{m-|\alpha''|}$.

(This holds for $|\alpha''| \leq m-1$, and it holds trivially for $|\alpha''| \geq m$, since $P_{\hat{Q}} - P_{\hat{Q}'}$ has degree at most $m-1$.)

Consequently, $|\partial^{\alpha''}(F_{\hat{Q}} - P_{\hat{Q}'})(x)| \leq CM\, \delta_{\hat{Q}}^{m-|\alpha''|} + CM(\lambda\delta_{\hat{Q}})^{m-|\alpha''|}$.

It follows that

$$|\partial^\alpha \tilde{F}(x)| \;\leq\; C \sum_{\alpha'+\alpha''=\alpha} \left(\delta_{\hat{Q}'}^{-|\alpha'|} \cdot M\, \delta_{\hat{Q}'}^{m-|\alpha''|} + \delta_{\hat{Q}'}^{-|\alpha'|} \;\cdot\; M \;\cdot (\lambda\delta_{\hat{Q}'})^{m-|\alpha''|}\right)$$

$$\leq C' M \delta_{\hat{Q}'}^{m-|\alpha|} + C' M \cdot (\lambda \delta_{\hat{Q}'})^{m-|\alpha|} \leq C''(\lambda \delta_{\hat{Q}'})^{-2} M ,$$

since $|\alpha| = m+2$. Thus, we have proven the following estimate:

$$|\partial^\alpha \tilde{F}(x)| \leq CM \cdot (\lambda \delta_{\hat{Q}'})^{-2} \text{ for } |\alpha| = m+2 , \ x \in \hat{Q}' , \ \hat{Q}' \in CZ .$$

Now let $Q \in CZ(\tau)$ be given.

Fix β, a multi-index of order $|\beta| = m$. Let $L_{\beta,Q}(\cdot)$ be the first-degree Taylor polynomial of $\partial^\beta \tilde{F}$ at x_Q; and let $\hat{Q}$ be a cube in CZ that contains Q. We know that $c\tau\delta_{\hat{Q}} \leq \delta_Q \leq C\tau\delta_{\hat{Q}}$. Hence, the estimate we have just proven for the $(m+2)^{\text{nd}}$ derivatives of $\tilde{F}$ tells us that

$$|\partial^\gamma [\partial^\beta \tilde{F}]| \leq CM \cdot (\lambda \delta_{\hat{Q}})^{-2} \leq C' M \left(\frac{\tau}{\lambda}\right)^2 \delta_Q^{-2} \text{ on } Q , \text{ for } |\gamma| = 2 .$$

Hence, by Taylor's theorem, we have

$$|\partial^\beta \tilde{F}(x) - L_{\beta,Q}(x)| \leq C(\tfrac{\tau}{\lambda})^2 M \text{ for all } x \in Q.$$

On the other hand, constraint (IV) tells us that

$$|\partial^\beta \tilde{F}(x)| \leq \left(1 + C\lambda + C\left(\tfrac{\tau}{\lambda}\right)^m\right) M \text{ at the vertices of } Q.$$

Therefore, (recall $m \geq 2$):

$$|L_{\beta,Q}(x)| \leq \left(1 + C\lambda + C\left(\tfrac{\tau}{\lambda}\right)^2\right) M \text{ at the vertices of } Q.$$

Since $L_{\beta,Q}$ is a first-degree polynomial, the maximum of $|L_{\beta,Q}(x)|$ over all $x \in Q$ is attained at a vertex of Q. Therefore,

$$|L_{\beta,Q}(x)| \leq \left(1 + C\lambda + C\left(\tfrac{\tau}{\lambda}\right)^2\right) M \text{ for all } x \in Q.$$

Together with our estimate for $|\partial^\beta \tilde{F}(x) - L_{\beta,Q}(x)|$, this yields:

$$|\partial^\beta \tilde{F}(x)| \leq \left(1 + C\lambda + C\left(\tfrac{\tau}{\lambda}\right)^2\right) M \text{ for all } x \in Q.$$

Since Q was an arbitrary cube in $CZ(\tau)$, and β is an arbitrary milti-index of order m, we conclude that

$$|\partial^\alpha \tilde{F}(x)| \leq \left(1 + C\lambda + C\left(\frac{\tau}{\lambda}\right)^2\right) M \text{ for } |\alpha| = m , x \in Q_0 .$$

We recall also the constraints (V):

$$|\partial^\alpha (\tilde{F} - P^y)(y)| \leq C\lambda M \delta_{\hat{Q}}^{m-|\alpha|} \text{ for } |\alpha| \leq m-1 , \ y \in \hat{Q} \cap E , \hat{Q} \in CZ .$$

For each $y \in E$, we pick $\hat{Q}_y \in CZ$ such that $y \in \hat{Q}_y$.

By definition of CZ, we have $\#(E \cap 3\hat{Q}_y) \leq 1$, hence y is the one and only point of E belonging to $3\hat{Q}_y$. Consequently, $|y - y'| > \delta_{\hat{Q}_y}$ for $y' \in E$, $y' \neq y$. Similarly, $|y - y'| > \delta_{\hat{Q}_{y'}}$ for $y, y' \in E$ distinct.

Hence, $|y - y'| > \frac{1}{2}\delta_{\hat{Q}_y} + \frac{1}{2}\delta_{\hat{Q}_{y'}}$ for $y, y' \in E$ distinct.

For each $y \in E$, we study the function $H_y(x) := [P^y(x) - J_y(\tilde{F})(x)] \cdot \varphi\left(\frac{x-y}{\frac{1}{2}\delta_{\hat{Q}_y}}\right)$.

Since φ is supported in $\{|x| \leq 1\}$, we know that H_y is supported in $\{x \in \mathbb{R}^n : |x-y| \leq \frac{1}{2}\delta_{\hat{Q}_y}\}$. Consequently, H_y and $H_{y'}$ have disjoint supports, for any $y, y' \in E$ distinct, since we have just seen that $|y - y'| > \frac{1}{2}\delta_{\hat{Q}_y} + \frac{1}{2}\delta_{\hat{Q}_{y'}}$.

Let us examine a particular H_y. Since $\partial^\alpha(\varphi - 1)(0) = 0$ for $|\alpha| \leq m - 1$, it follows that $J_y(H_y) = P^y - J_y(\tilde{F})$. Also, the polynomial $P = P^y - J_y(\tilde{F})$ satisfies

$$|\partial^\alpha P(y)| = |\partial^\alpha P^y(y) - \partial^\alpha \tilde{F}(y)| \leq C\lambda M \delta_{\hat{Q}_y}^{m-|\alpha|} \quad \text{for} \quad |\alpha| \leq m-1\,,$$

since $y \in \hat{Q}_y \cap E$ and $\hat{Q}_y \in CZ$.

Therefore, $|\partial^\alpha P(x)| \leq C'\lambda M\, \delta_{\hat{Q}_y}^{m-|\alpha|}$ for $|\alpha| \leq m-1$, $|x - y| \leq \frac{1}{2}\,\delta_{\hat{Q}_y}$. Since also $\varphi\left(\frac{x-y}{\frac{1}{2}\delta_{\hat{Q}_y}}\right)$ is supported in $\{|x-y| \leq \frac{1}{2}\delta_{\hat{Q}_y}\}$ and satisfies $|\partial_x^\alpha \varphi\left(\frac{x-y}{\frac{1}{2}\delta_{\hat{Q}_y}}\right)| \leq C\delta_{\hat{Q}_y}^{-|\alpha|}$, it follows that

$$|\partial_x^\alpha H_y(x)| = \left|\partial_x^\alpha \left\{P(x)\,\varphi\left(\frac{x-y}{\frac{1}{2}\delta_{\hat{Q}_y}}\right)\right\}\right| \leq C\lambda M \delta_{\hat{Q}_y}^{m-|\alpha|} \quad \text{for} \quad x \in \mathbb{R}^n\,,\ |\alpha| \leq m\,.$$

In particular, $|\partial_x^\alpha H_y(x)| \leq C\lambda M$ for $|\alpha| = m$, $x \in \mathbb{R}^n$.

Since the supports of the $H_y(y \in E)$ are pairwise disjoint, we now know that

$$J_y\left(\sum_{y' \in E} H_{y'}\right) = P^y - J_y(\tilde{F}) \text{ for each } y \in E; \text{ and}$$

$$|\partial^\alpha \left(\sum_{y' \in E} H_{y'}\right)(x)| \leq C\lambda M \quad \text{for} \quad |\alpha| = m\,,\ x \in \mathbb{R}^n\,.$$

We define

$$F^{\#}(x) := \tilde{F}(x) + \sum_{y' \in E} H_{y'}(x) \quad \text{for} \quad x \in Q_0\,.$$

We have

$$J_y(F^{\#}) = P^y \text{ for each } y \in E; \text{ and}$$

$$|\partial^\alpha F^{\#}(x)| \le \left(1 + C\lambda + C\left(\frac{\tau}{\lambda}\right)^2\right) M \text{ for } |\alpha| = m, x \in Q_0,$$

thanks to our estimate for the m^{th} derivatives of $\tilde{F}$.

Thus, we have proven the following.

Let $P^y(y \in E)$, $P_Q(Q \in CZ(\tau))$, M satisfy constraints (I)$\cdots$(V).

Then the function $F^{\#} \in C^m(Q_0)$, defined constructively as above, satisfies

$J_y(F^{\#}) = P^y$ for each $y \in E$, and

$|\partial^\alpha F^{\#}(x)| \le \left(1 + C\lambda + C\left(\frac{\tau}{\lambda}\right)^2\right) M$ for $|\alpha| = m$, $x \in Q_0$.

Proof of Theorems 1 and 2

Without loss of generality, we can take Q_0 to be dyadic in Theorems 1 and 2. We take $\lambda = c_1\epsilon$ and $\tau = c_1^2\epsilon^{\frac{3}{2}}$ for a small enough constant c_1 (depending only on m and n). We introduce the Calderón-Zygmund decompositions CZ and $CZ(\tau)$, the points x_Q (all $Q \in CZ(\tau)$), and the constraints (I)$\cdots$(V), as in the previous sections. We regard each P_Q as a jet associated to the point x_Q. Thus, (I)$\cdots$(V) is a family of constraints on a variable $(M, \vec{P}^+) \in \mathbb{R} \oplus Wh(E^+)$, where E^+ consists of E together with all the $x_Q(Q \in CZ(\tau))$. We define $K \subset \mathbb{R} \oplus Wh(E^+)$ as the set of all $(M, \vec{P}^+)$ that satisfy the constraints (I)$\cdots$(V).

For all the above E^+ and K, all the assertions of Theorems 1 and 2 have been proven in the previous sections.

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