Joint functional calculus for commuting differential operators on nilpotent groups: Schwartz kernels and multipliers

Fulvio Ricci

Scuola Normale Superiore di Pisa

Analysis and Applications: A Conference in Honor of E. M. Stein
Princeton University, May 15-21, 2011

D. Müller, F. R., E. M. Stein (1995)

D. Müller, F. R., E. M. Stein (1995)

- Heisenberg group $H_n \cong \mathbb{C}^n \times \mathbb{R}$

D. Müller, F. R., E. M. Stein (1995)

- Heisenberg group $H_n \cong \mathbb{C}^n \times \mathbb{R}$
- L the U_n-invariant sublaplacian, $T = \partial_t$ derivative in the central direction

D. Müller, F. R., E. M. Stein (1995)

- Heisenberg group $H_n \cong \mathbb{C}^n \times \mathbb{R}$
- L the U_n-invariant sublaplacian, $T = \partial_t$ derivative in the central direction
- multiplier operators $m(L, i^{-1}T)$, with m a bounded Borel function on $\mathbb{R}^2$

D. Müller, F. R., E. M. Stein (1995)

- Heisenberg group $H_n \cong \mathbb{C}^n \times \mathbb{R}$
- L the U_n-invariant sublaplacian, $T = \partial_t$ derivative in the central direction
- multiplier operators $m(L, i^{-1}T)$, with m a bounded Borel function on $\mathbb{R}^2$
- $m(L, i^{-1}T)f = f * K_m$, with K_m a U_n-invariant distribution

D. Müller, F. R., E. M. Stein (1995)

- Heisenberg group $H_n \cong \mathbb{C}^n \times \mathbb{R}$
- L the U_n-invariant sublaplacian, $T = \partial_t$ derivative in the central direction
- multiplier operators $m(L, i^{-1}T)$, with m a bounded Borel function on $\mathbb{R}^2$
- $m(L, i^{-1}T)f = f * K_m$, with K_m a U_n-invariant distribution

D. Müller, F. R., E. M. Stein (1995)

- Heisenberg group $H_n \cong \mathbb{C}^n \times \mathbb{R}$
- L the U_n-invariant sublaplacian, $T = \partial_t$ derivative in the central direction
- multiplier operators $m(L, i^{-1}T)$, with m a bounded Borel function on $\mathbb{R}^2$
- $m(L, i^{-1}T)f = f * K_m$, with K_m a U_n-invariant distribution

Theorem

D. Müller, F. R., E. M. Stein (1995)

- Heisenberg group $H_n \cong \mathbb{C}^n \times \mathbb{R}$
- L the U_n-invariant sublaplacian, $T = \partial_t$ derivative in the central direction
- multiplier operators $m(L, i^{-1}T)$, with m a bounded Borel function on $\mathbb{R}^2$
- $m(L, i^{-1}T)f = f * K_m$, with K_m a U_n-invariant distribution

Theorem

(i) *If m is a smooth Marcinkiewicz multiplier, then K_m is a smooth, U_n-invariant, flag kernel on H_n, adapted to the flag*

$$\{0\} \subset \{0\} \times \mathbb{R} \subset H_n .$$

D. Müller, F. R., E. M. Stein (1995)

- Heisenberg group $H_n \cong \mathbb{C}^n \times \mathbb{R}$
- L the U_n-invariant sublaplacian, $T = \partial_t$ derivative in the central direction
- multiplier operators $m(L, i^{-1}T)$, with m a bounded Borel function on $\mathbb{R}^2$
- $m(L, i^{-1}T)f = f * K_m$, with K_m a U_n-invariant distribution

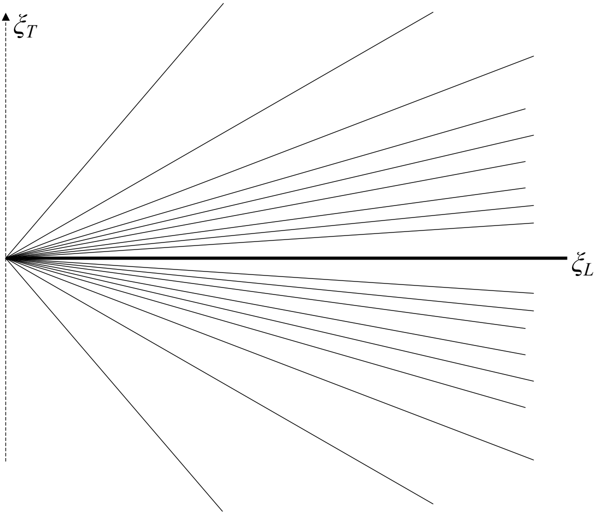
Theorem

(i) *If m is a smooth Marcinkiewicz multiplier, then K_m is a smooth, U_n-invariant, flag kernel on H_n, adapted to the flag*

$$\{0\} \subset \{0\} \times \mathbb{R} \subset H_n \,.$$

(ii) *If K is a U_n-invariant smooth flag kernel on H_n, adapted to the above flag, then there exists a smooth Marcinkiewicz multiplier m such that $K = K_m$.*

The Heisenberg fan



Schwartz kernels and multipliers

Theorem (Astengo, Di Blasio, R., 2007)

Schwartz kernels and multipliers

Theorem (Astengo, Di Blasio, R., 2007)

(i) *If m is a Schwartz function on $\mathbb{R}^2$, then K_m is a U_n-invariant Schwartz function on H_n.*

Schwartz kernels and multipliers

Theorem (Astengo, Di Blasio, R., 2007)

(i) *If m is a Schwartz function on $\mathbb{R}^2$, then K_m is a U_n-invariant Schwartz function on H_n.*

(ii) *If F is a U_n-invariant Schwartz function on H_n, then there exists a Schwartz function m on $\mathbb{R}^2$ such that $F = K_m$.*

Schwartz kernels and multipliers

Theorem (Astengo, Di Blasio, R., 2007)

(i) *If m is a Schwartz function on $\mathbb{R}^2$, then K_m is a U_n-invariant Schwartz function on H_n.*

(ii) *If F is a U_n-invariant Schwartz function on H_n, then there exists a Schwartz function m on $\mathbb{R}^2$ such that $F = K_m$.*

Schwartz kernels and multipliers

Theorem (Astengo, Di Blasio, R., 2007)

(i) *If m is a Schwartz function on $\mathbb{R}^2$, then K_m is a U_n-invariant Schwartz function on H_n.*

(ii) *If F is a U_n-invariant Schwartz function on H_n, then there exists a Schwartz function m on $\mathbb{R}^2$ such that $F = K_m$.*

$$\mathcal{S}(H_n)^{\mathrm{U}_n} \cong \mathcal{S}(\mathrm{fan}) \ .$$

Related results: Geller, Benson-Ratcliff

Nilpotent Gelfand pairs

Nilpotent Gelfand pairs

- N nilpotent Lie group;

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;
- $\mathbb{D}(N)^K$ algebra of left-invariant and K-invariant differential operators on N;

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;
- $\mathbb{D}(N)^K$ algebra of left-invariant and K-invariant differential operators on N;
- $L^1(N)^K, \mathcal{S}(N)^K$ etc., spaces of K-invariant functions on N.

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;
- $\mathbb{D}(N)^K$ algebra of left-invariant and K-invariant differential operators on N;
- $L^1(N)^K, \mathcal{S}(N)^K$ etc., spaces of K-invariant functions on N.

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;
- $\mathbb{D}(N)^K$ algebra of left-invariant and K-invariant differential operators on N;
- $L^1(N)^K, \mathcal{S}(N)^K$ etc., spaces of K-invariant functions on N.

The following are equivalent:

- $\mathbb{D}(N)^K$ is commutative under composition;

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;
- $\mathbb{D}(N)^K$ algebra of left-invariant and K-invariant differential operators on N;
- $L^1(N)^K, \mathcal{S}(N)^K$ etc., spaces of K-invariant functions on N.

The following are equivalent:

- $\mathbb{D}(N)^K$ is commutative under composition;
- $L^1(N)^K$ is commutative under convolution.

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;
- $\mathbb{D}(N)^K$ algebra of left-invariant and K-invariant differential operators on N;
- $L^1(N)^K, \mathcal{S}(N)^K$ etc., spaces of K-invariant functions on N.

The following are equivalent:

- $\mathbb{D}(N)^K$ is commutative under composition;
- $L^1(N)^K$ is commutative under convolution.

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;
- $\mathbb{D}(N)^K$ algebra of left-invariant and K-invariant differential operators on N;
- $L^1(N)^K, \mathcal{S}(N)^K$ etc., spaces of K-invariant functions on N.

The following are equivalent:

- $\mathbb{D}(N)^K$ is commutative under composition;
- $L^1(N)^K$ is commutative under convolution.

If these conditions are satisfied, we say that (N, K) is a *nilpotent Gelfand pair*.

Nilpotent Gelfand pairs

- N nilpotent Lie group;
- K compact group of automorphisms of N;
- $\mathbb{D}(N)^K$ algebra of left-invariant and K-invariant differential operators on N;
- $L^1(N)^K, \mathcal{S}(N)^K$ etc., spaces of K-invariant functions on N.

The following are equivalent:

- $\mathbb{D}(N)^K$ is commutative under composition;
- $L^1(N)^K$ is commutative under convolution.

If these conditions are satisfied, we say that (N, K) is a *nilpotent Gelfand pair*.

Example: $\qquad (H_n, \mathrm{U}_n)\ , \qquad \mathbb{D}(H_n)^{\mathrm{U}_n} = \mathbb{C}[L, T]$.

Spectral multipliers

- $\mathcal{D} = (D_1, \ldots, D_d)$ a d-tuple of formally self-adjoint differential operators generating $\mathbb{D}(N)^K$;

Spectral multipliers

- $\mathcal{D} = (D_1, \ldots, D_d)$ a d-tuple of formally self-adjoint differential operators generating $\mathbb{D}(N)^K$;
- $\Sigma_{\mathcal{D}} \subset \mathbb{R}^d$ the joint L^2-spectrum of $(D_1, \ldots, D_d)$.

Spectral multipliers

- $\mathcal{D} = (D_1, \ldots, D_d)$ a d-tuple of formally self-adjoint differential operators generating $\mathbb{D}(N)^K$;
- $\Sigma_{\mathcal{D}} \subset \mathbb{R}^d$ the joint L^2-spectrum of $(D_1, \ldots, D_d)$.

Spectral multipliers

- $\mathcal{D} = (D_1, \ldots, D_d)$ a d-tuple of formally self-adjoint differential operators generating $\mathbb{D}(N)^K$;
- $\Sigma_{\mathcal{D}} \subset \mathbb{R}^d$ the joint L^2-spectrum of $(D_1, \ldots, D_d)$.

Conjecture:

For every nilpotent Gelfand pair, $\mathcal{S}(N)^K \cong \mathcal{S}(\Sigma_{\mathcal{D}})$.

Spectral multipliers

- $\mathcal{D} = (D_1, \ldots, D_d)$ a d-tuple of formally self-adjoint differential operators generating $\mathbb{D}(N)^K$;
- $\Sigma_{\mathcal{D}} \subset \mathbb{R}^d$ the joint L^2-spectrum of $(D_1, \ldots, D_d)$.

Conjecture:

For every nilpotent Gelfand pair, $\mathcal{S}(N)^K \cong \mathcal{S}(\Sigma_{\mathcal{D}})$.

This is a theorem when

- (N, K) is a *Heisenberg pair*, i.e., $N = H_n$ (F. Astengo, B. Di Blasio, F. R.);

Spectral multipliers

- $\mathcal{D} = (D_1, \ldots, D_d)$ a d-tuple of formally self-adjoint differential operators generating $\mathbb{D}(N)^K$;
- $\Sigma_{\mathcal{D}} \subset \mathbb{R}^d$ the joint L^2-spectrum of $(D_1, \ldots, D_d)$.

Conjecture:

For every nilpotent Gelfand pair, $\mathcal{S}(N)^K \cong \mathcal{S}(\Sigma_{\mathcal{D}})$.

This is a theorem when

- (N, K) is a *Heisenberg pair*, i.e., $N = H_n$ (F. Astengo, B. Di Blasio, F. R.);
- (N, K) satisfies *Vinberg's condition* (V. Fischer, F. R., O. Yakimova).

Spherical functions

The (bounded) *spherical functions* of (N, K) are the (bounded) eigenfunctions of all operators in $\mathbb{D}(N)^K$.

Given $\mathcal{D} = \{D_1, \ldots, D_d)$ as above, a bounded spherical function can be labeled by the d-tuple

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\varphi) = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$$

of its eigenvalues relative to $(D_1, \ldots, D_d)$.

Spherical functions

The (bounded) *spherical functions* of (N, K) are the (bounded) eigenfunctions of all operators in $\mathbb{D}(N)^K$.

Given $\mathcal{D} = \{D_1, \ldots, D_d)$ as above, a bounded spherical function can be labeled by the d-tuple

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\varphi) = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$$

of its eigenvalues relative to $(D_1, \ldots, D_d)$.

Then

$$\Sigma_{\mathcal{D}} = \left\{\boldsymbol{\xi}(\varphi) : \varphi \text{ b.s.f.}\right\} ,$$

as topological spaces (with the compact-open topology on the space of b.s.f.).

Spherical transform

- Given $F \in L^1(N)^K$, its spherical transform is

$$\mathcal{G}F(\xi) = \int_N F(x)\varphi_\xi(x^{-1})\,dx \in C_0(\Sigma_{\mathcal{D}}) .$$

Spherical transform

- Given $F \in L^1(N)^K$, its spherical transform is

$$\mathcal{G}F(\xi) = \int_N F(x)\varphi_{\xi}(x^{-1})\,dx \in C_0(\Sigma_{\mathcal{D}}) .$$

- $\mathcal{G}(F * G) = (\mathcal{G}F)(\mathcal{G}G)$.

Spherical transform

- Given $F \in L^1(N)^K$, its spherical transform is

$$\mathcal{G}F(\xi) = \int_N F(x)\varphi_\xi(x^{-1})\,dx \in C_0(\Sigma_{\mathcal{D}})\ .$$

- $\mathcal{G}(F * G) = (\mathcal{G}F)(\mathcal{G}G)$.
- the map $m \longmapsto K_m$ coincides with $\mathcal{G}^{-1}$.

Spherical transform

- Given $F \in L^1(N)^K$, its spherical transform is

$$\mathcal{G}F(\xi) = \int_N F(x)\varphi_\xi(x^{-1})\,dx \in C_0(\Sigma_\mathcal{D})\ .$$

- $\mathcal{G}(F * G) = (\mathcal{G}F)(\mathcal{G}G)$.
- the map $m \longmapsto K_m$ coincides with $\mathcal{G}^{-1}$.

Spherical transform

- Given $F \in L^1(N)^K$, its spherical transform is

$$\mathcal{G}F(\xi) = \int_N F(x)\varphi_\xi(x^{-1})\, dx \in C_0(\Sigma_{\mathcal{D}}) \ .$$

- $\mathcal{G}(F * G) = (\mathcal{G}F)(\mathcal{G}G)$.
- the map $m \longmapsto K_m$ coincides with $\mathcal{G}^{-1}$.

Paradigm for a proof

The implication

$$m \in \mathcal{S}(\Sigma_{\mathcal{D}}) \implies F = \mathcal{G}^{-1} m \in \mathcal{S}(N)^K$$

is always true.

It can be derived starting from a result of A. Hulanicki, stating that, if D is a positive Rockland operator on a homogeneous group, then $m(D)$ is given by convolution with a Schwartz kernel for every $m \in C_c^\infty(\mathbb{R})$.

(The proof is in the Folland-Stein book.)

The other implication for the pair (H_n, U_n)

1. Consider first U_n-invariant Schwartz functions with vanishing moments of any order in the central variable t.

The other implication for the pair (H_n, U_n)

1. Consider first U_n-invariant Schwartz functions with vanishing moments of any order in the central variable t.
2. Observe that, for such a function F, the spherical transform $\mathcal{G}F$ vanishes of infinite order on the horizontal half-line $\xi_T = 0$ in the Heisenberg fan (the singular set).

The other implication for the pair (H_n, U_n)

1. Consider first U_n-invariant Schwartz functions with vanishing moments of any order in the central variable t.
2. Observe that, for such a function F, the spherical transform $\mathcal{G}F$ vanishes of infinite order on the horizontal half-line $\xi_T = 0$ in the Heisenberg fan (the singular set).
3. In this situation, extending $\mathcal{G}F$ to a function in $\mathcal{S}(\mathbb{R}^2)$ is very easy.

4. (D. Geller) For a general U_n-invariant Schwartz function F, show that $\mathcal{G}F$ admits Taylor expansions of any order at $(\xi_L, 0)$:

$$\mathcal{G}F(\xi_L, \xi_T) = \sum_{j=0}^{k} \frac{1}{j!} g_j(\xi_L)\xi_T^j + \xi_T^{k+1}\mathcal{G}R_k(\xi_L, \xi_T) \ ,$$

with $g_j \in \mathcal{S}(\mathbb{R})$ and $R_k \in \mathcal{S}(H_n)^{\mathrm{U}_n}$. In other words

$$F(v,t) = \sum_{j=0}^{k} \frac{1}{j!} T^j G_j + T^{k+1} R_k \ ,$$

with $G_j \in \mathcal{S}(H_n)^{\mathrm{U}_n}$ and $\mathcal{G}G_j(\xi_L, \xi_T) = i\, g_j(\xi_L)$.

4. (D. Geller) For a general U_n-invariant Schwartz function F, show that $\mathcal{G}F$ admits Taylor expansions of any order at $(\xi_L, 0)$:

$$\mathcal{G}F(\xi_L, \xi_T) = \sum_{j=0}^{k} \frac{1}{j!} g_j(\xi_L)\xi_T^j + \xi_T^{k+1}\mathcal{G}R_k(\xi_L, \xi_T) \ ,$$

with $g_j \in \mathcal{S}(\mathbb{R})$ and $R_k \in \mathcal{S}(H_n)^{\mathrm{U}_n}$. In other words

$$F(v,t) = \sum_{j=0}^{k} \frac{1}{j!} T^j G_j + T^{k+1} R_k \ ,$$

with $G_j \in \mathcal{S}(H_n)^{\mathrm{U}_n}$ and $\mathcal{G}G_j(\xi_L, \xi_T) = i\, g_j(\xi_L)$.

5. This gives a Schwartz jet $\{g_j\}_{j\in\mathbb{N}}$ on $\mathbb{R} \times \{0\}$ and the final extension of $\mathcal{G}F$ to $\mathbb{R}^2$ can be obtained via the Whitney extension theorem.

A second-level example

A second-level example

Take $N = \mathbb{C}^n \times H_n$, $K = \mathrm{U}_1 \times \mathrm{SU}_n \times \mathrm{U}_1$:

$$(e^{i\theta}, k, e^{i\varphi}) \cdot (z, v, t) = (e^{i\theta} k\, z, e^{i\varphi} k\, v, t) \ ,$$

$(z, v \in \mathbb{C}^n,\ t \in \mathbb{R})$.

A second-level example

Take $N = \mathbb{C}^n \times H_n$, $K = \mathrm{U}_1 \times \mathrm{SU}_n \times \mathrm{U}_1$:

$$(e^{i\theta}, k, e^{i\varphi}) \cdot (z, v, t) = (e^{i\theta} k\, z, e^{i\varphi} k\, v, t) \ ,$$

$(z, v \in \mathbb{C}^n, t \in \mathbb{R})$.

Fundamental invariants:

$$|z|^2 \ , \quad \big|\langle z, v\rangle\big|^2 \ , \quad |v|^2 \ , \quad t \ .$$

This gives a system $\mathcal{D}$ of 4 differential operators, where

- $D_1 = \Delta_z$ is the Laplacian on $\mathbb{C}^n$;
- D_2 is a 4-th order operator, mixing derivatives in $\mathbb{C}^n$ with vector fields on H_n;
- D_3 is the sublaplacian on H_n;
- $D_4 = \partial_t$ is the central derivative on H_n.

The regular set

For each point $\xi \in \Sigma_{\mathcal{D}}$, the coordinate ξ_1 (the eigenvalue of the spherical function under the action of $D_1 = \Delta_z$) has a particular relevance.

The regular set

For each point $\xi \in \Sigma_{\mathcal{D}}$, the coordinate ξ_1 (the eigenvalue of the spherical function under the action of $D_1 = \Delta_z$) has a particular relevance.

Assume that $\xi_1 \neq 0$. The following properties hold:

- in a neighborhood V of ξ, $\Sigma_{\mathcal{D}}$ is diffeomorphic to the spectrum $\Sigma'_{\mathcal{D}'}$ associated to another pair,

$$(N', K') = (\mathbb{R} \times H_n, \mathrm{SU}_{n-1} \times \mathrm{U}_1)\ ;$$

The regular set

For each point $\xi \in \Sigma_{\mathcal{D}}$, the coordinate ξ_1 (the eigenvalue of the spherical function under the action of $D_1 = \Delta_z$) has a particular relevance.

Assume that $\xi_1 \neq 0$. The following properties hold:

- in a neighborhood V of ξ, $\Sigma_{\mathcal{D}}$ is diffeomorphic to the spectrum $\Sigma'_{\mathcal{D}'}$ associated to another pair,

$$(N', K') = (\mathbb{R} \times H_n, \mathrm{SU}_{n-1} \times \mathrm{U}_1) ;$$

- modulo this diffeomorphism, the spherical transform $\mathcal{G}F$ of $F \in \mathcal{S}(N)^K$ coincides on V with $\mathcal{G}'F'$, where

$$F'(s, v, t) = \int_{\mathbb{R} \times \mathbb{C}^{n-1}} F(s + iu, z_2, \ldots, z_n, v, t)\, du\, dz_1 \ldots dz_n .$$

The singular set

If $\boldsymbol{\xi} \in \Sigma_{\mathcal{D}}$ has $\xi_1 = 0$, the corresponding spherical function does not depend on $z \in \mathbb{C}^n$ and is in fact a spherical function for the pair $(N'', K'') = (H_n, \mathrm{U}_n)$.
Then also $\xi_2 = 0$, and

$$\Sigma_{\mathcal{D}}^{\mathrm{sing}} = \{\boldsymbol{\xi} \in \Sigma_{\mathcal{D}} : \xi_1 = \xi_2 = 0\}$$

is a Heisenberg fan in the coordinate plane (ξ_3, ξ_4).

The singular set

If $\boldsymbol{\xi} \in \Sigma_{\mathcal{D}}$ has $\xi_1 = 0$, the corresponding spherical function does not depend on $z \in \mathbb{C}^n$ and is in fact a spherical function for the pair $(N'', K'') = (H_n, \mathrm{U}_n)$.
Then also $\xi_2 = 0$, and

$$\Sigma_{\mathcal{D}}^{\text{sing}} = \{\boldsymbol{\xi} \in \Sigma_{\mathcal{D}} : \xi_1 = \xi_2 = 0\}$$

is a Heisenberg fan in the coordinate plane (ξ_3, ξ_4).

Given $F \in \mathcal{S}(N)^K$, the restriction of $\mathcal{G}F$ to $\Sigma_{\mathcal{D}}^{\text{sing}}$ coincides with $\mathcal{G}''F''$, where

$$F''(v,t) = \int_{\mathbb{C}^n} F(z,v,t)\,dz\ .$$

Taylor expansion

We know then that $\mathcal{G}F$ admits a Schwartz extension to the (ξ_3, ξ_4) coordinate plane.
What we need now is a Schwartz jet $\{g_{j,k}\}_{j,k\in\mathbb{N}}$ on this plane, to describe the behaviour of $\mathcal{G}F$ if we move from $\Sigma_{\mathcal{D}}^{\text{sing}}$ in the (ξ_1, ξ_2) directions.

Taylor expansion

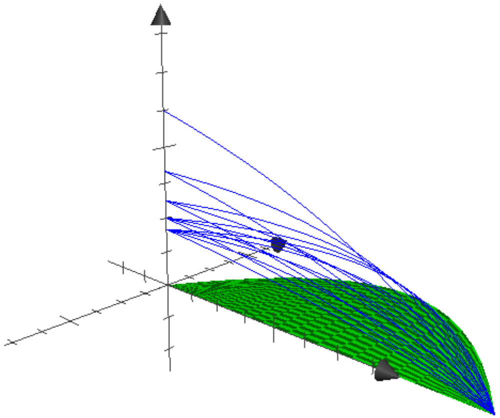
We know then that $\mathcal{G}F$ admits a Schwartz extension to the (ξ_3, ξ_4) coordinate plane.
What we need now is a Schwartz jet $\{g_{j,k}\}_{j,k\in\mathbb{N}}$ on this plane, to describe the behaviour of $\mathcal{G}F$ if we move from $\Sigma_{\mathcal{D}}^{\mathrm{sing}}$ in the (ξ_1, ξ_2) directions.

Proposition

There exist functions $G_{j,k} \in \mathcal{S}(N)^K$, with $\mathcal{G}G_{j,k}$ depending only on (ξ_3, ξ_4), such that, for every $p \in \mathbb{N}$,

$$F = \sum_{j+k\leq p} \frac{1}{j!k!} D_1^j D_2^k G_{j,k} + \sum_{|\alpha|=2p+2} \partial_z^\alpha R_\alpha \ ,$$

for appropriate $R_\alpha \in \mathcal{S}(N)$.



On the structure of nilpotent Gelfand pairs

Let (N, K) be a nilpotent Gelfand pair.

On the structure of nilpotent Gelfand pairs

Let (N, K) be a nilpotent Gelfand pair.

- N is at most two-step (Benson-Jenkins-Ratcliff). Then $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ is abelian and $\mathfrak{v}$ is a K-invariant complementary subspace.

On the structure of nilpotent Gelfand pairs

Let (N, K) be a nilpotent Gelfand pair.

- N is at most two-step (Benson-Jenkins-Ratcliff). Then $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ is abelian and $\mathfrak{v}$ is a K-invariant complementary subspace.
- Assume that $\mathfrak{v}$ is irreducible under the action of K (*Vinberg's condition*). Then $\mathfrak{z}$ is the center of $\mathfrak{n}$ and decomposes as $\check{\mathfrak{z}} \oplus \mathfrak{z}_0$, where
  (i) $\check{\mathfrak{z}}$ consists of the K-fixed elements of $\mathfrak{z}$;
  (ii) K acts irreducibly on $\mathfrak{z}_0$.

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;
- $K = \mathrm{U}_n$, $k \cdot (v, z) = (kv, kzk^*)$;

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;
- $K = \mathrm{U}_n$, $k \cdot (v, z) = (kv, kzk^*)$;
- then $\quad \check{\mathfrak{z}} = i\,\mathbb{R}, \quad \mathfrak{z}_0 = \mathfrak{su}_n$.

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;
- $K = \mathrm{U}_n$, $k \cdot (v, z) = (kv, kzk^*)$;
- then $\quad \check{\mathfrak{z}} = i\,\mathbb{R}, \quad \mathfrak{z}_0 = \mathfrak{su}_n$.

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;
- $K = \mathrm{U}_n$, $k \cdot (v, z) = (kv, kzk^*)$;
- then $\quad \check{\mathfrak{z}} = i\,\mathbb{R}, \quad \mathfrak{z}_0 = \mathfrak{su}_n$.

The natural choice of $\mathcal{D}$ consists of $2n$ free generators, obtained by symmetrization from:

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;
- $K = \mathrm{U}_n$, $k \cdot (v, z) = (kv, kzk^*)$;
- then $\check{\mathfrak{z}} = i\,\mathbb{R}$, $\mathfrak{z}_0 = \mathfrak{su}_n$.

The natural choice of $\mathcal{D}$ consists of $2n$ free generators, obtained by symmetrization from:

- $|v|^2$ (the sublaplacian);

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;
- $K = \mathrm{U}_n$, $k \cdot (v, z) = (kv, kzk^*)$;
- then $\check{\mathfrak{z}} = i\,\mathbb{R}$, $\mathfrak{z}_0 = \mathfrak{su}_n$.

The natural choice of $\mathcal{D}$ consists of $2n$ free generators, obtained by symmetrization from:

- $|v|^2$ (the sublaplacian);
- $q_k(v, z) = v^*(z^k)v$, $k = 1, \ldots, n-1$.

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;
- $K = \mathrm{U}_n$, $k \cdot (v, z) = (kv, kzk^*)$;
- then $\check{\mathfrak{z}} = i\mathbb{R}$, $\mathfrak{z}_0 = \mathfrak{su}_n$.

The natural choice of $\mathcal{D}$ consists of $2n$ free generators, obtained by symmetrization from:

- $|v|^2$ (the sublaplacian);
- $q_k(v, z) = v^*(z^k)v$, $k = 1, \dots, n-1$.
- the coordinate function on $\check{\mathfrak{z}}$;

The pair $(\mathbb{C}^n \oplus \mathfrak{u}_n, \mathrm{U}_n)$

- $\mathfrak{n} = \mathbb{C}^n \oplus \mathfrak{u}_n = \mathfrak{v} \oplus \mathfrak{z}$;
- for $v, v' \in \mathbb{C}^n$, set $[v, v'] = vv'^* - v'v^* \in \mathfrak{u}_n$;
- $K = \mathrm{U}_n$, $k \cdot (v, z) = (kv, kzk^*)$;
- then $\check{\mathfrak{z}} = i\,\mathbb{R}$, $\mathfrak{z}_0 = \mathfrak{su}_n$.

The natural choice of $\mathcal{D}$ consists of $2n$ free generators, obtained by symmetrization from:

- $|v|^2$ (the sublaplacian);
- $q_k(v, z) = v^*(z^k)v$, $k = 1, \ldots, n-1$.
- the coordinate function on $\check{\mathfrak{z}}$;
- $p_k(z_0) = \mathrm{tr}(z_0^k)$, $k = 2, \ldots, n$, $z_0 \in \mathfrak{z}_0$;

Regular set and quotient pairs

- To each point ξ of $\Sigma_{\mathcal{D}}$ we can associate a K-orbit in $\mathfrak{z}_0 = \mathfrak{su}_n$.
  Assume that this is not the trivial orbit, i.e. $(\xi_{n+2}, \ldots, \xi_{2n}) \neq (0, \ldots, 0)$.

Regular set and quotient pairs

- To each point ξ of $\Sigma_{\mathcal{D}}$ we can associate a K-orbit in $\mathfrak{z}_0 = \mathfrak{su}_n$.
  Assume that this is not the trivial orbit, i.e. $(\xi_{n+2}, \ldots, \xi_{2n}) \neq (0, \ldots, 0)$.
- The normal space $\mathfrak{z}_\xi$ to this orbit can be used to define a quotient Lie algebra $\mathfrak{n}_\xi = \mathfrak{v} \oplus (\check{\mathfrak{z}} \oplus \mathfrak{z}_\xi)$.

Regular set and quotient pairs

- To each point ξ of $\Sigma_{\mathcal{D}}$ we can associate a K-orbit in $\mathfrak{z}_0 = \mathfrak{su}_n$.
  Assume that this is not the trivial orbit, i.e. $(\xi_{n+2}, \ldots, \xi_{2n}) \neq (0, \ldots, 0)$.
- The normal space $\mathfrak{z}_\xi$ to this orbit can be used to define a quotient Lie algebra $\mathfrak{n}_\xi = \mathfrak{v} \oplus (\check{\mathfrak{z}} \oplus \mathfrak{z}_\xi)$.
- This Lie algebra is isomorphic to a product

$$(\mathbb{C}^{p_1} \oplus \mathfrak{u}_{p_1}) \oplus \cdots \oplus (\mathbb{C}^{p_k} \oplus \mathfrak{u}_{p_k}) \,, \qquad p_1 + \cdots + p_k = n \,.$$

Regular set and quotient pairs

- To each point ξ of $\Sigma_{\mathcal{D}}$ we can associate a K-orbit in $\mathfrak{z}_0 = \mathfrak{su}_n$.
  Assume that this is not the trivial orbit, i.e. $(\xi_{n+2}, \dots, \xi_{2n}) \neq (0, \dots, 0)$.
- The normal space $\mathfrak{z}_\xi$ to this orbit can be used to define a quotient Lie algebra $\mathfrak{n}_\xi = \mathfrak{v} \oplus (\check{\mathfrak{z}} \oplus \mathfrak{z}_\xi)$.
- This Lie algebra is isomorphic to a product

$$(\mathbb{C}^{p_1} \oplus \mathfrak{u}_{p_1}) \oplus \cdots \oplus (\mathbb{C}^{p_k} \oplus \mathfrak{u}_{p_k}) \ , \qquad p_1 + \cdots + p_k = n \ .$$

- The stabilizer K_ξ of the orbit is isomorphic to

$$\mathrm{U}_{p_1} \oplus \cdots \oplus \mathrm{U}_{p_k} \ .$$

Regular set and quotient pairs

- To each point ξ of $\Sigma_{\mathcal{D}}$ we can associate a K-orbit in $\mathfrak{z}_0 = \mathfrak{su}_n$.
  Assume that this is not the trivial orbit, i.e. $(\xi_{n+2}, \ldots, \xi_{2n}) \neq (0, \ldots, 0)$.
- The normal space $\mathfrak{z}_\xi$ to this orbit can be used to define a quotient Lie algebra $\mathfrak{n}_\xi = \mathfrak{v} \oplus (\check{\mathfrak{z}} \oplus \mathfrak{z}_\xi)$.
- This Lie algebra is isomorphic to a product

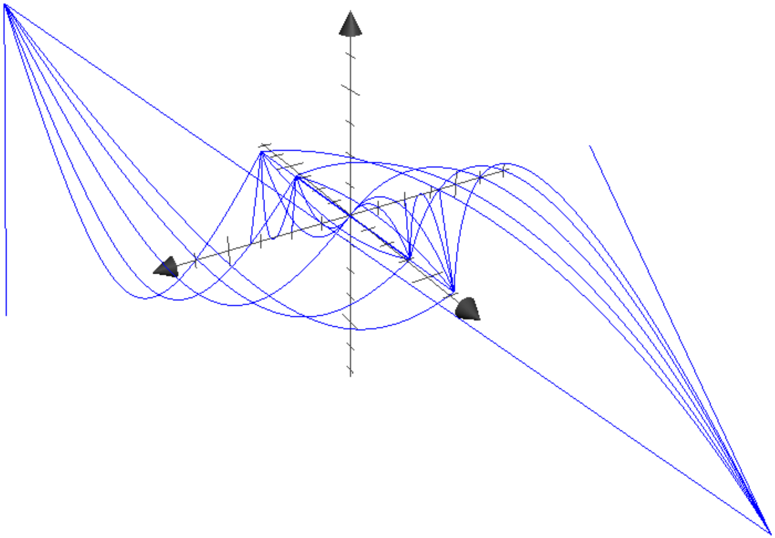
$$(\mathbb{C}^{p_1} \oplus \mathfrak{u}_{p_1}) \oplus \cdots \oplus (\mathbb{C}^{p_k} \oplus \mathfrak{u}_{p_k}) , \qquad p_1 + \cdots + p_k = n .$$

- The stabilizer K_ξ of the orbit is isomorphic to

$$\mathrm{U}_{p_1} \oplus \cdots \oplus \mathrm{U}_{p_k} .$$

- In the neighborhood of ξ, $\Sigma_{\mathcal{D}}$ is diffeomorphic to

$$\Sigma_{\mathcal{D}_1} \times \cdots \times \Sigma_{\mathcal{D}_k} .$$



The singular set

As before, $\Sigma_{\mathcal{D}}^{\mathrm{sing}}$ is naturally identified with the spectrum of another quotient pair, $(\check{N}, \mathrm{U}_n)$, where

$$\check{N} = \exp(\mathfrak{v} \oplus \check{\mathfrak{z}}) \cong H_n\ .$$

The singular set

As before, $\Sigma_{\mathcal{D}}^{\text{sing}}$ is naturally identified with the spectrum of another quotient pair, $(\check{N}, \mathrm{U}_n)$, where

$$\check{N} = \exp(\mathfrak{v} \oplus \check{\mathfrak{z}}) \cong H_n \ .$$

So $\Sigma_{\mathcal{D}}^{\text{sing}}$ is again a Heisenberg fan, in the coordinate plane $(\xi_1, \xi_{n+1}) = (\xi_L, \xi_T)$.
Again, we want a jet $\{g_\alpha(\xi_L, \xi_T)\}_{\alpha \in \mathbb{N}^{2n-2}}$ such that

$$\mathcal{G}F(\boldsymbol{\xi}) \sim \sum_\alpha \frac{1}{\alpha!} \xi'^\alpha g_\alpha(\xi_L, \xi_T) \ .$$

The inductive statement

Inductively, this reduces to proving the following:

If $F \in \mathcal{S}(N)^K$ has vanishing moments in the $\mathfrak{z}_0$-variables up to order p, i.e.,

$$F = \sum_{|\beta|=p} \partial_{z_0}^{\beta} F_{\beta} \ ,$$

then

$$F = \sum_{|\alpha|=p} D'^{\alpha} G_{\alpha} + \sum_{|\gamma|=p+1} \partial_{z_0}^{\gamma} F_{\gamma} \ ,$$

where $G_\alpha \in \mathcal{S}(N)^K$, $\mathcal{G} G_\alpha$ only depends on (ξ_L, ξ_T).

Representation-theoretic formulation

$$F = \sum_{|\beta|=p} \partial_{z_0}^{\beta} F_{\beta} \ ,$$

Representation-theoretic formulation

$$F = \sum_{|\beta|=p} \partial_{z_0}^{\beta} F_\beta \ ,$$

Identify F with the vector-valued function

$$\mathbf{F} = (F_\beta)_{|\beta|=p} : N \longrightarrow \mathcal{P}^p(\mathfrak{su}_n) \ .$$

Decompose

$$\mathcal{P}^p(\mathfrak{su}_n) \cong \sum_{\mu \in E_p} \mathcal{V}_\mu \ ,$$

with $\mathcal{V}_\mu$ irreducible under U_n. Then

$$\mathbf{F} = \sum_{\mu \in E_p} \mathbf{F}_\mu \ .$$

For $m \in \mathbb{N}$, let $\mathcal{P}^m(\mathbb{C}^n)$ the space of holomorphic homogeneous polynomials of degree m on $\mathbb{C}^n$.

For $m \in \mathbb{N}$, let $\mathcal{P}^m(\mathbb{C}^n)$ the space of holomorphic homogeneous polynomials of degree m on $\mathbb{C}^n$.

Equivalent condition:

For every m and $\mu \in E_p$, $\mathcal{P}^m(\mathbb{C}^n)$ is contained, as a representation space, with multiplicity at most one in $\mathcal{P}^m(\mathbb{C}^n) \otimes \mathcal{V}_\mu$.

Consequences

Assume that the conjecture is true for (N, K). The following one-to-one correspondences hold:

Consequences

Assume that the conjecture is true for (N, K). The following one-to-one correspondences hold:

- smooth K-invariant Calderón-Zygmund kernels on N

  $$\updownarrow$$

  restrictions to $\Sigma_{\mathcal{D}}$ of smooth Mihlin-Hörmander multipliers on $\mathbb{R}^d$;

Consequences

Assume that the conjecture is true for (N, K). The following one-to-one correspondences hold:

- smooth K-invariant Calderón-Zygmund kernels on N

  $$\updownarrow$$

  restrictions to $\Sigma_{\mathcal{D}}$ of smooth Mihlin-Hörmander multipliers on $\mathbb{R}^d$;

- smooth K-invariant flag kernels on N adapted to the flag $\{0\} \subset \mathfrak{z} \subset N$

  $$\updownarrow$$

  restrictions to $\Sigma_{\mathcal{D}}$ of smooth flag multipliers on $\mathbb{R}^d$ adapted to the flag $\{0\} \subset \mathbb{R}^k \subset \mathbb{R}^d$, where $\mathbb{R}^k$ is the subspace spanned by the coordinates ξ_D with D containing only $\mathfrak{z}$-derivatives.

Extension operators

The fact that $\mathcal{G} : \mathcal{S}(N)^K \longrightarrow \mathcal{S}(\Sigma_{\mathcal{D}})$ is an isomorphism means that

given any $p \in \mathbb{N}$, there is $q \in \mathbb{N}$ such that every $F \in \mathcal{S}(N)^K$ admits an extension $g_p \in \mathcal{S}(\mathbb{R}^d)$ with $\|g_p\|_{(p)} \leq C_p \|F\|_{(q)}$.

It does not say if the extension can be taken independent of p, or depending linearly on F.

Extension operators

The fact that $\mathcal{G} : \mathcal{S}(N)^K \longrightarrow \mathcal{S}(\Sigma_{\mathcal{D}})$ is an isomorphism means that

given any $p \in \mathbb{N}$, there is $q \in \mathbb{N}$ such that every $F \in \mathcal{S}(N)^K$ admits an extension $g_p \in \mathcal{S}(\mathbb{R}^d)$ with $\|g_p\|_{(p)} \leq C_p \|F\|_{(q)}$.

It does not say if the extension can be taken independent of p, or depending linearly on F.

A natural question is the existence of a continuous linear *extension operator* $\mathcal{E} : \mathcal{S}(\Sigma_{\mathcal{D}}) \longrightarrow \mathcal{S}(\mathbb{R}^d)$, i.e., such that $(\mathcal{E}g)_{|_{\Sigma_{\mathcal{D}}}} = g$.

Extension operators

The fact that $\mathcal{G} : \mathcal{S}(N)^K \longrightarrow \mathcal{S}(\Sigma_{\mathcal{D}})$ is an isomorphism means that

given any $p \in \mathbb{N}$, there is $q \in \mathbb{N}$ such that every $F \in \mathcal{S}(N)^K$ admits an extension $g_p \in \mathcal{S}(\mathbb{R}^d)$ with $\|g_p\|_{(p)} \le C_p\|F\|_{(q)}$.

It does not say if the extension can be taken independent of p, or depending linearly on F.

A natural question is the existence of a continuous linear *extension operator* $\mathcal{E} : \mathcal{S}(\Sigma_{\mathcal{D}}) \longrightarrow \mathcal{S}(\mathbb{R}^d)$, i.e., such that $(\mathcal{E}g)_{|_{\Sigma_{\mathcal{D}}}} = g$.

Such an operator exists for abelian pairs. This is a direct consequence of a theorem of J. Mather.

Extension operators

The fact that $\mathcal{G} : \mathcal{S}(N)^K \longrightarrow \mathcal{S}(\Sigma_{\mathcal{D}})$ is an isomorphism means that

given any $p \in \mathbb{N}$, there is $q \in \mathbb{N}$ such that every $F \in \mathcal{S}(N)^K$ admits an extension $g_p \in \mathcal{S}(\mathbb{R}^d)$ with $\|g_p\|_{(p)} \le C_p \|F\|_{(q)}$.

It does not say if the extension can be taken independent of p, or depending linearly on F.

A natural question is the existence of a continuous linear *extension operator* $\mathcal{E} : \mathcal{S}(\Sigma_{\mathcal{D}}) \longrightarrow \mathcal{S}(\mathbb{R}^d)$, i.e., such that $(\mathcal{E}g)_{|_{\Sigma_{\mathcal{D}}}} = g$.

Such an operator exists for abelian pairs. This is a direct consequence of a theorem of J. Mather.

Such an operator exists for the Heisenberg fan.
(C. Fefferman, F. R.)