# Eli's Impact:
# A Case Study

(Slides by Frances Wroblewski)

Major Ideas of Eli Include:

- Unexpected Irreducible Representations of Semisimple Lie Groups
- Cotlar-Stein Lemma on Almost Orthogonal Operators

- Kunze-Stein Phenomenon
- Stein Interpolation Theorem
- First Restriction Thm for Fourier Transforms
- Stein-Weiss and CF-Stein $H^p$ Theories

- $\overline{\partial}$ and $\overline{\partial}_b$ Problems,
  First on Strongly
  Pseudoconvex Domains,

  Then in greater generality.
  (Folland-Stein, Greiner-Stein,
  Nagel-Stein $\cdots$ )

- Multiparameter Singular Integrals on Flag Manifolds (Ricci-Stein)
- Many Others

## Analysis and Applications:

# A Conference in Honor of ELIAS M. STEIN

May 15-21, 2011
A02 McDonnell Hall
Princeton University
Princeton, New Jersey

BANQUET: THURSDAY, MAY 19, 2011 PROSPECT HOUSE, PRINCETON UNIVERSITY

Confirmed Speakers Include:

**JEAN BOURGAIN**, INSTITUTE FOR ADVANCED STUDY, PRINCETON | **LUIS CAFFARELLI**, UNIVERSITY OF TEXAS AT AUSTIN | **SUN-YUNG ALICE CHANG**, PRINCETON UNIVERSITY | **MICHAEL CHRIST**, UNIVERSITY OF CALIFORNIA, BERKELEY | **INGRID DAUBECHIES**, PRINCETON UNIVERSITY | **GUY DAVID**, UNIVERSITÉ DE PARIS-SUD – ORSAY | **CHARLES FEFFERMAN**, PRINCETON UNIVERSITY | **ALEX IONESCU**, PRINCETON UNIVERSITY | **DAVID JERISON**, MASSACHUSETTS INSTITUTE OF TECHNOLOGY | **PETER JONES**, YALE UNIVERSITY | **CARLOS KENIG**, UNIVERSITY OF CHICAGO | **SERGIU KLAINERMAN**, PRINCETON UNIVERSITY | **JOSEPH KOHN**, PRINCETON UNIVERSITY | **DETLEF MÜLLER**, CHRISTIAN-ALBRECHTS-UNIVERSITÄT ZU KIEL | **ALEX NAGEL**, UNIVERSITY OF WISCONSIN, MADISON | **LOUIS NIRENBERG**, COURANT INSTITUTE, NEW YORK UNIVERSITY | **DUONG PHONG**, COLUMBIA UNIVERSITY | **FULVIO RICCI**, SCUOLA NORMALE SUPERIORE DI PISA | **LINDA ROTHSCHILD**, UNIVERSITY OF CALIFORNIA, SAN DIEGO | **ANDREAS SEEGER**, UNIVERSITY OF WISCONSIN, MADISON | **YAKOV G. SINAI**, PRINCETON UNIVERSITY | **CHRISTOPHER SOGGE**, JOHNS HOPKINS UNIVERSITY | **JEREMY STEIN**, HARVARD UNIVERSITY | **CHRISTOPH THIELE**, UNIVERSITY OF CALIFORNIA, LOS ANGELES | **TERENCE TAO**, UNIVERSITY OF CALIFORNIA, LOS ANGELES | **STEPHEN WAINGER**, UNIVERSITY OF WISCONSIN, MADISON | **GREGG ZUCKERMAN**, YALE UNIVERSITY

For Additional Information and Registration: www.math.princeton.edu/conference/stein2011

- Littlewood-Paley Theory in Many Settings

- Littlewood-Paley Theory was one of the deepest parts of the classical study of Fourier Series in One Variable.

- Eli found the right viewpoint to develop Littlewood-Paley Theory on $\mathbb{R}^n$.

- He went on to develop Littlewood-Paley Theory on any compact Lie group, and then in any setting in which there is a heat kernel.

- Eli realized that there is a deep connection between ideas in Littlewood-Paley theory and the $\bar{\partial}$-problems in several complex variables.

Together with several co-authors (Folland, Greiner, Nagel, Ricci, Rothschild, . . . ) he carried out Analysis on Nilpotent Lie Groups and applied that analysis to PDE and Several Complex Variables.

- By his writing, his teaching, and his collaborations, Eli has disseminated those ideas, to the extent that they are now part of the viewpoint of most analysts.

Those ideas have had striking impact in unexpected places.

(Stay tuned!)

## Littlewood-Paley Theory

Start with a real-valued function $f(x)$ on $\mathbb{R}^n$.

Let $\hat{f}(\xi)$ be the Fourier transform of $f$.

# Partition of Unity

$$1 = \sum_{k=-\infty}^{\infty} \chi_k(\xi) \text{ on } \mathbb{R}^n \backslash \{0\}$$

- $\chi_k(\xi)$ supported on $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$
- $|\partial^\alpha \chi_k(\xi)| \leq C_\alpha 2^{-k|\alpha|}$ (each $\alpha$ )

Define $f_k$ by setting

$$\hat{f}_k(\xi) = \chi_k(\xi) \cdot \hat{f}(\xi)$$

Then define

$$G(f)(x) = \left( \sum_{k=-\infty}^{\infty} |f_k(x)|^2 \right)^{\frac{1}{2}} .$$

# Littlewood-Paley Theorem

For $1 < p < \infty$,

$f \in L^p(\mathbb{R}^n) \Leftrightarrow G(f) \in L^p(\mathbb{R}^n)$.

Moreover

$$c\|f\|_{L^p(\mathbb{R}^n)} \leq \|G(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

where $c$ and $C$ depend only on $p$ and $n$.

## Classical Version

(Littlewood, Paley, Marcinkiewicz, Zygmund) used complex variables. An essential tool was the Blaschke Product

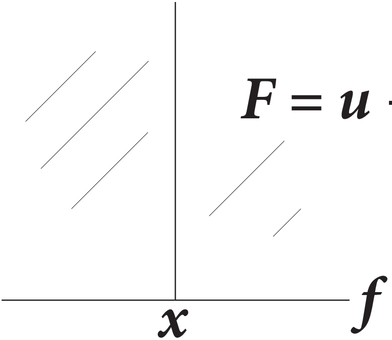
$$B(z) = \prod_{\nu} \left( e^{i\theta_\nu} \cdot \frac{z - z_\nu}{1 - \bar{z}_\nu z} \right)$$

$$F(z) \quad = \quad \tilde{F}(z) \, \cdot \, B(z)$$

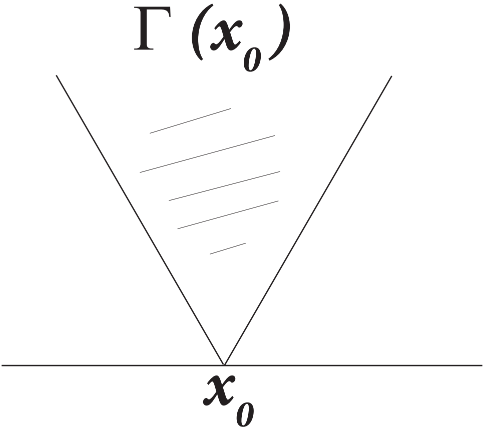
Given $f(x)$ on $\mathbb{R}$, pass to the Poisson integral $U(x+iy)$ on $\mathbb{R}^2_+$, and then to the conjugate harmonic function $V(x+iy)$.

$F = U + iV \quad$ is analytic in the upper half-plane.

# Littlewood-Paley Functions




$$g(f)(x) = \left( \int_0^\infty y \left| F'(x+iy) \right|^2 dy \right)^{\frac{1}{2}}$$




$$S(f)(x_0) =$$

$$\left( \int_{z=x+iy \in \Gamma(x_0)} |F'(z)|^2 \, dxdy \right)^{\frac{1}{2}}$$

Another variant
$g_\lambda^\star(f)$

The functions
$g(f), \quad S(f), \quad g_\lambda^\star(f)$
are strongly tied to complex variables.

They can be controlled using the Blaschke factorization.

The functions

$$g(f), \quad S(f), \quad g_\lambda^\star(f)$$

are close enough to $G(f)$ that they can be used to prove the Littlewood-Paley Theorem.

Enter Eli . . .

Eli viewed Littlewood-Paley theory as an application of the Theory of Singular Integrals.

# The Calderón-Zygmund Decomposition

(Credit also to Marcinkiewicz, Whitney)

Given $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, decompose $f$ into a "good" function and a "bad" function

$f = g + b, \quad$ where

- $g \in L^2(\mathbb{R}^n)$ with estimate

$$\int_{\mathbb{R}^n} |g(x)|^2 dx \leq C\lambda \|f\|_{L^1(\mathbb{R}^n)}.$$

- $b$ is supported in pairwise disjoint cubes $Q_\nu$, and has integral zero on each $Q_\nu$.

Moreover

$$\int_{Q_\nu} |b(x)|dx \leq C\lambda |Q_\nu| \text{ for each } \nu,$$

and

$$\sum_\nu |Q_\nu| \leq \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

The *CZ* Decomposition was used to analyze Singular Integral Operators, such as the Riesz transforms

$$\left(\frac{\partial}{\partial x_j}\right)(-\Delta_x)^{-\frac{1}{2}}$$

on $L^p(\mathbb{R}^n)$.

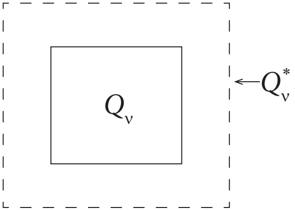
Eli saw that the $CZ$ decomposition can be used to understand

$$g(f), \quad S(f), \quad g^\star_\lambda(f), \quad G(f)$$

because

$$g(b), \quad S(b), \quad g^\star_\lambda(b), \quad G(b)$$

are easily estimated outside $\bigcup_\nu Q^\star_\nu$.



Eli's work gave the first real understanding (pun intended) of Littlewood-Paley theory.

In the late 60's, Eli showed that Littlewood-Paley Theory could be generalized further:

- Compact Lie Groups
- Any setting in which there is a heat kernel.

Eli then turned his attention to Littlewood-Paley Theory relevant to Complex Analysis on the unit ball in $\mathbb{C}^n$.

He saw the RIGHT POINT OF VIEW from which

Complex Analysis<br>on STRICTLY<br>PSEUDOCONVEX<br>DOMAINS

is closely analogous to

Basic Potential Theory<br>on $\mathbb{R}^n$.

After a linear fractional transf., the unit sphere in $\mathbb{C}^{n+1}$ can be viewed as a nilpotent Lie group $\mathbb{H}$. A point of $\mathbb{H}$ has the form $(z, t)$ with
$z \in \mathbb{C}^n, \; t \in \mathbb{R}$.

<u>Group law</u>:

$$(z, t)\cdot(z', t') = (z+z', t+t'+Im\, z\cdot\overline{z}')$$

Natural DILATIONS on $H^n$:

$$S_\lambda : (z, t) \to (\lambda z,\ \lambda^2 t).$$

Therefore, if

$$(z, t)^{-1} \cdot (z', t') = (z'', t'')$$

in $\mathbb{H}^n$, then the natural DISTANCE between $(z, t)$ and $(z', t')$ is

$$\boxed{d((z, t),\ (z', t')) \approx |z''| + |t''|^{\frac{1}{2}}}$$

Eli's ANALOGY between

Basic Potential Theory on $\mathbb{R}^n$

and

Complex Analysis on
on Strictly Pseudoconvex
Domains

# The Group

For basic pot. theory $\mathbb{R}^n$

For complex analysis $\mathbb{H}^n$

# The Basic PDE

For pot. theory $\quad \Delta u = f$

For complex analysis,
$\overline{\partial} u = \alpha, \qquad \overline{\partial}_b u = \alpha$

$\overline{\partial}$-Neumann problem, $\quad \Box_b$

# Fundamental Solution

- Basic Pot. Theory

$$\Delta u = f \qquad \text{solved by}$$

$$U(x) = c_n \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-2}}$$

- Complex Analysis

$$\Box_b w = \alpha \qquad \text{solved by}$$

$$w(x) = \int_{\mathbb{H}^n} K(x,y)\alpha(y)dy$$

where

$$K(x,y) \approx (d(x,y))^{-power}$$

# Sharp Estimates for Solutions

- Basic Pot. Theory
  Sharp estimates arise from SINGULAR INTEGRAL OPERATORS

- Complex Analysis

  Need analogues of singular integral operators on the Heisenberg group $\mathbb{H}^n$.

That's only the beginning of the story.

Eli's analogy extends to lots of other domains in $\mathbb{C}^n$, and to lots of other related PDE's.

Eli's ideas continue to exert a profound influence.

To illustrate, it would be natural to discuss:

- WAVELETS
- Coifman's ideas on imbedding large data sets into a low- dimensional Euclidean space;

- Use of additional info by Amit Singer

- The work of Klainerman-Rodnianski
  on
  General Relativity.

The rest of this lecture will be devoted to ...

# The Boltzmann Equation

Setup:

$$\begin{array}{ll} x \in \mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3 & \text{position} \\ v \in \mathbb{R}^3 & \text{velocity} \\ t \in [0, \infty) & \text{time} \end{array}$$

$F(v, x, t) =$

Density of particles per unit volume in $(v, x)$ – space $\mathbb{R}^3 \times \mathbb{T}^3$ at time $t$.

## What Happens to the Particles

- **Transport**: A particle with position $x$ and velocity $v$ at time $t$ will have position $x + v \cdot \Delta t$ and velocity $v$ at time $t + \Delta t$.

- **Elastic Binary Collisions**: A particle with position $x$ and velocity $v$ may collide at time $t$ with another particle with velocity $v_\star$ at position $x$. After the collision, the two particles at $x$ have velocities $v'$ and $v'_\star$, respectively.

## Conservation of Energy & Momentum:

$$v' = \frac{v + v_\star}{2} + \frac{1}{2}|v - v_\star|\sigma$$

$$v'_\star = \frac{v + v_\star}{2} - \frac{1}{2}|v - v_\star|\sigma,$$

where $\sigma \in S^2$.

Let $\theta$ be the angle between the vectors $v' - v'_\star$ and $v - v_\star$

(or, equivalently, between $\sigma$ and $v - v_\star$).

# Boltzmann Equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F).$$

For each fixed $(x, t)$, $Q(F, G)(v) =$

$$\int_{\mathbb{R}^3} dv_\star \int_{S^2} d\sigma B(v - v_\star, \sigma) \left[F'_\star G' - F_\star G\right],$$

where

$$G = G(v),\ G' = G(v'),\ F_\star = F(v_\star),\ F'_\star = F(v'_\star).$$

Maxwell computed

$$B(v - v_\star,\ \sigma),$$

assuming that particles interact by a potential

$$(Distance)^{-power}.$$

He found that

$$B(v - v_\star \cdot \sigma) \approx |v - v_\star|^\gamma \; |\theta|^{-2-2s},$$

with

$$\gamma > -3, \quad 0 < s < 1.$$

$\boxed{\text{THE SINGULARITY IN } \sigma \in S^2 \text{ IS NOT LOCALLY INTEGRABLE.}}$

$\boxed{\text{The factor } |v - v_x|^\gamma \text{ is not integrable at infinity.}}$

The vast majority of work on the Boltzmann equation before $\approx 2000$ assumed that $B(v - v^\star,\ \sigma)$ is (at least) integrable with respect to $\sigma \in S^2$.

We now know that the physically interesting case has fundamentally different behavior.

The Boltzmann Equation has a 5-parameter family of equilibrium solutions

$$F(x, v, t) =$$

$$\rho \cdot (2\pi T)^{-\frac{3}{2}} \exp\left(\frac{-|v - v_0|^2}{2T}\right).$$

$$\begin{aligned} \textit{Here}, \quad \rho &= \text{ particle density} &&\in (0, \infty) \\ v_0 &= \text{bulk velocity} &&\in \mathbb{R}^3 \\ T &= \text{temperature} &&\in (0, \infty). \end{aligned}$$

# Great Unsolved Problem

Prove (or disprove) that any physically reasonable initial $F_0(x, v)$ gives rise to a Boltzmann solution $F(x, v, t)$ that converges to one of the above equilibrium solutions as $t \to \infty$.

Decide how rapidly the convergence takes place.

Lots of work over many years:

*Aberyd, Carleman,*
*Desvillettes, Guo, Hilbert,*
*Levermore, Lions, Liu,*
*Mouhot, Ukai, Villani,*
*Wennberg*

# Dramatic Recent Progress:

(Gressman-Strain, PNAS 2010, JAMS 2011, ArXiv: 1011.5441v1, ArXiv: 1007.1276 v2)

Restrict attention here to the parameter range $\gamma + 2s \geq 0$.

Recall,

$$B(v - v_\star, \sigma) \approx |v - v_\star|^\gamma |\theta|^{-2-2s}.$$

For such $\gamma, s$, the following holds

## Thm (Gressman-Strain)

Let $F_0(x,v)$ be a positive initial particle density, close enough to $g = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{|v|^2}{2}\right)$ in a suitable norm.

Suppose that

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F_0(x,v)dxdv = 1$$

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} vF_0(x,v)dxdv = 0$$

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2F_0(x,v)dxdv = 1$$

Then there exists a positive solution $F(x, v, t)$ of the Boltzmann equation (with initial condition $F_0$) such that $F(\cdot,\ \cdot,\ t) \to g$ exponentially fast as $t \to \infty$.

> Thus, initial data close to equilibrium lead to a Boltzmann solution that tends exponentially fast to equilibrium as time $\to \infty$.

- For physically relevant $\gamma, s$ with $\gamma + 2s < 0$, there are analogous results, but they are more complicated to state, and the convergence to equilibrium is subexponential.
- See also Alexandre, Morimoto, Ukai, Xu, Yang

A fundamental idea in the proof of Gressman and Strain is to carry out analysis and define a Littlewood-Paley function in a non-Euclidean setting adapted to the Boltzmann equation, and to the particular equilibrium solution $g$.

To see why, we write

$$F = g + \sqrt{g}\ f$$

for small f.

The Boltzmann equation becomes

$$\boxed{\begin{array}{l} \partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f), \\ \text{where} \\ \Gamma(f, h) = g^{-\frac{1}{2}} Q(\sqrt{g}\ f, \quad \sqrt{g}\ h) \\ \text{and} \\ Lf = -\Gamma(f, \sqrt{g}) - \Gamma(\sqrt{g},\ f). \end{array}}$$

# Highly Oversimplified Discussion Follows!:

Want to use energy estimates - Multiply the Boltzmann equation by $f$ and integrate. Hope it does some good. We find that

$$\frac{1}{2}\frac{d}{dt}\|f(\cdot,\ \cdot,\ t)\|_{L^2}^2 + \int f(x,v,t)v\cdot\nabla_x f(x,v,t)dxdv + \int f\ Lf\ dxdv = \int f\Gamma(f,f)dxdv.$$

Now,

$$\int f(x,v,t) v \cdot \nabla_x f(x,v,t) dx dv =$$

$$\frac{1}{2} \int\limits_{\mathbb{R}^3 \times \mathbb{T}^3} v \cdot \nabla_x \left| f(x,v,t) \right|^2 dx dv = 0$$

So

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L^2}^2 + \int f \ Lf \ dx dv =$$
$$\int f \Gamma(f,f) dx dv.$$

Suppose we could find a norm $\|f\|_{\underline{X}}$ such that

$$\int f \; Lf \; dxdv \geq c\|f\|^2_{\underline{X}}$$

and

$$\int f\Gamma(f,f)dxdv \leq C\|f\|_{L^2} \; \|f\|^2_{\underline{X}}$$

Then our energy identity would tell us that

$$\frac{d}{dt}\|f\|^2_{L^2} + \left(c - C\,\|f\|_{L^2}\right)\|f\|^2_{\underline{X}} \leq 0.$$

$$\frac{d}{dt}\|f\|_{L^2}^2 + (c - C\,\|f\|_{L^2})\,\|f\|_{\underline{X}}^2 \le 0.$$

If $\quad C\,\|f\|_{L^2} < \frac{c}{2}$ initially,

and if $\|f\|_{\underline{X}} \ge c\|f\|_{L^2}$,

then we obtain the estimate

$$\frac{d}{dt}\|f\|_{L^2}^2 + c'\|f\|_{L^2}^2 \le 0,$$

hence Exponential Decay!

This discussion is

HIGHLY OVERSIMPLIFIED,

e.g.
$L$ has a 5-dimensional nullspace, so we can never have

$$\int f \; Lf \geq c\|f\|_{\underline{X}}^2 \quad \text{(all f)}.$$

NEVERTHELESS, one crucial remark in the preceding discussion is (more or less) correct: We need to find a norm $\|f\|_{\underline{X}}$ such that

$$\boxed{\begin{array}{l} \displaystyle\int_{\mathbb{R}^3} f\ Lf\ dv \geq c\|f\|_{\underline{X}}^2 - \text{Junk terms} \\[2em] \text{and} \\[1em] \displaystyle\left|\int_{\mathbb{R}^3} f\Gamma(f,f)dv\right| \leq C\|f\|_{\underline{X}}^2\|f\|_{L^2} \end{array}}$$

Here, we fix $x$ and regard $f$ as a function of $v$.

Before Gressman & Strain, people tried estimating

$$\int f\ Lf \quad \text{and} \quad \int f\,\Gamma(f,f)$$

in terms of (standard) Sobolev norms.

It doesn't work, because one needs <u>different</u> Sobolev norms to control these two integrals.

We need the SAME norm $\|f\|_{\underline{X}}$.

# Big Idea:

Identify $v \in \mathbb{R}^3$ with the point $(v, |v|^2) \in \mathbb{R}^4$. This identifies $\mathbb{R}^3$ with a paraboloid $P$ in $\mathbb{R}^4$.

We use the metric

$$d(v, v') = \left( |v - v'|^2 + \left| |v|^2 - |v'|^2 \right|^2 \right)^{\frac{1}{2}}$$

on $\mathbb{R}^3$, inherited from the above imbedding into $\mathbb{R}^4$.

Using the above distance $d(v, v')$, we define weighted $L^2$ and Sobolev norms by

$$|f|^2_{L^2_{\gamma+2s}} = \int_{\mathbb{R}^3} (1+|v|)^{\gamma+2s} |f(v)|^2 dv$$

and

$$|f|^2_{\dot{N}^{s,\gamma}} = \iint_{d(v,v')<1} (1+|v|)^{\gamma+2s+1} \frac{|f(v)-f(v')|^2}{\left(d(v,v')\right)^{3+2s}} dv\, dv'$$

Then set

$$\|f\|_{\underline{X}} = |f|_{L^2_{\gamma+2s}} + |f|_{\dot{N}^{s,\gamma}}.$$

With the above definition for $\|f\|_{\underline{X}}$, we find that

A.

$$\int_{R^3} f \;\, Lf \;\, dv \geq c\|f\|^2_{\underline{X}} - \text{ Junk}$$

and

B.

$$\left|\int_{R^3} f\Gamma(f,f)dv\right| \leq C\|f\|_{L^2} \;\|f\|^2_{\underline{X}}.$$

That's close enough to what we want to start a proof based on energy estimates.

The proof of B. is based on a Littlewood-Paley function adapted to the paraboloid $P$ introduced above.

This summary of Gressman-Strain is highly oversimplified, but I hope it conveys the correct spirit.

In particular, it is intended to show the fundamental role of

- Analysis in a non-Euclidean setting and
- Littlewood-Paley in such a setting.

This kind of analysis was invented and disseminated by

# Eli Stein

Enjoy the Conference!