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On the work
of
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Five Epic Years: 1969-1974

I. Thesis

II. Disc Multiplier

III. $H^1$-BMO Duality

IV. Mapping Theorem in $\mathbb{C}^n$ and Bergman Kernel

V. A Choice
I. **Thesis**

1. **Strongly singular integrals**

\[ T(f) = f * K \quad \text{on } \mathbb{R}^n. \]

**Typical example:**  \( K \) **distribution**

\[ K(x) = \frac{e^{ix}}{|x|^n}, \quad 0 < |x| \leq 1 \]

\[ = 0 \quad \text{if } |x| \geq 1 \]

Previously known:  \( T \) bounded on \( L^p \), \( 1 < p < \infty \)

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**Q1:** is \( T \) of weak-type \((1,1)\)?

**Q2:** (The "super" strongly singular case). Suppose:

\[ T_\lambda = f * K_\lambda, \text{ with } K_\lambda = \frac{e^{ix}}{|x|^{n+\lambda}}. \]

Is \( T_\lambda \) bounded on \( L^p \) when

\[ |1/p - 1/2| \leq \frac{1}{2} - \lambda/n, \text{ for } 0 < \lambda \leq n/2 \]

?
Reformulation of Q1: For fixed $\theta$, $0 \leq \theta < 1$:

\[
\begin{cases}
\hat{K}(\xi) = 0 \left(|\xi|^{-\theta n/2}\right), \quad \text{as } |\xi| \to \infty \\
\int_{|x| \geq |y|^{1-\theta}} |K(x - y) - K(x)| \, dx \leq A
\end{cases}
\]

(Above: $\theta = 1/2$. Note $\theta = 0$ is standard CZ situation.)

**Theorem** If $Tf = f \ast K$ as above, then $T$ is of weak-type $(1,1)$.

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Fix $\alpha > 0$; decompose $f = g + \sum_{j} b_{j}$, (CZ).

- Estimate $T(g)$ via Plancherel
- $b_{j}$ is supported in cube $Q_{j}$, and $\frac{1}{|Q_{j}|} \int_{Q_{j}} |b_{j}| \approx \alpha$. 


Need to estimate $T(b) = \sum_{j} T(b_j)$ outside $\bigcup_{j} B_j$.

(May assume diam $Q_j \leq 1$).

Problem is inside $\tilde{B}_j$, the ball concentric with $B_j$ but diam $\tilde{B}_j = (\text{diam } B_j)^{1-\theta}$.
• Idea: replace $b_j$ by $\tilde{b}_j = b_j \ast \varphi_j$,

$$\varphi_j(x) = \delta_{j}^{-n} \varphi\left(\frac{x}{\delta_{j}}\right), \quad \delta_{j} = \left(\text{diam}(B_{j})\right)^{1/(1-\theta)}.$$

• $$\int_{cB_{j}} |b_{j} \ast K - \tilde{b}_{j} \ast K|dx \leq c \int_{Q_{j}} |b_{j}|dx.$$

Suffices then to estimate $T(\tilde{b}) = \sum_{j} T(\tilde{b}_j)$.

Now

$$\| T(\tilde{b}) \|_{L^2} \lesssim \| (1 - \Delta)^{-n\theta/4} \tilde{b} \|_{L^2}.$$

But

• $$\| (1 - \Delta)^{-n\theta/4} \tilde{b} \|_{L^2} \lesssim \alpha \| b \|_{L^1}.$$

Because

• $$\| (1 - \Delta)^{-n\theta/4} \varphi_j \|_{L^2} \leq \frac{1}{m(Q_j)}.$$
2. **Square Functions**

Some familiar square functions:

- \( S^2(f)(x) = \int_{\Gamma(x)} |\nabla u(x - y, t)|^2 t^{1-n} \, dy \, dt \),

\[ u(x, t) = f * P_t = \text{Poisson integral of } f \]

- Littlewood-Paley type: \( \left( \sum |\Delta_k(f)|^2 \right)^{1/2} \)

\[ \Delta_k(f)(x) = \hat{f}(\xi) \eta(2^{-k}\xi). \]

These are all bounded on \( L^p \), \( 1 < p < \infty \).

A more intricate square function: \( g_\lambda^* \),

\[ (g_\lambda^*(f)(x))^2 = \int_{\mathbb{R}^{n+1}_+} |\nabla u(x - y, t)|^2 \left( \frac{t}{|y| + t} \right)^{n\lambda} t^{1-n} \, dy \, dt. \]

Majorizes both of the above.
What was known about \( f \rightarrow g^*_\lambda(f) \), \( \lambda > 1 \):

- maps \( L^p \rightarrow L^p \), if \( 1 < p < \infty \), and \( 2/\lambda < p \).

- fails when \( p \leq 2/\lambda \).

**Question:** What can be said about \( g^*_\lambda(f) \), when

\[
f \in L^p, \quad p = 2/\lambda, \quad 1 < p < 2,
\]

**Theorem** \( f \rightarrow g^*(f) \) is of weak-type \((p, p)\) in this range.

**NOTE:** this cannot hold for \( p = 1 \).
3. Bochner-Riesz

Let $S(f)$ be defined by

$$S(f)^\wedge = \hat{f}(\xi) \chi_B(\xi)$$

where $\chi_B = \text{characteristic function of unit ball } B$. Also

$$S^\delta(f)^\wedge = \hat{f}(\xi) \chi_B(\xi) (1 - |\xi|^2)^\delta.$$ 

**Question (1)** Is $S : L^p \to L^p$, when

$$\frac{2n}{n+1} < p < \frac{2n}{n-1}?$$

**Question (2)** Is $S^\delta : L^p \to L^p$ if

$$\frac{2n}{n + 1 + 2\delta} < p < \frac{2n}{n - 1 - 2\delta}?$$

(For radial functions, these were known to hold.)

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Restriction Phenomenom:

\[
(*) \left( \int_{S^{n-1}} |\hat{f}(\xi)|^2 d\sigma(\xi) \right)^{1/2} \leq A \| f \|_{L^p}
\]

holds for a range of \( p \)'s, \( 1 \leq p < p_0 \).

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**Theorem** Whenever the \( (L^p, L^2) \) restriction (*) holds, then \( S^\delta \) is bounded on \( L^p \) in the optimal range (i.e. \( \frac{2n}{n+1+2\delta} < p \leq 2 \), ultimately \( 1 \leq p \leq \frac{2(n+1)}{n+3} \)).

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After-thought: \( n = 2 \).

The restriction theorem:

\[
\left( \int_{S^1} |\hat{f}|^q d\sigma \right)^{1/q} \leq A \| f \|_{L^p(\mathbb{R}^2)}
\]

holds for

\[
1 \leq p < 4/3, \quad q = (1/3)p'.
\]

**NOTE:** as \( p \rightarrow 4/3 \), then \( q \rightarrow 4/3 \).

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Following this:

Carleson-Sjölin show that when $n = 2$,

$$S^\delta : L^p \longrightarrow L^p, \quad 4/3 \leq p \leq 4, \text{ and } \delta > 0.$$
II. Ball Multiplier

Question: Is $S$ bounded on $L^p$

(e.g. for $4/3 < p < 4$, when $n = 2$)?

**Theorem** No! (for $p \neq 2$, $n \geq 2$).

Background:

Let $R_j$ denote rectangles in the plane. Define

$R_j(f)$, by $R_j(f)(\xi) = \hat{f}(\xi) \chi_{R_j}(\xi)$.

**Question:** Does one have

$$
\left\| \left( \sum_j |R_j(f_j)|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p}
$$

with $R_j$ having arbitrary orientations?

Y. Meyer: If $S$ is bounded on $L^p$, then (**) holds for the same $p$. 
Counter-example sets (in $\mathbb{R}^2$)

- Nikodym Set
- Besicovitch (Kakeya) Set
Fefferman's observation:

Given $\epsilon > 0$, there is $N = N\epsilon$, and rectangles $R_1, R_2, \cdots, R_N$ each having side-length $(1, 1/N)$, so that

- $m\left(\bigcup_{j=1}^{N} R_j\right) < \epsilon$, but
- $R_1^*, R_2^*, \cdots, R_N^*$ are all disjoint

Now take $f_j = \chi_{R_j}$, then

$$|R_j(f_j)(x)| \geq 1/10 \text{ for } x \in R_j^*.$$  

Hence a contradiction to (***) whenever $p < 2$.

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Further results for Bochner-Riesz and related questions: ... Bourgain, Tao, ... .
III. $H_1$-BMO duality

John-Nirenberg: (1961): $f \in BMO$

If $\sup_Q \frac{1}{m(Q)} \int_Q |f - f_Q| \, dx = \| f \|_{BMO} < \infty$.

Inequality:

$$m\{x \in Q : |f(x) - f_Q| > \alpha\} \leq c_1 e^{-c_2 \alpha} m(Q),$$

all $\alpha > 0$, if $\| f \|_{BMO} \leq 1$.

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Related to work of John, Moser (the latter for Harnack-type inequality leading to DiGorgi-Nash estimates)

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Later observed: $BMO$ good substitute for $L^\infty$ in other settings:

Fact: $Tf = f \ast K$, and $K$ is a CZ kernel

then: $T : L^\infty \rightarrow BMO$

(In fact, $T : BMO \rightarrow BMO$.)

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The space $H^1$.

Classical $H^1$: Is $H^1$ of one complex variable (F. & M. Riesz, Hardy).

$F$ analytic in $z = x + iy$, $y > 0$ and

$$\sup_{y>0} \int_{\mathbb{R}} |F(x + iy)| \, dx < \infty.$$ 

$$H^1 = \left\{ F_0(x) = \lim_{y \to 0} F(x + iy) \right\}, \text{ with}$$

"Real" $H^1 = \{ f : f \in L^1 \text{ and } H(f) \in L^1 \}$. 

Next: $H^1$ in $\mathbb{R}^n$.

$$H^1 = \{ f \in L^1, \text{ and } R_j(f) \in L^1, \quad 1 \leq j \leq n \}$$
$H^1$ is a good substitute for $L^1$.  

Fact: $Tf = f * K$, $K$ is a CZ kernel

then: $T : H^1 \rightarrow L^1$ ,

in fact: $T : H^1 \rightarrow H^1$.  

(Here one used $g^*_\lambda$.)
Zygmund's Question:

- What is the Poisson integral characterization of \( f \in BMO \)?

\[
u(x,t) = f \ast P_t, \quad P_t \text{ the Poisson kernel.}
\]

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Note:

- For \( L^p \), \( f \in L^p \iff \sup_{t>0} \| u(\cdot,t) \|_{L^p} < \infty \)

\[1 < p \leq \infty.
\]

- For \( L^p \), \( f \in L^p \iff S(f) \in L^p, 1 < p < \infty.\)

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Theorem:

1. Dual space of \( H^1 \) is \( BMO \).

2. \( f \in BMO \iff \sup_B \frac{1}{m(B)} \int_{T(B)} |\nabla u(x,t)|^2 t dt < \infty \)

3. \( \iff f = f_0 + \sum_j R_j(f_j), f_0, f_1, \ldots f_n \in L^\infty. \)
Condition (2): \( d\mu = |\nabla (u(x, t))|^2 t \, dx \, dt \)

is a "Carleson measure" (on \( \mathbb{R}^{n+1} \)) i.e.

\[
\sup_B \frac{1}{m(B)} \int_{T(B)} d\mu = \| d\mu \|_C < \infty.
\]

**Fefferman duality:**

Let \( F, G \) be non-negative functions on \( \mathbb{R}^{n+1}_+ \).

Then

\[
\int_{\mathbb{R}^{n+1}_+} F(x, t) G(x, t) \frac{dx \, dt}{t} \leq c \| \tilde{S}(F) \|_{L^1(\mathbb{R}^n)} \| G \|_C
\]

where

\[
\tilde{S}(F)(x) = \left( \int_{\Gamma(x)} (F(y, t))^2 \frac{dy \, dt}{t^{1+n}} \right)^{1/2}.
\]

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Further Consequences

- Better understanding of $H^1$, $H^p$, $p \leq 1$, in particular, atomic decomposition.

- "sharp function":

$$f^\#(x) = \sup_{x \in Q} \frac{1}{m(Q)} \int_Q |f(x) - f_Q| \, dx.$$  

Then

$$f^\# \in L^p \implies f \in L^p, \quad p < \infty.$$  

- End-point estimates for $1 < p < 2$ for (super) strong-singular integrals.
IV. Mapping of Domains in $\mathbb{C}^n$:

**Question:** Suppose $\Omega_1$ and $\Omega_2$ are two bounded smooth domains in $\mathbb{C}^n$. Assume there is a holomorphic bijection $\Phi : \Omega_1 \rightarrow \Omega_2$. Does $\Phi$ extend to a diffeomorphism of $\partial \Omega_1$ to $\partial \Omega_2$?

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Some reasons $n > 1$ is different from $n = 1$.

1. When $n = 1$, the answer is yes. In fact, then there always exist "locally" good maps between any pair of smooth arcs.

2. When $n > 1$, and $\Omega_1$ unit ball, $\Omega_2$ is an "$\epsilon$" $C^\infty$ perturbation of $\Omega_1$, then in general such $\Phi$ does not exist (even, locally, near a boundary point of $\Omega_1$.)

3. Pseudo-convexity (for $n > 1$).
Theorem: If $\Omega_1$ and $\Omega_2$ are bounded smooth domains $\Phi : \Omega_1 \rightarrow \Omega_2$ holomorphic bijection. Then $\Phi$ extends smoothly to boundaries if both $\Omega_1$ and $\Omega_2$ are strongly pseudo-convex.

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(Later work: Boutet de Monvel-Sjöstrand, Webster, Bell-Ligocka, Nirenberg, P. Yang, ...)

Fefferman's Approach:

(B) Bergman kernel, $K_\Omega$, of domain $\Omega$.

$P_\Omega$ orthogonal projection: $L^2(\Omega) :\rightarrow L^2(\Omega) \cap (Hol)$

$$P_\Omega(f)(z) = \int_\Omega K_\Omega(z,w) f(w) \, dV(w)$$

Bergman metric, $g_{ij}^\Omega = \frac{\partial}{\partial \bar{z}_i \partial z_j} \log K_\Omega(z,z)$.

Fact: $\Phi : (\Omega_1, g_1^\Omega) \rightarrow (\Omega_2, g_2^\Omega)$ is an isometry.
(G) Follow the geodesics!

\[ \Phi: \quad \Omega_1 \rightarrow \Omega_2 \]

Main Issues:

(B) What does the Bergman kernel (and Bergman metric) look like near the boundary?

(G) Where do the geodesics lead to? and how?

Assume \( \Omega \) is bounded, smooth, and strongly pseudo-convex. Let \( r(z) \) defining function, and \( Q(z, w) \) holomorphic part of Taylor expansion of \( r(z) \) (up to second-order) at \( w \).

**Theorem\(^*\):**

\[
K_\Omega(z, w) = \frac{A(z, w)}{Q(z, w)^{n+1}} + B(z, w) \log Q(z, w) \quad \text{with} \quad A, B \in C^\infty.
\]

**Note:** For unit ball in \( \mathbb{C}^n \), \( r(z) = 1 - |z|^2 \),

\[
K(z, w) = c/(1 - z \cdot \bar{w})^{n+1}.
\]
Want $I = P_\Omega + Q_\Omega$, $Q_\Omega = P_\Omega^\perp$.

- Find "ball" $\bar{B}$ highly tangent (order 4) to $\Omega$ at $p$, but $\bar{B} \subset \Omega$

- From explicit identity

\[ I = P_{\bar{B}} + Q_{\bar{B}} \text{ on } \bar{B} \]

pass to approximate identity

\[ I + E = P_\Omega^0 + Q_\Omega^0 \quad \text{(explicit)} \]

($K_{\bar{B}}$ extends to $\Omega \times \Omega$ near $p$. Also one can correct by Kohn's $\bar{\partial}$-Neumann.)

Here $E \approx P_\Omega^0 \cdot \chi_{\Omega - \bar{B}}$

- $P_\Omega = P_\Omega^0 (1 + E)^{-1} = P_\Omega^0 - P_\Omega^0 E + P_\Omega^0 E^2 \ldots \ldots$.
(G) **Main Lemma**

Suppose \( X(t) = X(t, z_0, \xi_0) \) is geodesic starting at \( z_0 \) in direction \( \xi_0 \). Assume \( X(t), 0 \leq t < \infty \), does not lie in a compact set. Then

1. \( \lim_{t \to \infty} X(t, z_0, \xi_0) \) converges to a boundary point.

2. The same is true for the geodesic \( X(t, z_0, \xi) \) for \( \xi \) near \( \xi_0 \), and the resulting mapping: \( \xi \to \text{boundary} \), is a (local) diffeomorphism.

3. All boundary points can be reached this way.

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Requires several changes of variable:

- New “time” \( \tau, \frac{d\tau}{dt} = r(X(t)) \).

- Further desingularization because of “log term”.
V. Several Choices

1. Local solvability of linear p.d.e.

Consider the $m^{th}$ order linear partial differential equation, where $p$ is assumed to be of "principal type."

\[ (*) \quad p(x, D)u = f \]

**Theorem:** (R. Beals and C. Fefferman) Suppose $p$ satisfies the condition $\mathcal{P}$ of Nirenberg-Treves. Then $(* \equiv)$ is locally solvable.

**Note:** ($\mathcal{P}$) means: $\mathcal{S}p_m$ does not change sign on the null bicharacteristic curves of $\mathcal{R}p_m$.

Theorem was proved by N-T in the real-analytic case.

**Proof:** requires a refined phase-space decomposition of a transformed problem.

This involves a "stopping-time" argument, in terms of violation of any of three key properties.
2. Convergence of Fourier series

\[ f \rightarrow \sup_{\lambda} \left| \int_{\pi}^{\pi} \frac{e^{i\lambda y}}{y} f(x - y) \, dy \right| = C(f). \]

**Theorem:** A new proof that the Carleson operator \( C \) is a (weak-type) mapping \( L^2 \to L^2 \).

- Consider pairs: \((\omega, I)\), where \(\omega\) and \(I\) are dyadic intervals in \(\mathbb{R}\) and \([-\pi, \pi]\) respectively, with \(|\omega||I| = 1\) (These are later called "tiles"). Endow with ordering \((\omega, I) < (\omega', I')\), if \(I \subset I', \omega' \subset \omega\), and study collections of resulting "trees".

- Linearize \( C(f) \) as \( \int_{-\pi}^{\pi} \frac{e^{iN(x)y}}{y} f(x - y) \, dy \) and decompose according to \( \{x \in I, N(x) \in \omega\} \), with \( \frac{1}{y} = \sum_{k} \psi_k(y) \), and \(|I| = 2^{-k}\), where 
  \[ \psi_k(y) = 2^k \psi(2^k y). \]

  Similar ideas then play important role in "time-frequency" analysis of Lacey and Thiele, for bilinear Hilbert transform, etc.