



UNIVERSITÉ
DE GENÈVE

FACULTÉ DES SCIENCES

Quasiconformal maps and harmonic measure

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In part based on joint work with

Kari Astala & István Prause

quasiconformal maps

$$\varphi: \Omega \rightarrow \Omega' \quad W_{loc}^{1,2}\text{-homeomorphism}$$

Def 1 $\bar{\partial}\varphi(z) = \mu(z)\partial\varphi(z) \quad \text{a.e. } z \in \Omega \quad \|\mu\|_{\infty} \leq k < 1$

Def 2  eccentricity \leq
$$K = \frac{1+k}{1-k}$$

measurable Riemann mapping theorem:

- (unique up to Möbius) solution exists
- depends analytically on μ

distortion of dimension

Theorem [Astala 1994] for k – quasiconformal φ

$$\frac{1}{K} \left(\frac{1}{\dim E} - \frac{1}{2} \right) \leq \frac{1}{\dim \varphi(E)} - \frac{1}{2} \leq K \left(\frac{1}{\dim E} - \frac{1}{2} \right)$$

Rem result is sharp (easy from the proof)

In particular, $\dim E=1 \Rightarrow 1-k \leq \dim \varphi(E) \leq 1+k$

[Becker-Pommerenke 1987] $\dim \varphi(\mathbb{R}) \leq 1+37k^2$

Conjecture [Astala] $\dim \varphi(\mathbb{R}) \leq 1+k^2$

dimension of quasicircles

Thm [S] $\dim \varphi(\mathbb{R}) \leq 1 + k^2$

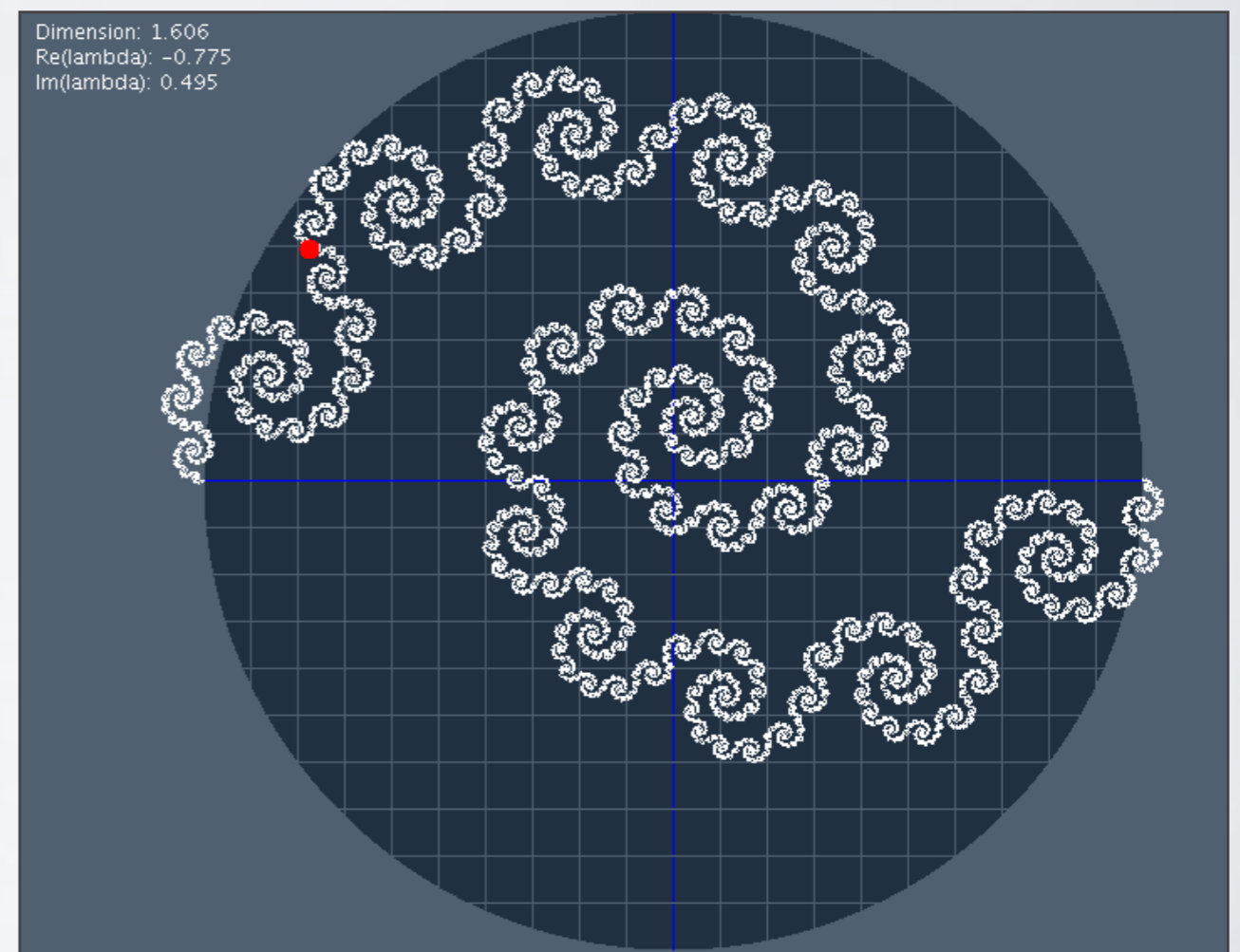
Dual statement:

φ symmetric wrt \mathbb{R} ,

$\text{spt } \nu \subset \mathbb{R}$
 $\dim \nu = 1$ } \Rightarrow

$\dim \varphi(\nu) \geq 1 - k^2$

Sharpness???



a nonrectifiable quasicircle

Proof: holomorphic motion

Any k - qc map φ_k can be embedded into a holomorphic motion of qc maps φ_λ , $\lambda \in \mathbb{D}$:

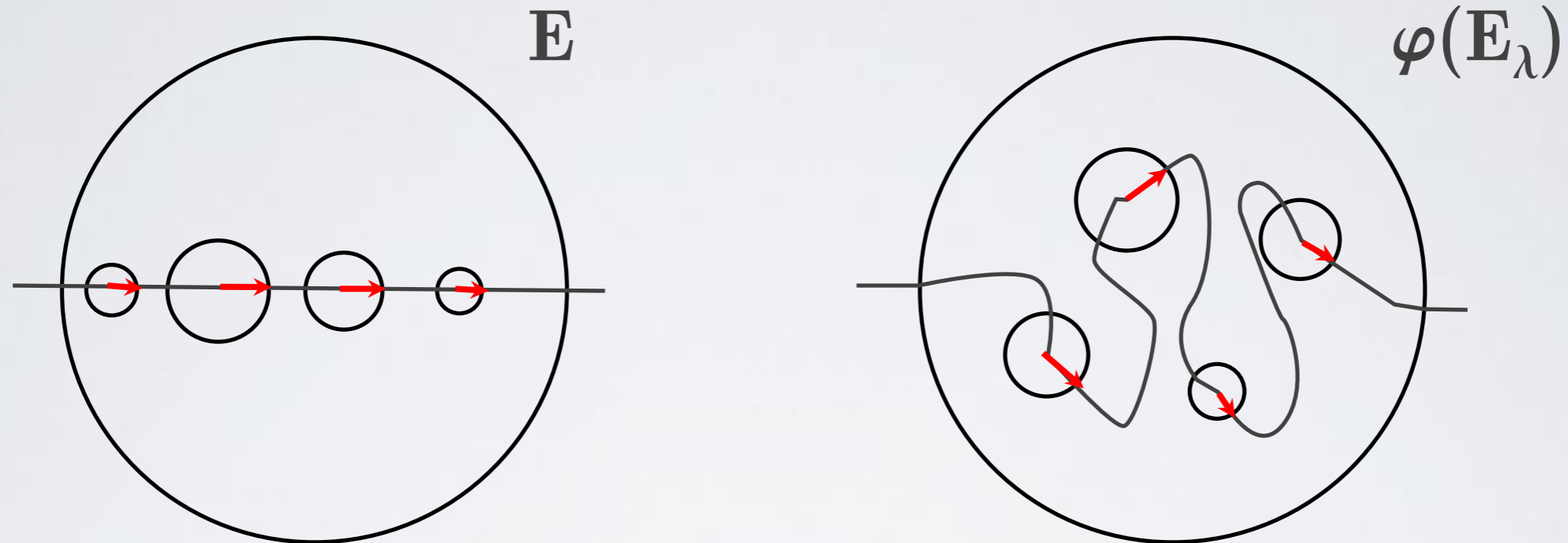
Define Beltrami coefficient $\mu = \mu_\varphi / \|\mu_\varphi\|$, $\|\mu\|=1$

$$\lambda \in \mathbb{D} \longrightarrow \lambda\mu \longrightarrow \varphi_\lambda \quad \text{which is } |\lambda|\text{-qc}$$

Mañé-Sad-Sullivan, Slodkowski :

A holomorphic motion of a set can be extended to a holomorphic motion of qc maps

Proof: fractal approximation



a packing of disks evolves in the motion

$\{B_\lambda\}$

“complex radii”

$\{r_\lambda\}$

Cantor sets

$C_\lambda \approx \varphi(E_\lambda)$

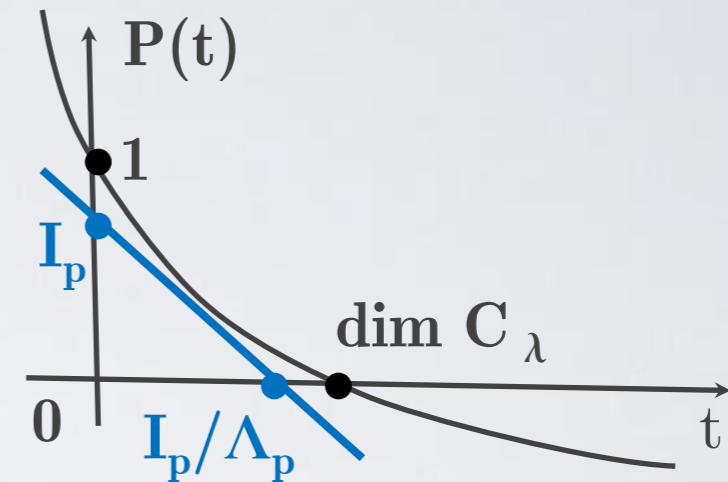
Proof: “thermodynamics”

Pressure [Ruelle, Bowen]

$$P_\lambda(t) := \log(\sum |r_j(\lambda)|^t)$$

“**Entropy**” $I_p := \sum p_j \log(1/p_j)$

“**Lyapunov exponent**” $\Lambda_p(\lambda) := \sum p_j \log(1/|r_j(\lambda)|)$
(harmonic in λ !)



Variational principle (Jensen’s inequality)

$$P_\lambda(t) = \sup_{p \in \text{Prob}} \sum p_j \log(|r_j(\lambda)|^t / p_j) = \sup_{p \in \text{Prob}} (I_p - t \Lambda_p(\lambda))$$

Bowen’s formula: $\dim C_\lambda = \text{root of } P_\lambda = \sup_{p \in \text{Prob}} I_p / \Lambda_p(\lambda)$

Proof: Harnack's inequality

- $\dim C_0 = 1 \implies I_p / \Lambda_p(0) \leq 1 \implies \Lambda_p(0) - I_p / 2 \geq I_p / 2$

- $\dim C_\lambda \leq 2 \implies I_p / \Lambda_p(\lambda) \leq 2 \implies \Lambda_p(\lambda) - I_p / 2 \geq 0$

- **Harnack** $\implies \Lambda_p(\lambda) - \frac{I_p}{2} \geq \frac{1 - |\lambda| I_p}{1 + |\lambda|} \frac{I_p}{2}$

$$\implies \Lambda_p(\lambda) \geq \frac{1}{1 + |\lambda|} I_p$$

$$\implies \dim C_\lambda = \sup_p I_p / \Lambda_p(\lambda) \leq 1 + |\lambda| \quad \blacksquare$$

- Quasicircle \implies **(anti)symmetric motion** \implies even Λ

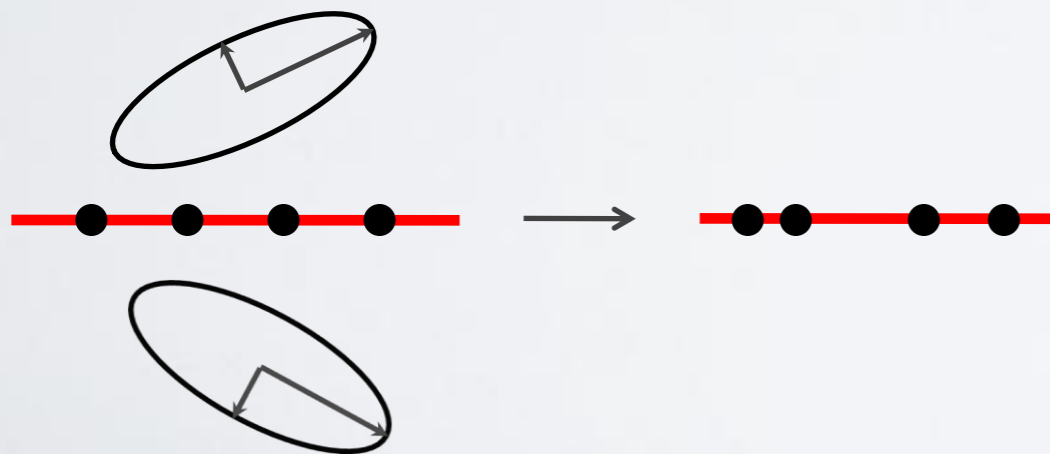
$$\implies \text{"quadratic" Harnack} \implies \dim C_\lambda \leq 1 + |\lambda|^2 \quad \blacksquare$$

Proof: symmetrization

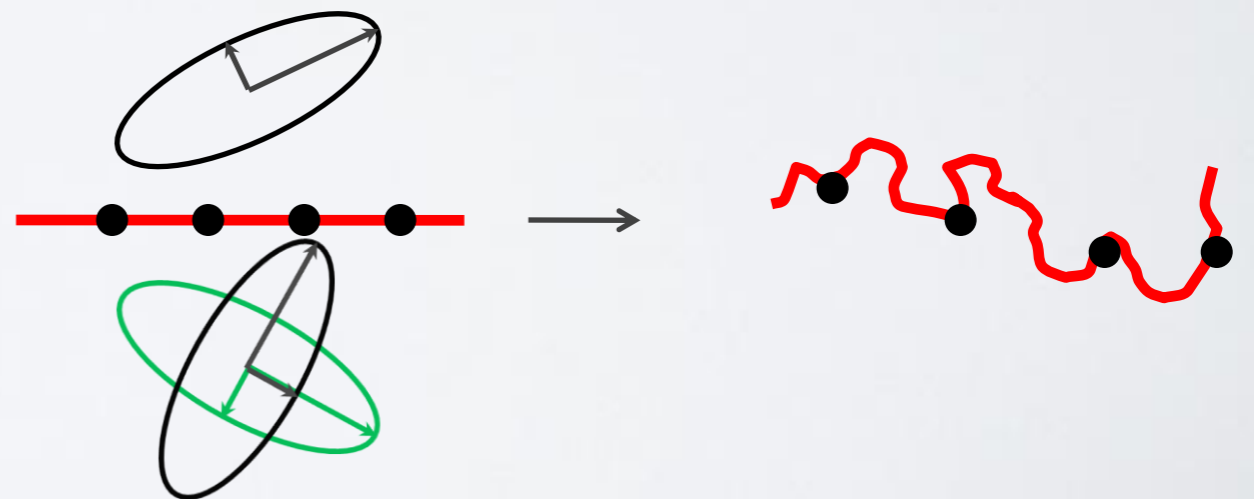
Thm [S] the following are equivalent:

- $\Gamma = \varphi(\mathbb{R})$ and φ is k -qc
- $\Gamma = \varphi(\mathbb{R})$ and φ is $\frac{2k}{1+k^2}$ qc in \mathbb{C}_+ and conformal in \mathbb{C}_-
- $\Gamma = \varphi(\mathbb{R})$ and φ is k -qc and antisymmetric

symmetric:



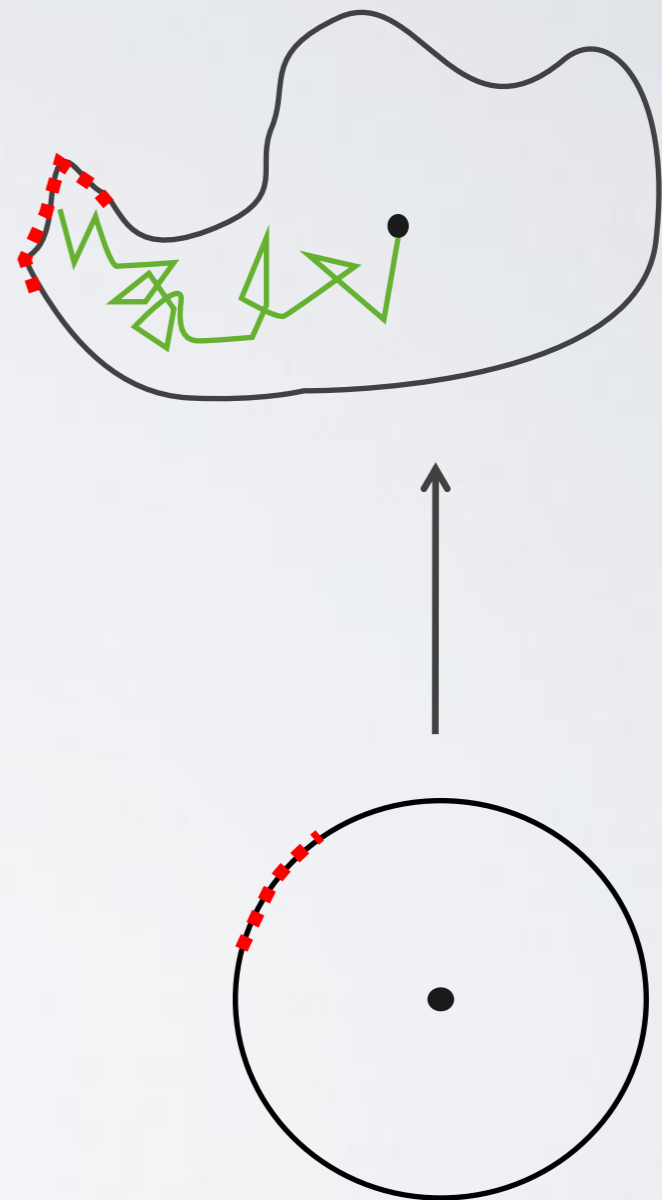
antisymmetric:



harmonic measure ω

- **Brownian motion**
exit probability
- **conformal map**
image of the length
- **potential theory**
equilibrium measure
- **Dirichlet problem for Δ**

$$u(z_0) = \int_{\partial\Omega} u(z) d\omega(z)$$

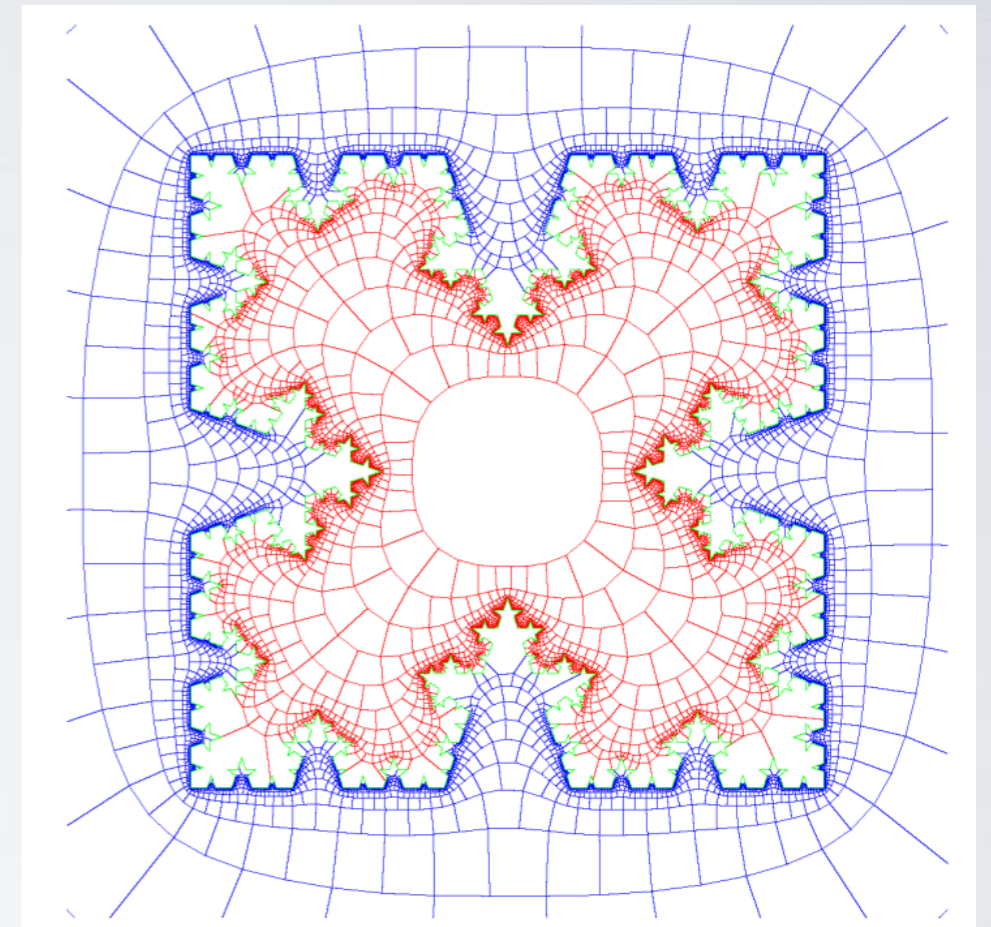
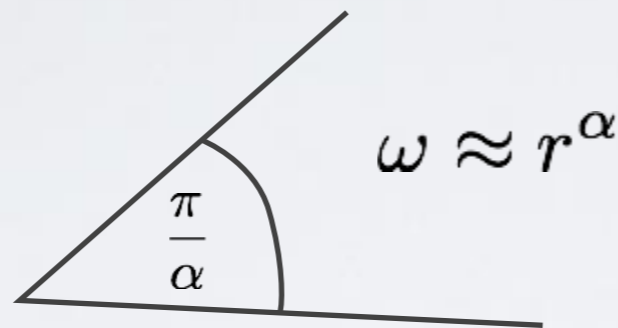


multifractality of ω

“fjords and spikes”

\mathcal{F}_α scaling: $\omega B(z, r) \approx r^\alpha$

geometric
Meaning :



Beurling's theorem: $\alpha \geq 1/2$

spectrum: $f(\alpha) = \dim \mathcal{F}_\alpha$

Courtesy of D. Marshall

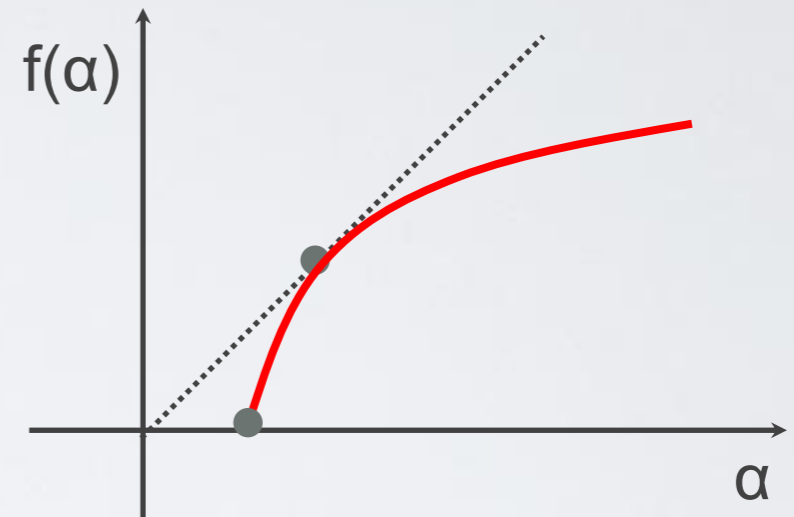
Makarov's theorem: Borel $\dim \omega = 1$, $f(1) = 1$

Many open problems reduce to estimating the

universal spectrum

$$f(\alpha) = \sup_{\Omega} f_{\Omega}(\alpha)$$

over all simply
connected domains



Conjecture : $f(\alpha) \stackrel{?}{=} 2 - \frac{1}{\alpha}$

[Brennan-Carleson-Jones-Krätzer-Makarov]

Legendre transform & pressure

Restrict pressure to **conformal maps** $\varphi : \mathbb{C}_+ \rightarrow \Omega$

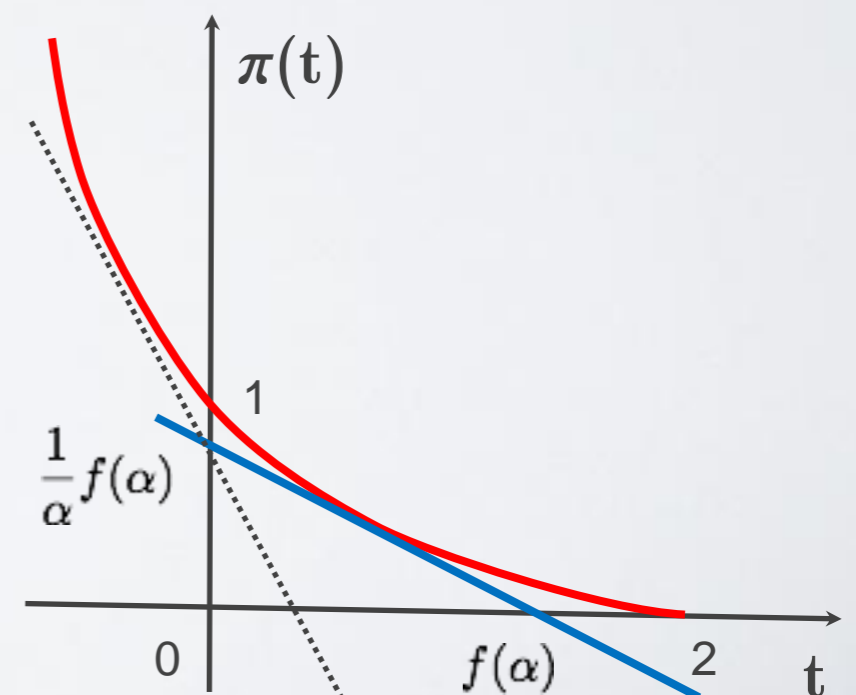
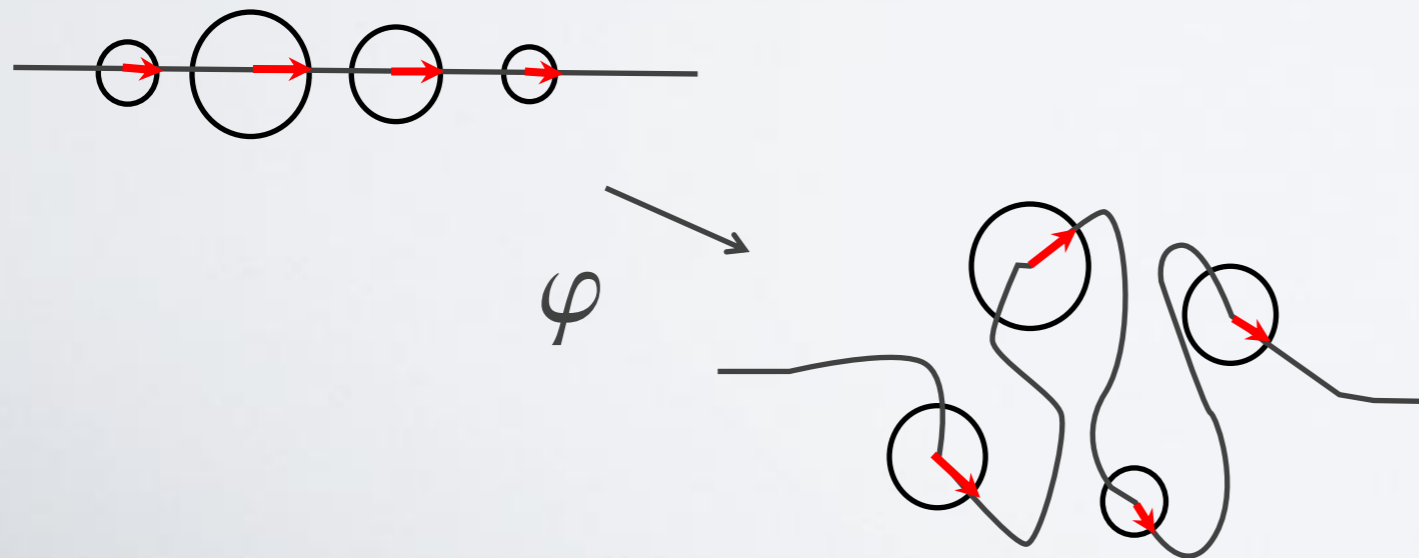
$$\pi_{\Omega}(t) := \log(\Sigma |r_j(\lambda)|^t)$$

Universal pressure $\pi(t) := \sup_{\Omega} \pi_{\Omega}(t)$

Thm [Makarov 1998] Legendre transforms:

$$f(\alpha) = \inf_t \{ \alpha \pi(t) + t \} \quad \pi(t) = \sup_{\alpha} \{ (f(\alpha) - t) / \alpha \}$$

Conjecture: $\pi(t) = (2-t)^2 / 4$



finding the universal spectrum

- no real intuition
- some numerical evidence
- only weak estimates

Example: $\pi(1)$ gives optimal

- coefficient decay rate for bounded conformal maps
- growth rate for the length of Green's lines

Conjecturally $\pi(1) = 0.25$, best known estimates:

$$0.23 \leq \pi(1) \leq 0.46$$

[Beliaev, Smirnov] [Hedenmalm, Shimorin]

fine structure of harmonic measure via the holomorphic motions

- I. qc deformations of conformal structure and harmonic measure**
- II. motions in bi-disk**
- III. welding conformal structures and Laplacian on 3-manifolds**

joint work with

Kari Astala and István Prause

I. deforming conf structure

Recall: $\text{spt } \nu \subset \mathbb{R}$ & $\dim \nu = 1 \implies \dim \varphi(\nu) \geq 1 - k^2$

Thm assume that the statement above is sharp:

$$\left. \begin{array}{l} \text{spt } \sigma \subset \mathbb{R} \\ \dim \sigma = 1 - k^2 \end{array} \right\} \implies \exists k\text{-qc } \varphi \text{ s.t. } \varphi(dx) = \sigma$$

then the universal spectrum conjecture holds

Rem in general no sharpness (e.g. any porous σ), but we need it only for relevant “Gibbs” measures

Question: how to deform? (use φ ?)

I. proof: deforming to ω

For “Gibbs” measures the **blue line** is tangent to $\pi(t)$

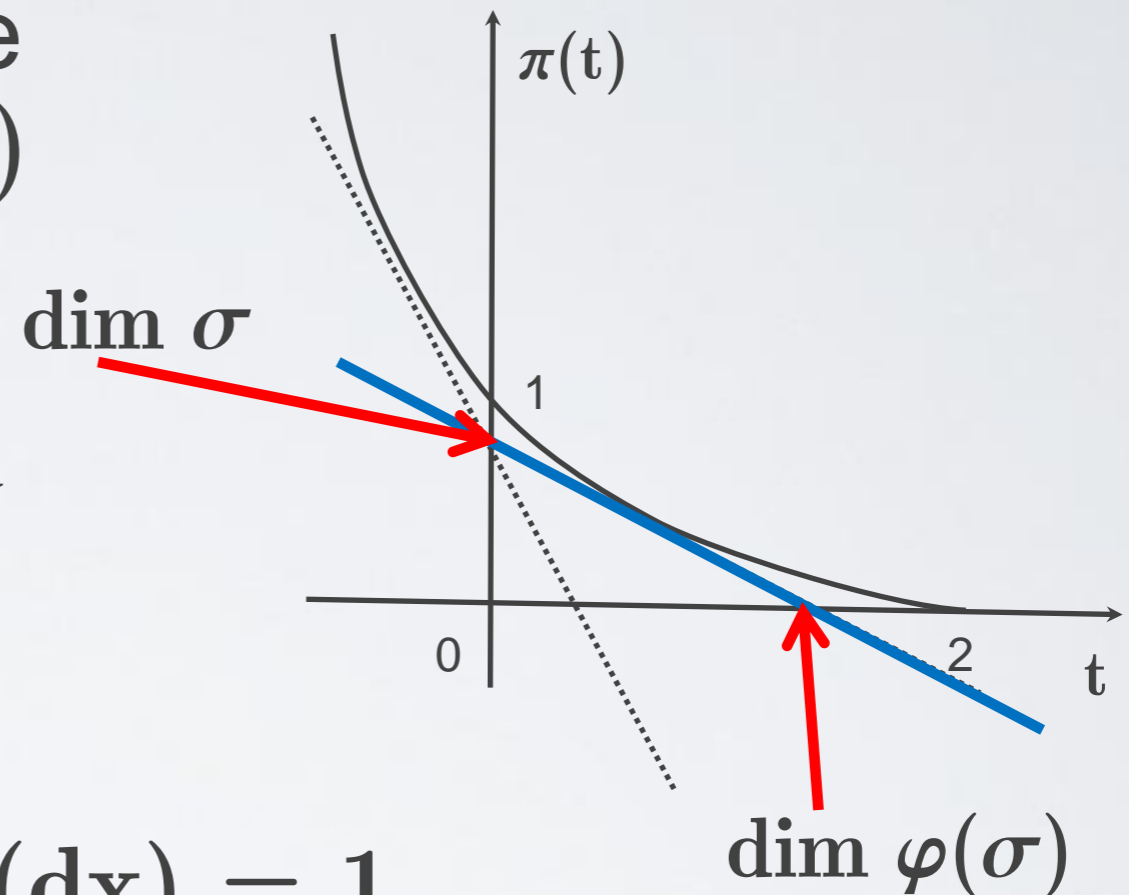
Set $1-k^2 := \dim \sigma$ and take holomorphic motion ψ such that $\psi_k(dx) = \sigma$

By Makarov’s theorem
 $\dim \varphi(\psi_k^{-1}(\sigma)) = \dim \varphi(dx) = 1$

By Astala’s theorem

$$\dim \varphi(\sigma) \leq 1+k$$

$$\implies \pi(t) \leq (2-t)^2 / 4$$



II. two-sided spectrum

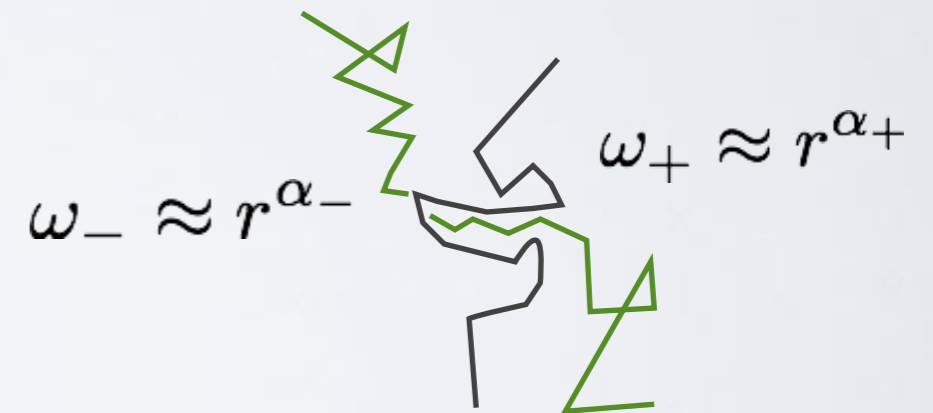
rotation [Binder]

$$f(\alpha, \gamma) = \dim \mathcal{F}_{\alpha, \gamma} \quad \omega \approx r^\alpha \text{ \& } \gamma\text{-spiraling}$$



two-sided spectrum

$$f(\alpha_-, \alpha_+, \gamma) = \dim \mathcal{F}_{\alpha_-, \alpha_+, \gamma}$$



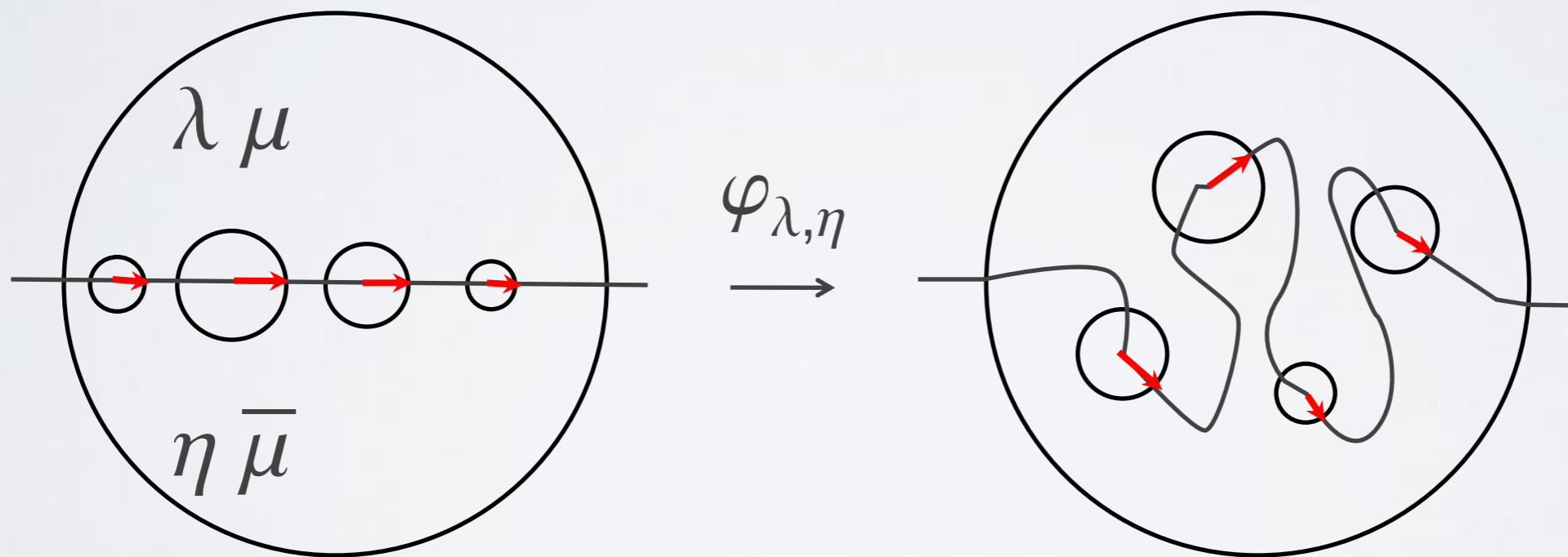
Beurling's estimate

$$\frac{1}{\alpha_-} + \frac{1}{\alpha_+} \leq \frac{2}{1 + \gamma^2}$$

II. bidisk motion

Take Beltrami μ in \mathbb{C}_+ of norm 1, symmetrize it

$$\mu_{\lambda,\eta} = \begin{cases} \lambda \mu(z) & \text{in } \mathbb{C}_+ \\ \eta \overline{\mu(\bar{z})} & \text{in } \mathbb{C}_- \end{cases} \longrightarrow \varphi_{\lambda,\eta}(z) \quad (\lambda, \eta) \in \mathbb{D}^2$$



symmetric for $\lambda = \bar{\eta}$, antisymmetric for $\lambda = -\bar{\eta}$

II. thermodynamics

$$P_{\lambda,\eta}(t) = \log \left(\sum |r(B_{\lambda,\eta})|^t \right) = \sup_p (\mathbb{I} - t \operatorname{Re} \Lambda_{\lambda,\eta})$$

$$\mathbb{I} = \sum p_i \log \frac{1}{p_i}$$

entropy

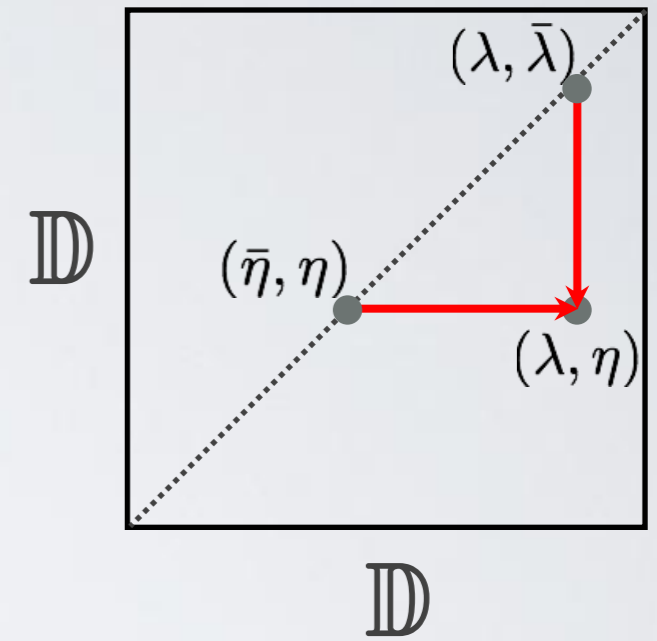
$$\Lambda_{\lambda,\eta} = \sum p_i \log \frac{1}{r_i(\lambda,\eta)}$$

(complex) Lyapunov exponent

$$\dim(C_{\lambda,\eta}) = \sup_p \dim p = \sup_p \frac{\mathbb{I}}{\operatorname{Re} \Lambda_{\lambda,\eta}}$$

II. “easy” estimates

- reflection symmetry $\varphi_{\lambda, \eta}(z) = \overline{\varphi_{\bar{\eta}, \bar{\lambda}}(\bar{z})}$
- diagonal $(\lambda, \bar{\lambda})$
- projections $(\lambda, \eta)_+ = (\lambda, \bar{\lambda}), (\lambda, \eta)_- = (\bar{\eta}, \eta)$

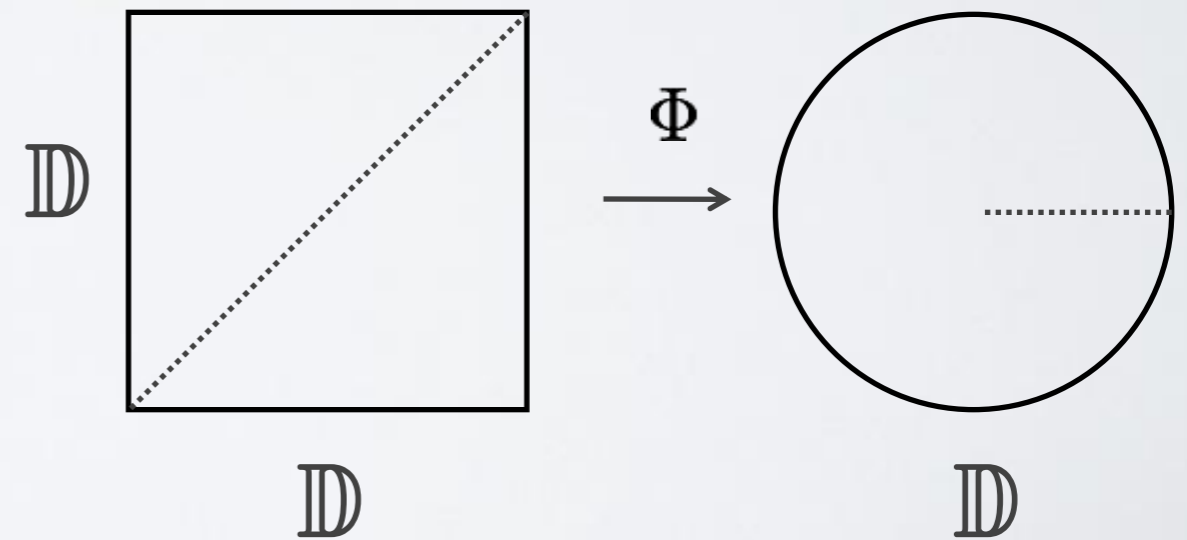


$$\Phi(\lambda, \eta) = 1 - \frac{1}{\Lambda_{\lambda, \eta}}$$

$$\Phi: \mathbb{D}^2 \rightarrow \mathbb{D} \quad \dim C_{\lambda, \eta} \leq 2$$

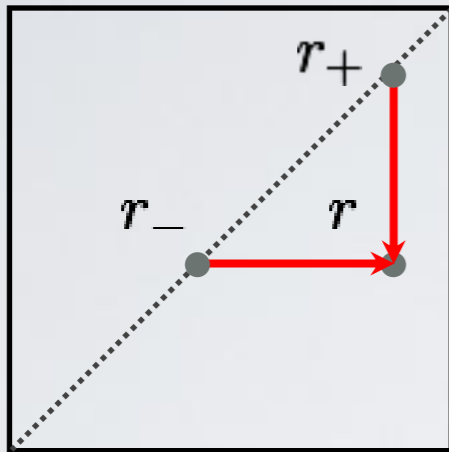
$$\Phi(\lambda, \bar{\lambda}) \geq 0 \quad \dim C_{\lambda, \bar{\lambda}} \leq 1$$

$$\Phi(\lambda, \eta) = \overline{\Phi(\bar{\eta}, \bar{\lambda})}$$

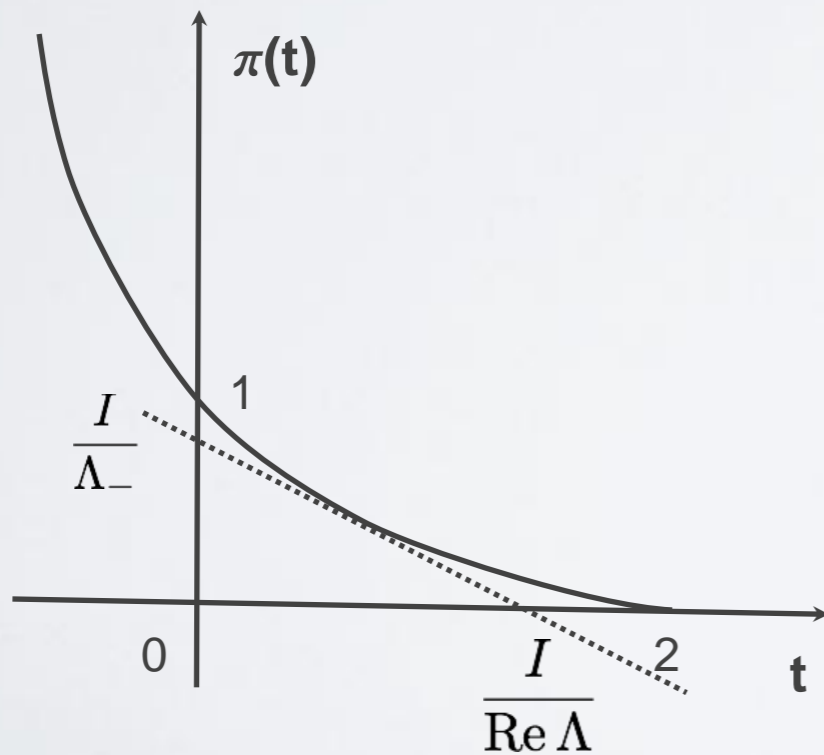


II. scaling relations

\mathbb{D}



\mathbb{D}



$$\omega_{\pm} B_{\lambda, \eta} \approx r_{\pm}$$

$$\log \frac{1}{r} = \frac{1 + i\gamma}{\alpha_{\pm}} \log \frac{1}{r_{\pm}}$$

$$\frac{1}{\Lambda} = \frac{f}{1 + i\gamma}, \quad \frac{1}{\Lambda_-} = \frac{f}{\alpha_-}, \quad \frac{1}{\Lambda_+} = \frac{f}{\alpha_+}$$

II. Beurling and Brennan

Beurling $\Rightarrow \frac{1}{\alpha_-} + \frac{1}{\alpha_+} \leq \frac{2}{1+\gamma^2} \Rightarrow 2 \operatorname{Re} \Phi(\lambda, \eta) \leq \Phi(\bar{\eta}, \eta) + \Phi(\lambda, \bar{\lambda})$

Corollary: $\lambda \mapsto \Phi(\lambda, \bar{\lambda})$ is subharmonic

Brennan's conjecture: $F: \Omega \rightarrow \mathbb{D}, \quad F' \in L^{4-\epsilon}$

Equivalent question: $f(\alpha) \leq 4(\alpha - \frac{1}{2})$?

Two-sided: $2 \operatorname{Re} \frac{1 - \Phi}{1 + \Phi} \geq \frac{1 - \Phi_-}{1 + \Phi_-} + \frac{1 - \Phi_+}{1 + \Phi_+}$?

II. two-sided spectrum

Conjecture: $|\Phi|^2 \leq \Phi_- \Phi_+$ or $\begin{pmatrix} \Phi(\lambda, \bar{\lambda}) & \Phi(\lambda, \bar{\eta}) \\ \Phi(\eta, \bar{\lambda}) & \Phi(\eta, \bar{\eta}) \end{pmatrix} \geq 0$

**Rem it is
equivalent to**

$$f(\alpha_-, \alpha_+, \gamma) \leq \frac{2 - (1 + \gamma^2) \left(\frac{1}{\alpha_-} + \frac{1}{\alpha_+} \right)}{1 - \frac{1 + \gamma^2}{\alpha_- \alpha_+}}$$

$$\alpha_+ \rightarrow \infty, \quad \gamma = 0$$

$$f(\alpha_-) \leq 2 - \frac{1}{\alpha_-}$$

II. the question

We know that

$$\Phi: \mathbb{D}^2 \rightarrow \mathbb{D}$$

$\Phi(\lambda, \bar{\lambda}) \geq 0$ and subharmonic

$$\Phi(\lambda, \eta) = \overline{\Phi(\bar{\eta}, \bar{\lambda})}$$

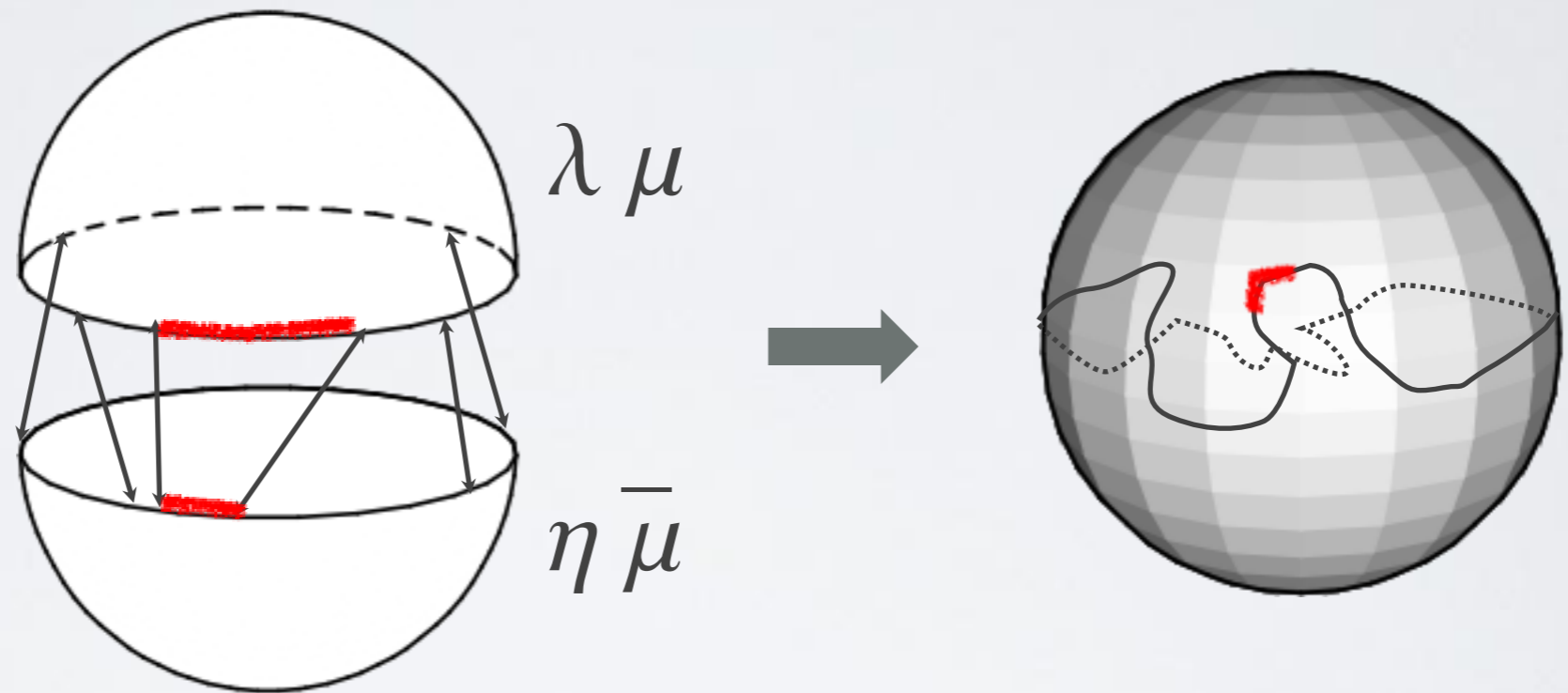
plus more...

**What do we need to deduce
the conjecture?**

$$\begin{pmatrix} \Phi(\lambda, \bar{\lambda}) & \Phi(\lambda, \bar{\eta}) \\ \Phi(\eta, \bar{\lambda}) & \Phi(\eta, \bar{\eta}) \end{pmatrix} \geq 0$$

III. conformal welding

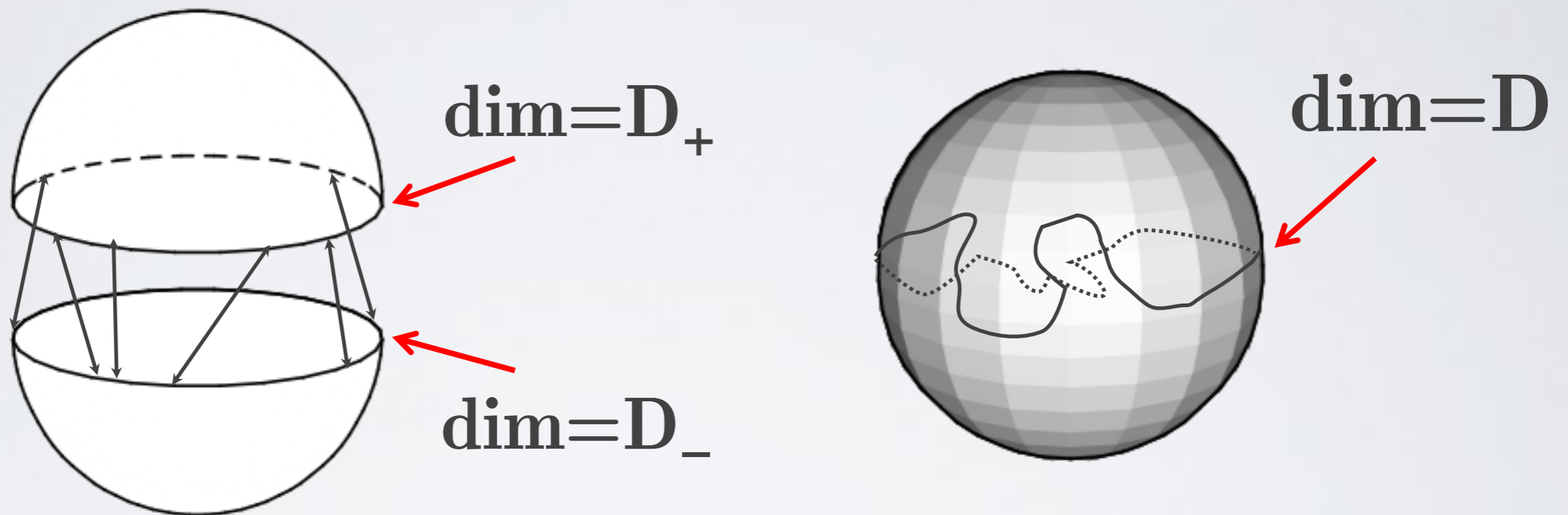
two perturbations of
conformal
structure



quasisymmetric welding \longleftrightarrow quasicircle

III. welding and dimensions

Take three images of the linear measure dx :



Then the conjectures before are equivalent to

$$(1-D)^2 \leq (1-D_-) (1-D_+)$$

III. Questions about $(1-D)^2 \leq (1-D_-)(1-D_+)$

Rem1 The inequality holds if $D_- = 1$.

Q1 Can one interpolate to prove it in general?

Rem2 For quasicircles arising in quasi-Fuchsian groups the base eigenvalue λ_0 of the Laplacian on the associated 3-manifold has $1-\lambda_0 = (1-D)^2$ for Patterson-Sullivan measure

Q2 Can one use 3D geometry ?

