Parabolic Invariants: From the Bergman Kernel to the AdS/CFT Correspondence

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Question. Can one express the Taylor expansions of $\varphi \mod \rho^{n+1}$ and ψ in terms of geometric invariants of $\partial\Omega$?

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In general, each a_j can be expressed as a linear combination of complete contractions

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The problem is difficult: the equation is degenerate at the boundary.

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$$u_2 \circ \Phi = \left|\det \Phi'\right|^{2/(n+1)} u_1.$$

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In 1982, Lee-Melrose showed that the Cheng-Yau solution has an asymptotic expansion at $\partial\Omega$ of the form

$$u \sim \rho \sum_{k=0}^{\infty} \eta_k \left(\rho^{n+1} \log \rho\right)^k, \qquad \eta_k \in C^\infty(\overline{\Omega}).$$

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$r_1, \ldots, r_L$ must be sufficiently small to avoid the indeterminacy of u at order $n+2$.

Expansion of Bergman Kernel

Theorem (CF, *Parabolic invariant theory in complex analysis*, 1979).

$$K_\Omega(z,z) = u^{-n-1}\sum_{j=0}^{n-19} I_j u^j + O(u^{-20}),$$

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The method of reasoning is that since u is invariant, the coefficients in the expansion in powers of u must be CR invariants. Charlie's invariant theory proves that all scalar CR invariants (up to the indicated power) are linear combinations of complete contractions of curvature of $\widetilde{g}$. Thus the result follows.

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The improved invariant theory enabled extension of the characterization of CR invariants and the expansion of the Bergman kernel up to the log term:

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The invariant theory required to prove this follows using the methods of Bailey-Eastwood-G.

Expansion of Log Coefficient

New ideas are required to handle the log term coefficient ψ. A main difficulty is the indeterminacy of the Monge-Ampère solution at order $n+2$. The solution was discovered by Hirachi in 2000.

Ingredients:

- Construction of a biholomorphically invariant family $\mathcal{F}$ of smooth defining functions u parametrized by $f \in C^\infty(\partial\Omega)$.
- Construction and characterization of all invariants $I(u)$ depending on $(\partial\Omega, u)$ by complete contractions of a metric $\widetilde{g}$ on $\mathbb{C}^* \times \mathbb{C}^n$.
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The invariant theory required to prove this follows using the methods of Bailey-Eastwood-G.
This completes the description of scalar CR invariants and the expansion of the Bergman kernel.

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The real analogue of CR geometry is conformal geometry. There is a version of the theory for conformal geometry. No analogue of the Bergman kernel, but there are applications in other directions.

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Conformal Group

The action of $O(n+1,1)$ preserves the Lorentz metric

$$\widetilde{g} = \sum_{i=1}^{n+1} \left(dX^i\right)^2 - \left(dX^0\right)^2 \quad \text{on } \mathbb{R}^{n+2}$$

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The Ricci-flat condition is the conformal analogue of the Monge-Ampère equation $J(u) = 1$.

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One reason: can view CR geometry as a subset of even dimensional conformal geometry. The "Fefferman metric" construction associates a Lorentz signature conformal structure on $\partial\Omega \times S^1$ to a strictly pseudoconvex boundary $\partial\Omega$.

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There are *exceptional* odd invariants (change sign under orientation reversal) if $n \equiv 0 \mod 4$. Examples: Pontrajagin invariants.

“Application”: Alexakis’ Theorem

The construction of scalar conformal invariants via the ambient metric plays an essential role in Alexakis’ recent structure theorem for conformally invariant integrals.

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Theorem. Up to a smooth homogeneous diffeomorphism, the inhomogeneous ambient metrics for $(M, [g])$ are parametrized by the choice of an arbitrary trace-free symmetric 2-tensor field A_{ij} on M. This "ambiguity tensor" corresponds to

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Also can construct a finite list of *basic exceptional invariants*.

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These results complete the construction of the ambient metric and the characterization of scalar conformal invariants for n even.

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Hyperbolic Space

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H^{n+1} also arises as a hyperboloid in Minkowski $\mathbb{R}^{n+2}$ with the induced metric.

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The restriction $g_+ \equiv \widetilde{g}|_{T\mathcal{H}}$ defines the (formal) Poincaré-Einstein metric associated to $(M, [g])$.

The condition $\mathrm{Ric}(\widetilde{g}) = 0$ is equivalent to $\mathrm{Ric}(g_+) = -ng_+$.

Suggests a global boundary value problem:

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Has led to much interaction and progress which continues now.

- Math $\rightarrow$ Physics — Holographic renormalization
- Physics $\rightarrow$ Math — Renormalized volume

Parabolic Geometries

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Complementary approach via Cartan connections: Cartan, Chern, Tanaka, ...