Onsager Equations, Nonlinear Fokker-Planck Equations, Navier-Stokes Equations

Peter Constantin

Department of Mathematics
The University of Chicago

Charlie Fefferman Fest, Princeton May 2009

In this talk:

- Equilibrium: Onsager Equation on Metric Spaces

In this talk:

- Equilibrium: Onsager Equation on Metric Spaces
- Kinetics: Nonlinear Fokker-Planck Equation

In this talk:

- Equilibrium: Onsager Equation on Metric Spaces
- Kinetics: Nonlinear Fokker-Planck Equation
- Dynamics: Coupling to Navier-Stokes Equation

Why?

- Nanoscale self-assembly
- Microfluidics
- Biomaterials
- Gels and Foams
- Soft Lattices, Jamming
- Pattern recognition

Major Problems

1. Modeling of interactions in the correct moduli space

Major Problems

1. Modeling of interactions in the correct moduli space
2. Derivation of Micro-Macro Effect

Major Problems

1. Modeling of interactions in the correct moduli space
2. Derivation of Micro-Macro Effect
3. PDE existence theory for coupled system

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.
- Reference measure: $d\mu$ – Borel Probability on M.

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.
- Reference measure: $d\mu$ – Borel Probability on M.
- Corpora state $f(m)d\mu(m)$ – Probabililty, AC w.r. $d\mu$.

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.
- Reference measure: $d\mu$ – Borel Probability on M.
- Corpora state $f(m)d\mu(m)$ – Probabililty, AC w.r. $d\mu$.
- Interaction kernel $k : M \times M \to \mathbb{R}_+$,

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.
- Reference measure: $d\mu$ – Borel Probability on M.
- Corpora state $f(m)d\mu(m)$ – Probabililty, AC w.r. $d\mu$.
- Interaction kernel $k : M \times M \to \mathbb{R}_+$, symmetric, bi-Lipschitz.

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.
- Reference measure: $d\mu$ – Borel Probability on M.
- Corpora state $f(m)d\mu(m)$ – Probabililty, AC w.r. $d\mu$.
- Interaction kernel $k : M \times M \to \mathbb{R}_+$, symmetric, bi-Lipschitz.
- Potential $U[f](m) = \int_M k(m,p)f(p)d\mu(p)$

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.
- Reference measure: $d\mu$ – Borel Probability on M.
- Corpora state $f(m)d\mu(m)$ – Probabililty, AC w.r. $d\mu$.
- Interaction kernel $k : M \times M \to \mathbb{R}_+$, symmetric, bi-Lipschitz.
- Potential $U[f](m) = \int_M k(m,p)f(p)d\mu(p)$
- Potential U = micro-micro interaction

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.
- Reference measure: $d\mu$ – Borel Probability on M.
- Corpora state $f(m)d\mu(m)$ – Probabililty, AC w.r. $d\mu$.
- Interaction kernel $k : M \times M \to \mathbb{R}_+$, symmetric, bi-Lipschitz.
- Potential $U[f](m) = \int_M k(m,p)f(p)d\mu(p)$
- Potential U = micro-micro interaction
- Free Energy

$$\mathcal{E}[f] = \int_M f \log f d\mu + \frac{1}{2}\int_M U[f] f d\mu$$

- Configuration space: M = compact, separable, metric space. $m \in M$ = corpus = doodad.
- Reference measure: $d\mu$ – Borel Probability on M.
- Corpora state $f(m)d\mu(m)$ – Probabililty, AC w.r. $d\mu$.
- Interaction kernel $k : M \times M \to \mathbb{R}_+$, symmetric, bi-Lipschitz.
- Potential $U[f](m) = \int_M k(m,p)f(p)d\mu(p)$
- Potential U = micro-micro interaction
- Free Energy

$$\mathcal{E}[f] = \int_M f \log f d\mu + \frac{1}{2}\int_M U[f] f d\mu$$

- Minima of Free Energy: Onsager Equation

$$\boxed{f = Z^{-1} e^{-U[f]}.}$$

Goals:

1. Existence theory for solutions of Onsager's equation

Goals:

1. Existence theory for solutions of Onsager's equation
2. Classification of high intensity limits

Goals:

1. Existence theory for solutions of Onsager's equation
2. Classification of high intensity limits
3. Selection mechanisms for high intensity limits

Example: Rods

$M = \mathbb{S}^{n-1}, \quad d\mu = \text{area}.$

$$\boxed{U[f](m) = \int_M k(m,p) f(p) d\mu(p)}$$

Example: Rods

$$M = \mathbb{S}^{n-1}, \quad d\mu = \text{area}.$$

$$\boxed{U[f](m) = \int_M k(m,p) f(p) d\mu(p)}$$

$$k(m,p) = k(p,m),$$

Example: Rods

$M = \mathbb{S}^{n-1}, \quad d\mu = \text{area}.$

$$\boxed{U[f](m) = \int_M k(m,p) f(p) d\mu(p)}$$

$k(m,p) = k(p,m)$, k smooth, real, given.

Example: Rods

$M = \mathbb{S}^{n-1}, \quad d\mu = \text{area}.$

$$\boxed{U[f](m) = \int_M k(m,p) f(p) d\mu(p)}$$

$k(m,p) = k(p,m)$, k smooth, real, given. Kernels:

Example: Rods

$M = \mathbb{S}^{n-1}, \quad d\mu = \text{area}.$

$$\boxed{U[f](m) = \int_M k(m,p)f(p)d\mu(p)}$$

$k(m,p) = k(p,m)$, k smooth, real, given. Kernels:

- Maier-Saupe: $M = \mathbb{S}^{n-1}$, $k(x,y) = b\sin^2(\phi(x,y))$

Example: Rods

$M = \mathbb{S}^{n-1}, \quad d\mu =$ area.

$$\boxed{U[f](m) = \int_M k(m,p)f(p)d\mu(p)}$$

$k(m,p) = k(p,m)$, k smooth, real, given. Kernels:

- Maier-Saupe: $M = \mathbb{S}^{n-1}$, $k(x,y) = b\sin^2(\phi(x,y))$
- Onsager model: $M = \mathbb{S}^{n-1}$, $k(x,y) = b|\sin(\phi(x,y))|$

b = intensity, inverse temperature.

Dimension Reduction, Maier-Saupe

$n \times n$ symmetric, traceless matrix S:

$$S \mapsto Z(S)$$

$$Z(S) = \int_{\mathbb{S}^{n-1}} e^{b(S^{ij}m_i m_j)} d\mu.$$

Dimension Reduction, Maier-Saupe

$n \times n$ symmetric, traceless matrix S:

$$S \mapsto Z(S)$$

$$Z(S) = \int_{\mathbb{S}^{n-1}} e^{b(S^{ij} m_i m_j)} d\mu.$$

$$f_S(m) = (Z(S))^{-1} e^{b(S^{ij} m_i m_j)}$$

Dimension Reduction, Maier-Saupe

$n \times n$ symmetric, traceless matrix S:

$$S \mapsto Z(S)$$

$$Z(S) = \int_{\mathbb{S}^{n-1}} e^{b(S^{ij} m_i m_j)} d\mu.$$

$$f_S(m) = (Z(S))^{-1} e^{b(S^{ij} m_i m_j)}$$

$$\sigma(S)_{ij} = \int_{\mathbb{S}^{n-1}} \left(m_i m_j - \frac{\delta_{ij}}{n} \right) f_S(m) d\mu.$$

Dimension Reduction, Maier-Saupe

$n \times n$ symmetric, traceless matrix S:

$$S \mapsto Z(S)$$

$$Z(S) = \int_{\mathbb{S}^{n-1}} e^{b(S^{ij}m_i m_j)} d\mu.$$

$$f_S(m) = (Z(S))^{-1} e^{b(S^{ij}m_i m_j)}$$

$$\sigma(S)_{ij} = \int_{\mathbb{S}^{n-1}} \left(m_i m_j - \frac{\delta_{ij}}{n} \right) f_S(m) d\mu.$$

Theorem

Onsager's equation with Maier-Saupe potential is equivalent to

$$\sigma(S) = S.$$

Limit $b \to \infty$

$$[\phi] = \int_{\mathbb{S}^2} \phi(m) f(m) d\mu.$$

Limit $b \to \infty$

$$[\phi] = \int_{\mathbb{S}^2} \phi(m) f(m) d\mu.$$

Isotropic:

$$\lim_{b \to \infty} [\phi] = \frac{1}{4\pi} \int_{\mathbb{S}^2} \phi(p) d\mu$$

Limit $b \to \infty$

$$[\phi] = \int_{\mathbb{S}^2} \phi(m) f(m) d\mu.$$

Isotropic:

$$\lim_{b\to\infty} [\phi] = \frac{1}{4\pi} \int_{\mathbb{S}^2} \phi(p) d\mu$$

Oblate:

$$\lim_{b\to\infty} [\phi] = \frac{1}{2\pi} \int_0^{2\pi} \phi(\cos\varphi, \sin\varphi, 0) d\varphi$$

Limit $b \to \infty$

$$[\phi] = \int_{\mathbb{S}^2} \phi(m) f(m) d\mu.$$

Isotropic:

$$\lim_{b\to\infty} [\phi] = \frac{1}{4\pi} \int_{\mathbb{S}^2} \phi(p) d\mu$$

Oblate:

$$\lim_{b\to\infty} [\phi] = \frac{1}{2\pi} \int_0^{2\pi} \phi(\cos\varphi, \sin\varphi, 0) d\varphi$$

Prolate:

$$\lim_{b\to\infty} [\phi] = \phi(m), \quad m \in \mathbb{S}^2.$$

Limit $b \to \infty$

$$[\phi] = \int_{\mathbb{S}^2} \phi(m) f(m) d\mu.$$

Isotropic:

$$\lim_{b\to\infty} [\phi] = \frac{1}{4\pi} \int_{\mathbb{S}^2} \phi(p) d\mu$$

Oblate:

$$\lim_{b\to\infty} [\phi] = \frac{1}{2\pi} \int_0^{2\pi} \phi(\cos\varphi, \sin\varphi, 0) d\varphi$$

Prolate:

$$\lim_{b\to\infty} [\phi] = \phi(m), \quad m \in \mathbb{S}^2.$$

The prolate state is selected. Why?

Freely Articulated N-corpora

$$\widetilde{M} = M_1 \times \cdots \times M_N, \quad d\mu = \Pi d\mu_j$$

Freely Articulated N-corpora

$$\widetilde{M} = M_1 \times \cdots \times M_N, \quad d\mu = \Pi d\mu_j$$

$$\widetilde{k}(p_1, q_1, p_2, q_2, \dots) = \sum_i k_i(p_i, q_i)$$

Freely Articulated N-corpora

$$\widetilde{M} = M_1 \times \cdots \times M_N, \quad d\mu = \Pi d\mu_j$$

$$\widetilde{k}(p_1, q_1, p_2, q_2, \dots) = \sum_i k_i(p_i, q_i)$$

$$\widetilde{U}[f] = \sum_{i=1}^{N} U_i[f],$$

Freely Articulated N-corpora

$$\widetilde{M} = M_1 \times \cdots \times M_N, \quad d\mu = \Pi d\mu_j$$

$$\widetilde{k}(p_1, q_1, p_2, q_2, \dots) = \sum_i k_i(p_i, q_i)$$

$$\widetilde{U}[f] = \sum_{i=1}^{N} U_i[f], \quad \text{with}$$

$$U_i[f](p_i) = \int_{\widetilde{M}} k_i(p_i, q_i) f(q_1, \dots q_N) d\mu(q)$$

Freely Articulated N-corpora

$$\widetilde{M} = M_1 \times \cdots \times M_N, \quad d\mu = \Pi d\mu_j$$

$$\widetilde{k}(p_1, q_1, p_2, q_2, \dots) = \sum_i k_i(p_i, q_i)$$

$$\widetilde{U}[f] = \sum_{i=1}^{N} U_i[f], \quad \text{with}$$

$$U_i[f](p_i) = \int_{\widetilde{M}} k_i(p_i, q_i) f(q_1, \dots q_N) d\mu(q)$$

Onsager Equation $\widetilde{f} = \widetilde{Z}^{-1} e^{-\widetilde{U}\widetilde{f}}$

Freely Articulated N-corpora

$$\widetilde{M} = M_1 \times \cdots \times M_N, \quad d\mu = \Pi d\mu_j$$

$$\widetilde{k}(p_1, q_1, p_2, q_2, \dots) = \sum_i k_i(p_i, q_i)$$

$$\widetilde{U}[f] = \sum_{i=1}^{N} U_i[f], \quad \text{with}$$

$$U_i[f](p_i) = \int_{\widetilde{M}} k_i(p_i, q_i) f(q_1, \dots q_N) d\mu(q)$$

Onsager Equation $\widetilde{f} = \widetilde{Z}^{-1} e^{-\widetilde{U}\widetilde{f}}$

$$\widetilde{Z} = \Pi_{j=1}^{N} Z_j, \quad \text{with } Z_j = \int_{M_j} e^{-U_j[f_j]} d\mu_j, \quad f_j = (Z_j)^{-1} e^{-U_j[f_j]}$$

Freely Articulated N-corpora

$$\widetilde{M} = M_1 \times \cdots \times M_N, \quad d\mu = \Pi d\mu_j$$

$$\widetilde{k}(p_1, q_1, p_2, q_2, \dots) = \sum_i k_i(p_i, q_i)$$

$$\widetilde{U}[f] = \sum_{i=1}^{N} U_i[f], \quad \text{with}$$

$$U_i[f](p_i) = \int_{\widetilde{M}} k_i(p_i, q_i) f(q_1, \dots q_N) d\mu(q)$$

Onsager Equation $\widetilde{f} = \widetilde{Z}^{-1} e^{-\widetilde{U}\widetilde{f}}$

$$\widetilde{Z} = \Pi_{j=1}^{N} Z_j, \quad \text{with } Z_j = \int_{M_j} e^{-U_j[f_j]} d\mu_j, \quad f_j = (Z_j)^{-1} e^{-U_j[f_j]}$$

$\widetilde{f}(p_1, \dots p_N) = f_1(p_1) f(p_2) \dots f_N(p_N)$ product measure

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

$$U[f](p_1, p_2) = \\ b\int_{\mathbb{T}^2} \|e(p_1)\wedge e(p_2) - e(q_1)\wedge e(q_2)\|^2 f(q_1, q_2)dq_1dq_2$$

with $e(p) = (\cos p, \sin p)$ if $p \in [0, 2\pi]$.

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

$$U[f](p_1, p_2) = \\ b \int_{\mathbb{T}^2} \|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 f(q_1, q_2) dq_1 dq_2$$

with $e(p) = (\cos p, \sin p)$ if $p \in [0, 2\pi]$.

$$\|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 = (\sin(p_1 - p_2) - \sin(q_1 - q_2))^2$$

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

$$U[f](p_1, p_2) = \\ b \int_{\mathbb{T}^2} \|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 f(q_1, q_2) dq_1 dq_2$$

with $e(p) = (\cos p, \sin p)$ if $p \in [0, 2\pi]$.

$\|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 = (\sin(p_1 - p_2) - \sin(q_1 - q_2))^2$

$$f_0(p_1, p_2) = Z^{-1} e^{-b \sin^2(p_1 - p_2)}.$$

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

$$U[f](p_1, p_2) = \\ b \int_{\mathbb{T}^2} \|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 f(q_1, q_2) dq_1 dq_2$$

with $e(p) = (\cos p, \sin p)$ if $p \in [0, 2\pi]$.

$\|e(p_1)\wedge e(p_2) - e(q_1)\wedge e(q_2)\|^2 = (\sin(p_1 - p_2) - \sin(q_1 - q_2))^2$

$$f_0(p_1, p_2) = Z^{-1} e^{-b\sin^2(p_1 - p_2)}.$$

As $b \to \infty$ this tends to $\delta((p_1 - p_2) \mathrm{mod} \pi)$.

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

$$U[f](p_1, p_2) = b\int_{\mathbb{T}^2} \|e(p_1)\wedge e(p_2) - e(q_1)\wedge e(q_2)\|^2 f(q_1, q_2)dq_1dq_2$$

with $e(p) = (\cos p, \sin p)$ if $p \in [0, 2\pi]$.

$\|e(p_1)\wedge e(p_2) - e(q_1)\wedge e(q_2)\|^2 = (\sin(p_1 - p_2) - \sin(q_1 - q_2))^2$

$$f_0(p_1, p_2) = Z^{-1}e^{-b\sin^2(p_1 - p_2)}.$$

As $b \to \infty$ this tends to $\delta((p_1 - p_2)\mathrm{mod}\pi)$. Additional solution $f_1(p_1, p_2)$.

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

$$U[f](p_1, p_2) = b \int_{\mathbb{T}^2} \|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 f(q_1, q_2) dq_1 dq_2$$

with $e(p) = (\cos p, \sin p)$ if $p \in [0, 2\pi]$.

$$\|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 = (\sin(p_1 - p_2) - \sin(q_1 - q_2))^2$$

$$f_0(p_1, p_2) = Z^{-1} e^{-b \sin^2(p_1 - p_2)}.$$

As $b \to \infty$ this tends to $\delta((p_1 - p_2) \mathrm{mod} \pi)$. Additional solution $f_1(p_1, p_2)$. As $b \to \infty$,

$$\lim_{b \to \infty} f_1(p_1, p_2) = \delta\left(\left(p_1 - p_2 - \frac{\pi}{2}\right) \mathrm{mod} \pi\right)$$

Example of Interacting Corpora

$M = \mathbb{S}^1$, $\widetilde{M} = \mathbb{S}^1 \times \mathbb{S}^1$.

$$U[f](p_1, p_2) = \\ b \int_{\mathbb{T}^2} \|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 f(q_1, q_2) dq_1 dq_2$$

with $e(p) = (\cos p, \sin p)$ if $p \in [0, 2\pi]$.

$$\|e(p_1) \wedge e(p_2) - e(q_1) \wedge e(q_2)\|^2 = (\sin(p_1 - p_2) - \sin(q_1 - q_2))^2$$

$$f_0(p_1, p_2) = Z^{-1} e^{-b \sin^2(p_1 - p_2)}.$$

As $b \to \infty$ this tends to $\delta((p_1 - p_2) \mathrm{mod} \pi)$. Additional solution $f_1(p_1, p_2)$. As $b \to \infty$,

$$\lim_{b \to \infty} f_1(p_1, p_2) = \delta\left(\left(p_1 - p_2 - \frac{\pi}{2}\right) \mathrm{mod} \pi\right)$$

This state is selected. Why ?

More degrees of freedom

$$M = [0, L] \times [0, L] \times [0, \pi], \; d\mu = \tfrac{1}{\pi L^2} dx_1 dx_2 d\theta$$

More degrees of freedom

$$M = [0, L] \times [0, L] \times [0, \pi],\ d\mu = \frac{1}{\pi L^2} dx_1 dx_2 d\theta$$

$$U[f](x_1, x_2, \theta) = \int_M (x_1 x_2 \sin\theta - y_1 y_2 \sin\phi)^2 f(y_1, y_2, \phi) d\mu$$

More degrees of freedom

$$M = [0, L] \times [0, L] \times [0, \pi], \; d\mu = \frac{1}{\pi L^2} dx_1 dx_2 d\theta$$

$$U[f](x_1, x_2, \theta) = \int_M (x_1 x_2 \sin\theta - y_1 y_2 \sin\phi)^2 f(y_1, y_2, \phi) d\mu$$

Let

$$A[f] = \int_M (x_1 x_2 \sin\theta) f(x_1, x_2, \theta) d\mu.$$

More degrees of freedom

$M = [0, L] \times [0, L] \times [0, \pi]$, $d\mu = \frac{1}{\pi L^2} dx_1 dx_2 d\theta$

$$U[f](x_1, x_2, \theta) = \int_M (x_1 x_2 \sin\theta - y_1 y_2 \sin\phi)^2 f(y_1, y_2, \phi) d\mu$$

Let

$$A[f] = \int_M (x_1 x_2 \sin\theta) f(x_1, x_2, \theta) d\mu.$$

Let g_b be any weakly convergent sequence of solutions of Onsager's equation, $b \to \infty$.

More degrees of freedom

$$M = [0, L] \times [0, L] \times [0, \pi],\ d\mu = \tfrac{1}{\pi L^2} dx_1 dx_2 d\theta$$

$$U[f](x_1, x_2, \theta) = \int_M (x_1 x_2 \sin\theta - y_1 y_2 \sin\phi)^2 f(y_1, y_2, \phi) d\mu$$

Let

$$A[f] = \int_M (x_1 x_2 \sin\theta) f(x_1, x_2, \theta) d\mu.$$

Let g_b be any weakly convergent sequence of solutions of Onsager's equation, $b \to \infty$. Then,

$$\lim_{b\to\infty} A[g_b] = 0.$$

More degrees of freedom

$M = [0, L] \times [0, L] \times [0, \pi]$, $d\mu = \frac{1}{\pi L^2} dx_1 dx_2 d\theta$

$$U[f](x_1, x_2, \theta) = \int_M (x_1 x_2 \sin\theta - y_1 y_2 \sin\phi)^2 f(y_1, y_2, \phi) d\mu$$

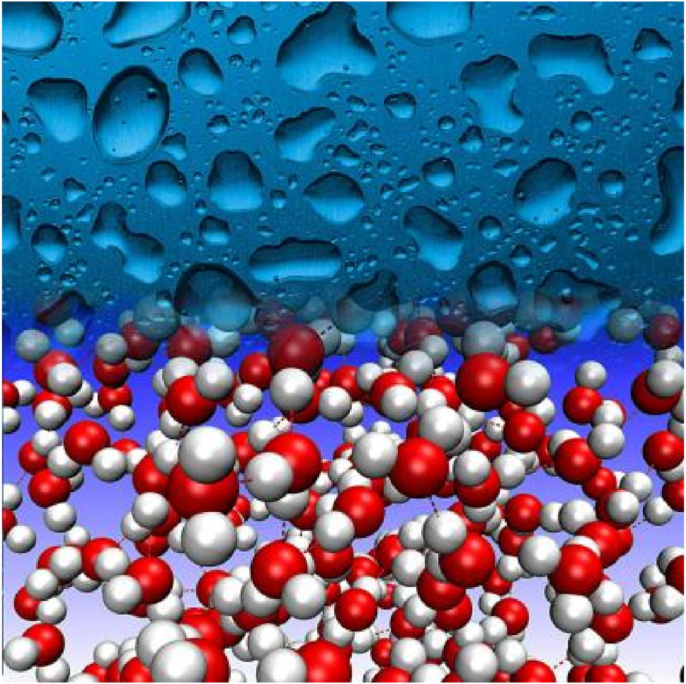
Let

$$A[f] = \int_M (x_1 x_2 \sin\theta) f(x_1, x_2, \theta) d\mu.$$

Let g_b be any weakly convergent sequence of solutions of Onsager's equation, $b \to \infty$. Then,

$$\lim_{b\to\infty} A[g_b] = 0.$$

Why is this different than before?



Packing

$V(r)$ nonnegative, nonincreasing, compactly supported.
$p = (x_1, \dots x_N)$, $x_i \in \Omega \subset \mathbb{R}^n$.

Packing

$V(r)$ nonnegative, nonincreasing, compactly supported.
$p = (x_1, \dots x_N)$, $x_i \in \Omega \subset \mathbb{R}^n$. Packing energy:

$$F(p) = \sum_{i<j} V(|x_i - x_j|).$$

Packing

$V(r)$ nonnegative, nonincreasing, compactly supported.
$p = (x_1, \dots x_N)$, $x_i \in \Omega \subset \mathbb{R}^n$. Packing energy:

$$F(p) = \sum_{i<j} V(|x_i - x_j|).$$

$$\widetilde{M} = \Omega \times \cdots \times \Omega \cap \{F \leq F_0\}.$$

Packing

$V(r)$ nonnegative, nonincreasing, compactly supported.
$p = (x_1, \dots x_N)$, $x_i \in \Omega \subset \mathbb{R}^n$. Packing energy:

$$F(p) = \sum_{i<j} V(|x_i - x_j|).$$

$$\widetilde{M} = \Omega \times \cdots \times \Omega \cap \{F \le F_0\}.$$

$$U[f](p) = \int_{\widetilde{M}} \left[|F(p) - F(q)|^2 + d^2(p,q) \right] f(q) d\mu$$

Packing

$V(r)$ nonnegative, nonincreasing, compactly supported.
$p = (x_1, \ldots x_N)$, $x_i \in \Omega \subset \mathbb{R}^n$. Packing energy:

$$F(p) = \sum_{i<j} V(|x_i - x_j|).$$

$$\widetilde{M} = \Omega \times \cdots \times \Omega \cap \{F \le F_0\}.$$

$$U[f](p) = \int_{\widetilde{M}} \left[|F(p) - F(q)|^2 + d^2(p, q)\right] f(q) d\mu$$

Connection to the example of freely articulated $2n$ corpora, jamming.

M compact metric space, d distance, μ Borel probability measure on M.

M compact metric space, d distance, μ Borel probability measure on M. Let

$$k : M \times M \to \mathbb{R}$$

- symmetric $k(m,p) = k(p,m)$
- bounded below $k(m,n) \geq 0$

M compact metric space, d distance, μ Borel probability measure on M. Let

$$k : M \times M \to \mathbb{R}$$

- symmetric $k(m,p) = k(p,m)$
- bounded below $k(m,n) \geq 0$
- uniformly bi-Lipschitz:

$$|k(m,n) - k(p,n)| \leq Ld(m,p)$$

If $f > 0$, $\int_M f d\mu = 1$, define

M compact metric space, d distance, μ Borel probability measure on M. Let

$$k : M \times M \to \mathbb{R}$$

- symmetric $k(m,p) = k(p,m)$
- bounded below $k(m,n) \geq 0$
- uniformly bi-Lipschitz:

$$|k(m,n) - k(p,n)| \leq Ld(m,p)$$

If $f > 0$, $\int_M f d\mu = 1$, define

$$\mathcal{E}[f] = \int_M f \log f d\mu + \frac{b}{2} \int_M \int_M k(p,q) f(p) d\mu(p) f(q) d\mu(q).$$

Theorem

For any $b > 0$ there exists a solution g that minimizes the energy:

Theorem

For any $b > 0$ there exists a solution g that minimizes the energy:

$$\mathcal{E}[g] = \min_{f \geq 0,\ \int_M f d\mu = 1} \mathcal{E}[f]$$

The function g solves the Onsager equation

Theorem

For any $b > 0$ there exists a solution g that minimizes the energy:

$$\mathcal{E}[g] = \min_{f \geq 0,\ \int_M f d\mu = 1} \mathcal{E}[f]$$

The function g solves the Onsager equation

$$g(x) = (Z(b))^{-1} e^{-bU(x)}$$

Theorem

For any $b > 0$ there exists a solution g that minimizes the energy:

$$\mathcal{E}[g] = \min_{f\geq 0,\ \int_M f d\mu = 1} \mathcal{E}[f]$$

The function g solves the Onsager equation

$$g(x) = (Z(b))^{-1} e^{-bU(x)}$$

with

$$Z(b) = \int_M e^{-bU(x)} d\mu(x)$$

Theorem

For any $b > 0$ there exists a solution g that minimizes the energy:

$$\mathcal{E}[g] = \min_{f \geq 0,\ \int_M f d\mu = 1} \mathcal{E}[f]$$

The function g solves the Onsager equation

$$g(x) = (Z(b))^{-1} e^{-bU(x)}$$

with

$$Z(b) = \int_M e^{-bU(x)} d\mu(x)$$

and

$$U(x) = \int_M k(x, y) g(y) d\mu(y).$$

Theorem

For any $b > 0$ there exists a solution g that minimizes the energy:

$$\mathcal{E}[g] = \min_{f\geq 0,\ \int_M f d\mu = 1} \mathcal{E}[f]$$

The function g solves the Onsager equation

$$g(x) = (Z(b))^{-1} e^{-bU(x)}$$

with

$$Z(b) = \int_M e^{-bU(x)} d\mu(x)$$

and

$$U(x) = \int_M k(x,y) g(y) d\mu(y).$$

The function g is normalized $\int g d\mu = 1$, strictly positive and Lipschitz continuous.

The ur-corpus

Let M be a compact metrizable space and let $k(x, y)$ be symmetric, bi-Lipschitz and bounded below.

The ur-corpus

Let M be a compact metrizable space and let $k(x, y)$ be symmetric, bi-Lipschitz and bounded below. In addition, assume:

$$k(x, x) = 0.$$

The ur-corpus

Let M be a compact metrizable space and let $k(x,y)$ be symmetric, bi-Lipschitz and bounded below. In addition, assume:

$$k(x,x) = 0.$$

Theorem

(C-Zlatos) *Let ν be a weak limit of a sequence $f_n d\mu$ of minima of the free energy $\mathcal{E}$ corresponding to $b_n \to \infty$. Then there exists $m \in M$ such that ν is concentrated on the level set $\Sigma(m) = \{p \,|\, k(m,p) = 0\}$.*

The ur-corpus

Let M be a compact metrizable space and let $k(x,y)$ be symmetric, bi-Lipschitz and bounded below. In addition, assume:

$$k(x,x) = 0.$$

Theorem

(C-Zlatos) *Let ν be a weak limit of a sequence $f_n d\mu$ of minima of the free energy $\mathcal{E}$ corresponding to $b_n \to \infty$. Then there exists $m \in M$ such that ν is concentrated on the level set $\Sigma(m) = \{p \mid k(m,p) = 0\}$.*

This explains why the prolate Maier-Saupe state is selected.

The ur-corpus

Let M be a compact metrizable space and let $k(x,y)$ be symmetric, bi-Lipschitz and bounded below. In addition, assume:

$$k(x,x) = 0.$$

Theorem

(C-Zlatos) *Let ν be a weak limit of a sequence $f_n d\mu$ of minima of the free energy $\mathcal{E}$ corresponding to $b_n \to \infty$. Then there exists $m \in M$ such that ν is concentrated on the level set $\Sigma(m) = \{p \mid k(m,p) = 0\}$.*

This explains why the prolate Maier-Saupe state is selected. Pattern recognition example: Rhombi centered at the origin, with k the area of the symmetric difference.

The ur-corpus

Let M be a compact metrizable space and let $k(x, y)$ be symmetric, bi-Lipschitz and bounded below. In addition, assume:

$$k(x, x) = 0.$$

Theorem

(C-Zlatos) Let ν be a weak limit of a sequence $f_n d\mu$ of minima of the free energy $\mathcal{E}$ corresponding to $b_n \to \infty$. Then there exists $m \in M$ such that ν is concentrated on the level set $\Sigma(m) = \{p \mid k(m, p) = 0\}$.

This explains why the prolate Maier-Saupe state is selected. Pattern recognition example: Rhombi centered at the origin, with k the area of the symmetric difference. The ur-rhombus is the square.

Principle: if a μ measure-preserving transformation T exists such that locally around $p = p_0$, $k(Tp, Tq) \le ck(p, q)$ with $c < 1$, then p_0 cannot be an ur-corpus.

Principle: if a μ measure-preserving transformation T exists such that locally around $p = p_0$,
$k(Tp, Tq) \leq ck(p, q)$ with $c < 1$, then p_0 cannot be an ur-corpus.
If a local k-preserving transformation around $p = p_0$ has the property that $\mu(T(B)) \geq C\mu(B)$ for small balls around p_0, with $C > 1$, then p_0 cannot be an ur-corpus.

Theorem

(Zlatos-C) Let $A_0, A_1 \subseteq M$ *be compacts such that* $k(p,q) = 0$ *for any* $p, q \in A_j$ *(*$j = 0, 1$*) and let* $B_j(\epsilon) = \{p \in M \mid d(p, A_j) < \epsilon\}$*. Assume that for some* $\epsilon_j > 0$ *there is a 1-1 map* $T : B_1(\epsilon_1) \to B_0(\epsilon_0)$

Theorem

(Zlatos-C) Let $A_0, A_1 \subseteq M$ *be compacts such that* $k(p,q) = 0$ *for any* $p, q \in A_j$ *(*$j = 0, 1$*) and let* $B_j(\epsilon) = \{p \in M \mid d(p, A_j) < \epsilon\}$. *Assume that for some* $\epsilon_j > 0$ *there is a 1-1 map* $T : B_1(\epsilon_1) \to B_0(\epsilon_0)$ *such that* T, T^{-1} *are measurable*

Theorem

(Zlatos-C) Let $A_0, A_1 \subseteq M$ *be compacts such that* $k(p,q) = 0$ *for any* $p, q \in A_j$ $(j = 0, 1)$ *and let* $B_j(\epsilon) = \{p \in M \mid d(p, A_j) < \epsilon\}$. *Assume that for some* $\epsilon_j > 0$ *there is a 1-1 map* $T : B_1(\epsilon_1) \to B_0(\epsilon_0)$ *such that* T, T^{-1} *are measurable and there is* $C > 1$ *such that*

$$\forall p, q \in B_1(\epsilon_1) \;:\; k(T(p), T(q)) \le k(p,q)$$

$$\forall B \subseteq B_1(\epsilon_1) \text{ measurable } \;:\; \mu(T(B)) \ge C\mu(B).$$

Theorem

(Zlatos-C) Let $A_0, A_1 \subseteq M$ *be compacts such that* $k(p,q) = 0$ *for any* $p, q \in A_j$ *(*$j = 0, 1$*) and let* $B_j(\epsilon) = \{p \in M \,|\, d(p, A_j) < \epsilon\}$*. Assume that for some* $\epsilon_j > 0$ *there is a 1-1 map* $T : B_1(\epsilon_1) \to B_0(\epsilon_0)$ *such that* T, T^{-1} *are measurable and there is* $C > 1$ *such that*

$$\forall p, q \in B_1(\epsilon_1) \;:\; k(T(p), T(q)) \le k(p,q)$$

$$\forall B \subseteq B_1(\epsilon_1) \text{ measurable } \;:\; \mu(T(B)) \ge C\mu(B).$$

Assume also that for each $p \in A_1$*,* $q \in M \setminus B_1(\epsilon_1)$ *we have* $k(p,q) > 0$*.*

Theorem

(Zlatos-C) Let $A_0, A_1 \subseteq M$ *be compacts such that* $k(p,q) = 0$ *for any* $p, q \in A_j$ $(j = 0, 1)$ *and let* $B_j(\epsilon) = \{p \in M \,|\, d(p, A_j) < \epsilon\}$. *Assume that for some* $\epsilon_j > 0$ *there is a 1-1 map* $T : B_1(\epsilon_1) \to B_0(\epsilon_0)$ *such that* T, T^{-1} *are measurable and there is* $C > 1$ *such that*

$$\forall p, q \in B_1(\epsilon_1) \; : \; k(T(p), T(q)) \le k(p,q)$$

$$\forall B \subseteq B_1(\epsilon_1) \text{ measurable } \; : \; \mu(T(B)) \ge C\mu(B).$$

Assume also that for each $p \in A_1$, $q \in M \setminus B_1(\epsilon_1)$ *we have* $k(p,q) > 0$. *Then* $\nu(A_1) < 1$ *for each measure* ν *that is a zero temperature weak limit of minimizers.*

Theorem

(Zlatos-C) Let $A_0, A_1 \subseteq M$ *be compacts such that* $k(p,q) = 0$ *for any* $p, q \in A_j$ $(j = 0, 1)$ *and let* $B_j(\epsilon) = \{p \in M \,|\, d(p, A_j) < \epsilon\}$. *Assume that for some* $\epsilon_j > 0$ *there is a 1-1 map* $T : B_1(\epsilon_1) \to B_0(\epsilon_0)$ *such that* T, T^{-1} *are measurable and there is* $C > 1$ *such that*

$$\forall p, q \in B_1(\epsilon_1) \ : \ k(T(p), T(q)) \leq k(p,q)$$

$$\forall B \subseteq B_1(\epsilon_1) \ \textit{measurable} \ : \ \mu(T(B)) \geq C\mu(B).$$

Assume also that for each $p \in A_1$, $q \in M \setminus B_1(\epsilon_1)$ *we have* $k(p,q) > 0$. *Then* $\nu(A_1) < 1$ *for each measure* ν *that is a zero temperature weak limit of minimizers.*

This explains the selection of the triangle of maximal area when the sides are of fixed length, and of minimal area when the sides are allowed to shrink.

Kinetics

M compact, connected Riemannian manifold of dimension d

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$,

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$,

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$,

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, volume element $d\mu = \sqrt{g}d\phi$ in local coordinates ϕ.

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, volume element $d\mu = \sqrt{g}d\phi$ in local coordinates ϕ. Generalized Doi-Smoluchowski equation

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, volume element $d\mu = \sqrt{g}d\phi$ in local coordinates ϕ. Generalized Doi-Smoluchowski equation

$$\boxed{\partial_t f = \Delta_g f + div_g(f\nabla_g U)}$$

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, volume element $d\mu = \sqrt{g}d\phi$ in local coordinates ϕ. Generalized Doi-Smoluchowski equation

$$\boxed{\partial_t f = \Delta_g f + div_g(f\nabla_g U)}$$

Δ_g Laplace-Beltrami,

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, volume element $d\mu = \sqrt{g}d\phi$ in local coordinates ϕ. Generalized Doi-Smoluchowski equation

$$\boxed{\partial_t f = \Delta_g f + div_g(f\nabla_g U)}$$

Δ_g Laplace-Beltrami,

$$\boxed{div_g(f\nabla_g U) = \frac{1}{\sqrt{g}}\partial_\alpha\left(\sqrt{g}g^{\alpha\beta}f\partial_\beta U\right).}$$

Kinetics

M compact, connected Riemannian manifold of dimension d with metric $g_{\alpha\beta}$, $g = \det(g_{\alpha\beta})$, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, volume element $d\mu = \sqrt{g}d\phi$ in local coordinates ϕ. Generalized Doi-Smoluchowski equation

$$\boxed{\partial_t f = \Delta_g f + div_g(f\nabla_g U)}$$

Δ_g Laplace-Beltrami,

$$\boxed{div_g(f\nabla_g U) = \frac{1}{\sqrt{g}}\partial_\alpha\left(\sqrt{g}g^{\alpha\beta}f\partial_\beta U\right).}$$

$$\boxed{U[f](p) = \int_M k(p,q)f(q)d\mu(q)}$$

Dissipative Structure

Free Energy:

Dissipative Structure

Free Energy:

$$\boxed{\mathcal{E}[f] = \int_M \left\{ \log f + \frac{1}{2} U[f] \right\} f d\mu}$$

Dissipative Structure

Free Energy:

$$\boxed{\mathcal{E}[f] = \int_M \left\{ \log f + \frac{1}{2} U[f] \right\} f d\mu}$$

NLFP:

Dissipative Structure

Free Energy:

$$\boxed{\mathcal{E}[f] = \int_M \left\{ \log f + \frac{1}{2} U[f] \right\} f d\mu}$$

NLFP:

$$\boxed{\partial_t f = \mathrm{div}_g \cdot \left(f \nabla_g \left(\frac{\delta \mathcal{E}[f]}{\delta f} \right) \right)}$$

Dissipative Structure

Free Energy:

$$\boxed{\mathcal{E}[f] = \int_M \left\{ \log f + \frac{1}{2} U[f] \right\} f d\mu}$$

NLFP:

$$\boxed{\partial_t f = \mathrm{div}_g \cdot \left(f \nabla_g \left(\frac{\delta \mathcal{E}[f]}{\delta f} \right) \right)}$$

Lyapunov functional:

$$\boxed{\frac{d}{dt}\mathcal{E}[f] = -\int_M f \left| \nabla_g \left(\frac{\delta \mathcal{E}[f]}{\delta f} \right) \right|^2 d\mu}$$

Connection to Onsager's equation

Connection to Onsager's equation

For Doi-Smoluchowski

$$\boxed{\frac{\delta \mathcal{E}[f]}{\delta f} = \log f + U[f]}$$

Connection to Onsager's equation

For Doi-Smoluchowski

$$\boxed{\frac{\delta \mathcal{E}[f]}{\delta f} = \log f + U[f]}$$

Time independent solutions:

Connection to Onsager's equation

For Doi-Smoluchowski

$$\boxed{\frac{\delta \mathcal{E}[f]}{\delta f} = \log f + U[f]}$$

Time independent solutions: Onsager equation:

$$\boxed{f = Z^{-1} e^{-U[f]}}$$

Connection to Onsager's equation

For Doi-Smoluchowski

$$\boxed{\frac{\delta\mathcal{E}[f]}{\delta f} = \log f + U[f]}$$

Time independent solutions: Onsager equation:

$$\boxed{f = Z^{-1}e^{-U[f]}}$$

Dynamics: nontrivial. Multiple steady states, gradient system, finite dimensional attractor.

Connection to Onsager's equation

For Doi-Smoluchowski

$$\boxed{\frac{\delta \mathcal{E}[f]}{\delta f} = \log f + U[f]}$$

Time independent solutions: Onsager equation:

$$\boxed{f = Z^{-1} e^{-U[f]}}$$

Dynamics: nontrivial. Multiple steady states, gradient system, finite dimensional attractor. Inertial Manifolds: Vukadinovic (2008-9).

Connection to Onsager's equation

For Doi-Smoluchowski

$$\boxed{\frac{\delta\mathcal{E}[f]}{\delta f} = \log f + U[f]}$$

Time independent solutions: Onsager equation:

$$\boxed{f = Z^{-1}e^{-U[f]}}$$

Dynamics: nontrivial. Multiple steady states, gradient system, finite dimensional attractor. Inertial Manifolds: Vukadinovic (2008-9). The non-smooth setting needs to be developed: gradient flow in space of probabilities on metric spaces, (see Ambrosio et al).

Embedding in Fluid: Passive

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(W f) = \mathrm{div}_g\left(f \nabla_g \left(\log f + U[f]\right)\right)$$

Embedding in Fluid: Passive

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(W f) = \mathrm{div}_g\left(f \nabla_g\left(\log f + U[f]\right)\right)$$

with

$$W(x,m,t) = \sum_{i,j=1}^{n} c_{ji}^{\alpha}(m)\frac{\partial u_i}{\partial x^j}(x,t)$$

Embedding in Fluid: Passive

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g\left(f \nabla_g\left(\log f + U[f]\right)\right)$$

with

$$W(x,m,t) = \sum_{i,j=1}^{n} c_{ji}^{\alpha}(m) \frac{\partial u_i}{\partial x^j}(x,t)$$

$$c_{ji}^{\cdot}(m) \in T_m^*(M).$$

Embedding in Fluid: Passive

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g\left(f \nabla_g \left(\log f + U[f]\right)\right)$$

with

$$W(x,m,t) = \sum_{i,j=1}^{n} c_{ji}^{\alpha}(m) \frac{\partial u_i}{\partial x^j}(x,t)$$

$$c_{ji}^{\cdot}(m) \in T_m^*(M).$$

Example, rods in 3D:

Embedding in Fluid: Passive

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g\left(f \nabla_g \left(\log f + U[f]\right)\right)$$

with

$$W(x,m,t) = \sum_{i,j=1}^{n} c_{ji}^{\alpha}(m) \frac{\partial u_i}{\partial x^j}(x,t)$$

$$c_{ji}^{\cdot}(m) \in T_m^*(M).$$

Example, rods in 3D: $M = \mathbb{S}^2$,

Embedding in Fluid: Passive

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g\left(f \nabla_g \left(\log f + U[f]\right)\right)$$

with

$$W(x, m, t) = \sum_{i,j=1}^{n} c_{ji}^{\alpha}(m) \frac{\partial u_i}{\partial x^j}(x, t)$$

$$c_{ji}^{\cdot}(m) \in T_m^*(M).$$

Example, rods in 3D: $M = \mathbb{S}^2$,

$$W(x, m, t) = (\nabla_x u(x, t))m - ((\nabla_x u(x, t))m \cdot m)m.$$

Embedding in Fluid: Passive

$$\partial_t f + u\cdot\nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g\left(f\nabla_g\left(\log f + U[f]\right)\right)$$

with

$$W(x,m,t) = \sum_{i,j=1}^{n} c_{ji}^{\alpha}(m)\frac{\partial u_i}{\partial x^j}(x,t)$$

$$c_{ji}^{\cdot}(m) \in T_m^*(M).$$

Example, rods in 3D: $M = \mathbb{S}^2$,

$$W(x,m,t) = (\nabla_x u(x,t))m - ((\nabla_x u(x,t))m\cdot m)m.$$

Macro-Micro Effect: from first principles, if scales are separated.

Active: Navier-Stokes

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \nabla \cdot \sigma$$
$$\nabla \cdot u = 0$$

Active: Navier-Stokes

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \nabla\cdot\sigma$$
$$\nabla\cdot u = 0$$

$$\sigma = \sigma_{ij}(x,t)$$

added stress tensor.

Active: Navier-Stokes

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \nabla \cdot \sigma$$
$$\nabla \cdot u = 0$$

$$\sigma = \sigma_{ij}(x,t)$$

added stress tensor.

Micro-Macro Effect

$$\boxed{\sigma(x) = \int_M \{c \cdot \nabla_g U[f](x,m) - \mathrm{div}_g c\} f(x,m) d\mu(m) \qquad *}$$

Active: Navier-Stokes

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \nabla\cdot\sigma$$
$$\nabla\cdot u = 0$$

$$\sigma = \sigma_{ij}(x,t)$$

added stress tensor.

Micro-Macro Effect

$$\boxed{\sigma(x) = \int_M \left\{c\cdot\nabla_g U[f](x,m) - \mathrm{div}_g c\right\} f(x,m)d\mu(m) \quad *}$$

$$f = Z^{-1}e^{-U[f]} \;\Rightarrow\; \sigma = 0$$

Active: Navier-Stokes

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \nabla\cdot\sigma$$
$$\nabla\cdot u = 0$$

$$\sigma = \sigma_{ij}(x,t)$$

added stress tensor.

Micro-Macro Effect

$$\boxed{\sigma(x) = \int_M \left\{c\cdot\nabla_g U[f](x,m) - \mathrm{div}_g c\right\} f(x,m)\,d\mu(m) \qquad *}$$

$f = Z^{-1}e^{-U[f]} \Rightarrow \sigma = 0 \;\; U = 0,\; W = (\nabla_x u)m - m((\nabla_x u)m\cdot m) \Rightarrow \sigma = \int_M (m\otimes m - \frac{1}{3})d\mu$

Theorem

*3DNS + NL Fokker-Planck eqns with *. Then*

$$\begin{aligned} E(t) = \frac{1}{2}\int |u|^2 dx + \\ + \int \left\{ f \log f + \frac{1}{2}(U[f])f \right\} dx d\mu. \end{aligned}$$

is nondecreasing on solutions.

Theorem

*3DNS + NL Fokker-Planck eqns with *. Then*

$$E(t) = \frac{1}{2}\int |u|^2 dx + \\ + \int \left\{ f \log f + \frac{1}{2}(U[f])f \right\} dx d\mu.$$

is nondecreasing on solutions. If (u, f) is a smooth solution then

$$\frac{dE}{dt} = -\nu \int |\nabla_x u|^2 dx - \\ - \int \int_M f \left|\nabla_g \left(\log f + U[f]\right)\right|^2 d\mu dx.$$

Theorem

*3DNS + NL Fokker-Planck eqns with *. Then*

$$E(t) = \frac{1}{2}\int |u|^2 dx + \\ + \int \left\{ f \log f + \frac{1}{2}(U[f])f \right\} dx d\mu.$$

is nondecreasing on solutions. If (u, f) is a smooth solution then

$$\frac{dE}{dt} = -\nu \int |\nabla_x u|^2 dx - \\ - \int \int_M f \left| \nabla_g \left(\log f + U[f] \right) \right|^2 d\mu dx.$$

If the smooth solution is time independent, then $u = 0$ and f solves the Onsager equation

$$f = Z^{-1} e^{-U[f]}.$$

NLFP + 3D time-dependent Stokes

$$
\begin{gathered}
\partial_t f + u\cdot\nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g(f\nabla_g(\log f + U[f])),\\
\partial_t u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma + F, \quad \nabla_x\cdot u = 0.
\end{gathered}
$$

NLFP + 3D time-dependent Stokes

$$\partial_t f + u\cdot\nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g(f\nabla_g(\log f + U[f])),$$
$$\partial_t u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma + F, \quad \nabla_x\cdot u = 0.$$

Theorem

Let u_0 divergence-free, in $W^{2,r}(\mathbb{T}^3)$, $r>3$, f_0 positive, $\int_M f_0(x,m)d\mu = 1$, $f_0 \in L^\infty(dx;\mathcal{C}(M)) \cap \nabla_x f_0 \in L^r(dx;H^{-s}(M))$, $s \le \frac{d}{2}+1$.

NLFP + 3D time-dependent Stokes

$$\partial_t f + u\cdot\nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g(f\nabla_g(\log f + U[f])),$$
$$\partial_t u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma + F, \quad \nabla_x\cdot u = 0.$$

Theorem

Let u_0 divergence-free, in $W^{2,r}(\mathbb{T}^3)$, $r > 3$, f_0 positive,
$\int_M f_0(x,m)d\mu = 1$,
$f_0 \in L^\infty(dx;\mathcal{C}(M)) \cap \nabla_x f_0 \in L^r(dx; H^{-s}(M))$, $s \le \frac{d}{2} + 1$.
Then the solution exists for all time and

$$\|u\|_{L^p[(0,T);W^{2,r}(dx)]} < \infty,$$
$$\|\nabla_x f\|_{L^\infty[(0,T);L^r(dx;H^{-s}(M))]} < \infty$$

for any $p > \frac{2r}{r-3}$, $T > 0$.

Time independent Stokes, counting:

$$-\nu\Delta u + \nabla p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0$$

Time independent Stokes, counting:

$$-\nu\Delta u + \nabla p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0$$

gives

Time independent Stokes, counting:

$$-\nu\Delta u + \nabla p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0$$

gives

$$\nabla_x u \sim \sigma$$

Time independent Stokes, counting:

$$-\nu\Delta u + \nabla p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0$$

gives

$$\nabla_x u \sim \sigma$$

$$D_t f \sim (\nabla_x u)\, \mathrm{div}_g (cf)$$

Time independent Stokes, counting:

$$-\nu\Delta u + \nabla p = \mathrm{div}_x\sigma, \quad \nabla_x \cdot u = 0$$

gives

$$\nabla_x u \sim \sigma$$

$$D_t f \sim (\nabla_x u)\,\mathrm{div}_g(cf) \ \sim \sigma \mathrm{div}_g(cf)$$

Time independent Stokes, counting:

$$-\nu\Delta u + \nabla p = \mathrm{div}_x\sigma, \quad \nabla_x\cdot u = 0$$

gives

$$\nabla_x u \sim \sigma$$

$$D_t f \sim (\nabla_x u)\,\mathrm{div}_g(cf) \ \sim \sigma\mathrm{div}_g(cf)$$

nonlocal Burgers

Time independent Stokes, counting:

$$-\nu\Delta u + \nabla p = \mathrm{div}_x\sigma, \quad \nabla_x \cdot u = 0$$

gives

$$\nabla_x u \sim \sigma$$

$$D_t f \sim (\nabla_x u)\,\mathrm{div}_g(cf) \;\sim \sigma \mathrm{div}_g(cf)$$

nonlocal Burgers

$$\sigma = \int_M \{c\cdot\nabla_g U[f] - \mathrm{div}_g c\}\, f d\mu$$

NLFP + 2D time dependent Navier-Stokes

Theorem

(C-Masmoudi) Let $u_0 \in \left(W^{\alpha,r} \cap L^2\right)(\mathbb{R}^2)$, *divergence-free,*

NLFP + 2D time dependent Navier-Stokes

Theorem

(C-Masmoudi) Let $u_0 \in \left(W^{\alpha,r} \cap L^2\right)(\mathbb{R}^2)$, *divergence-free,* $f_0 \in W^{1,r}(H^{-s}(M))$, *with* $r > 2$, $\alpha > 1$, $s \leq \frac{d}{2} + 1$ *and* $f_0 \geq 0$, $\int_M f_0 d\mu \in (L^1 \cap L^\infty)(\mathbb{R}^2)$.

NLFP + 2D time dependent Navier-Stokes

Theorem

(C-Masmoudi) Let $u_0 \in \left(W^{\alpha,r} \cap L^2\right)(\mathbb{R}^2)$*, divergence-free,* $f_0 \in W^{1,r}(H^{-s}(M))$*, with* $r > 2$, $\alpha > 1$, $s \le \frac{d}{2} + 1$ *and* $f_0 \ge 0$, $\int_M f_0 d\mu \in (L^1 \cap L^\infty)(\mathbb{R}^2)$*. Then the coupled NS and nonlinear Fokker-Planck system in 2D has a global solution* $u \in L^\infty_{loc}(W^{1,r}) \cap L^2_{loc}(W^{2,r})$ *and* $f \in L^\infty_{loc}(W^{1,r}(H^{-s}))$.

NLFP + 2D time dependent Navier-Stokes

Theorem

(C-Masmoudi) Let $u_0 \in \left(W^{\alpha,r} \cap L^2\right)(\mathbb{R}^2)$, *divergence-free,* $f_0 \in W^{1,r}(H^{-s}(M))$, *with* $r > 2$, $\alpha > 1$, $s \leq \frac{d}{2} + 1$ *and* $f_0 \geq 0$, $\int_M f_0 d\mu \in (L^1 \cap L^\infty)(\mathbb{R}^2)$. *Then the coupled NS and nonlinear Fokker-Planck system in 2D has a global solution* $u \in L^\infty_{loc}(W^{1,r}) \cap L^2_{loc}(W^{2,r})$ *and* $f \in L^\infty_{loc}(W^{1,r}(H^{-s}))$. *Moreover, for* $T > T_0 > 0$, *we have* $u \in L^\infty((T_0, T); W^{2-0,r})$.

No a priori bound.

$$\boxed{\sup_k \lambda_k^{\alpha - \frac{1}{k}\int_0^t \|\nabla_x S_{k-1}(u(s))\|_{L^\infty} ds} \|\Delta_k(u)(t)\|_{L^p}}$$

Let $q > 2$.

Let $q > 2$. We say that (u_0, f_0) are standard initial data if $\mathrm{div}_x u_0 = 0$, $u_0 \in W^{2,q}(\mathbb{T}^2)$,

Let $q > 2$. We say that (u_0, f_0) are standard initial data if $\mathrm{div}_x u_0 = 0$, $u_0 \in W^{2,q}(\mathbb{T}^2)$, $f_0 > 0$, $f_0(x,m) \in W^{1,q}(\mathbb{T}^2; L^2(M))$ and $\int_M f_0(x,m)d\mu = 1$.

Let $q > 2$. We say that (u_0, f_0) are standard initial data if $\mathrm{div}_x u_0 = 0$, $u_0 \in W^{2,q}(\mathbb{T}^2)$, $f_0 > 0$, $f_0(x,m) \in W^{1,q}(\mathbb{T}^2; L^2(M))$ and $\int_M f_0(x,m)d\mu = 1$.

Theorem

(C-Seregin) Let $q \geq 4$, (u_0, f_0) *be standard initial data and let* $T > 0$ *be arbitrary. Let* $p > \frac{2q}{q-2}$, $\alpha > \frac{d}{2} + 1$.

Let $q > 2$. We say that (u_0, f_0) are standard initial data if $\mathrm{div}_x u_0 = 0$, $u_0 \in W^{2,q}(\mathbb{T}^2)$, $f_0 > 0$, $f_0(x,m) \in W^{1,q}(\mathbb{T}^2; L^2(M))$ and $\int_M f_0(x,m)d\mu = 1$.

Theorem

(C-Seregin) Let $q \geq 4$, (u_0, f_0) be standard initial data and let $T > 0$ be arbitrary. Let $p > \frac{2q}{q-2}$, $\alpha > \frac{d}{2} + 1$. There exists a constant K depending only on the norms of the initial data, $T, \kappa, \nu, p, q, \alpha$, with K bounded by a double exponential of T, and a unique solution (u, f) of the coupled NS-NLFP system such that

Let $q > 2$. We say that (u_0, f_0) are standard initial data if $\mathrm{div}_x u_0 = 0$, $u_0 \in W^{2,q}(\mathbb{T}^2)$, $f_0 > 0$, $f_0(x,m) \in W^{1,q}(\mathbb{T}^2; L^2(M))$ and $\int_M f_0(x,m)d\mu = 1$.

Theorem

(C-Seregin) Let $q \geq 4$, (u_0, f_0) be standard initial data and let $T > 0$ be arbitrary. Let $p > \frac{2q}{q-2}$, $\alpha > \frac{d}{2} + 1$. There exists a constant K depending only on the norms of the initial data, $T, \kappa, \nu, p, q, \alpha$, with K bounded by a double exponential of T, and a unique solution (u, f) of the coupled NS-NLFP system such that

$$\|\nabla_x \nabla_x u\|_{L^p(0,T;L^q(\mathbb{T}^2))} \leq K,$$

$$\sup_{t\leq T} \|\nabla_x u(\cdot, t)\|_{L^\infty} \leq K,$$

and

$$\sup_{t\leq T} \|f(\cdot, t)\|_{W^{1,q}(\mathbb{T}^2; H^{-\alpha}(M)))} \leq K.$$

hold.

Navier Stokes Equation

$$\boxed{\begin{gathered}\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \nabla \cdot \sigma \\ \nabla \cdot u = 0\end{gathered}}$$

The tensor $\sigma_{ij}(x,t)$: driving stress.

Navier Stokes Equation

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \nabla\cdot\sigma$$
$$\nabla\cdot u = 0$$

The tensor $\sigma_{ij}(x,t)$: driving stress.

Sufficient for regularity, if σ smooth

$$\int_0^T \|u\|^2_{L^\infty(dx)}dt < \infty$$

Navier Stokes Equation

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \nabla\cdot\sigma$$
$$\nabla\cdot u = 0$$

The tensor $\sigma_{ij}(x,t)$: driving stress.

Sufficient for regularity, if σ smooth

$$\int_0^T \|u\|^2_{L^\infty(dx)}dt < \infty$$

Exponent of the amplification factor of tracers, key quantity

$$\int_0^T \|\nabla u\|_{L^\infty(dx)}dt < \infty$$

2D, Bounded stress

Theorem

Let $\sigma \in L^\infty(dtdx)$. Let $u_0 \in L^2(dx)$. There exists a unique weak solution of the forced 2D NS eqns, with

$$\boxed{u \in L^\infty(dt)(L^2(dx)) \cap L^2(dt)(W^{1,2}(dx))}$$

2D, Bounded stress

Theorem

Let $\sigma \in L^\infty(dtdx)$. Let $u_0 \in L^2(dx)$. There exists a unique weak solution of the forced 2D NS eqns, with

$$\boxed{u \in L^\infty(dt)(L^2(dx)) \cap L^2(dt)(W^{1,2}(dx))}$$

Moreover,

$$\boxed{\int_0^T \|\nabla u\|_{L^q(dx)}^{\frac{q}{q-1}} dt < \infty, \quad \forall\, q \geq 2}$$

2D, Bounded stress

Theorem

Let $\sigma \in L^\infty(dtdx)$. Let $u_0 \in L^2(dx)$. There exists a unique weak solution of the forced 2D NS eqns, with

$$\boxed{u \in L^\infty(dt)(L^2(dx)) \cap L^2(dt)(W^{1,2}(dx))}$$

Moreover,

$$\boxed{\int_0^T \|\nabla u\|_{L^q(dx)}^{\frac{q}{q-1}} dt < \infty, \quad \forall\, q \geq 2}$$

$$\boxed{\int_0^T \|u\|_{L^\infty(dx)}^{p} dt < \infty, \quad \forall\, p < 2.}$$

Navier-Stokes with nearly singular forces

$$\boxed{\partial_t u + u\cdot\nabla_x u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma, \quad \nabla_x\cdot u = 0}$$

Theorem

Let u be a solution of the 2D Navier-Stokes system with divergence-free initial data $u_0 \in W^{1,2}(\mathbb{R}^2)\cap W^{1,r}(\mathbb{R}^2)$. Let $T>0$ and let the forces $\nabla\cdot\sigma$ obey

$$\sigma \in L^1(0,T;L^\infty(\mathbb{R}^2))\cap L^2(0,T;L^2(\mathbb{R}^2))$$
$$\nabla\cdot\sigma \in L^1(0,T;L^r(\mathbb{R}^2))\cap L^2(0,T;L^2(\mathbb{R}^2))$$

with $r>2$.

Navier-Stokes with nearly singular forces

$$\boxed{\partial_t u + u\cdot\nabla_x u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma, \quad \nabla_x\cdot u = 0}$$

Theorem

Let u be a solution of the $2D$ Navier-Stokes system with divergence-free initial data $u_0 \in W^{1,2}(\mathbb{R}^2)\cap W^{1,r}(\mathbb{R}^2)$. Let $T > 0$ and let the forces $\nabla\cdot\sigma$ obey

$$\sigma \in L^1(0,T;L^\infty(\mathbb{R}^2))\cap L^2(0,T;L^2(\mathbb{R}^2))$$
$$\nabla\cdot\sigma \in L^1(0,T;L^r(\mathbb{R}^2))\cap L^2(0,T;L^2(\mathbb{R}^2))$$

with $r > 2$. Let

$$\boxed{\|\sigma\|_{L^\infty}\sim K, \quad \|\nabla\cdot\sigma\|_{L^r}\sim B}$$

Then

$$\boxed{\int_0^T \|\nabla u(t)\|_{L^\infty} dt \le K \log_*(B)}$$

Then

$$\boxed{\int_0^T \|\nabla u(t)\|_{L^\infty} dt \le K \log_*(B)}$$

and also

$$\frac{1}{M}\sum_{q=1}^{M}\int_0^T \|\Delta_q \nabla u(t)\|_{L^\infty} dt \le K$$

Then

$$\boxed{\int_0^T \|\nabla u(t)\|_{L^\infty} dt \le K \log_*(B)}$$

and also

$$\frac{1}{M}\sum_{q=1}^{M}\int_0^T \|\Delta_q \nabla u(t)\|_{L^\infty} dt \le K$$

with K depending on T, norms of σ and the initial velocity, but not on gradients of σ nor M, and B depending on norms of the spatial gradients of σ.

Then

$$\boxed{\int_0^T \|\nabla u(t)\|_{L^\infty} dt \le K \log_*(B)}$$

and also

$$\frac{1}{M}\sum_{q=1}^{M}\int_0^T \|\Delta_q \nabla u(t)\|_{L^\infty} dt \le K$$

with K depending on T, norms of σ and the initial velocity, but not on gradients of σ nor M, and B depending on norms of the spatial gradients of σ.

$$\boxed{u = \sum_{q=-1}^{\infty} \Delta_q(u)}$$

Littlewood-Paley decomposition

Natural questions for NS with singular forcing

$$\boxed{\int_0^T \|u\|^2_{L^\infty(dx)}dt < \infty \Rightarrow \int_0^T \|\nabla u\|_{L^\infty(dx)} < \infty \quad ?}$$

Natural questions for NS with singular forcing

$$\boxed{\int_0^T \|u\|_{L^\infty(dx)}^2 dt < \infty \Rightarrow \int_0^T \|\nabla u\|_{L^\infty(dx)} < \infty \quad ?}$$

$$\boxed{\int_0^T \|u\|_{L^\infty(dx)}^2 dt < \infty \quad ?}$$

Natural questions for NS with singular forcing

$$\boxed{\int_0^T \|u\|^2_{L^\infty(dx)}dt < \infty \Rightarrow \int_0^T \|\nabla u\|_{L^\infty(dx)} < \infty \quad ?}$$

$$\boxed{\int_0^T \|u\|^2_{L^\infty(dx)}dt < \infty \quad ?}$$

$$\boxed{\int_0^T \|\nabla u\|_{L^\infty(dx)}dt < \infty \quad ?}$$

Natural questions for NS with singular forcing

$$\boxed{\int_0^T \|u\|_{L^\infty(dx)}^2 dt < \infty \Rightarrow \int_0^T \|\nabla u\|_{L^\infty(dx)} < \infty \quad ?}$$

$$\boxed{\int_0^T \|u\|_{L^\infty(dx)}^2 dt < \infty \quad ?}$$

$$\boxed{\int_0^T \|\nabla u\|_{L^\infty(dx)} dt < \infty \quad ?}$$

Partial regularity?

Hölder regularity of weak solution

$\Omega_1 \Subset \Omega$ domains in $\mathbb{R}^2$, $0 < T_1 < T$.

Hölder regularity of weak solution

$\Omega_1 \Subset \Omega$ domains in $\mathbb{R}^2$, $0 < T_1 < T$.

$$Q = \Omega \times (-T, 0), \qquad Q_1 = \Omega_1 \times (-T_1, 0).$$

Hölder regularity of weak solution

$\Omega_1 \Subset \Omega$ domains in $\mathbb{R}^2$, $0 < T_1 < T$.

$$Q = \Omega \times (-T, 0), \qquad Q_1 = \Omega_1 \times (-T_1, 0).$$

Parabolic balls:

Hölder regularity of weak solution

$\Omega_1 \Subset \Omega$ domains in $\mathbb{R}^2$, $0 < T_1 < T$.

$$Q = \Omega \times (-T, 0), \qquad Q_1 = \Omega_1 \times (-T_1, 0).$$

Parabolic balls: $Q(z_0, R) = B(x_0, R) \times (t_0 - R^2, t_0)$, where $z_0 = (x_0, t_0)$, $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$.

Hölder regularity of weak solution

$\Omega_1 \Subset \Omega$ domains in $\mathbb{R}^2$, $0 < T_1 < T$.

$$Q = \Omega \times (-T, 0), \qquad Q_1 = \Omega_1 \times (-T_1, 0).$$

Parabolic balls: $Q(z_0, R) = B(x_0, R) \times (t_0 - R^2, t_0)$, where $z_0 = (x_0, t_0)$, $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$. Means:

Hölder regularity of weak solution

$\Omega_1 \Subset \Omega$ domains in $\mathbb{R}^2$, $0 < T_1 < T$.

$$Q = \Omega \times (-T, 0), \qquad Q_1 = \Omega_1 \times (-T_1, 0).$$

Parabolic balls: $Q(z_0, R) = B(x_0, R) \times (t_0 - R^2, t_0)$, where $z_0 = (x_0, t_0)$, $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$. Means:

$$(f)_{z_0,R} = \frac{1}{|Q(z_0,R)|} \int\limits_{Q(z_0,R)} f(z)dz,$$

$$[p]_{x_0,R} = \frac{1}{|B(x_0,R)|} \int\limits_{B(x_0,R)} p(x)dx.$$

Spaces

For $0 < \gamma < 1$,

Spaces

For $0 < \gamma < 1$, space $M_{2,\gamma}(Q)$,

Spaces

For $0 < \gamma < 1$, space $M_{2,\gamma}(Q)$, seminorm

$$\|\sigma\|_{M_{2,\gamma}(Q)} = \sup_{Q(z_0,R)\subset Q} R^{1-\gamma}\left(\frac{1}{|Q(z_0,R)|}\int_{Q(z_0,R)} |\sigma(z) - (\sigma)_{z_0,R}|^2 dz\right)^{\frac{1}{2}} < \infty.$$

Spaces

For $0<\gamma<1$, space $M_{2,\gamma}(Q)$, seminorm

$$\|\sigma\|_{M_{2,\gamma}(Q)} = \sup_{Q(z_0,R)\subset Q} R^{1-\gamma}\left(\frac{1}{|Q(z_0,R)|}\int_{Q(z_0,R)}|\sigma(z)-(\sigma)_{z_0,R}|^2dz\right)^{\frac{1}{2}}<\infty.$$

Note

$$L^r(Q)\subset M_{2,\gamma}(Q)$$

for $r\ge\frac{4}{1-\gamma}$.

Theorem

(C-Seregin) Let

$$u \in L^4(Q;\mathbb{R}^2),\ p \in L^2(Q),\ \sigma \in M_{2,\gamma}(Q;\mathbb{M}^{2\times 2})$$

with $0 \le \gamma < 1$, *satisfying the Navier-Stokes equations*

$$\partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p + \mathrm{div}\,\sigma, \quad \mathrm{div}\, u = 0$$

in Q *in the sense of distributions.*

Theorem

(C-Seregin) Let

$$u \in L^4(Q;\mathbb{R}^2),\ p \in L^2(Q),\ \sigma \in M_{2,\gamma}(Q;\mathbb{M}^{2\times 2})$$

with $0 \le \gamma < 1$, *satisfying the Navier-Stokes equations*

$$\partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p + \operatorname{div}\sigma, \quad \operatorname{div} u = 0$$

in Q *in the sense of distributions. Then*

$$u \in C^{\gamma}(\overline{Q}_1)$$

if $0 < \gamma < 1$ *and*

$$u \in BMO(Q_1)$$

if $\gamma = 0$.

Additional results

Let H and V be the L^2 and H^1 spaces of divergence-free functions.

Additional results

Let H and V be the L^2 and H^1 spaces of divergence-free functions.

Theorem

(C-S) Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $p \in L^2(0,T;L^2(\mathbb{T}^2))$ *be a solution of the initial value problem*

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = \operatorname{div} \sigma, \qquad \operatorname{div} u = 0,$$

$$u(\cdot,0) = u_0(\cdot) \in H,$$

where $\sigma \in L^r(\mathbb{T}^2 \times (0,T); \mathbb{M}^{2\times 2})$ *with* $r \geq 4$.

Additional results

Let H and V be the L^2 and H^1 spaces of divergence-free functions.

Theorem

(C-S) Let $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $p \in L^2(0, T; L^2(\mathbb{T}^2))$ *be a solution of the initial value problem*

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = \operatorname{div} \sigma, \qquad \operatorname{div} u = 0,$$

$$u(\cdot, 0) = u_0(\cdot) \in H,$$

where $\sigma \in L^r(\mathbb{T}^2 \times (0, T); \mathbb{M}^{2\times 2})$ *with* $r \geq 4$. *Then, given* $s > 0$, *there exists a constant* C_s *depending only on* s, ν, *the norm of* u_0 *in* H, *the norm of* σ *in* $L^r(\mathbb{T}^2)$, *such that*

$$\|u\|_{L^\infty(\mathbb{T}^2 \times (s,T))} \leq C_s.$$

Moreover, the function u *is Hölder continuous in* $\mathbb{T}^2 \times [s, T]$ *with exponent* $\gamma = 1 - \frac{4}{r}$.

Idea of proof of Hölder continuity for NS

Local iterative estimates for L^4 space-time integrals of the velocity, in the spirit of De Giorgi, Campanato.

Idea of proof of Hölder continuity for NS

Local iterative estimates for L^4 space-time integrals of the velocity, in the spirit of De Giorgi, Campanato. Pressure

$$p = R_iR_j(\sigma_{ij} - u_iu_j),$$

The iteration relates integrals on smaller parabolic cubes to integrals on larger ones.

Idea of proof of Hölder continuity for NS

Local iterative estimates for L^4 space-time integrals of the velocity, in the spirit of De Giorgi, Campanato. Pressure

$$p = R_iR_j(\sigma_{ij} - u_iu_j),$$

The iteration relates integrals on smaller parabolic cubes to integrals on larger ones. For the iterative procedure to succeed, the modulus of absolute continuity of the map

$$\Omega \subset \left\{\mathbb{T}^2 \times (0, T)\right\} \mapsto \int_{\Omega} |u(x,t)|^4 dxdt,$$

needs to be controlled uniformly apriori, to guarantee that such an integral is arbitrarily small, if the parabolic Lebesgue measure of Ω is small enough.

$$\Phi(u; z_0, \varrho) = \left(\int_{Q(z_0,\varrho)} |u - (u)_{z_0,\varrho}|^4 dz \right)^{\frac{1}{2}},$$

$$\Phi(u; z_0, \varrho) = \left(\int_{Q(z_0,\varrho)} |u - (u)_{z_0,\varrho}|^4 dz \right)^{\frac{1}{2}},$$

$$\Psi(u; z_0, \varrho) = \left(\int_{Q(z_0,\varrho)} |u|^4 dz \right)^{\frac{1}{2}},$$

$$\Phi(u; z_0, \varrho) = \left(\int_{Q(z_0,\varrho)} |u - (u)_{z_0,\varrho}|^4 dz \right)^{\frac{1}{2}},$$

$$\Psi(u; z_0, \varrho) = \left(\int_{Q(z_0,\varrho)} |u|^4 dz \right)^{\frac{1}{2}},$$

$$D(p; z_0, \varrho) = \int\limits_{Q(z_0,\varrho)} |p - [p]_{x_0,\varrho}|^2 dz.$$

$$\Phi(u; z_0, \varrho) = \left(\int_{Q(z_0,\varrho)} |u - (u)_{z_0,\varrho}|^4 dz \right)^{\frac{1}{2}},$$

$$\Psi(u; z_0, \varrho) = \left(\int_{Q(z_0,\varrho)} |u|^4 dz \right)^{\frac{1}{2}},$$

$$D(p; z_0, \varrho) = \int\limits_{Q(z_0,\varrho)} |p - [p]_{x_0,\varrho}|^2 dz.$$

Lemma

Let the function $v \in L^4(Q(z_0, R))$ satisfy the heat equation

$$\partial_t v - \Delta v = 0$$

in $Q(z_0, R)$.

$$\Phi(u;z_0,\varrho) = \left(\int_{Q(z_0,\varrho)} |u-(u)_{z_0,\varrho}|^4 dz\right)^{\frac{1}{2}},$$

$$\Psi(u;z_0,\varrho) = \left(\int_{Q(z_0,\varrho)} |u|^4 dz\right)^{\frac{1}{2}},$$

$$D(p;z_0,\varrho) = \int\limits_{Q(z_0,\varrho)} |p-[p]_{x_0,\varrho}|^2 dz.$$

Lemma

Let the function $v \in L^4(Q(z_0,R))$ *satisfy the heat equation*

$$\partial_t v - \Delta v = 0$$

in $Q(z_0,R)$. *Then*

$$\Phi(v;z_0,\varrho) \le c\left(\frac{\varrho}{R}\right)^4 \Phi(v;z_0,R)$$

for all $0 < \varrho \le R$.

Lemma

Given $G \in L^2(Q(z_0, R); \mathbb{M}^{2\times 2})$, *there exists a unique function*

$$w \in C([t_0 - R^2, t_0]; L^2(B(x_0, R); \mathbb{R}^2)) \cap$$
$$L^2([t_0 - R^2, t_0]; W^{1,2}(B(x_0, R); \mathbb{R}^2))$$

Lemma

Given $G \in L^2(Q(z_0,R);\mathbb{M}^{2\times 2})$, *there exists a unique function*

$$w \in C([t_0 - R^2, t_0]; L^2(B(x_0,R);\mathbb{R}^2)) \cap \\ L^2([t_0 - R^2, t_0]; W^{1,2}(B(x_0,R);\mathbb{R}^2))$$

such that

$$\partial_t w - \Delta w = -\mathrm{div}\, G$$

in $Q(z_0,R)$

Lemma

Given $G \in L^2(Q(z_0,R);\mathbb{M}^{2\times 2})$, *there exists a unique function*

$$w \in C([t_0-R^2,t_0];L^2(B(x_0,R);\mathbb{R}^2))\cap \\ L^2([t_0-R^2,t_0];W^{1,2}(B(x_0,R);\mathbb{R}^2))$$

such that

$$\partial_t w - \Delta w = -\mathrm{div}\, G$$

in $Q(z_0,R)$ *and*

$$w = 0$$

on the parabolic boundary of $Q(z_0,R)$.

Lemma

Given $G \in L^2(Q(z_0,R);\mathbb{M}^{2\times 2})$*, there exists a unique function*

$$w \in C([t_0-R^2,t_0];L^2(B(x_0,R);\mathbb{R}^2))\cap L^2([t_0-R^2,t_0];W^{1,2}(B(x_0,R);\mathbb{R}^2))$$

such that

$$\partial_t w - \Delta w = -\mathrm{div}\, G$$

in $Q(z_0,R)$ *and*

$$w = 0$$

on the parabolic boundary of $Q(z_0,R)$*. Moreover, the function* w *satisfies the estimates:*

$$|w|^2_{2,Q(z_0,R)} \equiv \sup_{t_0-R^2<t<t_0} \|w(\cdot,t)\|^2_{2,B(x_0,R)} + \|\nabla w\|^2_{2,Q(z_0,R)}$$

$$\le 2\|G\|^2_{2,Q(z_0,R)},$$

$$\Phi(w;z_0,R) \le c|w|^2_{2,Q(z_0,R)}.$$

Lemma

For solutions of NSE, we have

$$\Phi(u;z_0,\varrho) \le c\left\{\left[\left(\frac{\varrho}{R}\right)^4 + \Psi(u;z_0,R)\right]\Phi(u;z_0,R) + \right.$$

$$\left. + D(p;z_0,R) + MR^{2+2\gamma}\right\}$$

Lemma

For solutions of NSE, we have

$$\Phi(u;z_0,\varrho) \le c\left\{\left[\left(\frac{\varrho}{R}\right)^4 + \Psi(u;z_0,R)\right]\Phi(u;z_0,R) + \right.$$

$$\left. + D(p;z_0,R) + MR^{2+2\gamma}\right\}$$

whenever $Q(z_0,R) \subset Q$ *and* $0 < \varrho \le R$.

Lemma

For solutions of NSE, we have

$$\Phi(u; z_0, \varrho) \le c\left\{\left[\left(\frac{\varrho}{R}\right)^4 + \Psi(u; z_0, R)\right]\Phi(u; z_0, R) + \right.$$

$$\left. + D(p; z_0, R) + MR^{2+2\gamma}\right\}$$

whenever $Q(z_0, R) \subset Q$ *and* $0 < \varrho \le R$. *Here,* $M = \|\sigma\|^2_{M_{2,\gamma}(Q)}$.

Lemma

$$D(p; z_0, \varrho) \le$$

$$c\left[\left(\frac{\varrho}{R}\right)^4 D(p; z_0, R) + \Psi(u; z_0, R)\Phi(u; z_0, R) + MR^{2+2\gamma}\right]$$

Lemma

For solutions of NSE, we have

$$\Phi(u; z_0, \varrho) \le c\left\{\left[\left(\frac{\varrho}{R}\right)^4 + \Psi(u; z_0, R)\right]\Phi(u; z_0, R) + \right.$$

$$\left. + D(p; z_0, R) + MR^{2+2\gamma}\right\}$$

whenever $Q(z_0, R) \subset Q$ *and* $0 < \varrho \le R$. *Here,* $M = \|\sigma\|^2_{M_{2,\gamma}(Q)}$.

Lemma

$$D(p; z_0, \varrho) \le$$

$$c\left[\left(\frac{\varrho}{R}\right)^4 D(p; z_0, R) + \Psi(u; z_0, R)\Phi(u; z_0, R) + MR^{2+2\gamma}\right]$$

whenever $Q(z_0, R) \subset Q$ *and* $0 < \varrho \le R$.

The following result is used to control the modulus of absolute continuity.

Theorem

Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ *be a solution of the 2D Navier-Stokes equations with initial data* $u_0 \in H \cap L^r(\mathbb{T}^2)$ *and* $\sigma \in L^r(\mathbb{T}^2 \times (0,T); \mathbb{M}^{2\times 2})$ *with* $r \geq 4$*. There exists a constant* K *depending only on the norm* $\|\sigma\|_{L^r(\mathbb{T}^2\times(0,T))}, \nu$, T *and the norm of* u_0 *in* $H \cap L^r(\mathbb{T}^2)$ *such that*

$$\sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^r(\mathbb{T}^2)} \leq K.$$

Generalized Ladyzhenskaya inequalities

The previous result is based on the inequality

$$\|f\|^2_{L^{2r}(\mathbb{R}^2)} \le C_r \|f\|_{L^r(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)}$$

Generalized Ladyzhenskaya inequalities

The previous result is based on the inequality

$$\|f\|_{L^{2r}(\mathbb{R}^2)}^2 \le C_r \|f\|_{L^r(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)}$$

that holds for all $f \in L^r(\mathbb{R}^2)$ with $\nabla f \in L^2(\mathbb{R}^2)$,

Generalized Ladyzhenskaya inequalities

The previous result is based on the inequality

$$\|f\|_{L^{2r}(\mathbb{R}^2)}^2 \le C_r \|f\|_{L^r(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)}$$

that holds for all $f \in L^r(\mathbb{R}^2)$ with $\nabla f \in L^2(\mathbb{R}^2)$, generalizing the $r = 2$ inequality due to Ladyzhenskaya.

Generalized Ladyzhenskaya inequalities

The previous result is based on the inequality

$$\|f\|^2_{L^{2r}(\mathbb{R}^2)} \le C_r \|f\|_{L^r(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)}$$

that holds for all $f \in L^r(\mathbb{R}^2)$ with $\nabla f \in L^2(\mathbb{R}^2)$, generalizing the $r = 2$ inequality due to Ladyzhenskaya. In turn, the inequality above is the $n = 2$ particular case of

$$\|f\|^2_{L^{2r}(\mathbb{R}^n)} \le C \|f\|_{L^r(\mathbb{R}^n)} \|\nabla f\|_{B_2^{0,n}(\mathbb{R}^n)}$$

Generalized Ladyzhenskaya inequalities

The previous result is based on the inequality

$$\|f\|_{L^{2r}(\mathbb{R}^2)}^2 \le C_r \|f\|_{L^r(\mathbb{R}^2)} \|\nabla f\|_{L^2(\mathbb{R}^2)}$$

that holds for all $f \in L^r(\mathbb{R}^2)$ with $\nabla f \in L^2(\mathbb{R}^2)$, generalizing the $r = 2$ inequality due to Ladyzhenskaya. In turn, the inequality above is the $n = 2$ particular case of

$$\|f\|_{L^{2r}(\mathbb{R}^n)}^2 \le C \|f\|_{L^r(\mathbb{R}^n)} \|\nabla f\|_{B_2^{0,n}(\mathbb{R}^n)}$$

valid for all $r \ge \frac{n}{2}$, with $B_2^{0,n}$ the Besov space with norm

$$\|f\|_{B_q^{s,p}(\mathbb{R}^n)} = \left[\sum_{j=-\infty}^{\infty} \lambda_j^{qs} \|\Delta_j f\|_{L^p(\mathbb{R}^n)}^q\right]^{\frac{1}{q}}$$

References

- P. Constantin, Smoluchowski Navier-Stokes systems, Contemporary Mathematics **429** G-Q Chen, E. Hsu, M. Pinsky editors, AMS, Providence (2007), 85-109
- P. Constantin, N. Masmoudi, Global well-posedness for a Smoluchowski equation coupled with Navier-Stokes equations in 2D, Commun. Math. Phys. **278** (2008), 179-191.
- P. Constantin, G. Seregin, Hölder Continuity of Solutions of 2D Navier-Stokes Equations with Singular Forcing, to appear
- P. Constantin, G. Seregin, Global regularity of solutions of coupled Navier-Stokes equations and nonlinear Fokker Planck equations, to appear
- P. Constantin, The Onsager equation for corpora, J. Comp. Theor. Nanoscience, 2009, to appear.
- P. Constantin, A. Zlatos, On the high intensity limit of interacting corpora,CMS, to appear.