The regularity theory for solutions to non-local, fully nonlinear equation

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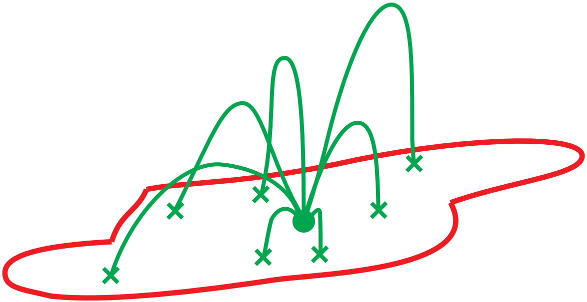
Part I

- Non-divergence second order equations and fully nonlinear equations from optimal control
- The non local version
- The regularity theory for second order equations (a review)
  - A.B.P
  - Krylov Safanov
  - Initial regularity theory
  - Evans-Krylov
  - Classical solutions



$$x_0 \xrightarrow{\Delta t} \varphi_{\varepsilon,x_0}(x_0 + y)\, dy = \frac{1}{\varepsilon^n}\varphi_{1,x_0}(x_0 + epy)\, dy$$

with φ a symmetric probability density continuous in x.

If we let Δt, ε go to zero at the appropriate rate

$$\Delta t \sim \varepsilon^2$$

the φ_ε concentrates as a density in the unit sphere and the probability density $u(x,t)$ of a particle being at position x at time t evolves according to an equation

$$\begin{aligned} D_t u(x,t) &= \int_{S^1} \theta_x(\sigma) D_{\sigma\sigma} u(x,t) d\sigma \\ &= \lim_{\varepsilon \to 0} \int d\sigma \int \ell^2 \varphi_{\varepsilon,x}(\ell\sigma) \left[\frac{u(x+\ell\sigma) + u(x-\ell\sigma) - 2u(x)}{\ell^2} \right] d\ell \end{aligned}$$

(i.e., $\theta(\sigma)$ is the second radial moment of φ

$$\theta(\sigma) = \int_0^\infty \ell^2 \varphi(\ell\sigma) d\ell$$

In the non local case, when the process is not continuous and “non infinitesimal” jumps take place

$$\int \varphi_{\varepsilon,x}(y) \left[\frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2} \right] dy$$

does not converge anymore to a distribution on the sphere.

For instance, if $\varphi_{\varepsilon,x}(y)$ is a “Cauchy-like” distribution

$$\varphi(y) = \frac{1}{(1+|y|^2)^{\frac{n+\alpha}{2}}}$$

and $\alpha < 2$,

$$\varphi_\varepsilon(y) \sim \frac{\varepsilon^\alpha}{(\varepsilon^2 + |y|^2)^{\frac{n+\alpha}{2}}}$$

the natural time-space scaling becomes $\Delta t = \varepsilon^\alpha$ and the corresponding diffusion equation becomes

$$D_t u(x,t) = \underbrace{(2-\alpha) \int \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+\alpha}}\, dy}_{\text{“}\Delta^\alpha(u)\text{”}}$$

More generally, we will be interested in the equation

$$D_t u = \int [u(x+y) + u(x-y) - 2u(x)] K(x,y)$$

Back to the second order case

The family of equations

$$L(u) = \int \theta_x(\sigma) D_{\sigma\sigma} u$$

$(\lambda < \theta < \Lambda)$ appears to be a very large one, but in fact, for any given weight θ

$$\int \theta(\sigma) D_{\sigma\sigma} u$$

defines a standard second order elliptic equation, since

$$D_{\sigma\sigma} u = \sum \sigma_i \sigma_j u_{ij}$$

so

$$Lu = \sum a_{ij} D_{ij} u \ , \quad \text{with} \ \ a_{ij} = \int \theta \cdot \sigma_i \sigma_j$$

Optimal control

The equations of optimal control arise when we can choose in the continuous case the weight θ at each point x, among a family of weights $\theta_\alpha \in \mathcal{A}$, in order to enhance the diffusion process:

$$D_t u = \sup_{\theta_{\alpha,x} \in \mathcal{A}} \int \theta_{\alpha,x} u_{\sigma\sigma} d\sigma =$$
$$= \sup[a_{ij,\alpha}(x) D_{ij} u] = F(D^2 u)$$

and from a family of kernels $K \subset \mathbb{K}$:

$$D_t u = \sup_{K \in \mathbb{K}} \int [u(x+y) + u(x-y) - 2u(x)] K(x,y)\, dy = \Phi(u)$$

for the non local case.

In this lecture we will consider only the stationary case, i.e.,

$$D_t(u) = 0 = F(D^2u) \quad \text{(resp. } \phi(u))$$

Heuristically, we may think that u solves the maximal equation above if, for each x, there exists a particular θ_α (resp. K) such that

$$L_{\theta_\alpha}(u)(x) = 0 \quad \text{(resp. } L_K(u)(x) = 0)$$

and for any other θ, (K), $L_\theta(u)(x)$ (resp. $L_K(u)(x)$) ≤ 0

Three basic properties of the resulting function F under the sup. process are

a) Ellipticity: If M, N are symmetric matrices, and N is positive

$$F(M+N) - F(M) \sim \|N\|$$

In particular if

$$F(M_1) = F(M_2) = 0$$

we have

$$F(M_1) = F(M_1 - M_2 + M_2) = F((M_1 - M_2)^+ - (M_1 - M_2)^- + M_2)$$

and this implies that

$$\|(M_1 - M_2)^+\| \sim \|M_1 - M_2\|^-$$

b) From a) it follows that the differential of F with respect to M,

$$F_{ij}(M) = \frac{\partial F}{\partial M_{ij}}(M)$$

is an “elliptic matrix”

$$\lambda I \leq F_{ij} \leq \Lambda I \; .$$

In particular, if u is a solution of $F(D^2u) = 0$ and we differentiate the equation with respect to a direction e, we get

$$F_{ij}(D^2u)D_{ij}(\underbrace{D_e u}_{w}) = 0$$

“first derivatives, w, of the solution u, satisfy an elliptic equation with measurable coefficients.”

c) F is a convex function of D^2, since it is the supremum of linear functions

$$\int \theta_\alpha D_{\sigma\sigma} u$$

d) From c) if we take a second derivative of the equation in the direction e

$$0 = D_{ee}F = F_{ij}(D^2u)D_{x_i x_j} \underbrace{D_{ee}u}_{\omega^*} + \underbrace{D_{ije}}_{\text{vector}} \underbrace{F_{ij}, k\ell}_{\substack{\text{positive}\\ \text{matrix}}} \underbrace{D_{k\ell e}}_{\text{vector}}$$

$D_{ee}u$ (and all its convex combinations) are all supersolutions of the same linearized equation

$$F_{ij}D_{ij} \ldots \leq 0$$

Regularity theory of solutions to linear equations with measurable coeffients and to optimal control

- Linear equations: Solutions of a linear equation u satisfy the Harnack inequality (Krylov-Safanov):

$$u \geq 0 \text{ in } B_1 ,$$

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u$$

and therefore are Hölder continuous:

If

$$\operatorname*{osc}_{B_r} u \leq C(\operatorname*{osc}_{B_1} u) r^{\alpha}$$

Two important properties implicit in the proof

a) weak-L^{ε}-estimate

1) u a supersolution in B_1

2) $u \geq 0, \quad u(0) = 1.$

Then

$$|\{u > t\}| \leq Ct^{-\varepsilon}$$

b) (oscillation theorem)

1) u a subsolution in B_1, $u \leq 1$, then

$$u(0) \leq C|\{u > 0\} \cap B_{1/2}|$$

(compare with mean value theorem for harmonic functions)

Regularity for optimal control

1) Bounded solutions are $C^{1,\alpha}$:

Why?: First derivatives, $\omega = D_e u$, of the solution u, satisfy the linear equation

$$F_{ij}(D^2u)D_{ij}\omega = 0$$

At this point we know nothing about D^2u,

but $a_{ij}(x) = F_{ij}(D^2u(x))$ is an elliptic matrix.

From Krylov-Safanov $\boxed{\omega \text{ is } C^\alpha}$

(no convexity assumption)

2) (Evans-Krylov)

Bounded solutions are $C^{2,\alpha}$ (classical) solutions

$$(\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)})$$

Why?: All second derivatives $\omega = D_{ee}u$ are supersolutions of the same linearized operator,

$$F_{ij}(D^2u)D_{ij}w \leq 0$$

That means that the value of w at a point, controls the "average" of w by above (L^ε lemma).

But $D^2u = M$ "lives" in the Lipschitz, elliptic surface

$$F(M) = 0$$

providing the remaining control, from the opposite side that forces oscillation of D^2u to decay.

Part II

Non local linear equations and optimal control

(in collaboration with Luis Silvestre)

Typical linear equation

$$L_K(u)(x) = \int [u(x+y) - u(x-y) - 2u(x)]K(y)\,dy$$

(or more generally $K(x,y)$)

The kernel K is non negative, integrable at infinity and in order to make sense, near zero we "can afford" a divergent singularity as long as it is integrable against $|y|^2$ (at least if u is C^2), i.e.,

$$\int K(y)\min(|y|^2, 1) < \infty$$

A family of examples is given by the "fractional Laplacians"

$$\Delta^\tau u = (1-\alpha) \int \frac{[u(x+y) + u(x-y) - 2u(x)]}{|y|^{n+2\tau}} \, dy$$

so called because its symbol is $-|\xi|^{2\tau}$ (i.e., has the homogeneity of a 2τ-derivative and $-\Delta^\tau$ is a positive operator, and also because it is the first variation of minimizing the H_τ norm ("τ derivatives in L^2")

As in the second order equations, solutions of

$$\Delta^{\tau} u = f$$

are expected to be "2τ derivatives" better than f and this indicates that if we expect the equation $L_K(u)$ to be "regularizing", K should be divergent at the origin.

(If K is in L^1, $L_K(u)$ is "at least as good" as u, so it will not force regularity of u!)

In fact, to expect "h-derivatives regularization" by $(L_K)^{-1}f$, the kernel $K(y)$ should diverge at zero by a factor of $|y|^{-h}$. So, we fix the "order of the integrate equation $L_K(u)$" to be h by asking the kernel $K_x(y)$ to satisfy

$$\lambda(2-h)|y|^{-(n+h)} \leq K_x(y) \leq \Lambda(2-h)|y|^{-(n+h)}$$

Regularity of linear equations

We consider a "linear equation of order h, with measurable coefficients" to be

$$Lu = \int_{\mathbb{R}^n} [u(x+y) + u(x-y) - 2u(x)] K(x,y)$$

with K having no regularity in χ and

$$\lambda(2-h)|y|^{-(n+h)} \leq K(x,y) \leq \Lambda(2-h)|y|^{-(n+h)}$$

Note that to compute Lu we need the values of u in all of $\mathbb{R}^n$, so in order to study

$$L(u) = 0 \text{ in } B_1$$

we still need to prescribe (as Dirichlet data) the values of u in all of $\mathbb{R}^n$. (For a jump process the particle does not "hit the boundary". It may jump to anywhere in $\mathbb{R}^n$.)

Harnack inequality

If u is a non negative solution in B_1 of

$$0 = L_{K_x}(u) = \int [u(x+y) + u(x-y) - 2u(x)]K(x,y)$$

with K of order $h \geq h_0 > 0$ and u outside B_1 is “reasonable”.

Then

$$\sup_{B_{1/2}} u \leq C[u(0) + \text{“nice correction from the data outside” } B_1]$$

- The constant C is uniform all the way to second order equations
- L^{ε} lemma and oscillation lemma hold
- Main difficulty is to find the corresponding Alexandrov-Bakelmann-Pucci theorem, and in particular the estimates for the gradient map.

Regularity for optimal control equations

$$0 = \phi(u) = \sup_{K \in \mathbb{K}} \int [u(x+y) + u(x-y) - 2u(x)] K(y) \, dy \quad \text{for } x \text{ in } B_1$$

All $K \in \mathbb{K}$ uniformly of order $h \geq h_0 > 0$. Then

a) Initial regularity: u is of class $C^{1,\alpha}(B_{1/2})$ and

$$|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)} + \text{ “term from outside”}$$

(all uniform in $h_0 \leq h \leq 2$)

(Again, “first derivatives are solutions of a linearized operator” and as such they are Hölder continuous.)

$\phi(u)$ does not have to be a $\sup L_K$, can be an $\inf \sup L_K$

(Isaacs equation from game theory)

Higher regularity (non local version of Evans-Krylov)

If $\phi(u) = 0$, ϕ convex, of order h, then u is a classical solution, with regularity

$$\|u\|_{C^{h+\sigma}(B_{1/2})} \leq \|u\|_{L^\infty(B_1)} + \text{ outside}$$

The σ and the constants are uniform for $0 < h_0 \leq h \leq 2$, all the way to second order.

Heuristic of the proof

The proof is based on the following idea.

For every x we consider the set $A(x)$ of those y for which

$$h(x,y) = [u(x+y) + u(x-y) - 2u(x) - [u(y) + u(-y) - 2u(0)] \leq 0$$

This set $A(x)$ is in some sense extremal.

Indeed for any other set B

$$0 \geq \int_B h(x,y)|x-y|^{-(n+s)}dy \geq \int_{A(x)} h(x,y)|x-y|^{-(n+s)}dy \qquad (*)$$

We want to show that as x goes to zero, $(*)$ goes to zero Hölder continuously. If we could keep A fix, $(*)$ would be a supersolution and the L^ε estimate applies.

We then do that by showing the convergence "for all A's" at the same time.

Second order case

Let u be a C^2 solution of

$$F(D^2u) = \sup_{\theta\in\mathcal{A}} \int \theta(\sigma(u_{\sigma\sigma}d\sigma = 0$$

(F convex!).

Then

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{C^{1,1}(B_1)}$$

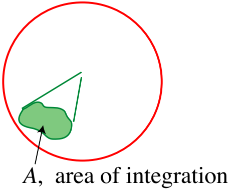
Proof. We will prove that

$$|\Delta u(x) - \Delta u(0)| \leq \left| \int_S [u_{\sigma\sigma}(x) - u_{\sigma\sigma}(0)]\, d\sigma \right| \leq |x|^\alpha$$

This follows after rescaling from the following inductive argument:

For any measurable set A of the unit sphere S^1, define

$$h(x, A) = \int_A [u_{\sigma\sigma}(x) - u_{\sigma\sigma}(0)]\, d\sigma$$



Assume that for any A,

$$h(x, A)|_{B_1} \geq -1$$

There exists a $\tau_0 > 0$ such that for any A

$$h(x, A)|_{B_{1/2}} \geq -1 + \tau_0$$

Indeed, this would imply, by choosing $A(x) = \chi_{u_{\sigma\sigma}>0}$, that

$$\int |u_{\sigma\sigma}(x) - u_{\sigma\sigma}(0)| \leq |x|^{\alpha}$$

and in particular $|\Delta u(x) - \Delta u(0)| \leq |x|^{\alpha}$.

To prove the inductive statement, let me recall the consequences from Krylov-Safanov:

a) weak-L^ε-estimate

1) u a supersolution in B_1

2) $u \geq 0, \quad u(0) = 1.$

Then

$$|\{u > t\}| \leq Ct^{-\varepsilon}$$

b) (oscillation theorem)

1) u a subsolution in B_1, $u \leq 1$, then

$$u(0) \leq C|\{u > 0\} \cap B_{1/2}|$$

The proof is then as follows: Assume that for some $x_0 \in B_{1/4}$, for some A_0,

$$h(x_0, A_0) \leq -1 + \tau_0$$

By convexity of F, $h(x, A_0) + 1$ is a (non negative) supersolution in B_1. Since

$$h(x_0, A_0) + 1 \leq \tau_0$$

from the L^ε estimate, for τ_0 very small, this forces

$$h(x, A_0) + 1 \leq (\tau_0)^{1/2}$$

in “almost all” of, say, $B_{3/4}$:

$$|\{h(x, A_0) + 1\} \geq \tau_0^{1/2}| \leq C\tau_0^{-\varepsilon/2}$$

Also, if $C \subset A$ (resp. $C \subset B = \mathcal{C}A$) and $x \in \Omega = \{h(x,A) \le \tau^{1/2}\}$ we must have

$$h(x,C) \le (\tau_0)^{1/2} \qquad (\text{resp. } h(x,C) \ge -\tau_0^{1/2})$$

(If no, by deleting (or adding) C from A we would make

$$h(x,A \setminus C) < -1 \qquad (\text{resp. } h(x,A \cup C) < -1)$$

a contradiction.)

Therefore, in Ω,

$$|h(x,A) - \Big(- \int (u_{\sigma\sigma})^{-}(x) \Big)| \le C\tau^{1/2-}$$

and also

$$|h(x,B) - \int (u_{\sigma\sigma})^{+}(x)| \le C\tau^{1/2}$$

But remember that ellipticity implies that

$$\int (u_{\sigma\sigma})^{-} \sim \int (u_{\sigma\sigma})^{+}$$

Then, since in Ω, $\int u_{\sigma\sigma}^{-} \sim -h(x, A) \sim -1$, this implies that

$$h(x, B) \sim \int u_{\sigma\sigma}^{+} \geq n_0 > 0 \quad (\text{say } 10^{-4})$$

But $h(x, B)$ is also a supersolution.

Hence $v(x) = n_0 - h(x, B)$ is a subsolution,

$$v \leq n_0 + 1 \quad (\text{since } \; h \geq -1)$$

$$v \leq 0 \; \text{ in almost all of } \; B_{3/4} \quad (\text{i.e., in } \; \Omega)$$

and $v(0) = n_0$.

This contradicts the oscillation lemma, since $\nu(0)$ must satisfy $\nu(0) \leq |\{v \geq 0\} \cap B_{3/4}|$, "almost nowhere" on $B_{3/4}$.

In the non local case the spirit of the proof is the same:

In this case for A a symmetric set in $\mathbb{R}^n$ (instead of the sphere) we define

$$h(x,A) = \int \Big\{ [u(x+y) + u(x-y) - 2u(x)] - [u(0+y) + u(0-y) - 2u(0)] \Big\} K(y) \chi_A(y)\, dy$$

and the same ideas apply.