Actions of the group of outer automorphisms of a free group

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Outer automorphisms of free groups

Free groups are the most simple, basic infinite groups.

Their automorphism groups are surprisingly complicated and resistant to understanding.

Studied since early last century, e.g. by Nielsen, Whitehead, Magnus, Stallings.

Want to use geometric and topological methods to study $Out(F_n)$.

- I will supply descriptions (pictures) of spaces on which $Out(F_n)$ acts.
- Will also mention some theorems about these spaces and consequences for $Out(F_n)$.
- Then I will talk about spaces that can't have interesting $Out(F_n)$ actions, and some open questions

Outer automorphisms of free groups

Many of these spaces relate mapping class theory to the theory of $\mathrm{Out}(F_n)$.

Basic observation: A homeomorphism of a punctured surface induces an automorphism of its fundamental group, giving a map

$$\pi_0(\mathrm{Homeo}^+(S_{g,s})) \to \mathrm{Out}(F_n)$$

This map is injective [Magus, Zieschang]; in fact

$$\mathrm{MCG}(S_{g,s}) = \mathrm{stab}(\text{periperal conjugacy classes})$$

There is a second important relationship

Second observation: Abelianization $F_n \to \mathbb{Z}^n$ induces a map on (outer) automorphism groups

$$\mathrm{Out}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$$

This map is easily seen to be surjective. Its kernel is quite mysterious.

I. Spaces with interesting Out(F_n) actions

Outer space

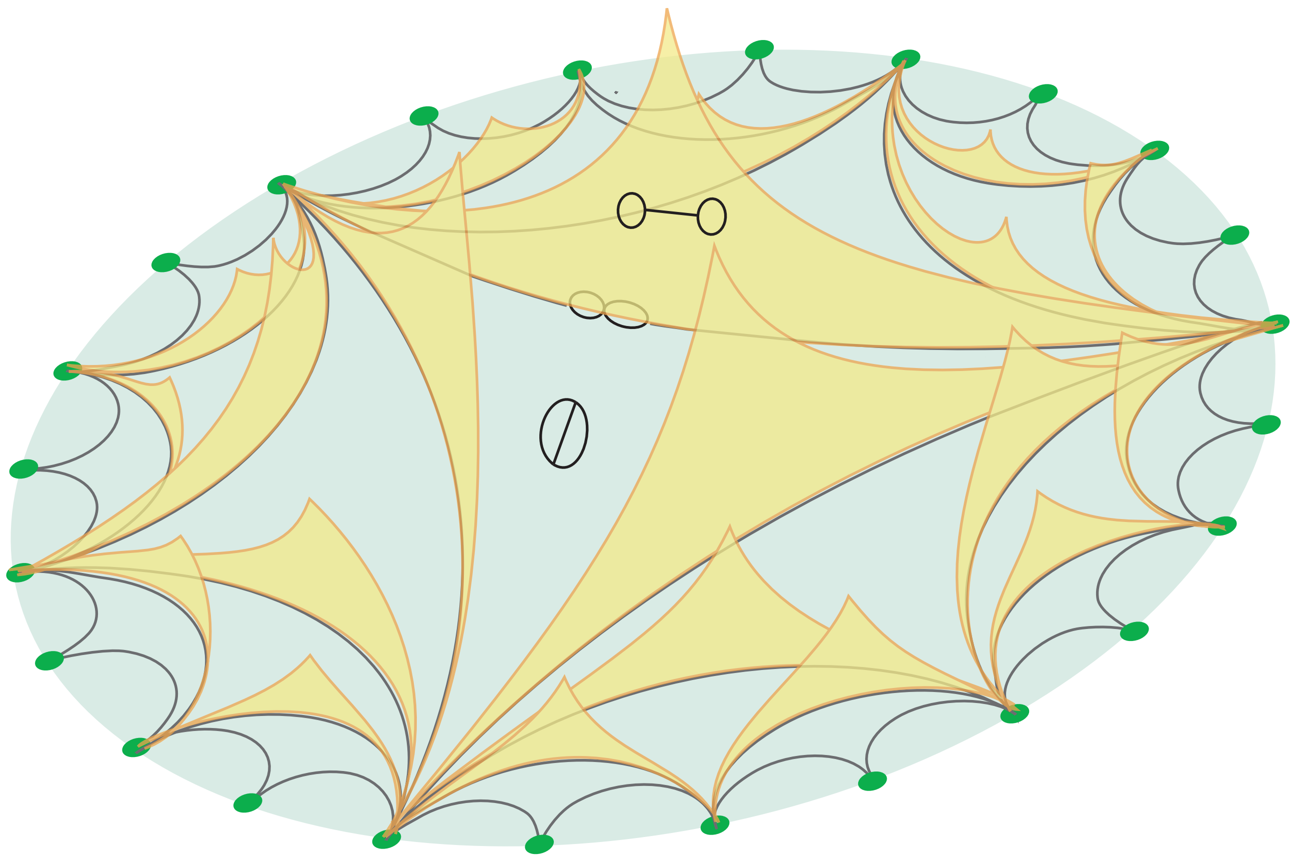
Outer space

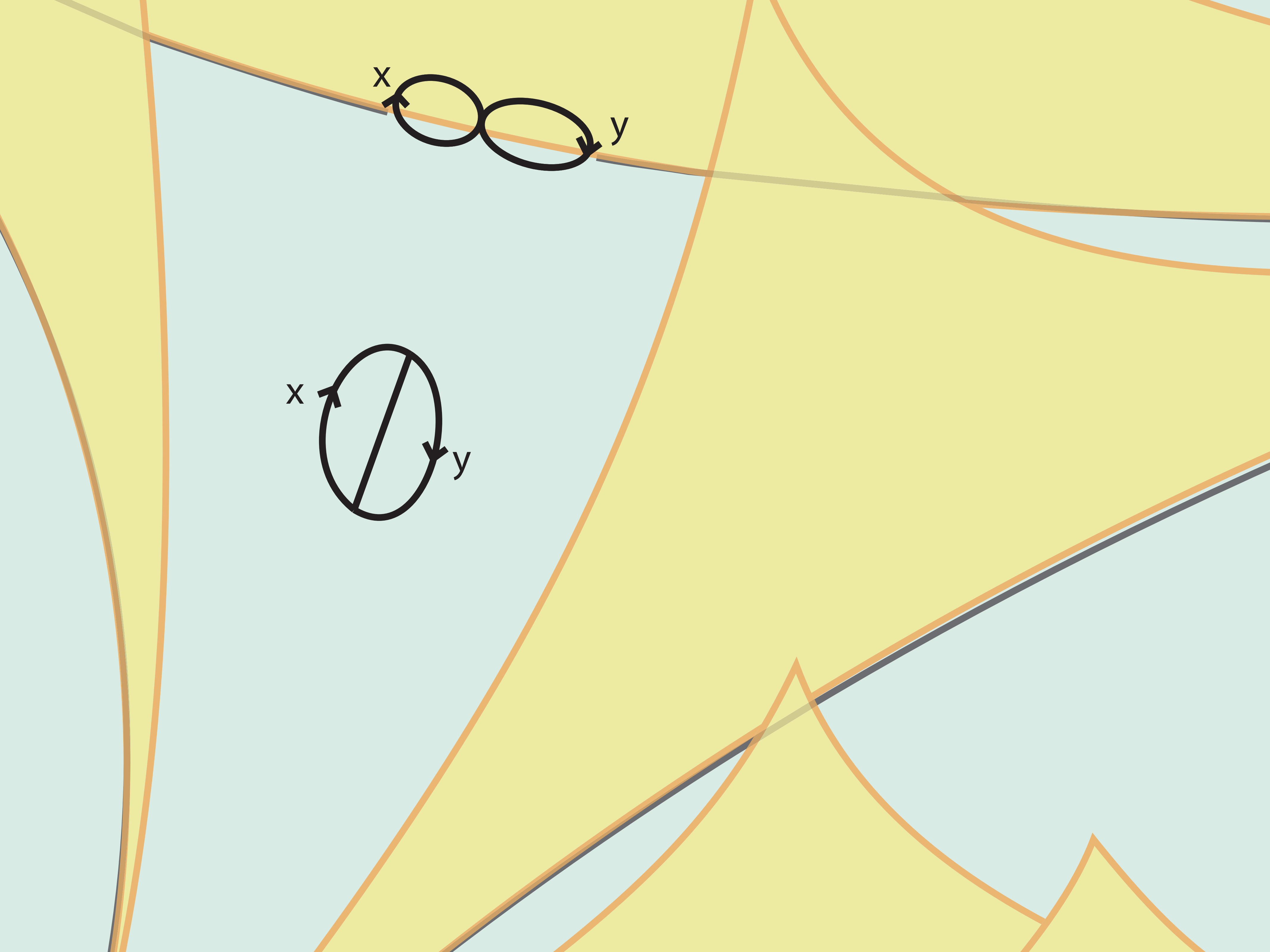
Definition motivated by Thurston's study of mapping classes of a surface via their action on Teichmüller space.

A point in Teichmüller space is a marked Riemann surface of area one. The action of the MCG changes the marking.

A point in Outer space is a marked metric graph, sum of edge-lengths equal to one. $Out(F_n)$ changes the marking.

Outer space (n=2)




x
y
x
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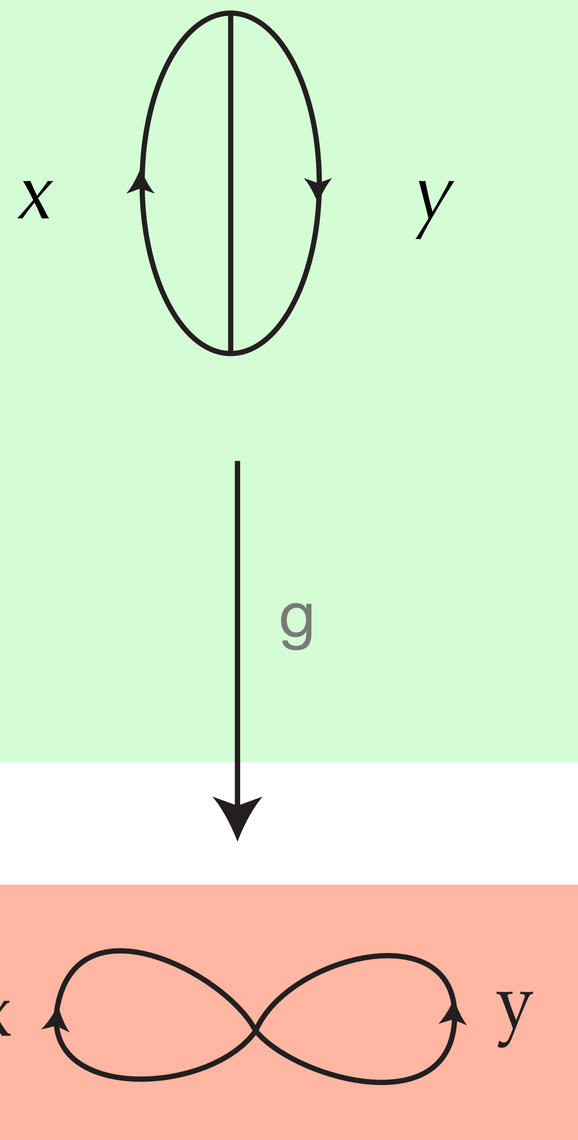
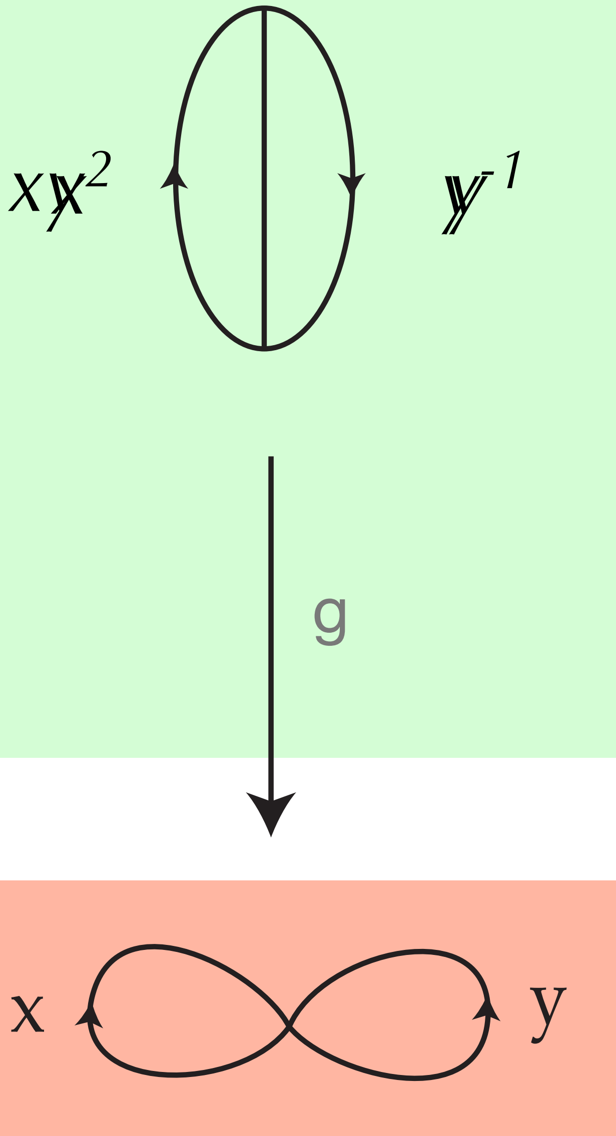
Outer space is a union of open simplices
$\sigma(\Gamma, g)$



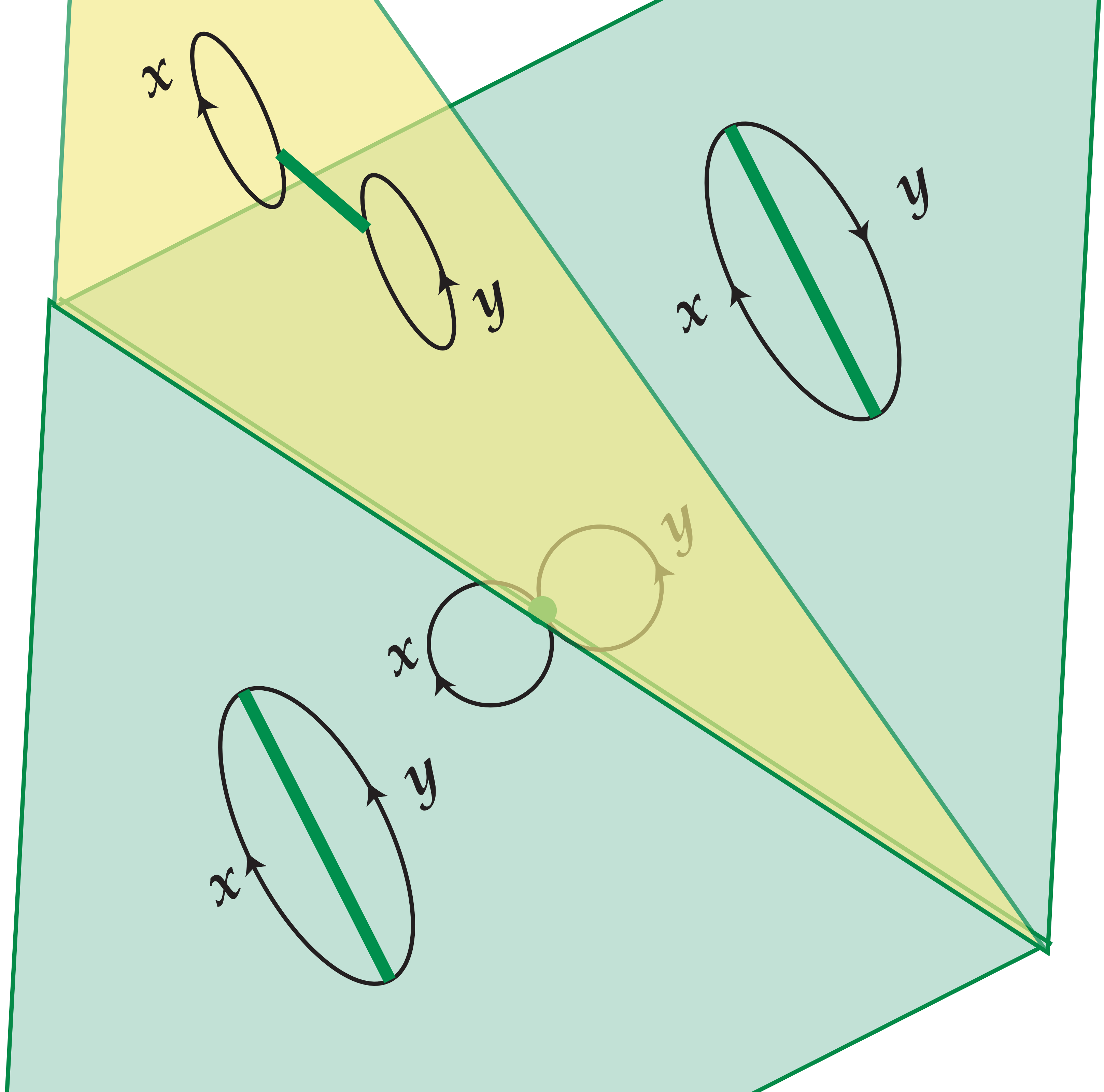
- Γ is a connected graph, with all vertices of valence at least 3.
- The dimension of $\sigma(\Gamma, g)$ is the number of edges of Γ minus 1.
- The markings on the edges of G describe a homotopy equivalence $g : \Gamma \to R_n$
- R_n = fixed rose with n petals
  F_n is identified with $\pi_1(R_n)$



Out(F_n) acts by changing the marking:

$$\varphi(x) = xy^2$$
$$\varphi(y) = y^{-1}$$

Then $\boldsymbol{\varphi} \cdot (\boldsymbol{\Gamma}, g) = (\boldsymbol{\Gamma}, \boldsymbol{\varphi} g)$



x
y
y
x
y
x
x
y

Outer space

Teichmüller space is homeomorphic to Euclidean space. The MCG acts properly discontinuously.

Outer space is not a manifold. But:

Theorem [Culler-V 86] Outer space is contractible. Out(F_n) acts properly discontinuously.

The MCG is the full group of isometries of Teichmüller space [Royden]

Outer space decomposes as a union of ideal simplices, Giving each simplex an equilateral Euclidean metric we obtain.

Theorem [Bridson-V 01] Out(F_n) is the full group of isometries of Outer space.

Outer space and mapping class groups

Outer space is the union of pieces which are invariant under $MCG(S_{g,s})$:

Choose an identification $\pi_1(S_{g,s}) \cong F_n$

Given any graph $\Gamma \subset S_{g,s}$, $\Gamma \simeq S_{g,s}$, this gives a marking on Γ.

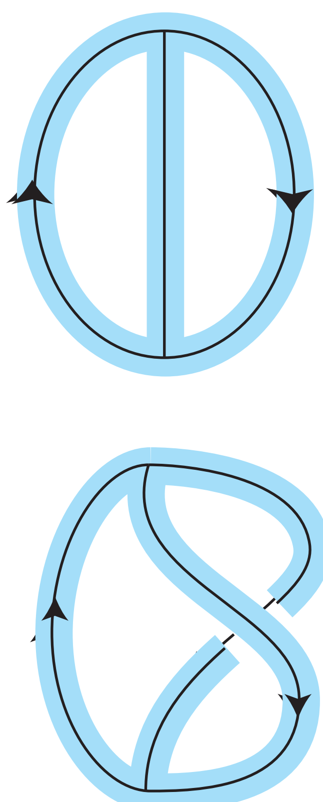
The set of such marked graphs, with all possible metrics, is a subspace of Outer space, called a **ribbon graph subspace** $\mathcal{O}(S_{g,s})$. These are easily seen to be manifolds, and contractions of Outer space restrict to prove these are contractible.

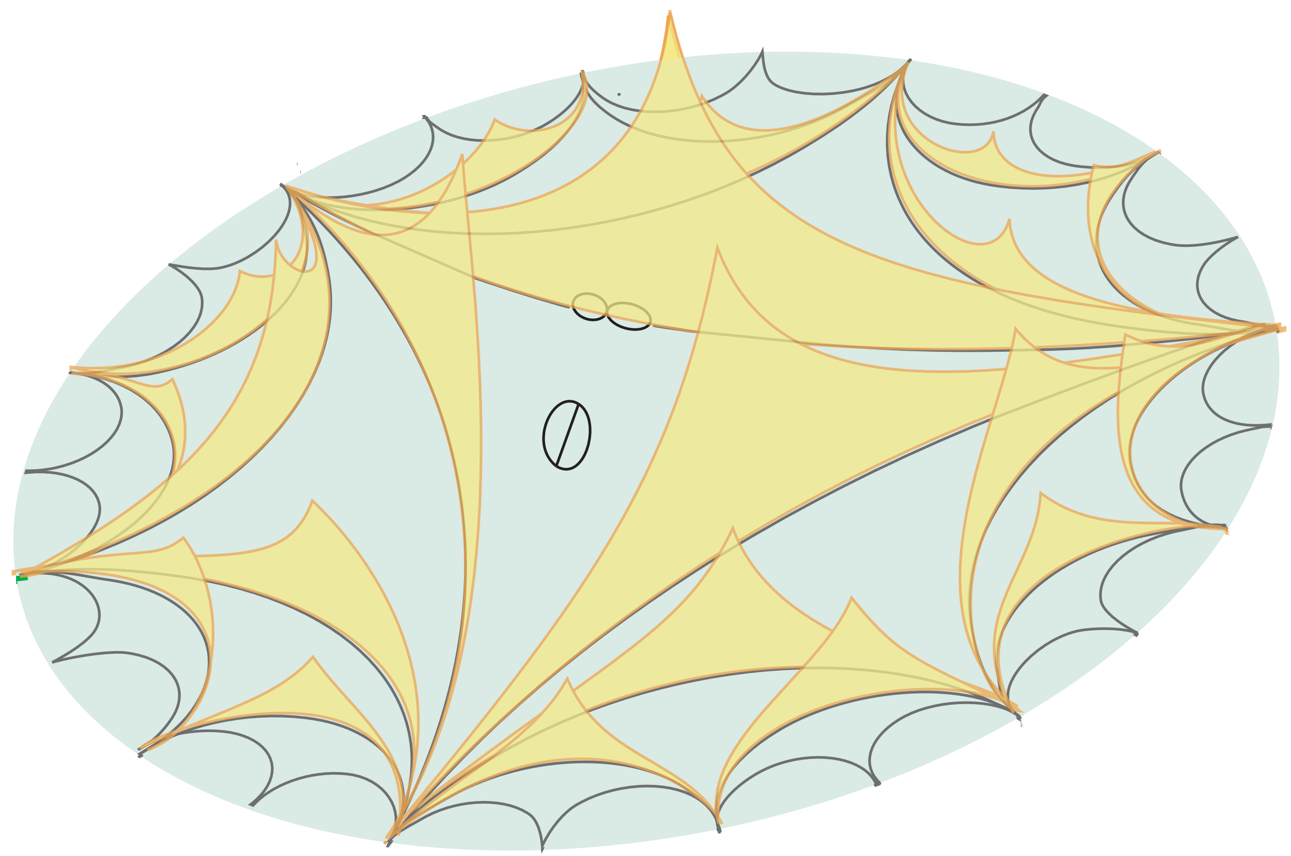
Penner, Harer gave coordinates for $\mathcal{O}(S_{g,s})$, identifying it equivariantly with $\mathcal{T}(S_{g,s}) \times \Delta^{s-1}$.

Outer space and mapping class groups

These ribbon graph subspaces cover Outer space:

e.g. n=2:





Compactifications

Thurston compactified Teichmüller space by embedding it into a (finite-dimensional) space of projective length functions and taking the closure.

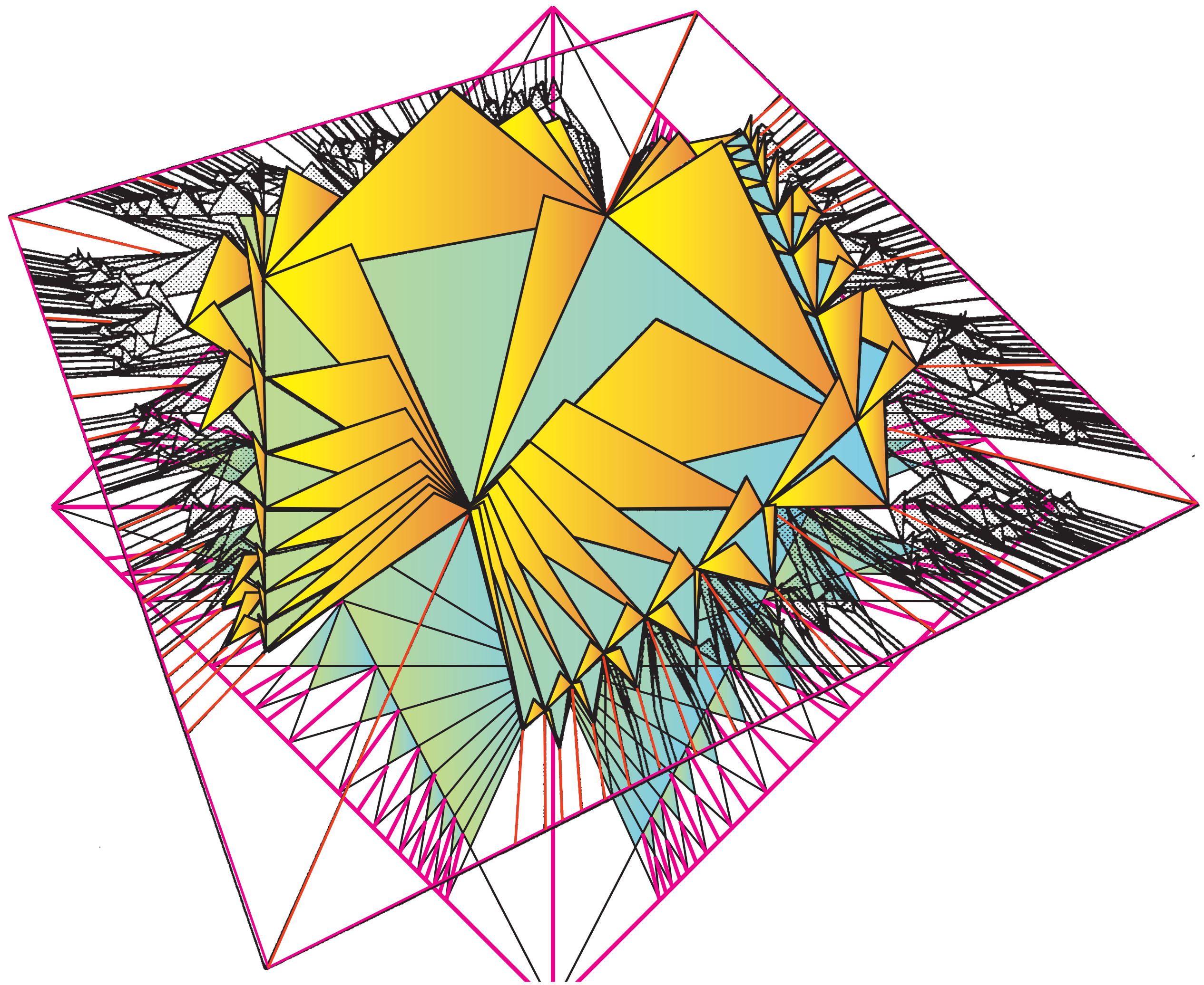
The closure is a ball. The sphere at infinity is the space of projective measured laminations. The action of a mapping class extends to the closure, so has (at least one) fixed point. Analysis of the fixed points classifies the mapping class.

Outer space also embeds into a space of projective length functions, but not into any finite-dimensional subspace [Smillie-V].

Outer space is not homeomorphic to $\mathbb{R}^N$...it's not even a manifold. Its boundary is not a sphere...it doesn't even behave like a boundary.

Compactifications

n=2 pictures:



Virtual cohomological dimension

Thurston program had to wait for Bestvina-Feighn-Handel to introduce new ideas, including a substitute for the space of projective measured laminations.

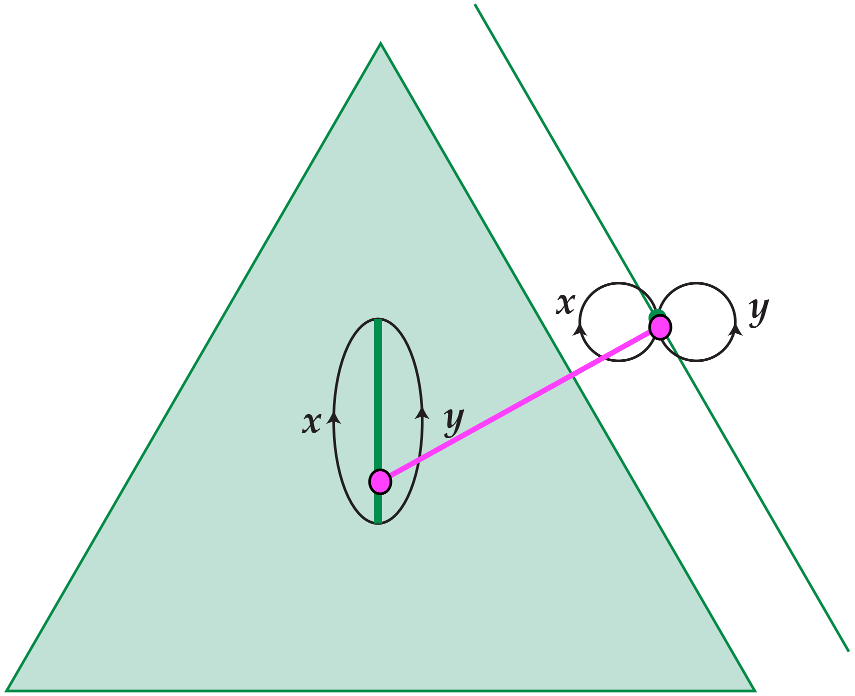
What we can do is compute the virtual cohomological dimension of $Out(F_n)$ by considering the combinatorial structure of Outer space.

The spine of outer space

The spine of outer space

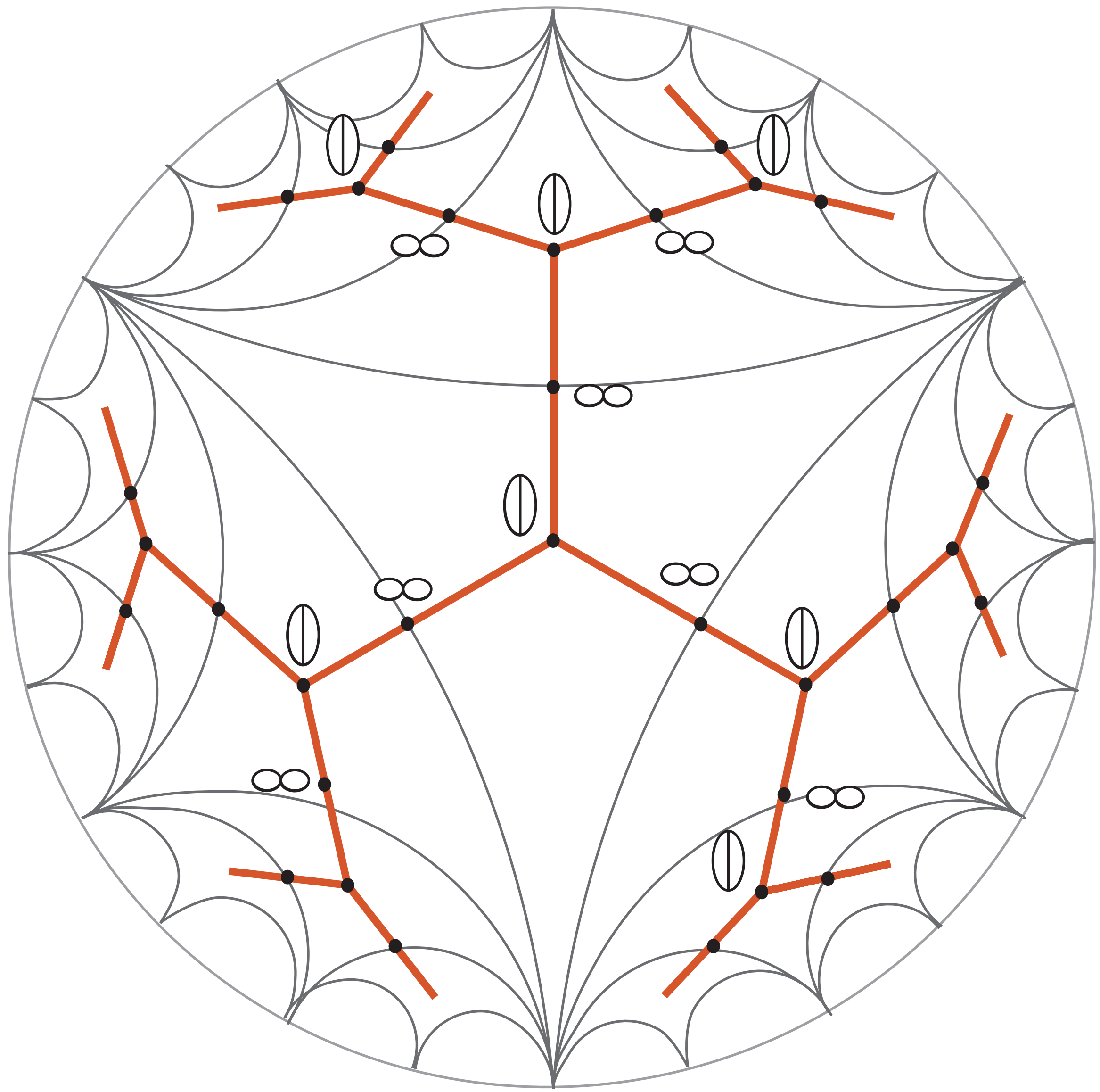
The set of open simplices σ(G,g) is partially ordered by the face relation.

Any partially ordered set gives rise to a simplicial complex. The k-simplices are totally ordered chains of length k+1.

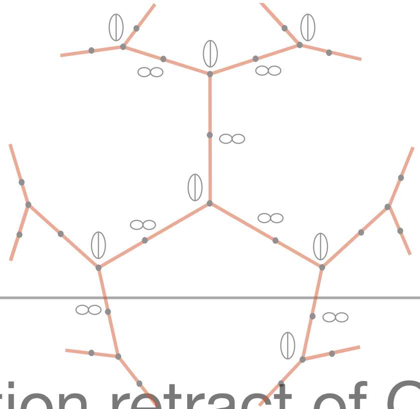


The spine of outer space

n=2
(reduced)



The spine of outer space



Theorem: The spine is an equivariant deformation retract of Outer space, of dimension 2n-3, with compact quotient.

Algebraic fact: There are free abelian subgroups of Out(F_n) of rank 2n-3.

Corollary: The virtual cohomological dimension of Out(F_n) is equal to 2n-3.

Culler, Khramtsov, Zimmerman: Every finite subgroup of Out(F_n) stabilizes some vertex of the spine.

Corollary: There are only finitely many conjugacy classes of finite subgroups of Out(F_n).

The spine of outer space and mapping class groups

The dimension of the spine gives an upper bound on the virtual cohomological dimension of mapping class groups of punctured surfaces. This is exact if there is 1 puncture, off by (s-1) in general.

To get the lower bound, Harer needed to prove MCG($S_{g,s}$) is a virtual duality group.

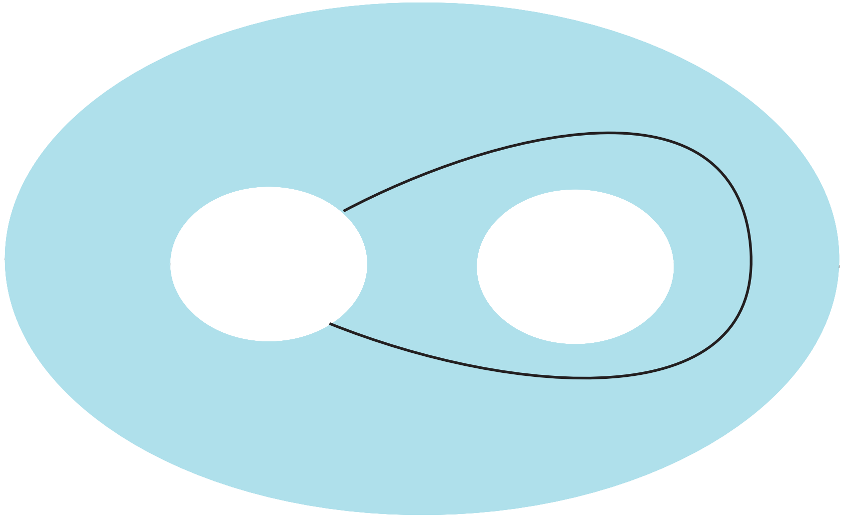
The restriction of the spine to $\mathcal{O}(S_{g,s})$ is a partially ordered set of simplices which correspond to graphs in $S_{g,s}$.

Dual to these graphs are arcs.

Arc complexes

Arc complexes

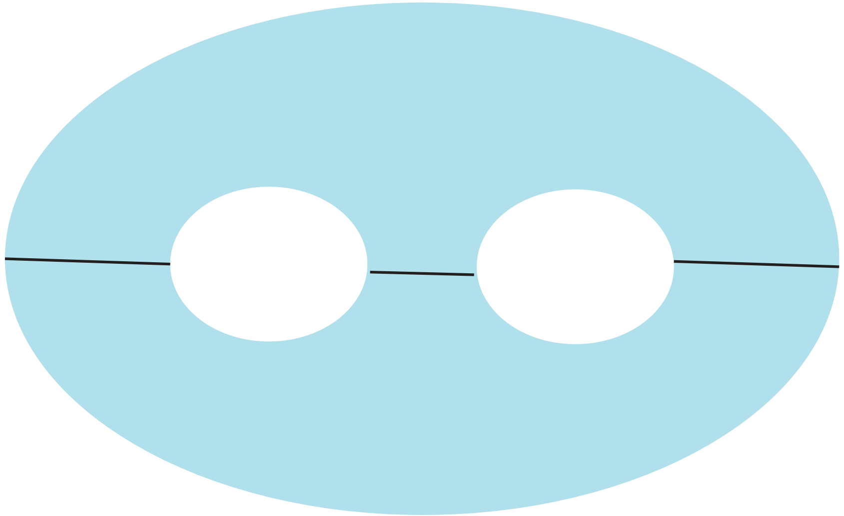
The spine restricted to $\mathcal{O}(S_{g,s})$ can be re-interpreted as an arc complex (arcs which fill $S_{g,s}$):



If you choose a boundary component, insist that all arcs begin and end on that component, and allow annuli, you obtain a smaller (still contractible) arc complex, giving the exact VCD.

Arc complexes

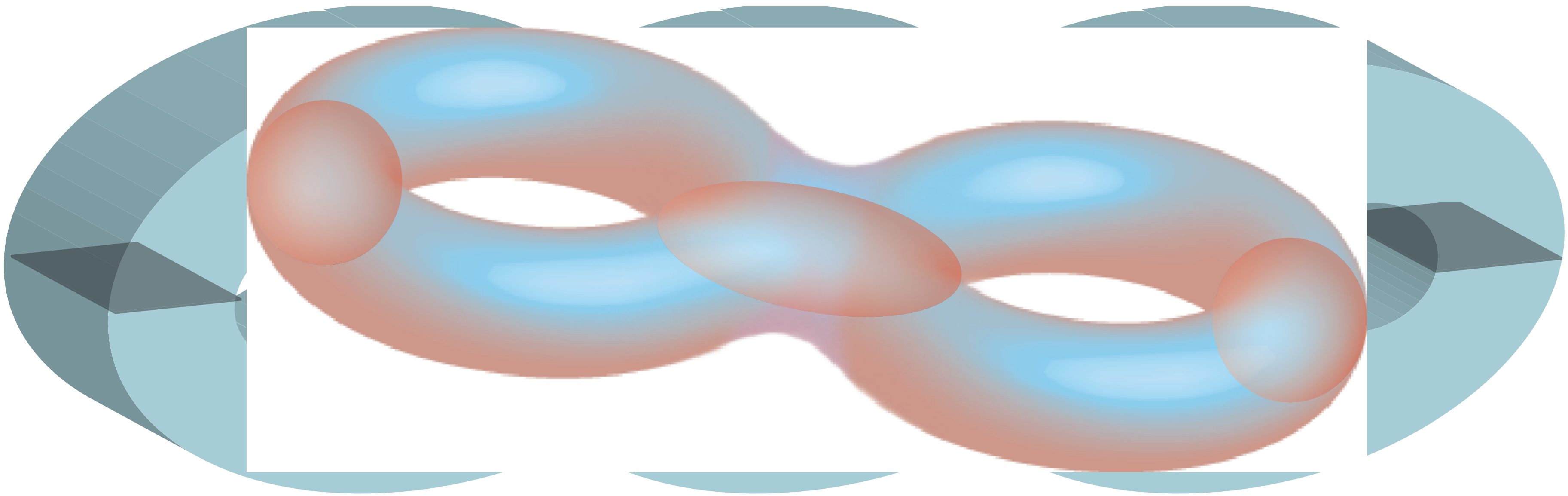
These arc complexes can in turn be interpreted as complexes of **spheres**



Sphere complexes

Sphere complexes

Fatten and double an arc system



Glue by the identity to obtain spheres in a connected sum of $S^1 \times S^2$'s

Sphere complexes

- Laudenbach: $M = \#(S^1 \times S^2)$. There is a split short exact sequence
$$1 \to D \to \pi_0 \mathrm{Diff}(M) \to \mathrm{Out}(F_n) \to 1$$
where D is a 2-group generated by Dehn twists on embedded 2-spheres in M.

- $\pi_0\mathrm{Diff}(M)$ acts on spheres in M. Descends to an action of $\mathrm{Out}(F_n)$ on the sphere complex since D acts trivially on embedded 2-spheres.

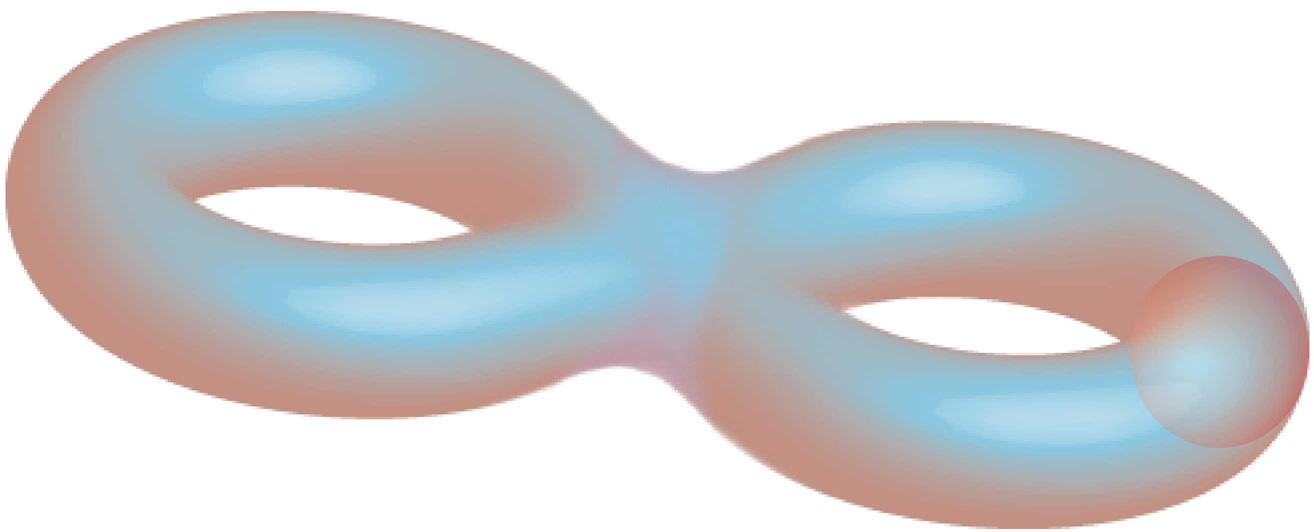
- The subcomplex of sphere systems with simply connected complementary pieces is identified with the spine of Outer space.

Sphere complexes

Other subcomplexes and variations of the sphere complex that have proved useful include:

- The subcomplex of coconnected sphere systems with non-simply-connected complement.

Any such sphere system gives a free factor of F_n.

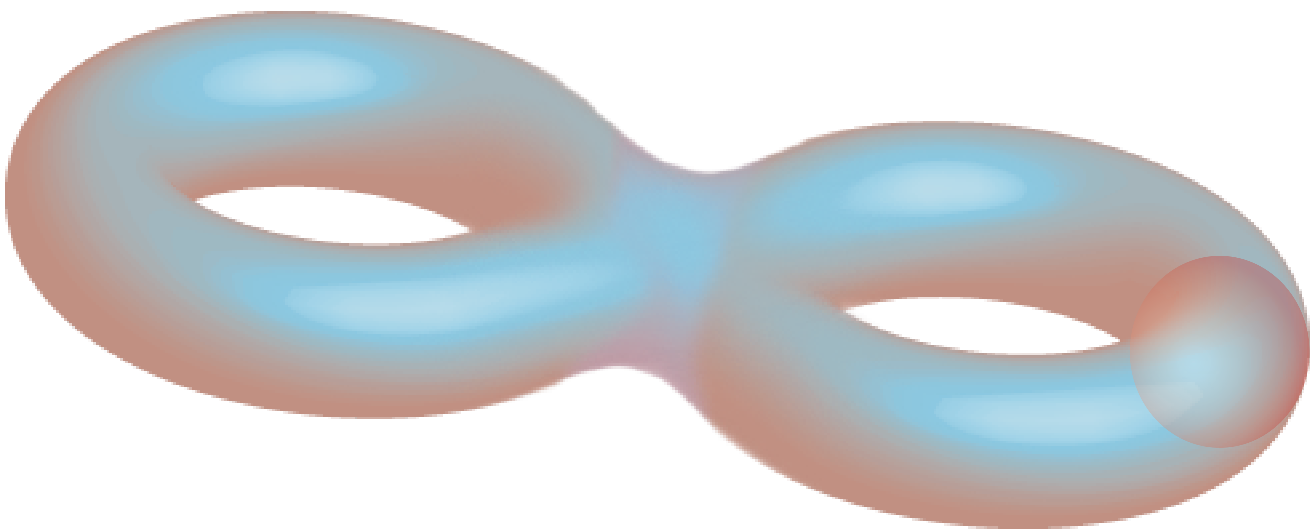


Free factor complexes

The resulting map was used to prove:

Theorem. [Hatcher-V 98] The geometric realization of the partially ordered set of free factors of F_n is homotopy equivalent to a wedge of spheres of dimension n-2.

This is analogous to the classical Solomon-Tits theorem about the geometric realization of the poset of subspaces of a vector space.



Free factor complexes

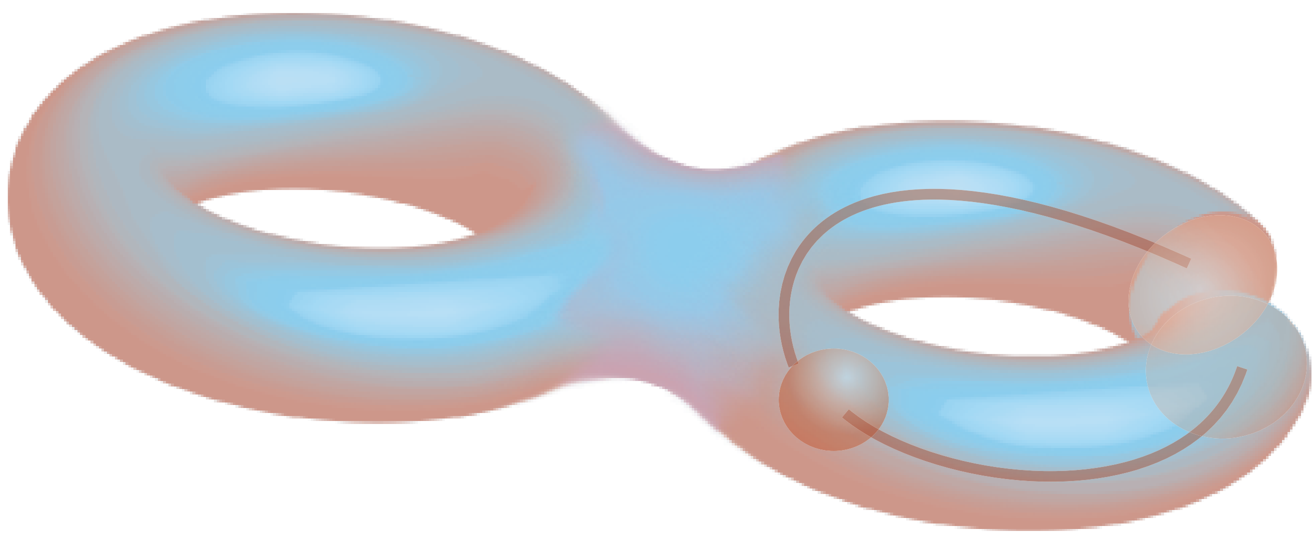
A closely related complex, of **free factorizations** of F_n is spherical and was used to prove homology stability for $Aut(F_n)$:

Theorem. [Hatcher-V 98] The natural inclusions $Aut(F_n) \to Aut(F_{n+1})$ induce isomorphisms $H_i(Aut(F_n)) \cong H_i(Aut(F_{n+1}))$ for $n >> i$.

Another closely related complex is the complex of **partial bases** of F_n, which is also spherical.

Tethered spheres

A simpler proof of homology stability can be given by considering a complex of **tethered** spheres [Hatcher-V, Hatcher-Wahl]:



Similarly, the proof of homology stability for mapping class groups can be considerably simplified by using complexes of tethered curves.

Sphere complexes

Historical note: 2-spheres in $\#(S^1 \times S^2)$ were used by J.H.C. Whitehead in his work on automorphisms of free groups.

But he didn't have Laudenbach's theorem.


And now for
something
completely
different...

II. Spaces with no actions

Proper cocompact actions

This is what you look for first in geometric group theory. (If G acts properly and cocompactly on a metric space X, then G is quasi-isometric to X.)

$Out(F_n)$ acts properly and cocompactly on the spine of Outer space. Can this be given a metric of non-positive curvature?

Theorem [Bridson-V 95] $Out(F_n)$ cannot act properly and cocompactly on any CAT(0) space.

Proof: First show $Out(F_3)$ has an exponential Dehn function. This uses fact [ECHLPT(hurston)] that $GL(3, \mathbb{Z})$ has exponential Dehn function. Then use centralizers and induction arguments to show that $Out(F_n)$ is not bicombable. (Groups which act properly and cocompactly on CAT(0) spaces ARE bicombable.)

Actions on trees

Classical way to split a group into simpler pieces...find an action on a simplicial tree, then use Bass-Serre theory.

Theorem [Culler-V 96] For $n>2$, any action of $Out(F_n)$ on an $\mathbb{R}$-tree has a global fixed point (Out has *property* $F\mathbb{R}$).

Proof is elementary, uses the existence of a well-behaved generating set. But also uses fact that the abelianization of $Out(F_n)$ is finite.

Proof applies also to mapping class groups of closed surfaces.

Question: Do finite index subgroups of $Out(F_n)$ have $F\mathbb{R}$?

Answer: No for $n=3$, open for $n > 3$. "No" answer implies $Out(F_n)$ does not have Kazhdan's Property (T)

Actions on CAT(0) manifolds

The map $\mathrm{Out}(F_n) \to GL(n, \mathbb{Z})$ provides actions of $\mathrm{Out}(F_n)$ on CAT(0) manifolds, like $\mathbb{R}^n$ and the homogeneous space for $GL(n,\mathbb{R})$.

To make statements simpler, replace $\mathrm{Out}(F_n)$ by $\mathrm{SAut}(F_n)$, the preimage in $\mathrm{Aut}(F_n)$ of $SL(n, \mathbb{Z})$.

Theorem A. [Bridson-V 07] For $n>2$, any action of $\mathrm{SAut}(F_n)$ by isometries on a CAT(0) manifold M of dimension $< n$ is trivial.

The proof uses torsion in $\mathrm{SAut}(F_n)$ in an essential way, and the question is still open for torsion-free subgroups of finite index.

The proof proceeds by considering metric spheres around a fixed point for an involution. A metric sphere in a CAT(0) manifold need not be a manifold - but it is at least a $\mathbb{Z}/2$-homology sphere.

Actions on spheres

The proof proceeds by considering metric spheres around a fixed point for an involution. A metric sphere in a CAT(0) manifold need not be a manifold - but it is at least a $\mathbb{Z}/2$-homology sphere, and we also prove.

Theorem B. If $n > 2$, any action of $SAut(F_n)$ by homeomorphisms on a $\mathbb{Z}/2$-homology sphere of dimension $< n-1$ is trivial.

Here actions need not be discrete, semisimple,...

Immediate corollaries are Theorems A and B for any quotient of $SAut(F_n)$, including $S(Out(F_n))$ and $SL(n, \mathbb{Z})$.

For $SL(n, \mathbb{Z})$ Theorem B was proved by [Zimmerman 07] completing a program of [Parwani 06]

Using the torsion in $\mathrm{Aut}(F_n)$

The largest finite subgroup in $\mathrm{Aut}(F_n)$ is the signed symmetric group $W_n = N \rtimes S_n$. Here $N \cong (\mathbb{Z}/2)^n$ is generated by $\{e_i\}$ and W has center $\Delta = e_1 e_2 \ldots e_n$. Set $SW_n = W_n \cap \mathrm{SAut}(F_n)$, $SN = S \cap \mathrm{SAut}(F_n)$.

Proposition. Let G be any group and $f\colon \mathrm{SAut}(F_n) \to G$ a homomorphism. Let f_0 be the restriction of f to SW_n Then either

$\mathrm{Ker}(f_0) = 1$

n is even, $\mathrm{Ker}(f_0) = \langle \Delta \rangle \cong \mathbb{Z}/2$ and f factors through $PSL(n, \mathbb{Z})$

$\mathrm{Ker}(f_0) = SN$ and $\mathrm{Im}(f) \cong SL(n, \mathbb{Z}/2)$

$\mathrm{Ker}(f_0) = SW_n$ and $\mathrm{Im}(f) = \{1\}$

Proof for smooth manifolds

Suppose M is complete, simply-connected and non-positively curved, and that $SAut(F_n)$ acts on M, i.e. we have a homomorphism

$$SAut(F_n) \to Isom(M).$$

If e_1e_2 is in the kernel of this homomorphism, then by the proposition the image is trivial or isomorphic to $SL(n, \mathbb{Z}/2)$.

We can rule the second possibility out using Smith theory and the fact that $SL(n, \mathbb{Z}/2)$ is simple.

Therefore e_1e_2 acts by a nontrivial involution of M. Since $SAut(F_n)$ is perfect, it acts by orientation-preserving homeomorphisms, so the fixed point set F of e_1e_2 is a smooth manifold of codimension at least 2.

Proof for smooth manifolds

Therefore e_1e_2 acts by a nontrivial involution of M. Since SAut(F_n) is perfect, it acts by orientation-preserving homeomorphisms, so the fixed point set F of e_1e_2 is a smooth manifold of codimension at least 2.

There is a natural SAut(F_{n-2}) in the centralizer of e_1e_2, which by induction acts trivially on F and on its normal bundle, and therefore on all of M. But the normal closure of this SAut(F_{n-2}) is all of SAut(F_n), so the whole group acts trivially on M.

Remarks about the CAT(0) situation

If M is not smooth, but only CAT(0), Fix(e_1e_2) may not be a manifold, and we don't have a normal bundle.

This is why instead we consider actions on $\mathbb{Z}/2$ homology spheres; we also need some CAT(0) geometry.

Theorem B is proved using the Proposition and more Smith theory.

Questions

Questions raised by Cohen, Thurston, Calegari at 2007 Topology Festival:

Can Out be realized as a group of homeomorphisms of a manifold with fundamental group F_n? Or diffeomorphisms?

Same question for mapping class group. [Morita 87] proved the map Diffeo(S) to MCG(S) does not have a section for genus > 5, by cohomology considerations. [Markovic 07] proved that the map Homeo(S) to MCG(S) does not have a section for g > 5.

Both proofs rely on the torsion in MCG. What about torsion-free subgroups of finite index? Can you lift them?

More questions on torsion-free subgroups of finite index: Can finite-index subgroups of Out act nontrivially on trees, n>3?

Can they act on spheres, CAT(0) manifolds,...

Happy birthday!