Combinatorial link Floer homology
and transverse knots

Dylan Thurston

Joint with/work of Sucharit Sarkar
Ciprian Manolescu
Peter Ozsváth
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June 10, 2007, Princeton, NJ
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The invariant called knot Heegaard-Floer
Determines the genus—and more.
To distinguish transverse knots
(and it turns out there are lots!)
HFK opens up a new door.

June 10, 2007, Princeton, NJ
Outline

▶ Introduction

Computing $HFK$

Variants

Grid moves

Transverse knots
What is Heegaard-Floer homology?

\[ \dim(\hat{HFK}_i(K; s)) : \]

\[ \begin{array}{ccc}
& 1 & 1 \\
1 & & \\
1 & 2 & \\
1 & 2 & \\
& \text{Maslov} & 1 \\
& 1 & 2 & \text{Alexander} \\
& 1 & & \\
\end{array} \]

Characteristics of \( \hat{HFK} \):

- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus; (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing \( HFK \)...

... and so the world’s simplest algorithm for knot genus!
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Setting: Grid diagrams

Grid diagram: square diagram with one $X$ and one $O$ per row and column.

Turn it into a knot: connect $X$ to $O$ in each column; $O$ to $X$ in each row. Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under cyclic rotations:
Move top segment to bottom.
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Computing the Alexander polynomial

We categorify the following formula:

\[ \pm t^*(1 - t)^{n-1}\Delta(K; t) \]

▶ Make matrix of \( t^{-\text{winding \#}} \)
(with extra row/column of 1’s);

▶ det determines the Conway-Alexander polynomial \( \Delta \)
\( (n = \text{size of diagram}; \text{here } 6) \)
Computing the Alexander polynomial

We categorify the following formula:

\[
\begin{vmatrix}
1 & 1 & 1 & t & t & t \\
1 & 1 & t^{-1} & 1 & t & t \\
1 & t & 1 & 1 & t & t \\
1 & t & t & t & t^2 & t \\
1 & t & t & t & t & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{vmatrix}
= \pm t^*(1 - t)^{n-1} \Delta(K; t)
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► Computing HFK

Variants

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Transverse knots
Computing \( HFK \): Chain complex \( \tilde{CK} \)

Define a chain complex \( \tilde{CK} \) over \( \mathbb{Z}/2 \).

- Generated by matchings between horizontal and vertical grid circles (as counted in \( \det \) for Alexander).
- Boundary \( \partial \) switches corners on empty rectangles:

Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (Empty means: no X’s, O’s, or other points in generator.)
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Computing $HFK$: $\partial^2 = 0$

Each term in $\partial^2$ must have a mate:

- If rectangles are disjoint, take rectangles in either order.
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Computing *HFK*: Gradings on $\tilde{CK}$

In the plane, $\mapsto\rightarrow$ removes one inversion.

For $A, B, C \subset \mathbb{R}^2$,

$$I(A, B) := \#\{a \square b \mid a \in A, b \in B\}$$

$$I(A - B, C) := I(A, C) - I(B, C)$$

For $x$ a generator, $\mathbb{X}$ the set of $X$’s, $\mathcal{O}$ the set of $O$’s, the gradings are:

- **Maslov:** $M(x) := I(x - \mathcal{O}, x - \mathcal{O}) + 1$.

- **Alexander:**
  $$A(x) := \frac{1}{2}(I(x - \mathcal{O}, x - \mathcal{O}) - I(x - \mathbb{X}, x - \mathbb{X}) - (n - 1)).$$
Computing $HFK$: The answer

Theorem (Manolescu-Ozsváth-Sarkar)

For $G$ a grid diagram for $K$,

$$H_*(\tilde{CK}(G)) \cong \widehat{HFK}(K) \otimes V \otimes^{n-1}$$

where $V := (\mathbb{Z}/2)_{0,0} \otimes (\mathbb{Z}/2)_{-1,-1}$.

Gillam and Baldwin used this to compute $\widehat{HFK}$ for all knots with $\leq 11$ crossings, including new values of knot genus.
Outline

Introduction

Computing \( \text{HFK} \)

▶ Variants

Grid moves

Transverse knots
Improving the answer

To remove factors of $V^\otimes n^{-1}$:

$HFK^-$: variant of $\widehat{HFK}$

Module over $\mathbb{Z}/2[U]$

$U$ has degree $(-1, -2)$

Related to $\widehat{HFK}$ by Univ. Coeff. Thm.

To compute: Add one $U_i$ for each $O$

Complex $CK^-(G)$ over $\mathbb{Z}/2[U_1, \ldots, U_n]$

$\partial$ counts rect. that contain only $O$'s, weighted by corresponding $U_i$.

Theorem
(Manolescu-Ozsváth-Sarkar)

$$H_*(CK^-(G)) \simeq HFK^-(K),$$

Each $U_i$ acts by $U$ on the homology.
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$$H_*(CK^-(G)) \simeq HFK^-(K),$$

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Further variants

Can also:

- Allow rectangles to cross $X$’s to get a filtered complex, and
- Add signs (in essentially unique way) to work over $\mathbb{Z}[U]$. 
Outline

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Computing $HFK$

Variants

- Grid moves

Transverse knots
Combinatorial invariance

Theorem (Manolescu-Ozsváth-Szábo-T.)

For any sequence of elementary grid moves, there is an explicit chain map exhibiting invariance of $HFK^-$. 

Conjecture (Naturality or Functoriality)

The chain map depends only on isotopy class of sequence of elementary grid moves. That is, oriented mapping class group of $K$ acts on $HFK^-(K)$. 
Elementary grid moves

- **Cycle:** Move left column to right, or top row to bottom.
- **Commute:** Switch two non-interfering columns or rows.
- **Stabilize:** Introduce a notch at a corner.

(Cromwell ’95, Dynnikov ’06)
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Where’s Reidemeister III?

(Cromwell ’95, Dynnikov ’06)
Chain map for commutation counts pentagons

To construct a chain map for commutation, draw two versions of the middle gridcircle on a single diagram.

The chain map counts empty pentagons going between the two gridcircles.
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• Transverse knots
Contact structures and knots

A contact structure is a twisted 2-plane field:
if $\alpha$ is a 1-form defining the plane field, $\alpha \wedge d\alpha$ is positive.
(Warning: above contact structure is reversed.)

A Legendrian knot is a knot that is tangent to the plane field.
A transverse knot is a knot that is transverse to the plane field.

Transverse knots have one easy invariant, the self-linking number.

Question. Can we find transverse knots with the same classical knot type and self-linking number?
Ways to stabilize

Four ways to stabilize: Where to leave the empty square?

- Two diagonal opposite ways preserve Legendrian knot.
- Two adjacent ways preserve closed braid.
- Three ways preserve transverse knot.

Warning: The Legendrian/transverse knots are mirrored.
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Transverse invariant: Definition

Definition
The canonical generator \( x^+(G) \) is given by the upper-right corner of each \( X \).

Facts:
- \( \partial x^+ = 0 \). (The \( X \)'s block any rectangles.)
- \([x^+(G)]\) maps to \([x^+(G')]\) under commutation and 3 out of 4 stabilizations.

Theorem (Ozsváth-Szabó-T.)
\([x^+(G)]\) in \( \text{HFK}^- (m(K)) \) is an invariant of the transverse knot represented by \( G \), up to quasi-isomorphism of filtered complexes.
Transverse invariant: Properties

Let $G$ be a grid diagram representing the transverse knot $T$.

- $x^+(G)$ lives in bigrading $(s, 2s)$, where $s = \frac{sl(T)+1}{2}$.

- If $T'$ differs from $T$ by a positive stabilization, then $[x^+(T')] = U[x^+(T)]$.

- $[x^+(T)] \neq 0$ in $HFK^-$.

Corollary

For any transverse knot $T$ of topological type $K$,

$$\frac{sl(T) + 1}{2} \leq \tau(K) \leq g_4(K)$$

where $\tau(K)$ is the largest Alexander grading which has an element which is not $U$ torsion.
Transverse invariant: Examples

Let $\theta(T)$ (resp. $\hat{\theta}(T)$) be the transverse invariant in $HFK^-(m(K))$ (resp. $HFK(m(K))$).

$\hat{\theta}(T) = 0$ iff $\theta(T)$ is divisible by $U$.

Theorem (Ng-Ozsváth-T.)

The knots $m(10_{132})$ and $m(12n_{200})$ have two trans. reps. with same sl, one with $\hat{\theta} = 0$ and one with $\hat{\theta} \neq 0$.

This technique also works for the $(2,3)$ cable of the $(2,3)$ torus knot, originally found by Etnyre-Honda and Menasco-Matsuda.

Let $\delta_1$ be the next differential in the spectral sequence on $\hat{HFK}$.

Theorem (Ng-Ozsváth-T.)

The pretzel knots $P(-4, -3, 3)$ and $P(-6, -3, 3)$ have two trans. reps. with same sl, one with $\delta_1 \circ \hat{\theta} = 0$ and one with $\delta_1 \circ \hat{\theta} \neq 0$. 
Transverse invariant: Going further

Theorem (Ng-Ozsváth-T.)

*If the Naturality Conjecture is true, then the twist knot $7_2$ has two trans. reps. with the same sl, with $\hat{\theta}$ in different orbits of the mapping class group.*

But $\theta$ is not a complete invariant: Birman and Menasco have classified closed 3-braids up to transverse isotopy. In their small examples of distinct transverse knots, $\theta$ lives in a 1-dimensional space, so cannot distinguish them.