Our work with Bill on lengths of graph transformations

Or

How Bill's insights applied to problems in data structure design

Danny Sleator (Carnegie Mellon)

Bob Tarjan (Princeton, HP)

Outline

Part 1: Binary Trees

Lower and upper bounds on the rotation distance between trees. (Punch line: the lower bound makes use of hyperbolic geometry)

[D. D. Sleator, R. E. Tarjan, W. P. Thurston, Rotation Distance, Triangulations, and Hyperbolic Geometry, Journal of the American Mathematical Society, Vo.1, No.3., 1988]

Part 2: Plane Triangulations

We extend the above concept of diagonal flips to the case of planar graphs and derive lower and upper bounds, by completely different, but equally elegant techniques.

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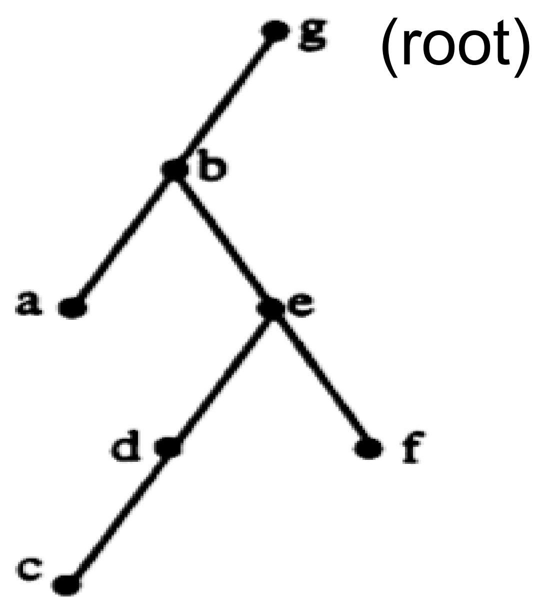
Part 1: Binary Search Trees

A standard well-studied data structure used to represent an ordered list of items.

Each node has a key, left child (possibly empty) and right child (possibly empty)

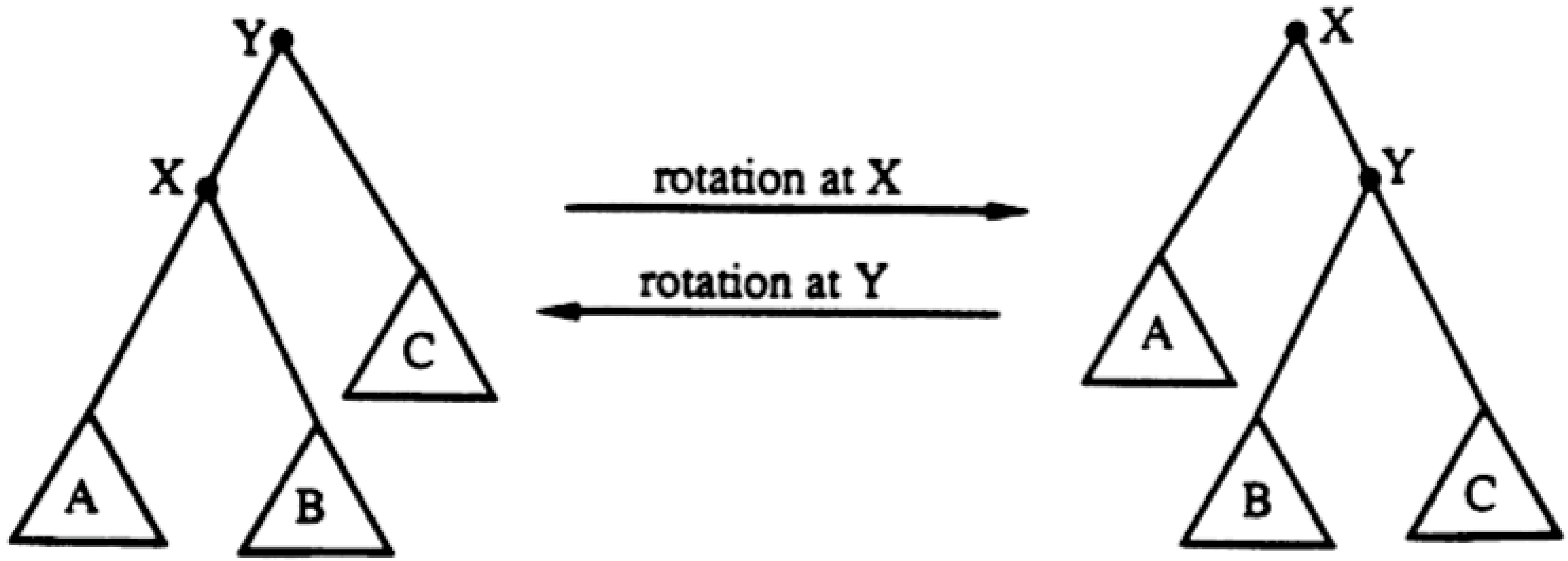
e.g. the list (a,b,c,d,e,f,g) could be represented as follows:



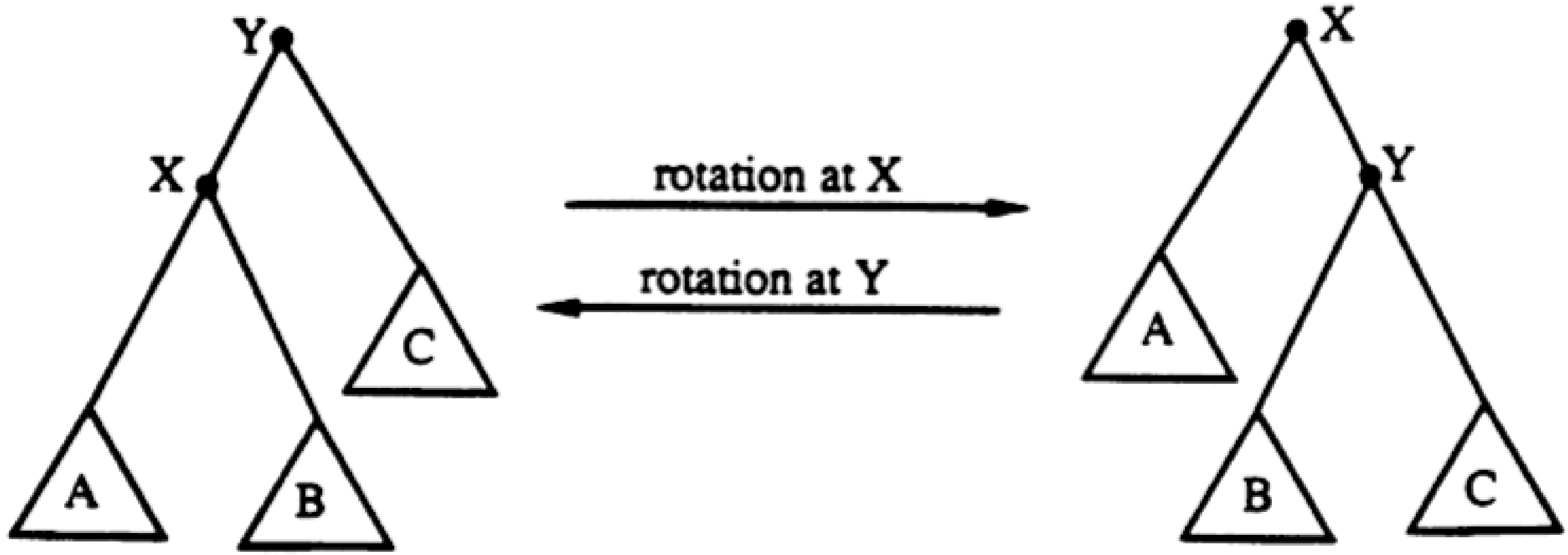
Many different trees can represent the same list of items.

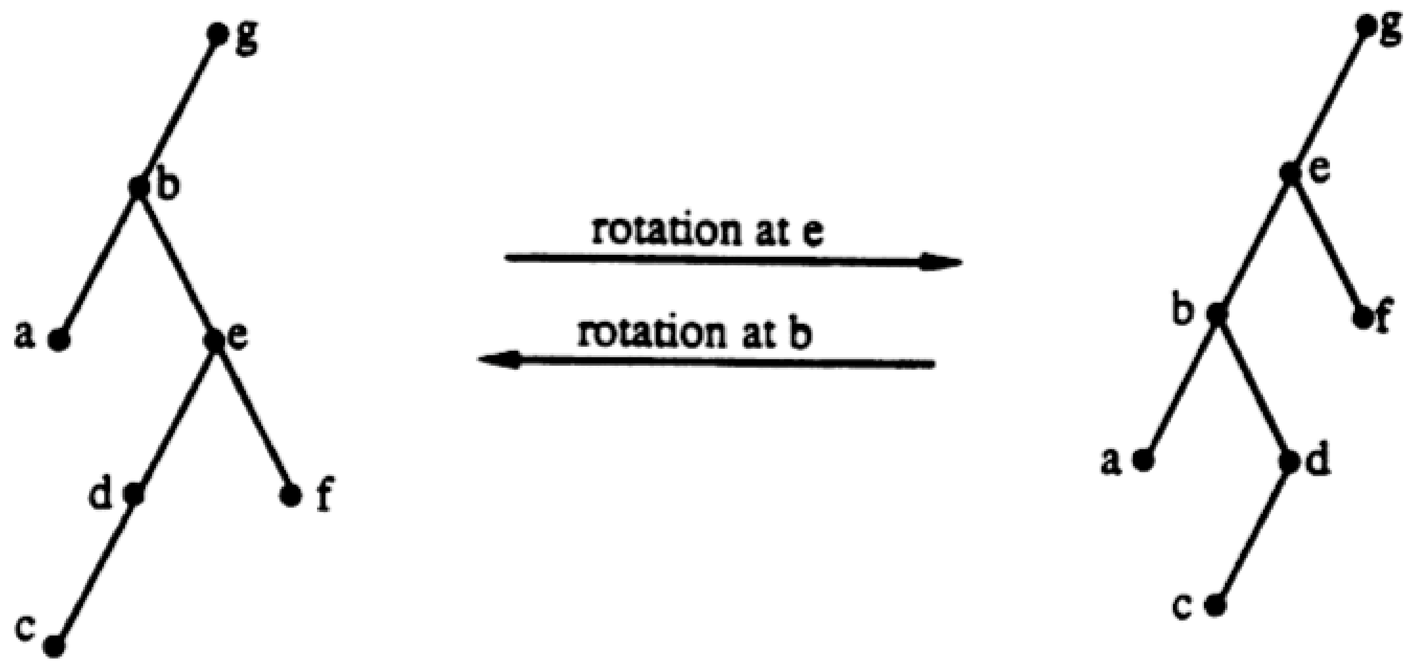
The purpose of a binary search tree is to facilitate searching, insertions, and deletions in O(*log n*) time.

To achieve this, it is necessary to have a way to adjust shape of the tree. The standard way to do this is with *rotations*.



Note that the depths of some nodes change, but the list represented remains the same.


Y
X
rotation at X
rotation at Y
C
A
B
X
Y
A
B
C


g
b
a
e
d
f
c
rotation at e
rotation at b
g
e
b
f
a
d
c

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Question1: Given two trees, how many rotations are required to convert one into the other?

And an even more basic question:

Question 2: What is the maximum distance two trees can be apart?

Our answer: the maximum distance between two *n*-node trees is 2*n*-6 for n>10. We construct such trees. And the construction is interesting.

In our construction we go through a series of reductions from one problem to another. The solution to the last problem gives us our answer.

Pair of trees with large rotation distance

⇕ 1

Pair of polygon triangulations with large flip distance

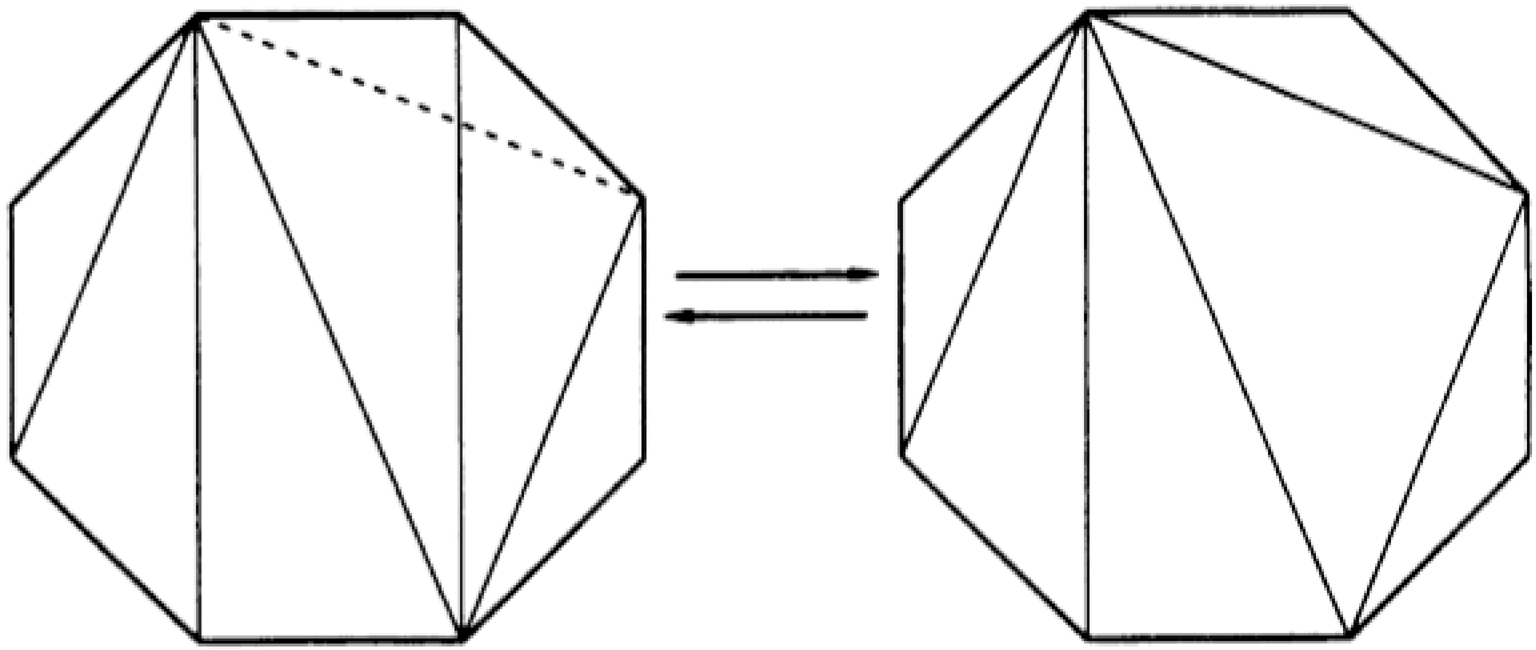
⇑ 2

Polyhedra requiring many tetrahedra

⇑ 3

Hyperbolic polyhedra with large volume

Triangulation of a Polygon: a collection of non-crossing diagonals dividing the polygon into triangles.

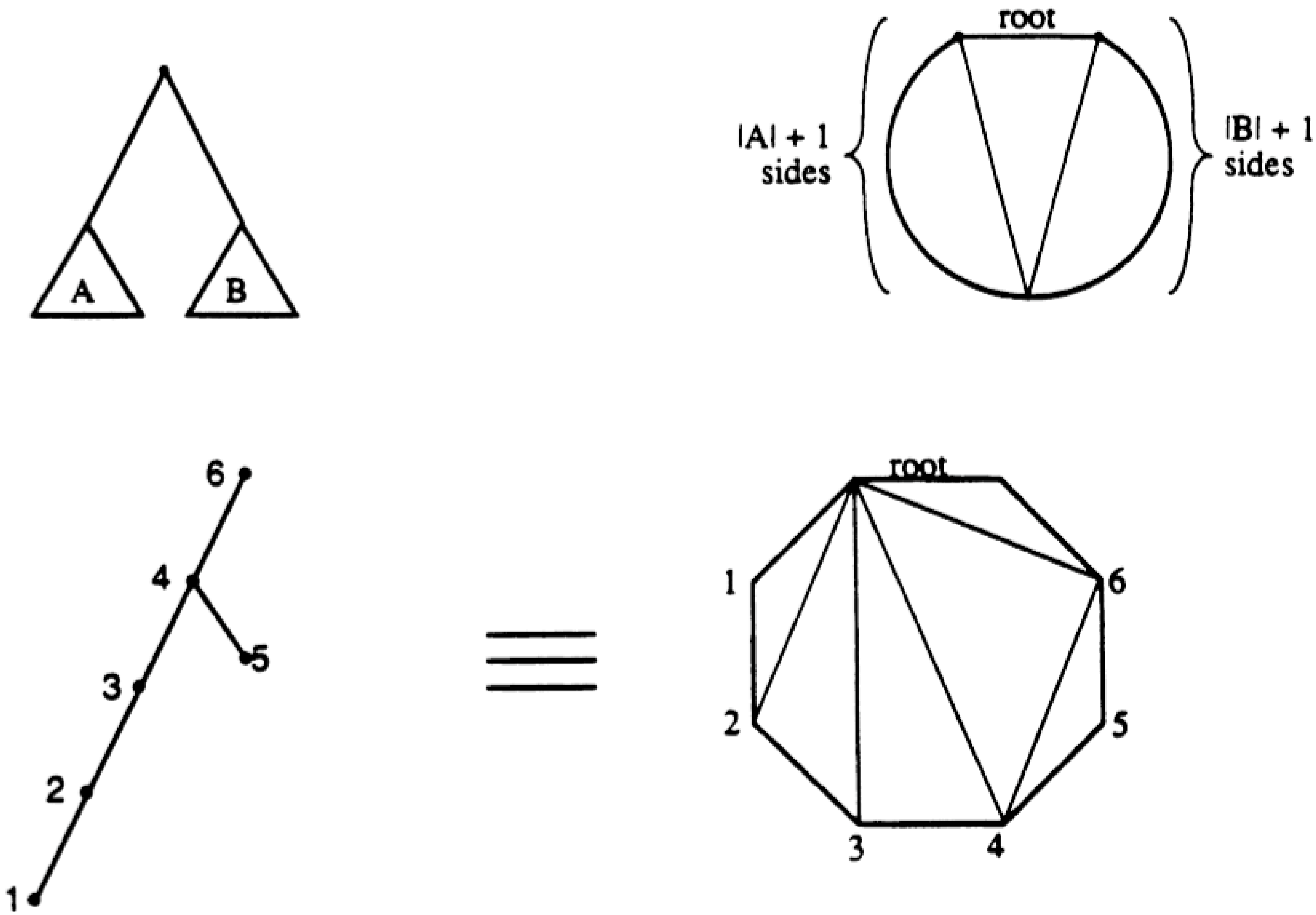


A Diagonal Flip removes a diagonal and divides the resulting quadrilateral in the opposite way.

First Connection

Triangulations of an n-gon under diagonal flips

n-2 node binary trees under rotations


A
B
root
|A| + 1 sides
|B| + 1 sides
6
4
5
3
2
1
root
1
6
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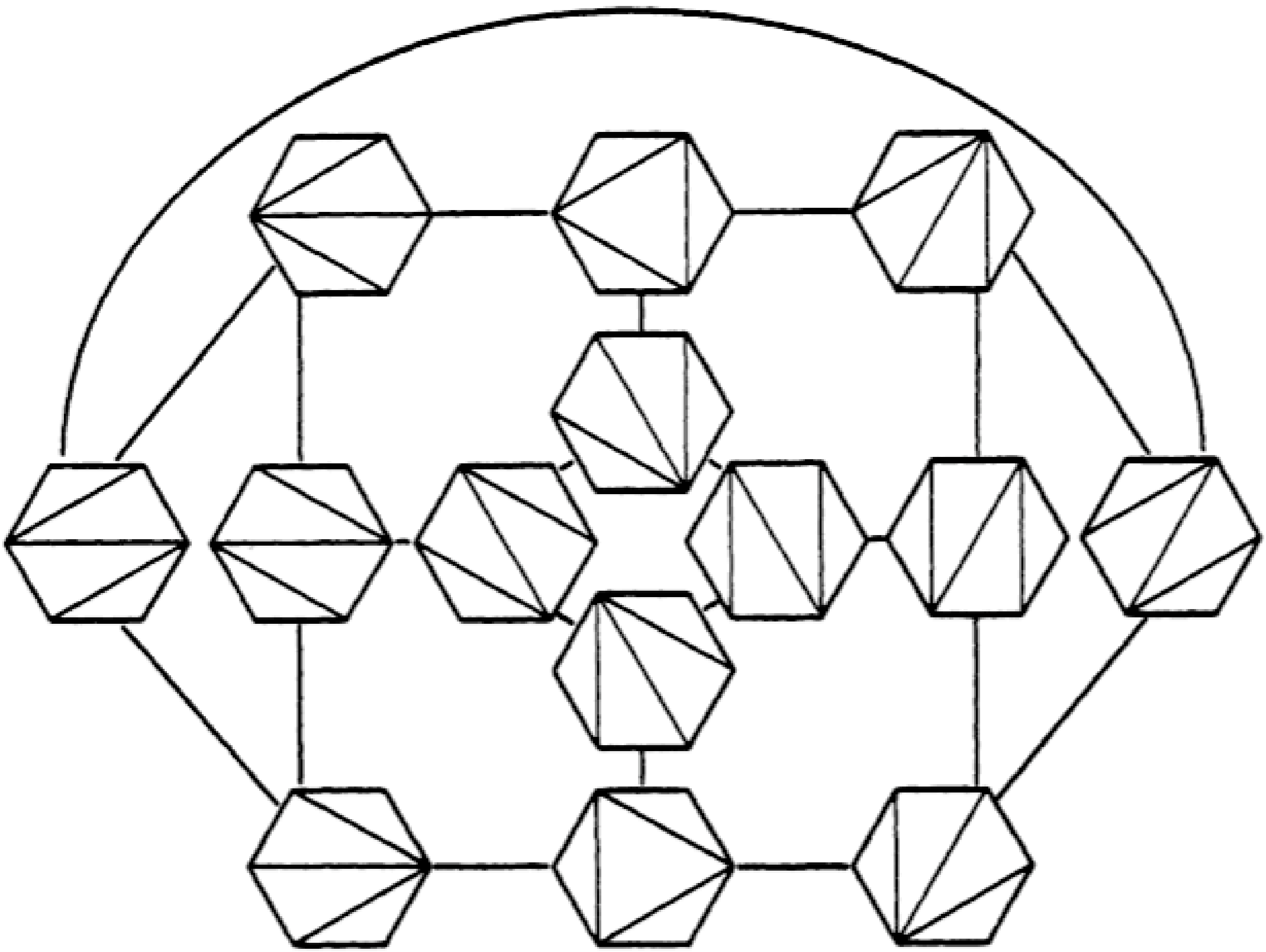
5
3
4

Let d(n) be the distance between the farthest pair of triangulated n-gons. We're interested in analyzing d(n), the diameter of this "rotation graph".



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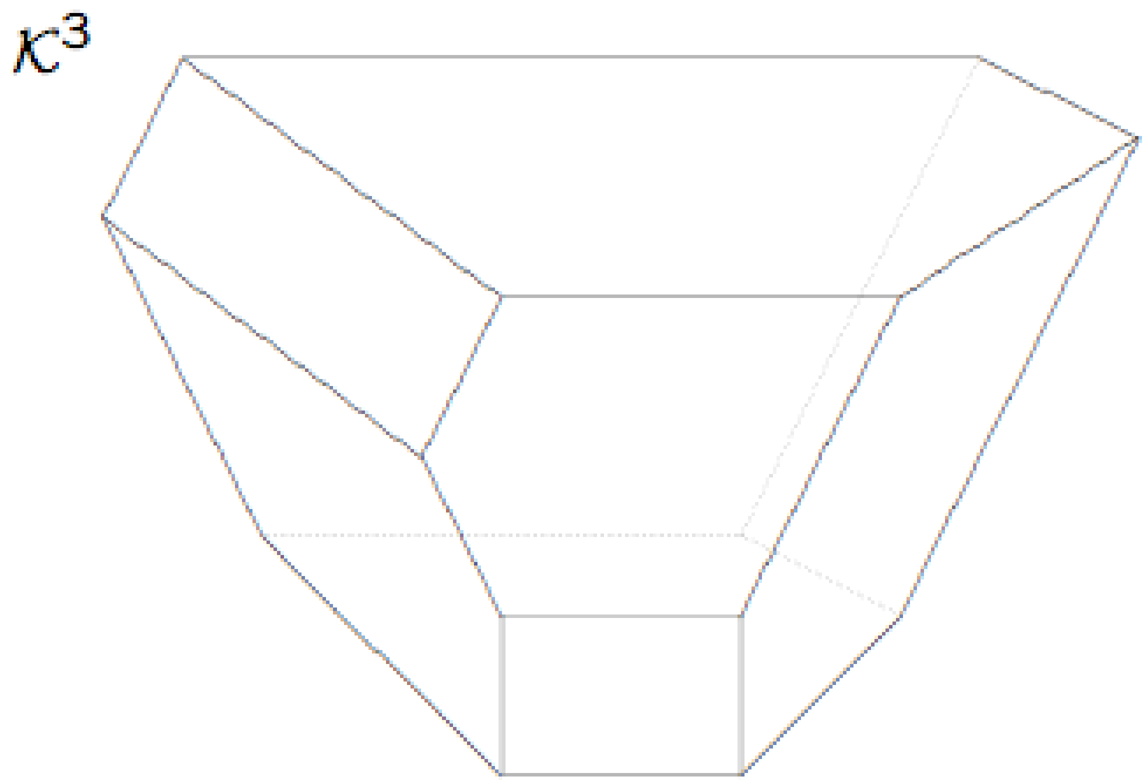
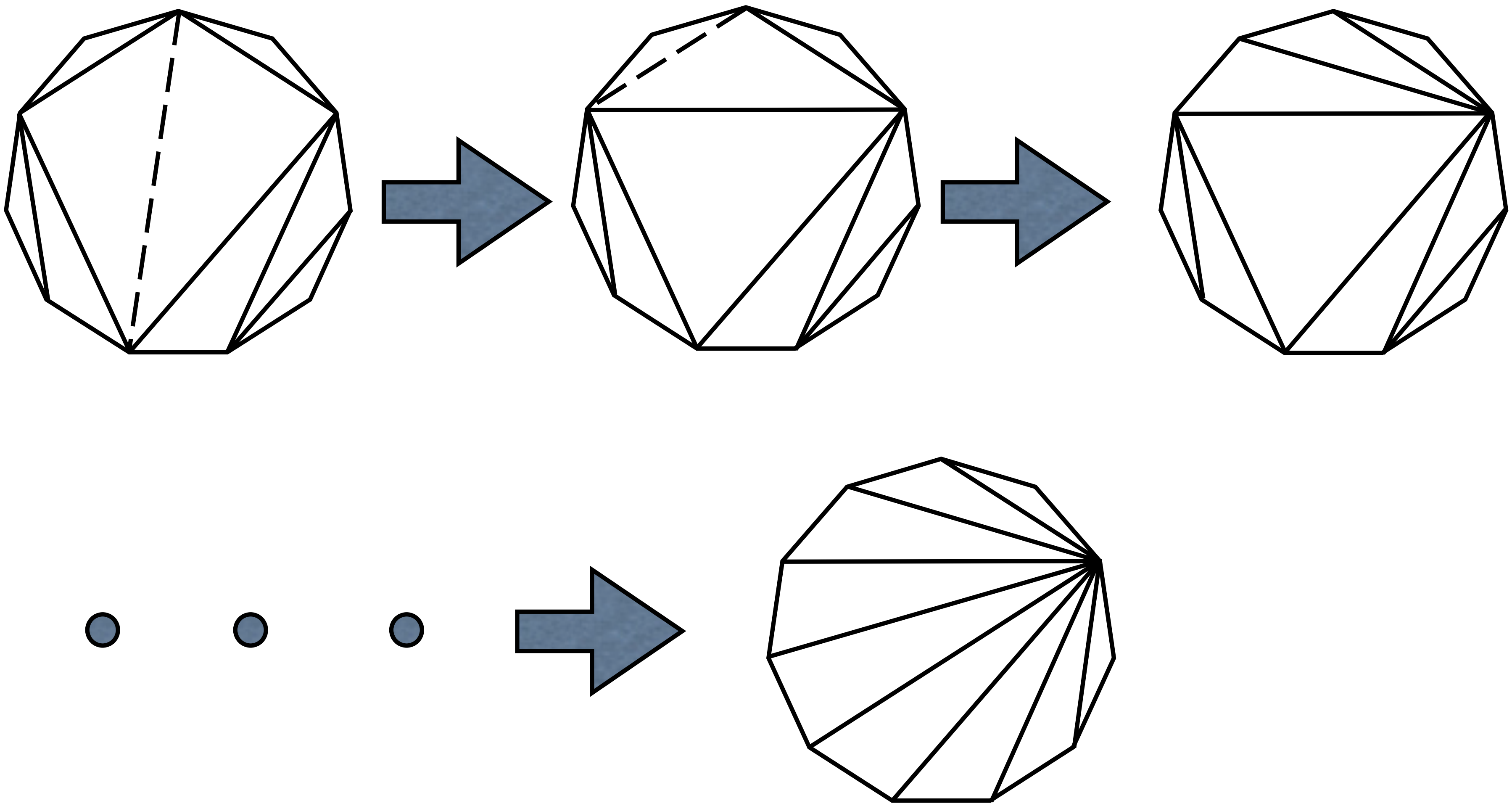


(figure borrowed from Jean-Louis Loday)

Proof that $d(n) \leq 2n-10$ for $n > 12$.



Proof contd.

Pick a vertex.
Cone to that vertex
Reverse the process to get the desired triangulation

Number of flips = $2n-6-d_1(v)-d_2(v)$

The average of $d_1(v)+d_2(v)$ over all v is $4-12/n$.
So for $n>12$ there is a v such that $d_1(v)+d_2(v) \geq 4$.

➔ # of flips is $\leq 2n-10$.

Results of a computer search:

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$d(n)$	0	1	2	4	5	7	9	11	12	15	16	18	20	22	24	26

Recall that our goal is to get lower bounds on d(n).

We move to our next transformation.

Generalized Triangulations

Triangulation of a ball:

A collection of “tetrahedra” and “gluing rules” such that they are homeomorphic to a ball ($r \leq 1$). (A topological or combinatorial object, not geometric.)

Triangulation of a sphere: (three definitions)

1. A collection of triangles and gluing rules homeomorphic to a sphere ($r = 1$).
2. The boundary of a triangulation of a ball.
3. A triangulated embedded planar graph.

Triangulations Continued

Given σ (a triangulation of the sphere) we consider triangulations of the ball whose boundary is σ and all of whose vertices are on σ. We denote such a triangulation by $E(\sigma)$.

One such triangulation is the cone triangulation to a single vertex.

Given σ, let $t(\sigma)$ be the minimum # of tetrahedra in any $E(\sigma)$.

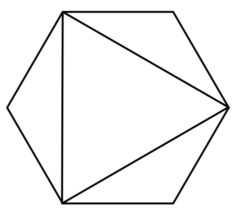
(Note that in 2D the number of triangles needed to triangulate a given polygon is fixed. Not so in 3D.)

Triangulations Continued

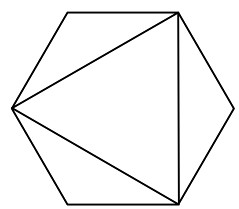
Let τ_1 and τ_2 be triangulations of an n-gon.

Let $U(\tau_1, \tau_2)$ be the triangulation of the sphere obtained by gluing τ_1 and τ_2 together around their boundaries.

Example:

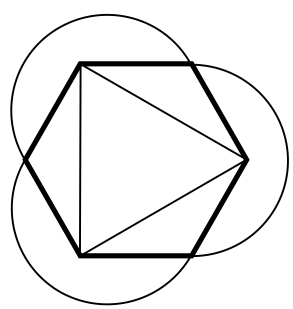


τ_1



τ_2

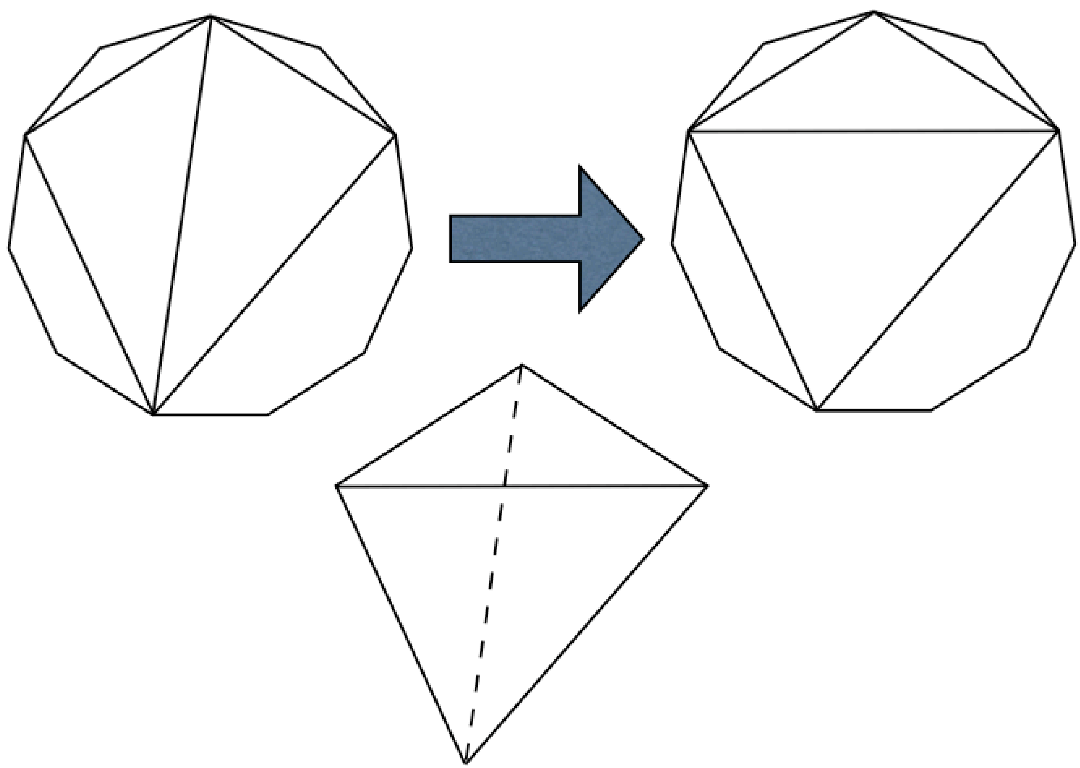
$U(\tau_1, \tau_2)$ =



Second Reduction

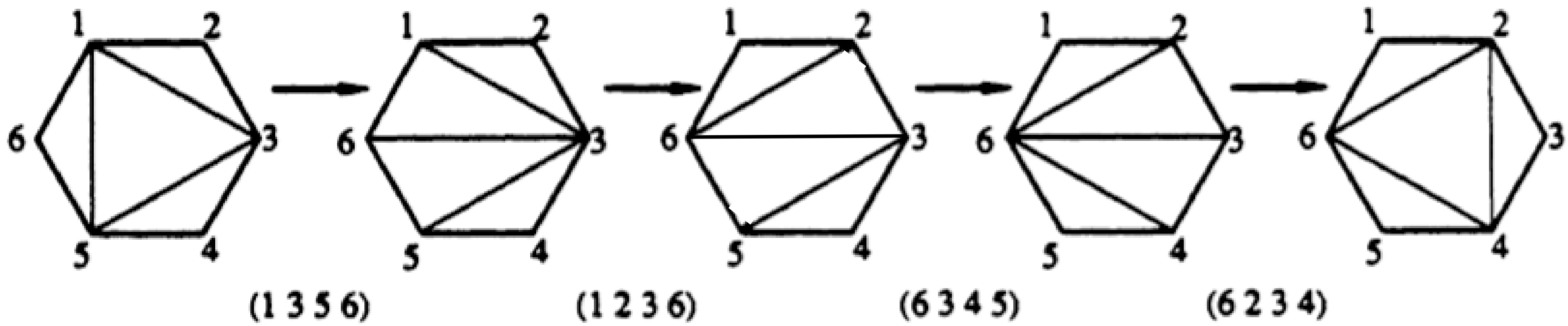
Lemma: $t(U(\tau_1, \tau_2)) \leq d(\tau_1, \tau_2)$

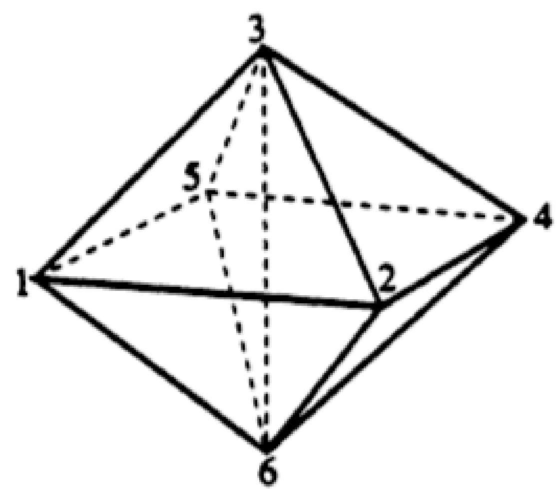
Proof: Given a sequence of *k* diagonal flips converting τ_1 into τ_2 we construct an $E(U(\tau_1, \tau_2))$ using *k* tetrahedra. Here's how:



■

Example





The four tetrahedra: 3 6 1 5, 3 6 2 1, 3 6 4 2, 3 6 5 4

Second Reduction contd.

Let $t(n) = \max_{\substack{\text{n-vertex Hamiltonian} \\ \text{triangulation of the sphere } \sigma}} t(\sigma)$

Corollary: $t(n) \leq d(n)$

Proof: Choose σ s.t. $t(\sigma) = t(n)$.
Cut σ along a Hamiltonian cycle.
This forms τ_1 and τ_2 s.t. $\sigma = U(\tau_1,\tau_2)$

$$t(n) = t(\sigma) = t(U(\tau_1,\tau_2)) \leq d(\tau_1,\tau_2) \leq d(n)$$

■

Now, to obtain lower bounds on d(n) we obtain them on t(n).

Third Reduction: Volumetric Arguments

Let σ be a 4-connected (i.e. Hamiltonian) triangulation of the sphere.

Suppose σ is the boundary of a polyhedron P with n vertices, all on the unit sphere.

Choose T = some $E(\sigma)$.

Draw the tetrahedra of T with straight lines in 3-space.

The union of these tetrahedra = P.

Third Reduction contd.

Let V_Δ be the volume of the biggest tetrahedron inscribed in a unit sphere. Then:

$$\frac{\mathrm{Vol}(P)}{V_\Delta} \leq \begin{array}{l}\text{\# tetrahedra required}\\ \text{To cover P}\end{array} \leq t(n)$$

This gives a lower bound on t(n).

Useless because V_Δ is large compared to Vol(P).

UNLESS we use a weighted form of volume -- heavy near the sphere.

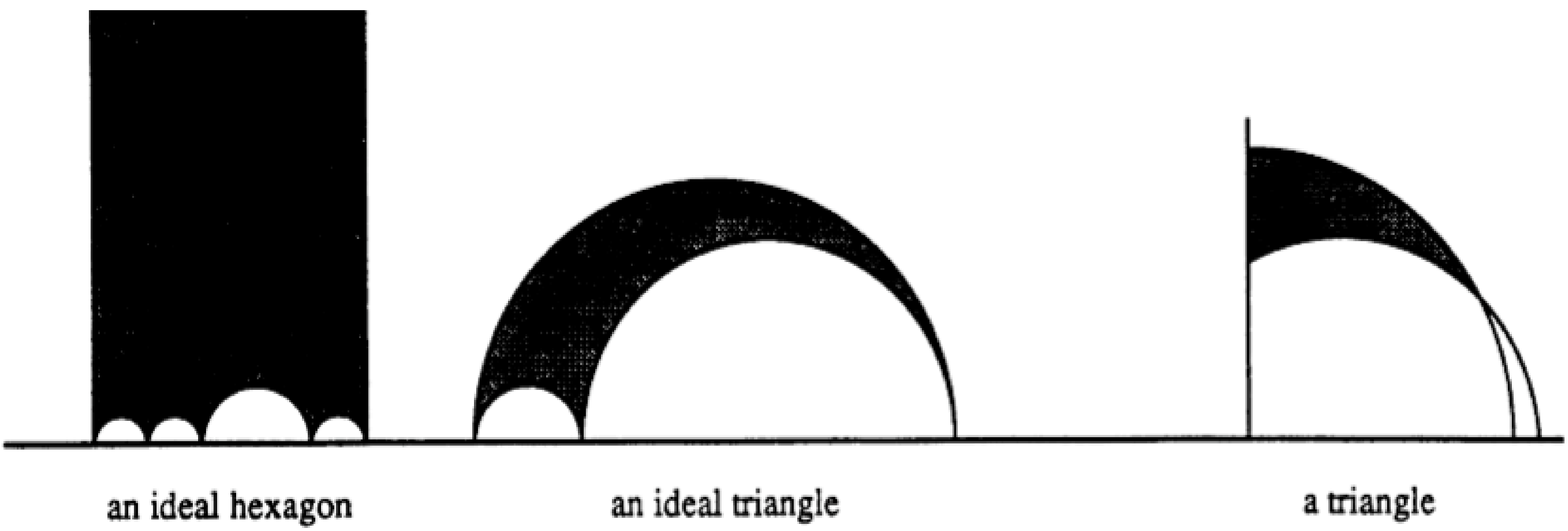
The answer: Use hyperbolic geometry

Hyperbolic Geometry

Many lines parallel to a given line.

Triangle angles sum to < 180^o

Upper half space model:



Geodesics are semi-circles perpendicular to the real axis.

Area of Triangle = π - Σ Angles.

All *ideal* triangles are congruent and have area = π

In 3 Dimensions

An ideal tetrahedron is one with all its vertices on the complex plane or at the point at infinity. All ideal tetrahedra are NOT congruent.

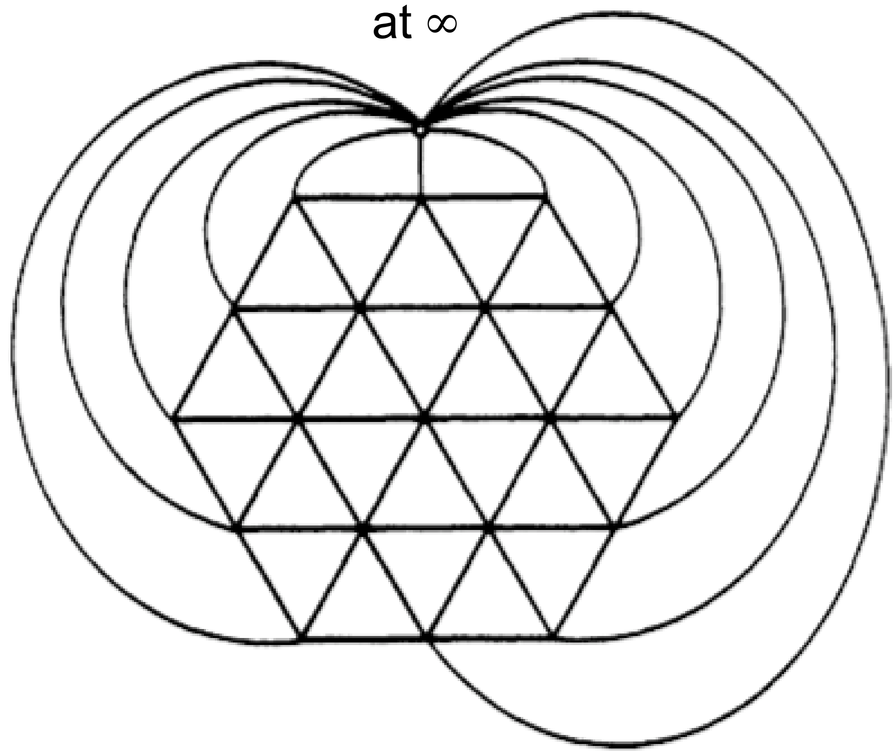
The tetrahedron of maximum volume is the most symmetrical ideal one. It’s volume is $V_0 = 1.0149\ldots$

Finding n-vertex polyhedra of large volume

First solution: volume = $(2n-O(n^{1/2}))\ V_0$

at ∞

In complex plane



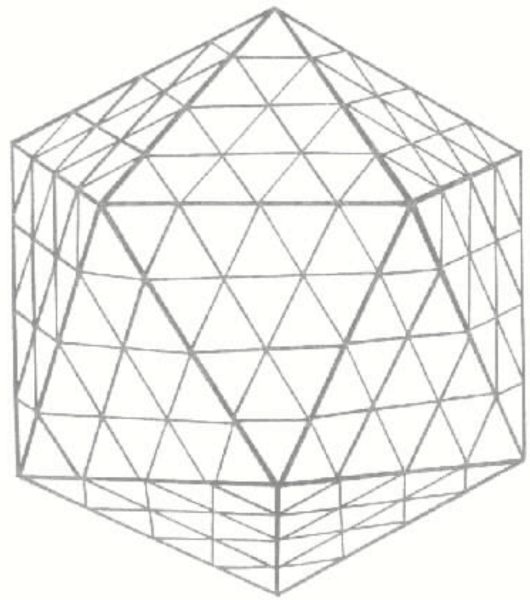
n vertices

2n - 4 faces   (Euler says V-E+F=2. E=3F/2 ➔ F=2V-4)

Volume = $[2n - 4 - (\text{deg at } \infty)]\ V_0 = [2n - O(n^{1/2})]\ V_0$

$2n - O(n^{1/2}) \leq t(n)$

Larger volumes are obtained by subdividing the icosahedron.



This gives a bound of $2n-O(\log(n)) \leq t(n)$

A final induction shows that coneing to a node of degree 6 is minimal. This gives a bound of $2n-4-6 = 2n-10$.

Theorem: $2n-10 \leq t(n)$ for all large n of the form $10k^2+2$.

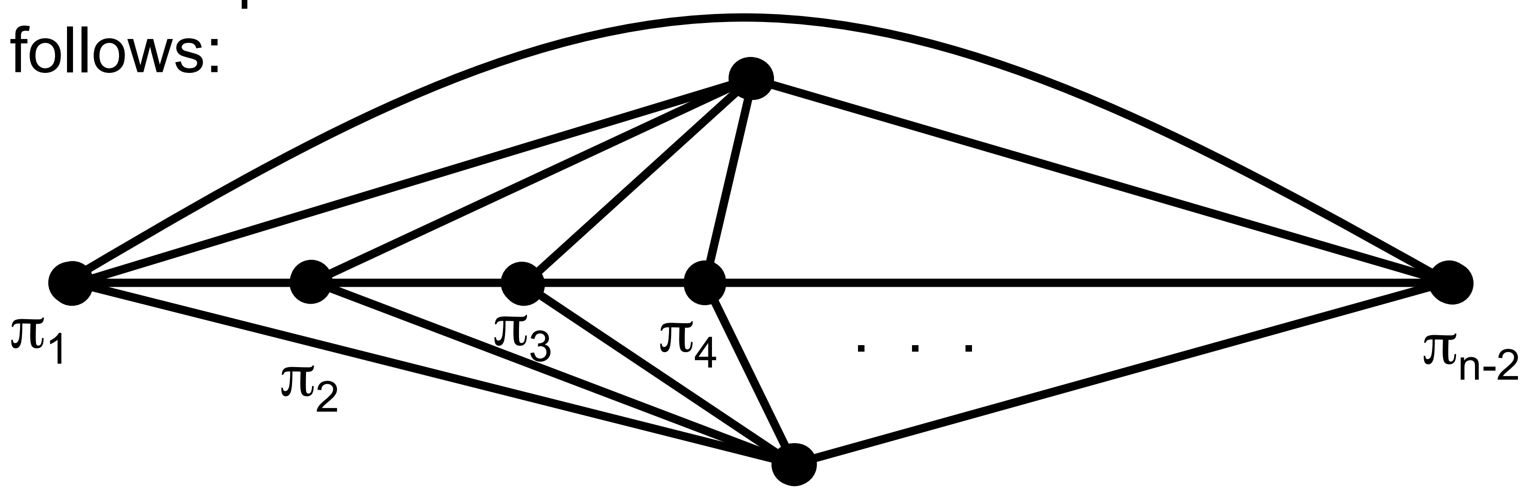
Part 2: Plane Triangulations

Consider the problem of extending our result to *plane triangulations* -- numbered planar triangulated graphs with no self-loops.

We seek lower and upper bounds on the maximum distance between them.

Trivial Lower Bound: *Ω(n) flips are required to transform one n-vertex plane triangulation into another.*

Proof: The number of plane triangulations is $\Omega((n-2)!)$ because a permutation of n-2 elements can be encoded as follows:



By Euler's formula there are 3n-6 edges. Thus, the number of flips is at least

$$\log_{3n-6}(n-2)! = \Omega(n)$$

■

Theorem (Wagner): *$O(n^2)$ flips are sufficient to transform any plane triangulation into any other.*

Theorem: *$O(n \log n)$ flips are sufficient to transform any plane triangulation into any other.*

Proof Outline: Analogous to a sorting algorithm

1. Transform the starting triangulation into one with a Hamiltonian cycle (a wheel).
2. Define the notion of a double wheel. Develop an algorithm to shuffle a double wheel into a wheel in O(n) flips.
3. Sort the vertices in O(log n) shuffle/unshuffle steps

■

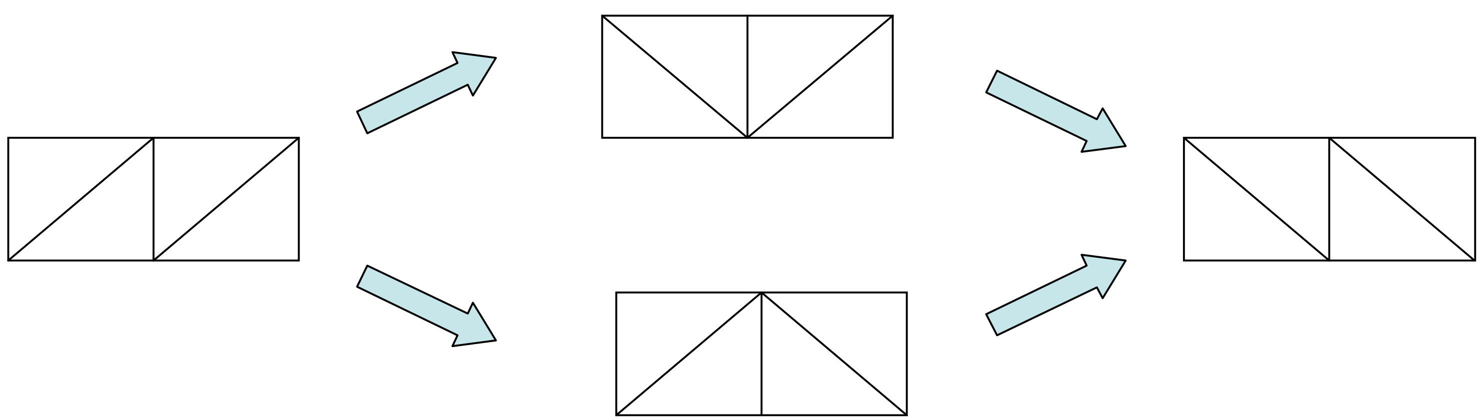
Theorem: *There exist plane triangulations that require Ω(n* log *n) flips to transform one to the other*.

Proof:

KEY OBSERVATION

Consider transformations (sequences) of length m that transform G to G'.

The number of different ones is $(3n-6)^m$. However, many of these sequences may lead to the same outcome. Consider:



KEY OBSERVATION contd

Two sequences are *equivalent* if one can be transformed into the other by reordering *commutative* flips.

All equivalent sequences transform G to G'.

We *encode* G' by encoding a *canonical* sequence transforming G into G'.

Few bits are needed to do this.

Simple counting can then be used to give our lower bound.

Proof contd.

A flip f_i in a sequence of flips $f_1, f_2, \ldots, f_m$ is *ready* if neither f_i nor any of its 4 neighbor edges are involved in a prior flip.

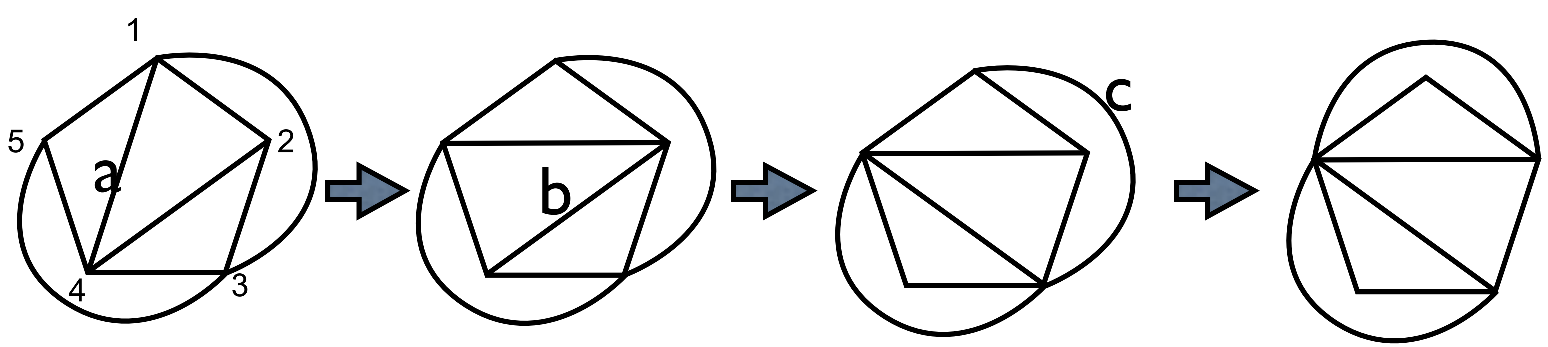
The canonical sequence is formed as follows:

1. Find all the ready flips.

2. Find the lexicographically least one f.
   This is the first of the canonical sequence.

3. Recursively reorder the rest of the sequence.

The new sequence transforms G into G', by induction.



a,c are ready

c, a, b is canonical

Proof contd.

If at all times we knew which flips were ready, we could reconstruct the canonical sequence. (Flip the least ready edge.)

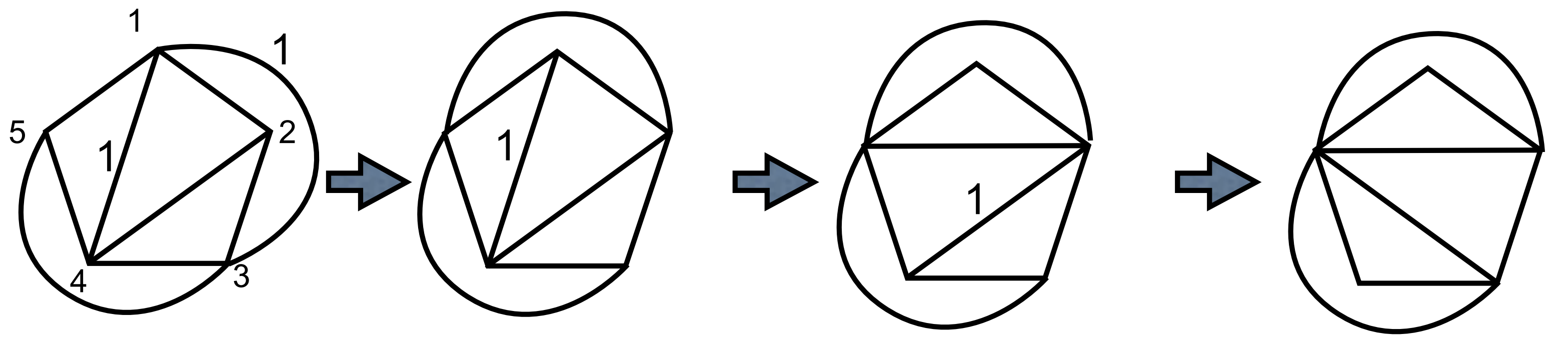
We can encode this information as follows:

- At each time let the *label* of the ready edges be 1, and the others be 0.
- After a flip is done, the labeling is the same, except in the vicinity of the flip. The five edges involved may change.
- The encoding consists of 3n-6+5m bits: 3n-6 for the initial labeling, and 5m for relabeling the neighborhoods of flips.

Given G and these 3n-6+5m bits, G' can be reconstructed.

Example

(0 labels not shown)



Initial labels:	New labels:	New labels:	New labels:
011000000	00000	00001	00000

Proof completed

The number of graphs reachable from G in m (or fewer) flips is at most $2^{3n-6+5m}$.

In order to reach all graphs, m must be such that

$$2^{3n-6+5m} \geq (n-2)!$$

$$3n-6+5m \geq \log(n-2)! = \Omega(n \log n)$$

$$m = \Omega(n \log n)$$

■

Final comments on part 2

The techniques described here are actually far more general.

For example, consider fully parenthesized expressions of n variables $x_1, x_2, \ldots, x_n$ with a binary associative commutative operation. How many moves (applying associativity or commutativity) are required to transform one expression into another? Our techniques allow us to prove that $\Omega(n \log n)$ operators are sometimes required.

References:

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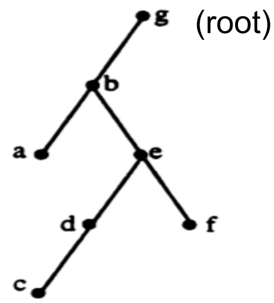

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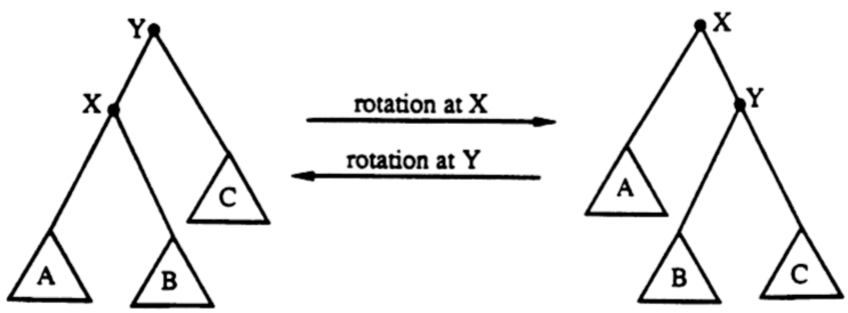

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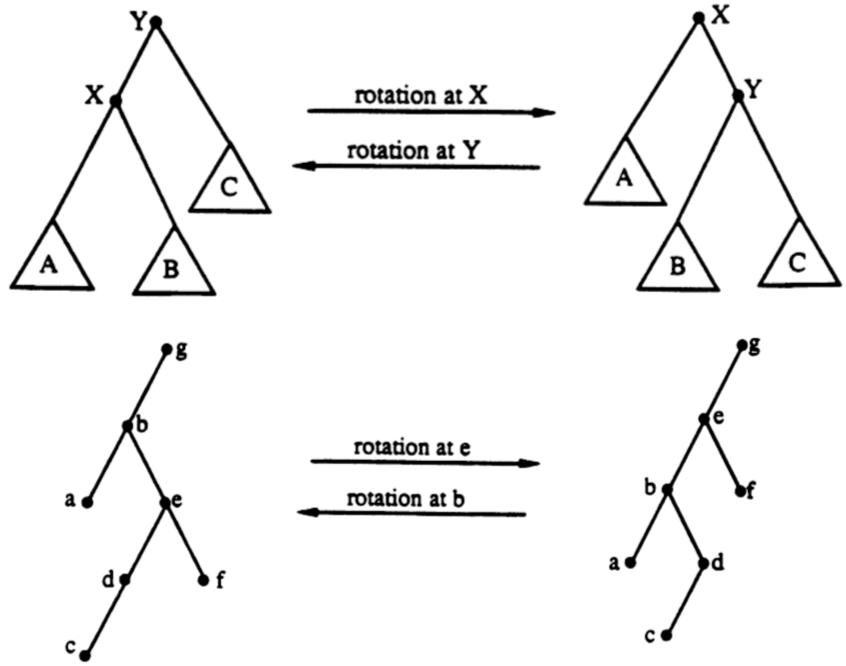

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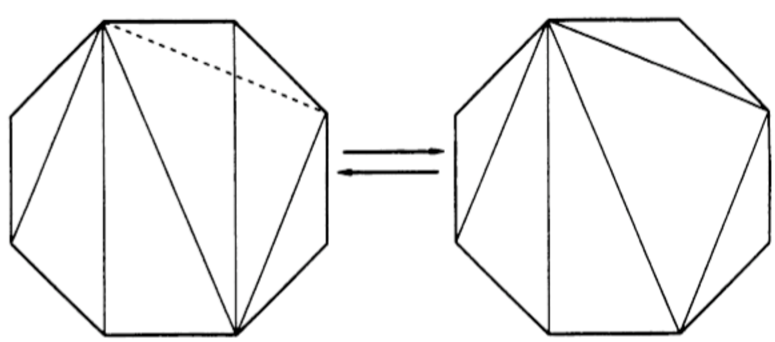

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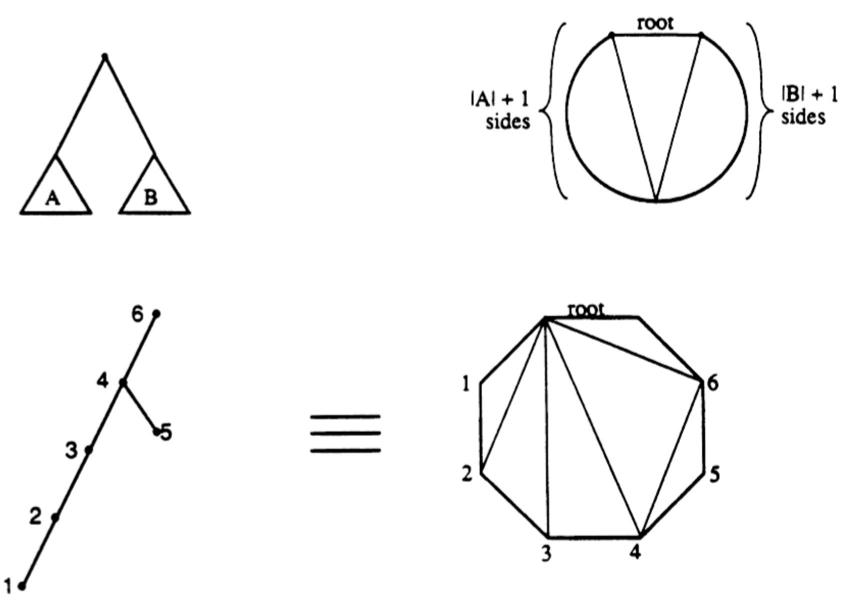
Triangulations of an n-gon under diagonal flips ⟺ n-2 node binary trees under rotations



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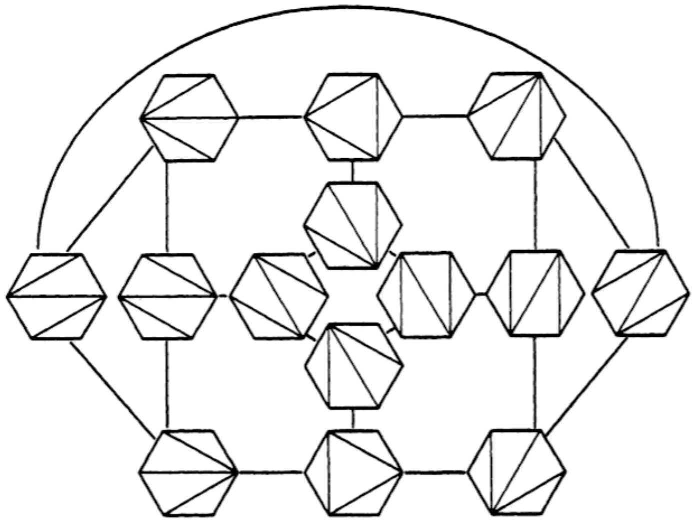

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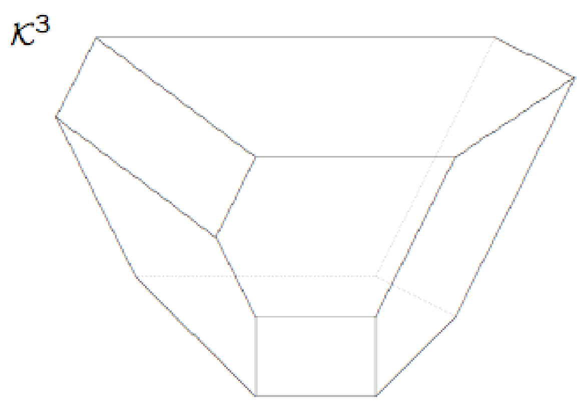

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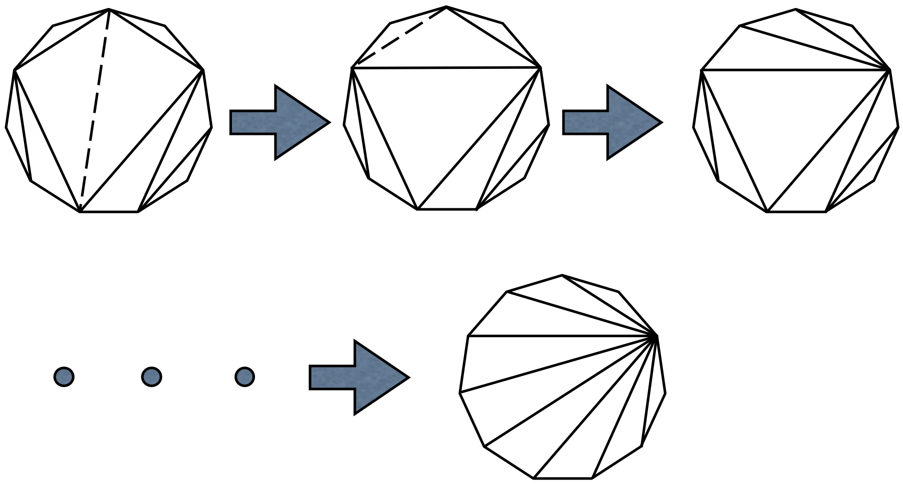

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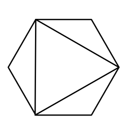
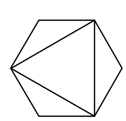
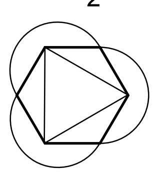


τ_1



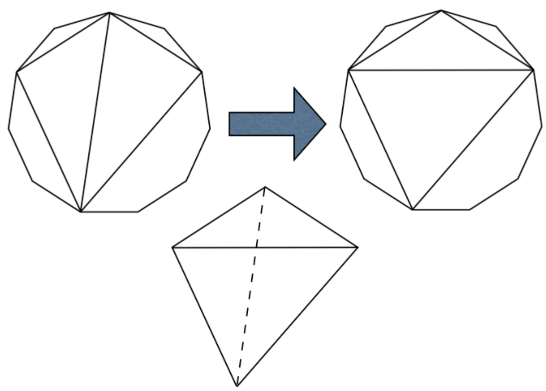
τ_2

$U(\tau_1, \tau_2) =$ 

Second Reduction

Lemma: $t(U(\tau_1, \tau_2)) \leq d(\tau_1, \tau_2)$

Proof: Given a sequence of *k* diagonal flips converting τ_1 into τ_2 we construct an $E(U(\tau_1, \tau_2))$ using *k* tetrahedra. Here's how:



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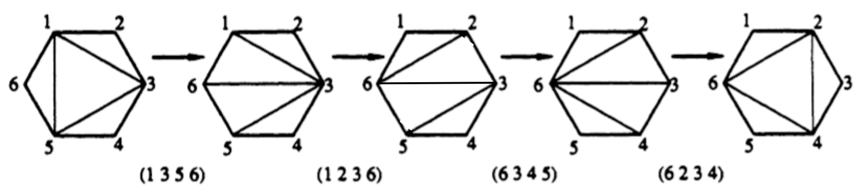
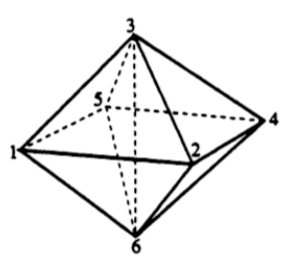
Example





The four tetrahedra: 3 6 1 5, 3 6 2 1, 3 6 4 2, 3 6 5 4

Second Reduction contd.

Let $t(n) = \max_{\substack{\text{n-vertex Hamiltonian} \\ \text{triangulation of the sphere } \sigma}} t(\sigma)$

Corollary: $t(n) \leq d(n)$

Proof: Choose σ s.t. $t(\sigma) = t(n)$.
Cut σ along a Hamiltonian cycle.
This forms τ_1 and τ_2 s.t. $\sigma = U(\tau_1,\tau_2)$

$$t(n) = t(\sigma) = t(U(\tau_1,\tau_2)) \leq d(\tau_1,\tau_2) \leq d(n)$$

■

Now, to obtain lower bounds on d(n) we obtain them on t(n).

Third Reduction: Volumetric Arguments

Let σ be a 4-connected (i.e. Hamiltonian) triangulation of the sphere.

Suppose σ is the boundary of a polyhedron P with n vertices, all on the unit sphere.

Choose T = some $E(\sigma)$.

Draw the tetrahedra of T with straight lines in 3-space.

The union of these tetrahedra = P.

Third Reduction contd.

Let V_Δ be the volume of the biggest tetrahedron inscribed in a unit sphere. Then:

$$\frac{Vol(P)}{V_\Delta} \leq \text{\# tetrahedra required To cover P} \leq t(n)$$

This gives a lower bound on t(n).

Useless because V_Δ is large compared to Vol(P).

UNLESS we use a weighted form of volume -- heavy near the sphere.

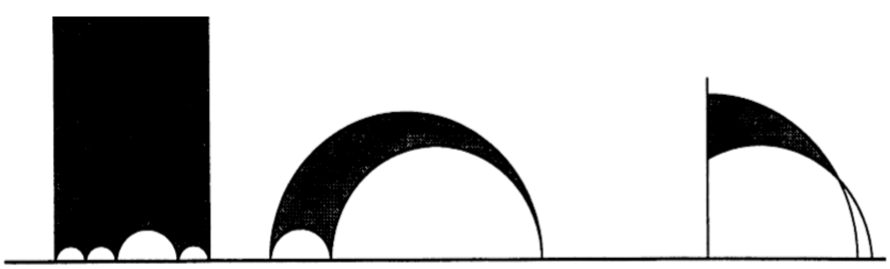
The answer: Use hyperbolic geometry

Hyperbolic Geometry

Many lines parallel to a given line.

Triangle angles sum to < 180^o

Upper half space model:



Geodesics are semi-circles perpendicular to the real axis.

Area of Triangle = $\pi - \Sigma$ Angles.

All *ideal* triangles are congruent and have area = π

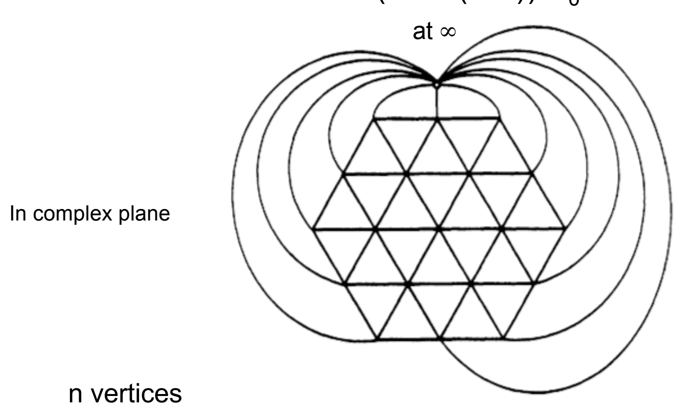
In 3 Dimensions

An ideal tetrahedron is one with all its vertices on the complex plane or at the point at infinity. All ideal tetrahedra are NOT congruent.

The tetrahedron of maximum volume is the most symmetrical ideal one. It's volume is $V_0 = 1.0149\ldots$

Finding n-vertex polyhedra of large volume

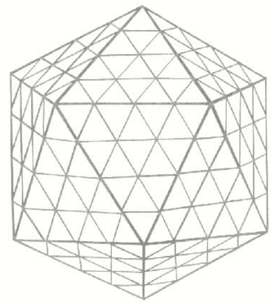
First solution: volume = $(2n-O(n^{1/2}))\ V_0$



n vertices
2n - 4 faces (Euler says V-E+F=2. E=3F/2 ➔ F=2V-4)
Volume = $[2n - 4 - (\text{deg at } \infty)]\ V_0 = [2n - O(n^{1/2})]\ V_0$

$2n - O(n^{1/2}) \leq t(n)$

Larger volumes are obtained by subdividing the icosahedron.



This gives a bound of $2n-O(\log(n)) \leq t(n)$

A final induction shows that coneing to a node of degree 6 is minimal. This gives a bound of $2n-4-6 = 2n-10$.

Theorem: $2n-10 \leq t(n)$ for all large n of the form $10k^2+2$.

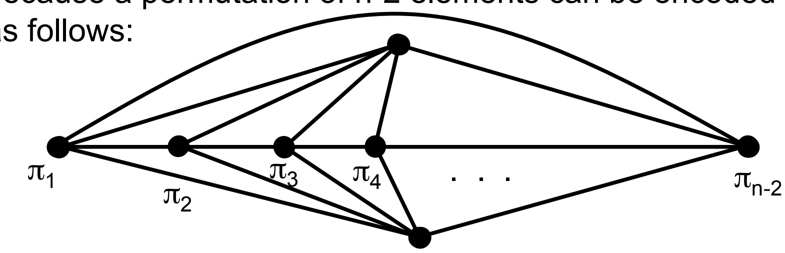
Part 2: Plane Triangulations

Consider the problem of extending our result to *plane triangulations* -- numbered planar triangulated graphs with no self-loops.

We seek lower and upper bounds on the maximum distance between them.

Trivial Lower Bound: *Ω(n) flips are required to transform one n-vertex plane triangulation into another.*

Proof: The number of plane triangulations is $\Omega((n-2)!)$ because a permutation of n-2 elements can be encoded as follows:



By Euler's formula there are 3n-6 edges. Thus, the number of flips is at least

$$\log_{3n-6}(n-2)! = \Omega(n)$$

■

Theorem (Wagner): *$O(n^2)$ flips are sufficient to transform any plane triangulation into any other.*

Theorem: *O(n* log *n) flips are sufficient to transform any plane triangulation into any other.*

Proof Outline: Analogous to a sorting algorithm

1. Transform the starting triangulation into one with a Hamiltonian cycle (a wheel).
2. Define the notion of a double wheel. Develop an algorithm to shuffle a double wheel into a wheel in O(n) flips.
3. Sort the vertices in O(log n) shuffle/unshuffle steps

■

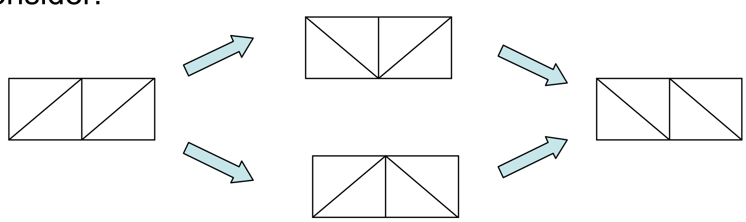
Theorem: *There exist plane triangulations that require Ω(n* log *n) flips to transform one to the other*.

Proof:

KEY OBSERVATION

Consider transformations (sequences) of length m that transform G to G'.

The number of different ones is $(3n-6)^m$. However, many of these sequences may lead to the same outcome. Consider:



KEY OBSERVATION contd

Two sequences are *equivalent* if one can be transformed into the other by reordering *commutative* flips.

All equivalent sequences transform G to G'.

We *encode* G' by encoding a *canonical* sequence transforming G into G'.

Few bits are needed to do this.

Simple counting can then be used to give our lower bound.

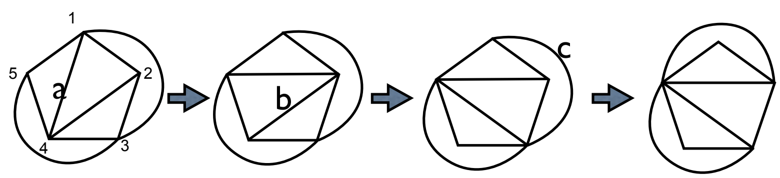
Proof contd.

A flip f_i in a sequence of flips $f_1, f_2, \ldots, f_m$ is *ready* if neither f_i nor any of its 4 neighbor edges are involved in a prior flip.

The canonical sequence is formed as follows:

1. Find all the ready flips.
2. Find the lexicographically least one f.
   This is the first of the canonical sequence.
3. Recursively reorder the rest of the sequence.

The new sequence transforms G into G', by induction.



a,c are ready   c, a, b is canonical

Proof contd.

If at all times we knew which flips were ready, we could reconstruct the canonical sequence. (Flip the least ready edge.)

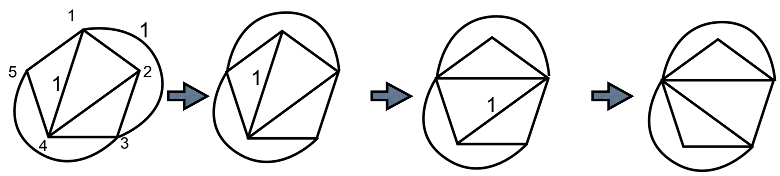
We can encode this information as follows:

- At each time let the *label* of the ready edges be 1, and the others be 0.
- After a flip is done, the labeling is the same, except in the vicinity of the flip. The five edges involved may change.
- The encoding consists of 3n-6+5m bits: 3n-6 for the initial labeling, and 5m for relabeling the neighborhoods of flips.

Given G and these 3n-6+5m bits, G' can be reconstructed.

Example

(0 labels not shown)



Initial labels: 011000000 | New labels: 00000 | New labels: 00001 | New labels: 00000

Proof completed

The number of graphs reachable from G in m (or fewer) flips is at most $2^{3n-6+5m}$.

In order to reach all graphs, m must be such that

$$2^{3n-6+5m} \geq (n-2)!$$

$$3n-6+5m \geq \log(n-2)! = \Omega(n \log n)$$

$$m = \Omega(n \log n)$$

■

Final comments on part 2

The techniques described here are actually far more general.

For example, consider fully parenthesized expressions of n variables $x_1, x_2, \ldots, x_n$ with a binary associative commutative operation. How many moves (applying associativity or commutativity) are required to transform one expression into another? Our techniques allow us to prove that $\Omega(n \log n)$ operators are sometimes required.

References:

[D. D. Sleator, R. E. Tarjan, W. P. Thurston, Rotation Distance, Triangulations, and Hyperbolic Geometry, Journal of the American Mathematical Society, Vo.1, No.3., 1988]

[D. D. Sleator, R. E. Tarjan, W. P. Thurston, Short Encodings of Evolving Structures, SIAM J. DISC. MATH., Vol. 5, No. 3, pp. 428-450, 1992]

Available at www.cs.cmu.edu/~sleator/papers