The crossing number of composite knots

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Let $K$ be a knot in $S^3$.

Its **crossing number** $c(K)$ is the minimal number of crossings in a diagram for $K$.

If $K_1$ and $K_2$ are oriented knots, their **connected sum** $K_1 \# K_2$ is defined by:
Old conjecture: $c(K_1\#K_2) = c(K_1) + c(K_2)$.

- $c(K_1\#K_2) \leq c(K_1) + c(K_2)$ is trivial.
- True when $K_1$ and $K_2$ are alternating [Kauffman], [Murasugi], [Thistlethwaite] - follows from the fact that a reduced alternating diagram has minimal crossing number, which is proved using the Jones polynomial.
- Very little is known in general.

**Theorem:** $[L]$

$$\frac{c(K_1) + c(K_2)}{281} \leq c(K_1\#K_2) \leq c(K_1) + c(K_2).$$
Theorem: [L]

\[
\frac{c(K_1) + \ldots + c(K_n)}{281} \leq c(K_1^\# \ldots \# K_n) \leq c(K_1) + \ldots + c(K_n).
\]

The advantage of this more general formulation is:

We may assume that each \( K_i \) is prime.

Write \( K_i = K_{i,1}^\# \ldots \# K_{i,m(i)} \).

Assuming the theorem for a sum of prime knots:

\[
c(K_1^\# \ldots \# K_n) \geq \frac{\sum_{i=1}^{n} \sum_{j=1}^{m(i)} c(K_{i,j})}{281} \geq \frac{\sum_{i=1}^{n} c(K_i)}{281}.
\]

We may also assume that each \( K_i \) is non-trivial.
**Distant unions**

The distant union $K_1 ⊔ \ldots ⊔ K_n$ of knots $K_1, \ldots, K_n$:

\[ K_1 ⊔ \ldots ⊔ K_n \]

**Lemma:** \[ c(K_1 ⊔ \ldots ⊔ K_n) = c(K_1) + \ldots + c(K_n). \]
Lemma: \( c(K_1 \sqcup \ldots \sqcup K_n) = c(K_1) + \ldots + c(K_n) \).

Proof:

(\( \leq \)): Use minimal crossing number diagrams for \( K_1, \ldots, K_n \) to construct a diagram for \( K_1 \sqcup \ldots \sqcup K_n \).

(\( \geq \)): Let \( D \) be a minimal crossing number diagram of \( K_1 \sqcup \ldots \sqcup K_n \). Use this to construct a diagram \( D_i \) of \( K_i \) by discarding the remaining components. Then

\[
c(K_1 \sqcup \ldots \sqcup K_n) = c(D) \\
\geq c(D_1) + \ldots + c(D_n) \\
\geq c(K_1) + \ldots + c(K_n).
\]
Strategy for the main theorem

Let $D$ be a minimal crossing number diagram of $K_1\# \ldots \# K_n$.

Use this to construct a diagram $D'$ for $K_1 \sqcup \ldots \sqcup K_n$ such that $c(D') \leq 281 \, c(D)$. Then

$$c(K_1) + \ldots + c(K_n) = c(K_1 \sqcup \ldots \sqcup K_n)$$
$$\leq c(D')$$
$$\leq 281 \, c(D)$$
$$= 281 \, c(K_1\# \ldots \# K_n).$$
Creating $K_1 \sqcup \ldots \sqcup K_n$ from $K_1 \# \ldots \# K_n$

Let $K = K_1 \# \ldots \# K_n$.
Let $X = \text{exterior of } K$.
Let $A_1, \ldots, A_n =$ the following annuli:
Creating $K_1 \sqcup \ldots \sqcup K_n$ from $K_1\# \ldots \# K_n$

Let $K = K_1\# \ldots \# K_n$.
Let $X =$ exterior of $K$.
Let $A_1, \ldots, A_n =$ the following annuli:

\[
\begin{array}{c}
A_i \\
\end{array}
\quad
\begin{array}{c}
\quad \\
\end{array}
\quad
\begin{array}{c}
A_n \\
\end{array}
\]

Remove the sub-arcs of $K$ running from $A_i$ to $A_{i+1}$ (mod $n$).
CREATING $K_1 \sqcup \ldots \sqcup K_n$ FROM $K_1\# \ldots \# K_n$

Let $K = K_1\# \ldots \# K_n$.
Let $X = \text{exterior of } K$.
Let $A_1, \ldots, A_n = \text{the following annuli:}$

Remove the sub-arcs of $K$ running from $A_i$ to $A_{i+1}$ (mod $n$).
Add an arc $\alpha_i$ on $A_i$, running between the two boundary components.
(In fact, we do something a bit more complicated than this.)
Creating $D'$ from $D$

$D$ may be complicated.
Creating $D'$ from $D$

$D$ may be complicated.

Hence, the annuli $A_1 \cup \ldots \cup A_n$ might be embedded in a ‘twisted way’
Creating $D'$ from $D$

$D$ may complicated.

Hence, the annuli $A_1 \cup \ldots \cup A_n$ might be embedded in a ‘twisted way’

So, when we add the arcs $\alpha_1 \cup \ldots \cup \alpha_n$, we may introduce new crossings.
We need to control the arcs $\alpha_1 \cup \ldots \cup \alpha_n$.

So, we need to control how the annuli $A_i$ are embedded in $X$ (the exterior of $K$).

For this, we use **normal surface theory**.

This requires a triangulation of $X$.

In fact, we’ll use a handle structure.
A handle structure on $X$ from $D$

Place four 0-handles near each crossing.
Place four 0-handles near each crossing.

Add four 1-handles.
A handle structure from $D$

Place four 0-handles near each crossing.

Add four 1-handles.

Near each edge of the diagram, add two 1-handles.
Add a 2-handle at each crossing.
Add a 2-handle in each region.
Add 2-handles along each ‘over-arc’ and ‘under-arc’ of the diagram. Finally, add 3-handles above and below the diagram.
We now have a handle structure on $X$, the exterior of $K$.
Let $A = A_1 \cup \ldots \cup A_n$.
We want to ambient isotope $A$ into normal form ...
Because $A$ is incompressible and $\partial$-incompressible, we may isotope it into normal form.

ie. each component of intersection with the handles looks like:

![Diagram showing 0-handle, 1-handle, and 2-handle](image)

In addition, each component of intersection with the 0-handles satisfies certain conditions.

→ only finitely many normal ‘disc types’.
An example of a normal disc type:
Recall that we must add an embedded arc $\alpha_i$ on each annulus $A_i$, running between the two boundary components.

Any such arc will do.

We may arrange that

- $\alpha_1 \cup \ldots \cup \alpha_n$ misses the 2-handles
- respects the product structure on the 1-handles.

Pick $\alpha_i$ so that it has minimal length (subject also to an extra condition).

This implies that it intersects each normal disc in at most one arc.
The diagram $D'$

Inserting the arcs $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$ gives a diagram $D'$ for $K_1 \sqcup \ldots \sqcup K_n$:

Its crossings come in 3 types.
What we’re aiming for

Inserting the arcs $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$ gives a diagram $D'$ for $K_1 \sqcup \ldots \sqcup K_n$:

$$= D'$$

- closely follows a subset of $K$
- bounded number of parallel arcs $K$-
- $\alpha$-$K$ crossings
- $K$-$K$ crossing
- one of finitely many tangles, $\alpha$-$\alpha$ crossings
Proof of the main theorem

$D'$ has 3 types of crossings:

- $K - K$ crossings $\leq c(D)$
- $\alpha - K$ crossings $\leq 4c(D) \times 6$
- $\alpha - \alpha$ crossings $\leq 4c(D) \times 64$

TOTAL $\leq 281 \ c(D)$
Wishful thinking: We’d be done if we could arrange that $A$ intersected each handle in one of finitely many possible configurations.

But this probably isn’t possible.

However, there are only finitely many configurations of disc types.

Key Claim 1. $\alpha$ can be chosen to run over at most 2 normal discs of each disc type in each handle.

This $\Rightarrow \alpha$ intersects each handle in one of finitely many possible configurations, and we’re done.
**Parallelity bundles**

What if a handle contains more than one copy of a normal disc? Then, any two such discs are normally parallel.

Between any two normally parallel adjacent discs, there is a copy of $D^2 \times I$.

These patch together to form an $I$-bundle embedded in the exterior of $A$, called a ‘parallelity bundle’ $B$. 
**Some terminology**

Cut $X$ along the annuli $A$.

Throw away the component with a copy of $A$ in its boundary.

Let $M$ be the rest.

Then $M = X_1 \sqcup \ldots \sqcup X_n$, where each $X_i$ is the exterior of $K_i$.

Let $S$ be the copy of $A$ in $M$.

$M$ inherits a handle structure from the handle structure on $X$.

The space between two adjacent normal discs of $A$ becomes a ‘parallelity handle’ of $M$. 
Claim 2. We may pick $\alpha$ so that it misses the parallelity bundle $\mathcal{B}$.

This $\Rightarrow$ Claim 1, because if a normal disc of $A$ has parallel copies on both sides, it lies in a parallelity handle of $M$.

In fact, it is convenient to enlarge $\mathcal{B}$ to a larger $I$-bundle $\mathcal{B}'$.

We’ll arrange for $\alpha$ to miss $\mathcal{B}'$ and hence $\mathcal{B}$. 
A generalised parallelity bundle

is a 3-dimensional submanifold $\mathcal{B}'$ of $(M, S)$ such that

- $\mathcal{B}'$ is an $I$-bundle over a compact surface $F$;
- the $\partial I$-bundle is $\mathcal{B}' \cap S$;
- $\mathcal{B}'$ is a union of handles;
- any handle in $\mathcal{B}'$ that intersects the $I$-bundle over $\partial F$ is a parallelity handle;
- $\text{cl}(M - \mathcal{B}')$ inherits a handle structure.

The $\partial I$-bundle is the horizontal boundary of $\mathcal{B}'$.
The $I$-bundle over $\partial F$ is the vertical boundary.
Claim 3: Possibly after modifying its handle structure, \((M, S)\) contains a generalised parallelity bundle \(\mathcal{B}'\) such that:

- \(\mathcal{B}'\) contains every parallelity handle of \(M\);
- \(\mathcal{B}'\) is a collection of \(I\)-bundles over discs.

The horizontal boundary is a union of discs in the annuli \(A\).
\[\Rightarrow\] it cannot separate the two components of \(\partial A_i\).
\[\Rightarrow\] \(\alpha\) can be chosen to miss \(\mathcal{B}'\)
\[\Rightarrow\] Claim 2.
How to prove Claim 3

Recall: $\mathcal{B}$ is the parallelity bundle.

Enlarge this to a maximal generalised parallelity bundle $\mathcal{B}'$.

**Claim 4:** The $\partial I$-bundle of a maximal generalised parallelity bundle is incompressible.

**Main idea of proof:**

If the $\partial I$-bundle were compressible, then we would (probably) get an arrangement like:
\[\rightarrow\text{Enlarge } \mathcal{B}' \text{ over } (\text{disc}) \times I \text{ region.}\]

Contradicts maximality of \(\mathcal{B}'\). \(\square\) Claim 4.

Because the horizontal boundary of \(\mathcal{B}'\) is an incompressible subsurface of \(A\), it is a union of discs and annuli parallel to core curves.
The corresponding component of $\mathcal{B}'$ is an $I$-bundle over an annulus. Its vertical boundary is two incompressible annuli in $M$.

Since each $K_i$ is prime, these are boundary parallel in $M$.

Keep applying this sort of modification to the handle structure $\rightarrow$

Claim 3. □ Main Theorem.
**Satellite knots**

Conjecture: \( c(K) \geq c(L) \).

Theorem: [L] There is a universal computable constant \( N \geq 1 \) with following property. If \( K \) is a non-trivial satellite knot, with companion knot \( L \), then \( c(K) \geq c(L)/N \).