Complete affine 3-manifolds and hyperbolic surfaces

Dedicated to Bill Thurston on his 60th birthday

William M. Goldman

Department of Mathematics University of Maryland

Geometry and the Imagination, June 11, 2007

Three-dimensional affine space forms

- When can a group G act on Euclidean space with quotient a manifold?
- When G acts by isometries, it is a finite extension of a free abelian group, and the various actions are easily classified.
- However when the action of G is only assumed to be affine, the classification is still open.
- The most interesting cases were discovered by Margulis in the early 1980's and occur when the quotient M is noncompact and G is a nonabelian free group. G acts by Lorentz isometries of $\mathbb{E}^{2+1}$ and the clasification closely relates to *hyperbolic structures* on noncompact surfaces.

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Milnor's Question (1977)

"On fundamental groups of complete affinely flat manifolds" (Adv. Math. 25, 178–187.)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

If not, a complete affine 3-manifold is an iterated fibration where the fibers are either cells or circles. In particular every compact 3-manifold quotient $\mathbb{R}^3/\Gamma$, where $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ is finitely covered by a torus bundle over S^1, that is, a geometric 3-manifold of type **Euc**, **Nil** or **Sol**.

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Evidence?

Milnor offers the following results as possible "evidence" for a negative answer to this question.

- *A connected Lie group admits a proper affine action $\Longleftrightarrow$ it is amenable (compact-by-solvable).*
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Idea for a counterexample

- Clearly a geometric problem, since free groups act properly by isometries on H^3 (Schottky 1907), and hence by diffeomorphisms on $\mathbb{E}^3$
- — but these actions are *not* affine.
- Milnor suggests:
  - *"Start with a free discrete subgroup of* $O(2,1)$ *and add translation components to obtain a group of affine transformations which acts freely. However it seems difficult to decide whether the resulting group action is properly discontinuous."*

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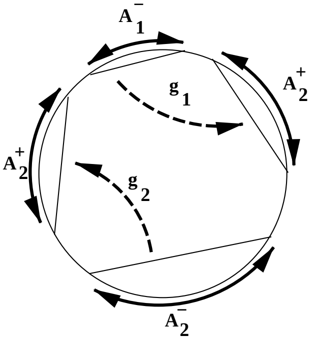
A Schottky group



- Generators g_1, g_2 pair half-spaces $A_i^- \longrightarrow H^2 \setminus A_i^+$.
- g_1, g_2 *freely* generate *discrete* group.
- Action proper with fundamental domain $H^2 \setminus \bigcup A_i^\pm$.

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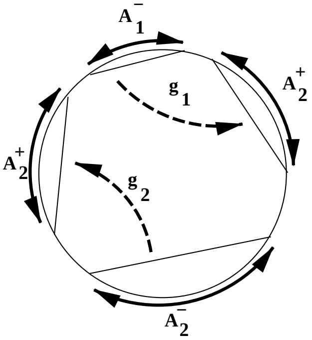

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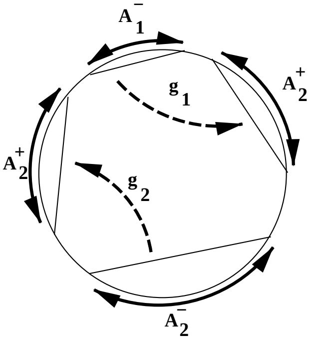

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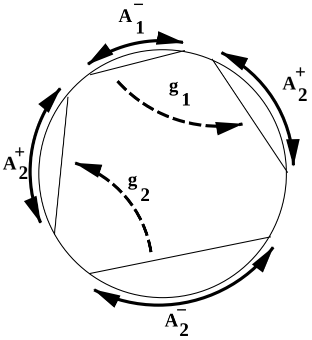

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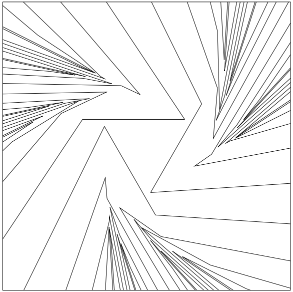

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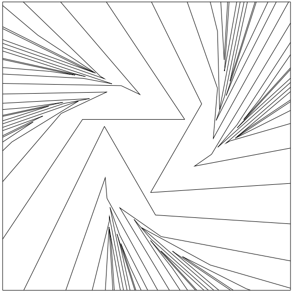

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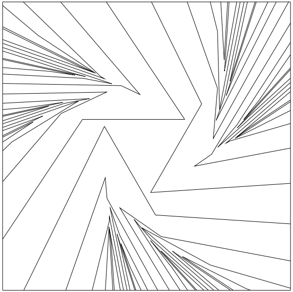

Flat Lorentz manifolds

Suppose that $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ acts properly and is not polycyclic.

- Let $\Gamma \xrightarrow{\mathbb{L}} \mathrm{GL}(3,\mathbb{R})$ be the *linear holonomy homomorphism.* Then:
  - $\mathbb{L}(\Gamma)$ is (conjugate to) a *discrete* subgroup of $\mathrm{O}(2,1)$;
  - $\mathbb{L}$ is injective. (Fried-Goldman 1983).
- Thus the associated complete hyperbolic surface.

$$\Sigma := \mathrm{H}^2/\mathbb{L}(\Gamma)$$

is homotopy-equivalent to $M = \mathbb{E}^{2,1}/\Gamma$.
  - Σ is not compact (Mess 1990).
- Thus Γ must be a free group and Milnor's construction is the *only way* to construct examples.

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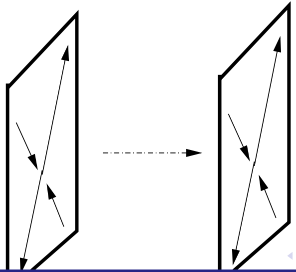
Cyclic groups

- Most elements $\gamma \in \Gamma$ are *boosts*, affine deformations of hyperbolic elements of $O(2,1)$. A fundamental domain is the *slab* bounded by two parallel planes.
- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a *closed geodesic*. Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.



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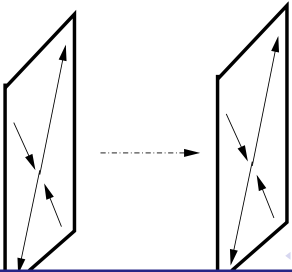

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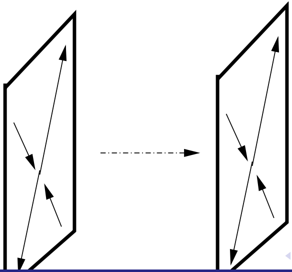

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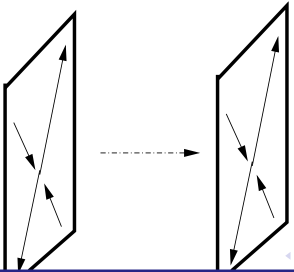
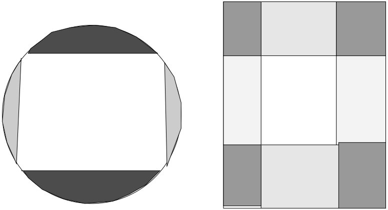


Slabs don't work!



- In H^2, the half-spaces $A_i^{\pm}$ are disjoint;
- Their complement is a fundamental domain.
- In affine space, half-spaces disjoint $\Rightarrow$ parallel!
- Complements of slabs always intersect,
- Unsuitable for building Schottky groups!

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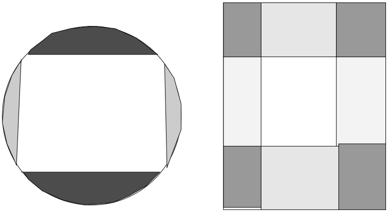

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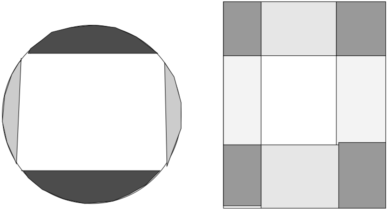

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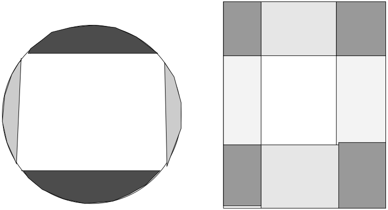

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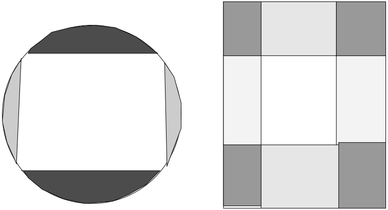

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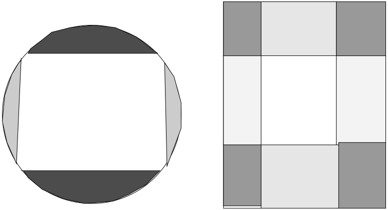

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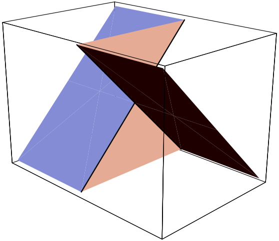
Crooked Planes (Drumm 1990)

- *Crooked Planes:* Flexible polyhedral surfaces bound fundamental polyhedra for free affine groups.



- Two null half-planes connected by lines inside light-cone.

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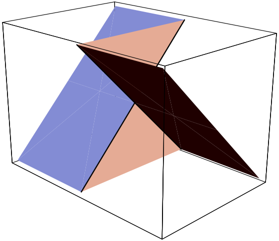

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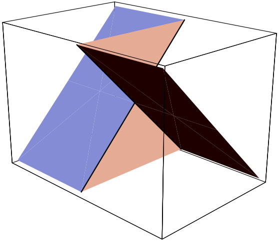

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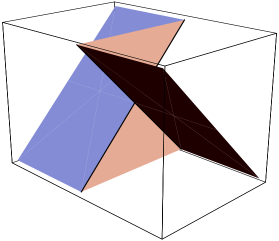

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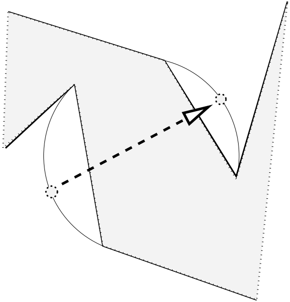
Crooked polyhedron for a boost



- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $\mathbb{E}^{2,1}$;
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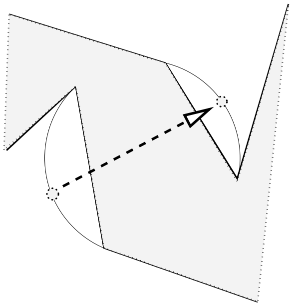

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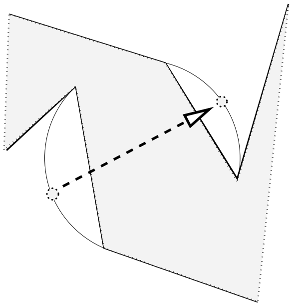

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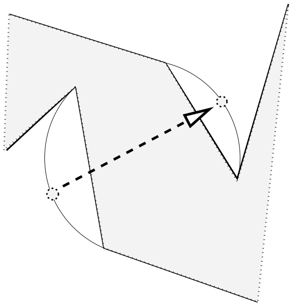
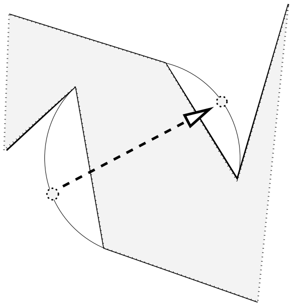


- Start with a *hyperbolic slab* in H^2.
- Extend into light cone in $\mathbb{E}^{2,1}$;
- Extend outside light cone in $\mathbb{E}^{2,1}$;
- Action proper except at the origin and two null half-planes.

Crooked polyhedron for a boost



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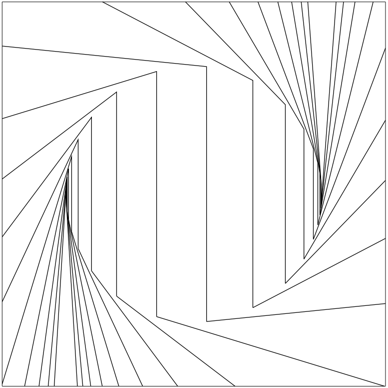
Images of crooked planes under a linear cyclic group



The resulting tesselation for a linear boost.

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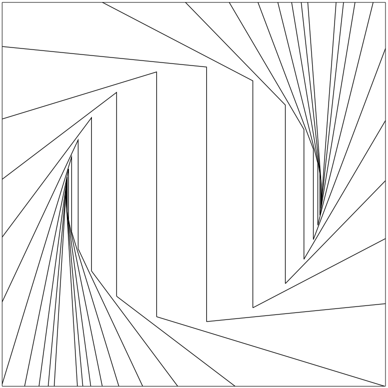

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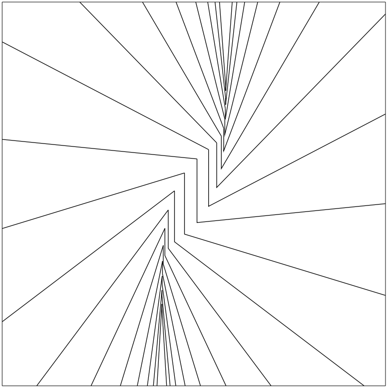
Images of crooked planes under an affine deformation



- Adding translations frees up the action
- — which is now proper on *all* of $\mathbb{E}^{2,1}$.

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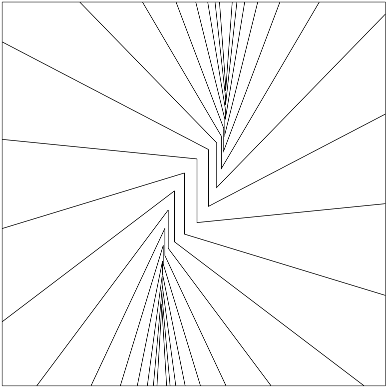

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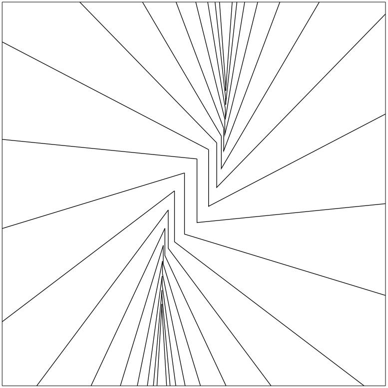

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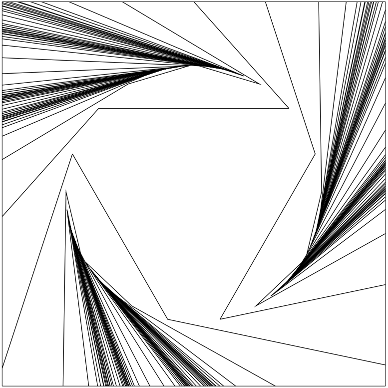
Linear action of Schottky group



Crooked polyhedra tile H^2 for subgroup of $O(2, 1)$.

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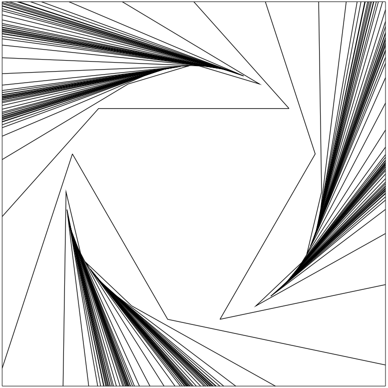

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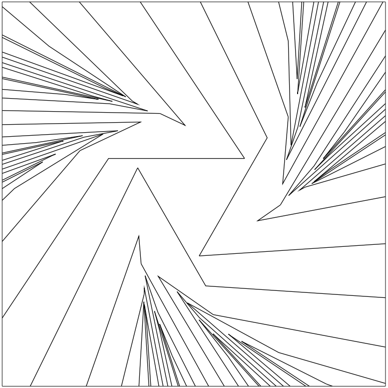

Carefully chosen affine deformation acts properly on $\mathbb{E}^{2,1}$.

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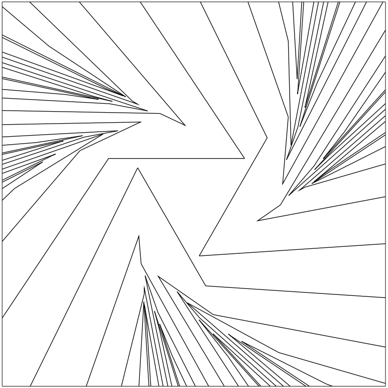

Carefully chosen affine deformation acts properly on $\mathbb{E}^{2,1}$.

The linear part

- Mess's theorem is the only obstruction for the existence of a proper affine deformation:
- (Drumm) Let Σ be a *noncompact* complete hyperbolic surface. Then its holonomy group admits a proper affine deformation and M^3 is a solid handlebody.
- Proof: Extend Schottky fundamental domains for Σ to crooked fundamental domains on $\mathbb{E}^{2,1}$.
- Characterize all proper affine deformations of a non-cocompact Fuchsian group

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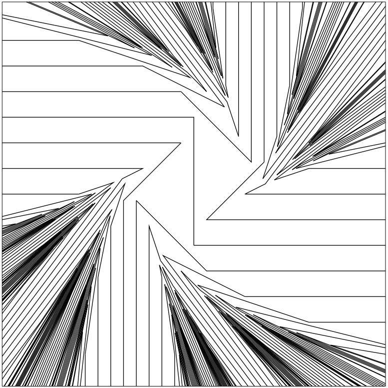

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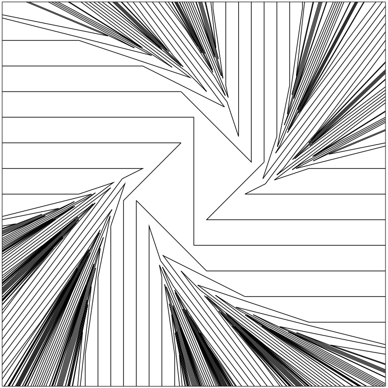

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Margulis's invariant

$\forall$ affine deformation $\Gamma \xrightarrow{\rho} \mathrm{Isom}(\mathbb{E}^{2,1})^0$, $\exists$ fixed eigenvector x^0_γ for $\mathbb{L}(\gamma)$ such that

$$\Gamma \xrightarrow{\alpha_u} \mathbb{R}$$
$$\gamma \longmapsto \langle u(\gamma), x^0_\gamma \rangle$$

satisfies:

- α_u is a class function on Γ;
- $\alpha_u(\gamma^n) = |n|\alpha_u(\gamma)$;
- When ρ acts properly, $|\alpha_u(\gamma)|$ is the *Lorentzian length* of the closed geodesic in M^3 corresponding to γ;
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Affine deformations

- Start with a Fuchsian group $\Gamma_0 \subset \mathsf{O}(2,1)$. An *affine deformation* is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$

determined by its translational part

$$u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}).$$

- Conjugating ρ by a translation $\Longleftrightarrow$ adding a coboundary to u.
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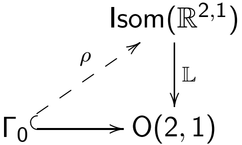

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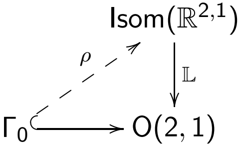

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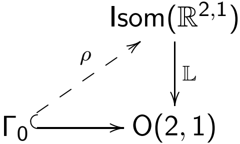

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Deformations of hyperbolic structures

- $[u]$ corresponds to an *infinitesimal deformation* Σ_t of the hyperbolic structure on Σ.
- Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\left.\frac{d}{dt}\right|_{t=0} \ell_{\Sigma_t}(\gamma)$$

where $\ell_{\Sigma_t}(\gamma)$ is the length of the closed geodesic on Σ_t corresponding to γ.
- Γ_u is proper $\Longrightarrow$ all closed geodesics lengthen (or shorten) under the deformation Σ_t.
- The converse is true $\Longleftrightarrow$ Σ is homeomorphic to a three-holed sphere, one-holed Klein bottle or two-holed projective plane. (Charette-Drumm-Goldman-Jones)

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Σ is a three-holed sphere

- For each $\gamma \in \Gamma_0$, the functional

$$H^1(\Gamma_0, \mathbb{R}^{2,1}) \xrightarrow{\alpha^\gamma} \mathbb{R}$$
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detects the rate of lengthening of γ under the deformation corresponding to $[u]$.

- If $\alpha^\gamma(u) > 0 \ \forall \gamma \subset \partial\Sigma$, then a crooked fundamental domain exists and Γ_u is proper.
- M^3 is a solid handlebody of genus two.
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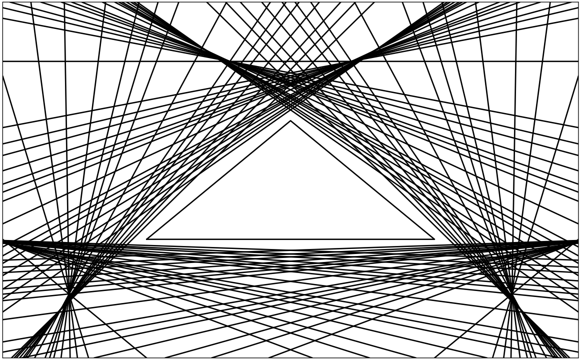
Lines defined by the linear functionals α^γ



The triangle is bounded by the lines corresponding to $\gamma \subset \partial\Sigma$.
Its interior parametrizes proper affine deformations.

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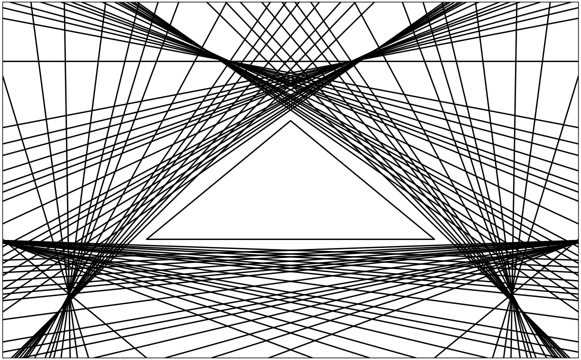

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Extension of Margulis's invariant to geodesic currents and measured laminations

- α_u *extends* from a class function on Γ_0 to a function defined on the convex set $\mathcal{C}(\Sigma)$ of *geodesic currents* on Σ:
- (Goldman-Labourie-Margulis) $\exists$ continuous biaffine map

$$\mathcal{C}(\Sigma) \times H^1(\Gamma_0, \mathbb{R}^{2,1}) \xrightarrow{\Psi} \mathbb{R}.$$

- If $\gamma \in \Gamma_0$, and μ is the corresponding geodesic current, then

$$\Psi(\mu, [u]) = \frac{\alpha_{[u]}(\gamma)}{\ell_\Sigma(\gamma)}$$

where $\ell_\Sigma(\gamma)$ is the length of the closed geodesic on Σ corresponding to γ.
- $\Gamma_{[u]}$ acts properly on $\mathbb{E}^{2,1} \iff \Psi(\mu, [u]) \neq 0$ for all $\mu \in \mathcal{C}(\Sigma)$.

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The Deformation Space

- The set of proper affine deformations of Γ_0 is the open convex cone in $H^1(\Gamma_0, \mathbb{R}^{2,1})$ defined by the functionals
$$[u] \overset{\alpha^\mu}{\longmapsto} \Psi(\mu, [u])$$
for $\mu \in \mathcal{C}(\Sigma)$.
- Sufficient to test *measured geodesic laminations* μ. (Thurston "Minimal stretch maps...")
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$$[u] \xmapsto{\alpha^\mu} \Psi(\mu, [u])$$

for $\mu \in \mathcal{C}(\Sigma)$.
- Sufficient to test *measured geodesic laminations* μ. (Thurston "Minimal stretch maps...")
- Proper affine deformations correspond to infinitesimal deformations of Σ which *lengthen* all measured geodesic laminations.

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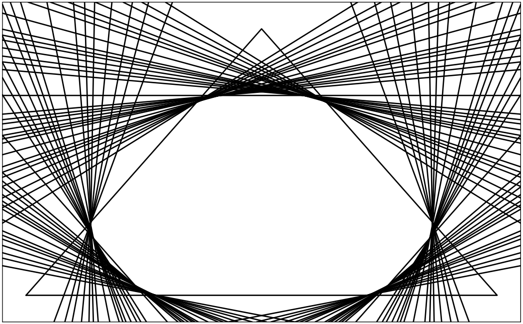
Linear functionals α^γ when Σ is a one-holed torus



The properness region is bounded by infinitely many intervals, each corresponding to a simple nonseparating loop on Σ. Boundary points lie on intervals or are points of strict convexity (irrational laminations) (Goldman-Margulis-Minsky).

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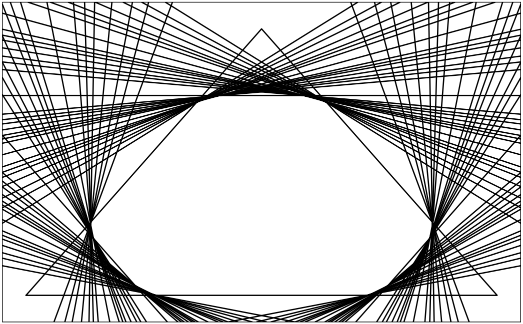

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