Moduli spaces and locally symmetric spaces

Benson Farb

University of Chicago

Once it is possible to translate any particular proof from one theory to another, then the analogy has ceased to be productive for this purpose; it would cease to be at all productive if at one point we had a meaningful and natural way of deriving both theories from a single one . . . . Gone is the analogy: gone are the two theories, their conflicts and their delicious reciprocal reflections, their furtive caresses, their inexplicable quarrels; alas, all is just one theory, whose majestic beauty can no longer excite us.

— *letter from André to Simone Weil, 1940*

[stolen from Kent-Leininger]

Prelude: A useful viewpoint

$\mathrm{Mod}_g := \mathrm{Homeo}^+(S_g)/\,\mathrm{Homeo}^0(S_g)$

Problem: Classify elements $\phi \in \mathrm{Mod}_g$.

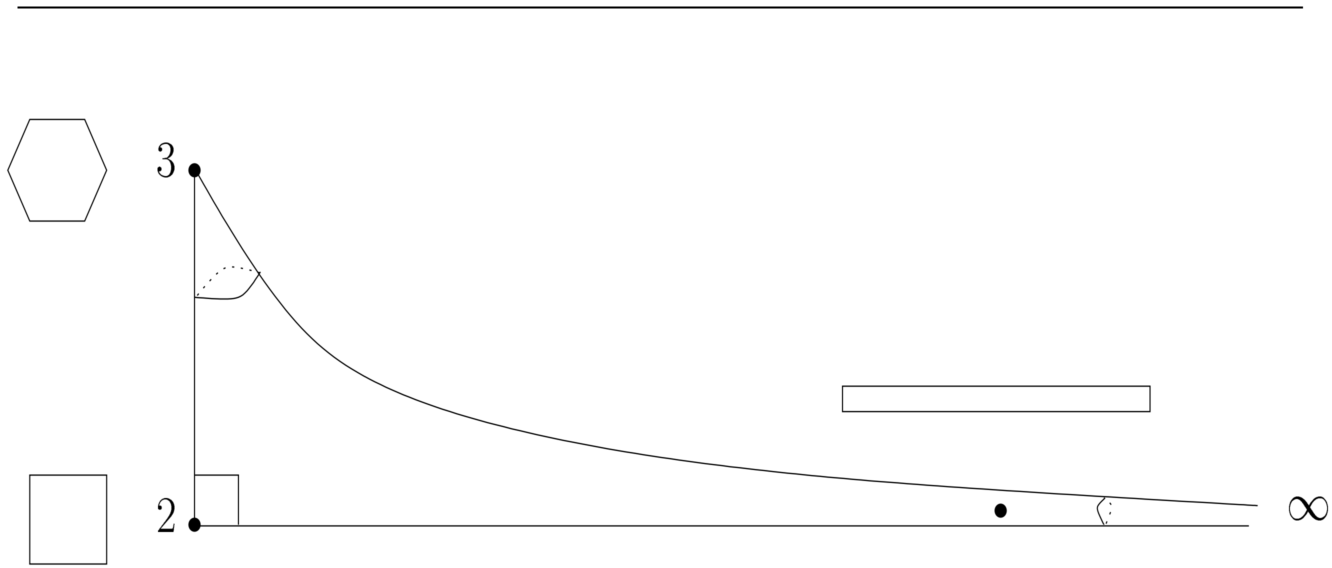
Solution (Thurston, Bers):

STEP 1. Look at action of Mod_g on Teichmüller space $\mathcal{T}_g$.

$\mathcal{T}_1 \,/\, \mathrm{Mod}_1 = \mathbf{H}^2 \,/\, \mathrm{SL}(2, \mathbf{Z})$

[Picture stolen from C. McMullen]

Reprise: **Moduli space for genus $g = 1$**



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- $\mathbf{H}^n$ negatively curved (so $d_{\mathbf{H}^n}$ is convex)
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STEP 3. Two proof endings

- Bers: Analyze $\mathbf{Minset}(\phi)$
- Thurston: Apply Brouwer Fixed Point Theorem; analyze

Prelude: The truth interferes

$\mathcal{T}_g$ is **NOT** negatively curved (unless $g = 1$)!

Nontrivial problems

1. How to analyze $\mathrm{Minset}(\phi)$?
2. How to compactify $\mathcal{T}_g$?

Prelude: $\mathcal{T}_g$ **vs. symmetric spaces**

How close is Teichmüller space to being a symmetric space?

How much of the formal geometry of a symmetric space does Teichmüller space have?

– J. Harer, 1988

Some pioneers: Dehn, Bers, Thurston, Harvey, Harer, Ivanov, Masur, and many others.

Talk Outline

Theme: Think of moduli space $\mathcal{M}_g := \mathcal{T}_g \,/\, \mathrm{Mod}_g$ as a locally symmetric orbifold.

- Use this philosophy to discover conjectures/theorems.
- Find universal constraints on this philosophy.

Today

1. Symmetry and homogeneity
2. Reduction theory

Other aspects

- Curvature (Riemannian, holomorphic, coarse)
- Convex cocompact groups
- Rank invariants ($\mathbf{R}$-rank, $\mathbf{Q}$-rank, geometric ranks)
- Rank one / higher rank dichotomy
- Compactifications

Locally symmetric spaces: Examples

- Finite volume hyperbolic manifolds $M = \Gamma \backslash \mathbf{H}^n$
- $\mathcal{V}_g = \mathrm{SL}(n, \mathbf{Z}) \backslash \mathrm{SL}(n, \mathbf{R}) / \mathrm{SO}(n)$

  $=$ moduli space of flat, unit volume, n-dimensional tori
- $\mathcal{A}_g = \mathrm{Sp}(2g, \mathbf{Z}) \backslash \mathrm{Sp}(2g, \mathbf{R}) / \mathrm{U}(g)$

  $=$ moduli space of g-dimensional, principally polarized abelian varieties
- $M = \Gamma \backslash G / K$, G semisimple, K max compact, Γ lattice.

Remarks

- $\mathcal{M}_1 = \mathcal{V}_2$
- $\mathcal{M}_g \hookrightarrow \mathcal{A}_g$ (Torelli Theorem)

Locally symmetric spaces M: First Properties

- M finite volume Riemannian
- Symmetry: $\widetilde{M}$ is symmetric at every point: the flip map

$$\gamma(t) \mapsto \gamma(-t)$$

is an isometry.
- Homogeneity: $\mathrm{Isom}(\widetilde{M})$ acts transitively on $\widetilde{M}$.
- Curvature: $K(M) \leq 0$
- Algebraic system: $M = \Gamma \backslash G / K$
  - Algebraic formulas for differential-geometric quantities
- Rigidity properties
  - $\pi_1(M)$ usually determines M (Mostow, Prasad, Margulis)

Dictionary: First entries

Nonlinear	Linear
Moduli space $\mathcal{M}_g$	loc. sym. space M
Mod_g	$\pi_1(M)$
$\mathcal{T}_g$	symmetric space $\widetilde{M}$

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Remark. Gives no info on WP metric. But $\mathrm{Isom}(\mathcal{T}_g, \mathrm{WP}) = \mathrm{Mod}_g^{\pm}$ (Masur-Wolf).

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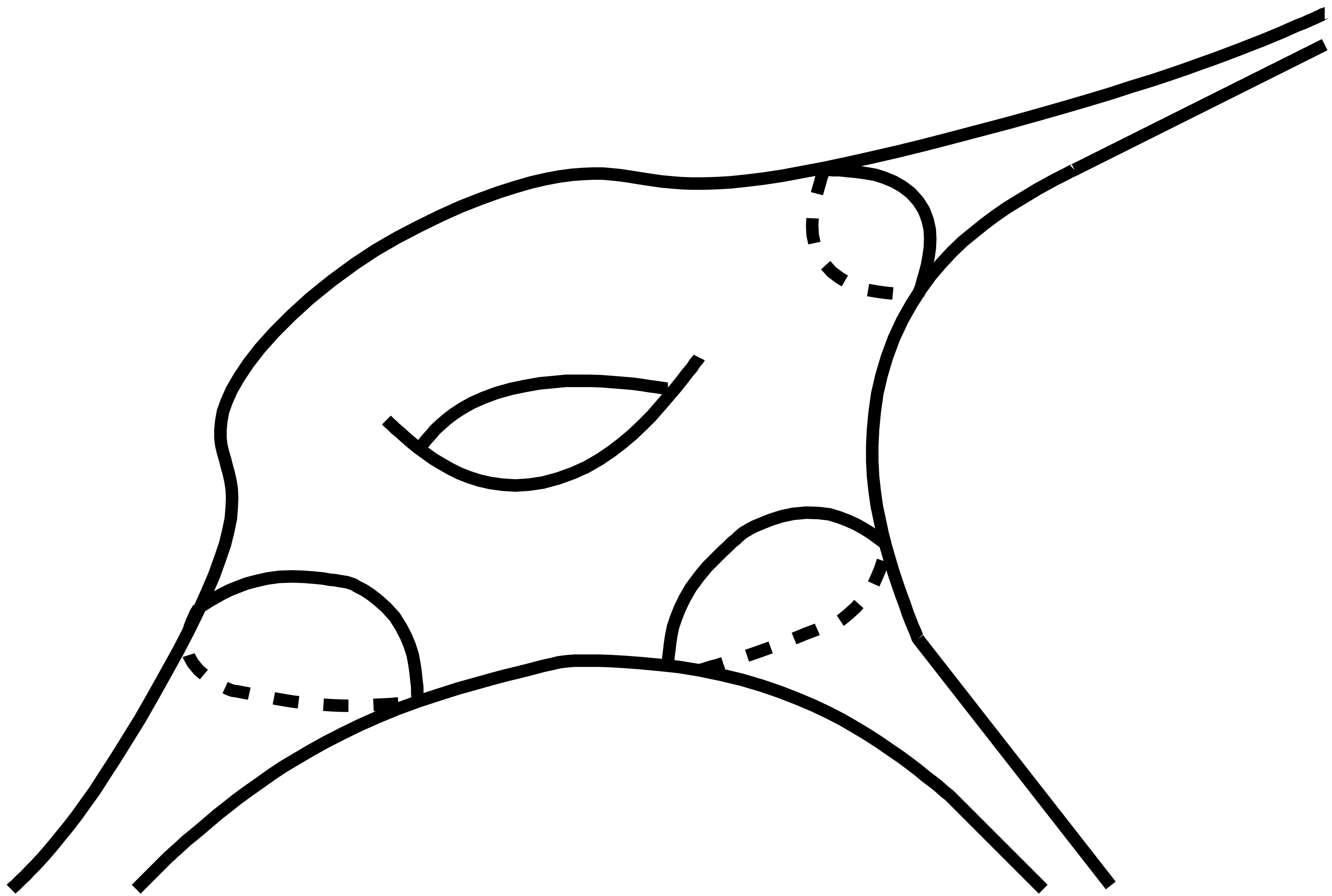
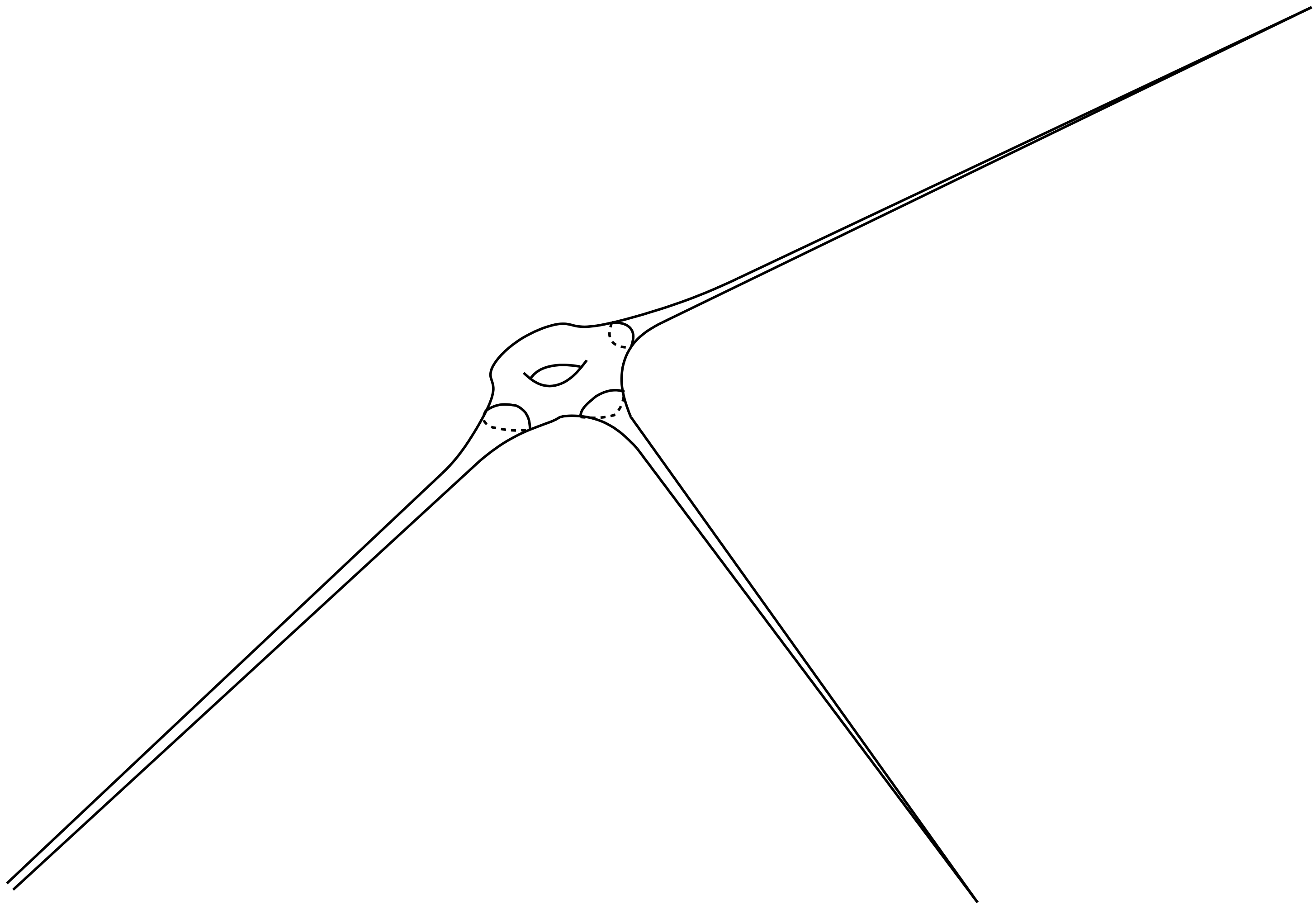
Finite volume hyperbolic manifold M



$\frac{1}{10}M$



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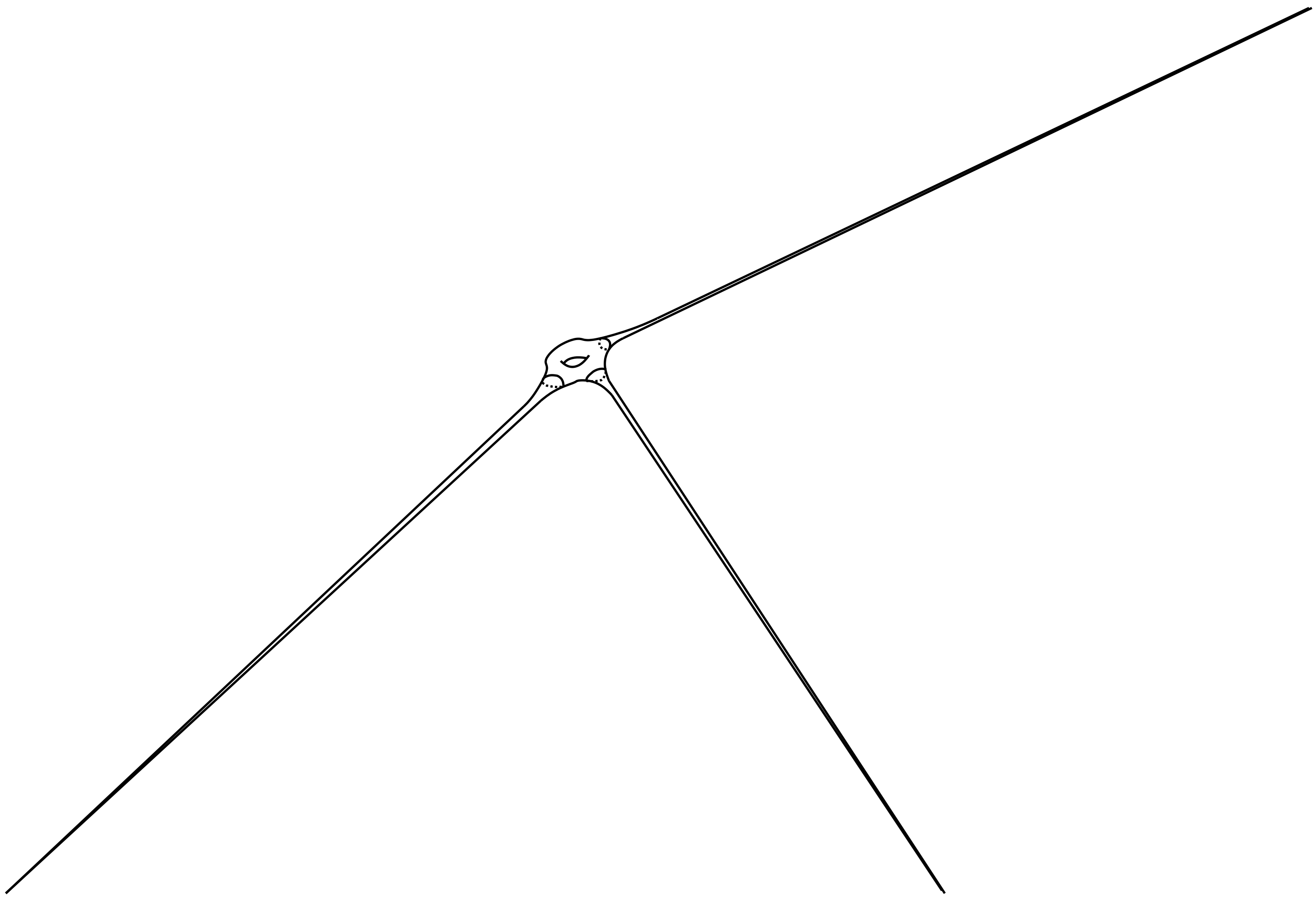
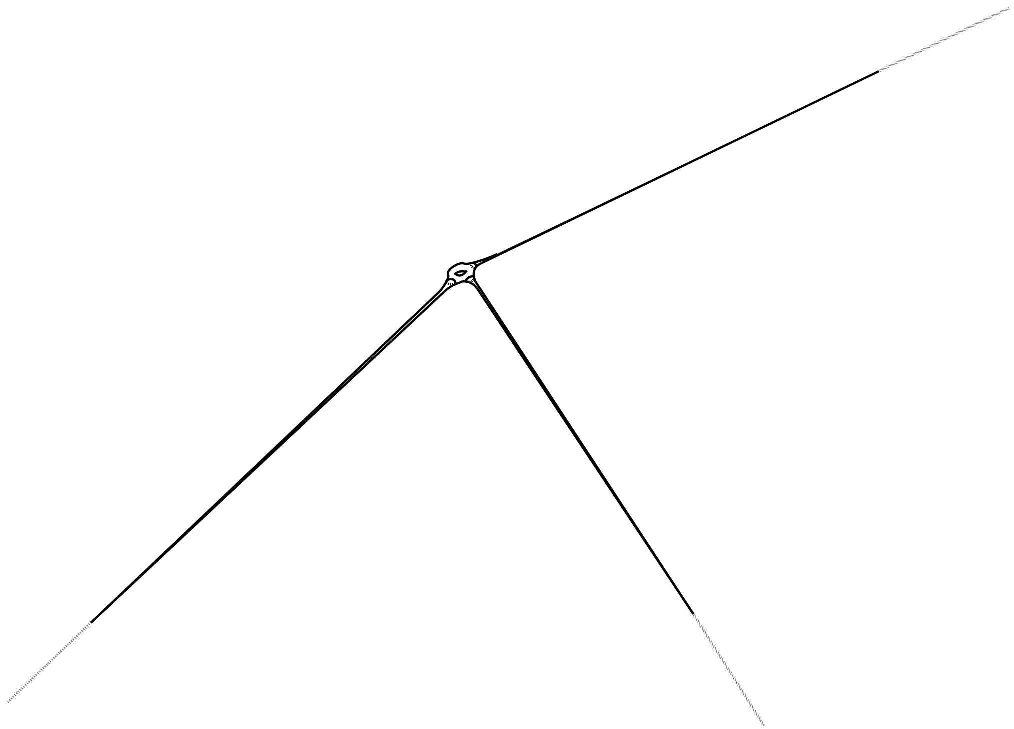
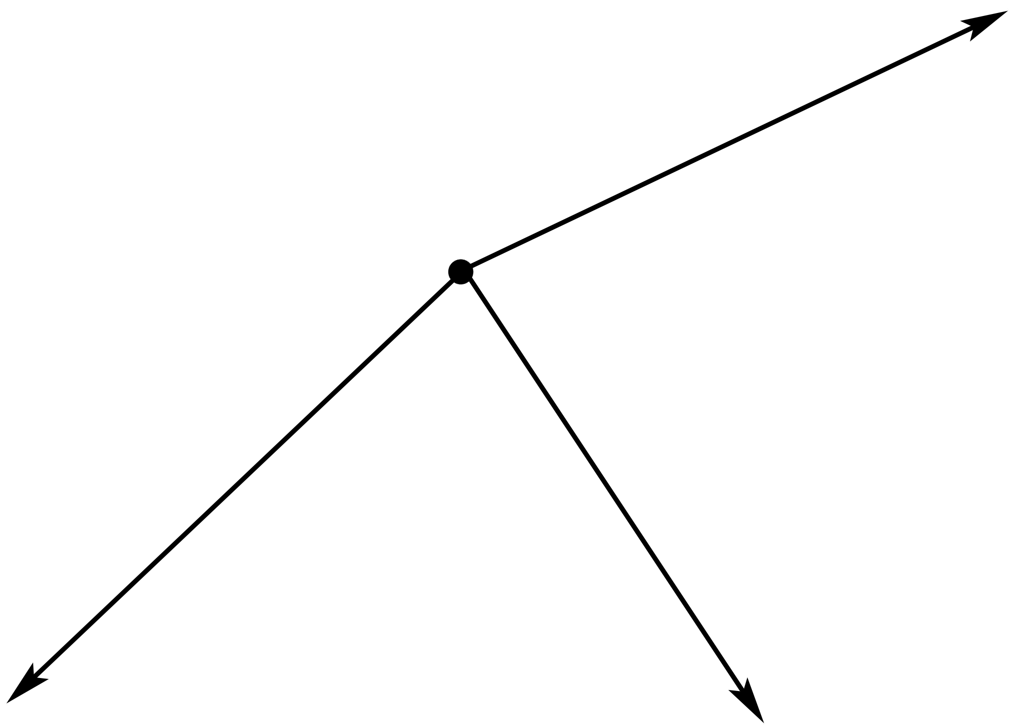


$\frac{1}{40}M$



$\frac{1}{1000000}M$



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Question. What is $\mathrm{Cone}(\mathcal{M}_g)$?

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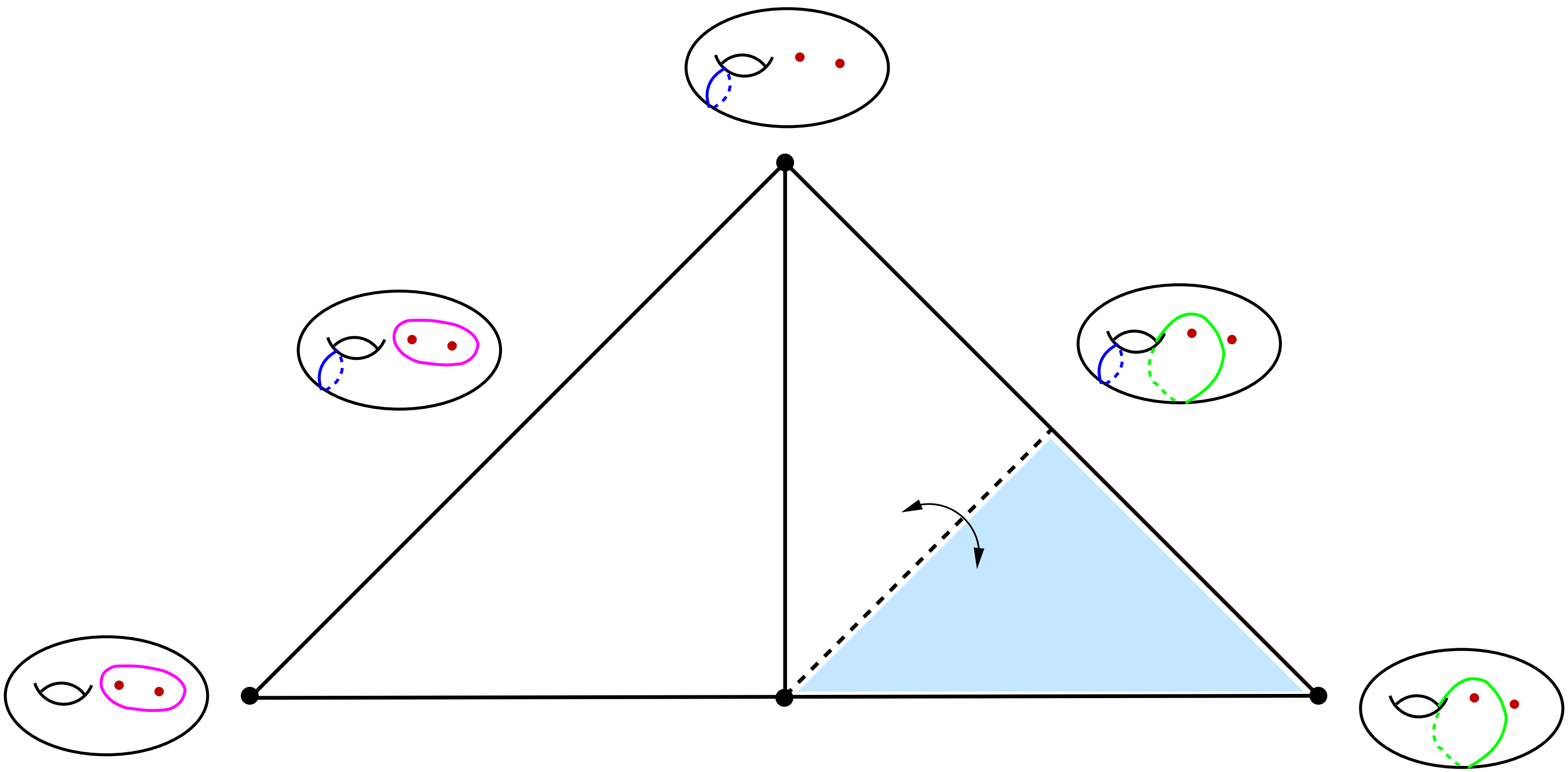
4. Endow $\mathcal{V}_g$ with induced path metric.

The 2-punctured torus case: $\mathcal{V}_{1,2}$



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Key ingredient in proof: Minsky Product Regions Theorem.

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Idea.

$$\left\{ \begin{array}{c} \text{Group theory and} \\ \text{geometry of } \Gamma_G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Geometry of the} \\ \rho(G)\text{-orbit in } \mathcal{T}_g \end{array} \right\}$$

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- (Genericity)

  For any $\{\phi_1, \ldots, \phi_r\}$ pseudo-Anosovs, $\exists N > 0$ so that the group

  $$< \{\phi_1^N, \ldots, \phi_r^N\} >$$

  is convex cocompact.

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  - Applied to prove commensurator and quasi-isometric rigidity theorems for Γ_G.

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