POINCARÉ INEQUALITIES, EMBEDDINGS, AND WILD GROUPS

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ABSTRACT. We present geometric conditions on a metric space (Y, d_Y) ensuring that almost surely, any isometric action on Y by Gromov's expander-based random group has a common fixed point. These geometric conditions involve uniform convexity and the validity of nonlinear Poincaré inequalities, and they are stable under natural operations such as scaling, Gromov-Hausdorff limits, and Cartesian products. We use methods from metric embedding theory to establish the validity of these conditions for a variety of classes of metric spaces, thus establishing new fixed point results for actions of Gromov's "wild groups".

1. INTRODUCTION

We establish the existence of finitely generated groups with strong fixed point properties. The seminal work on this topic is Gromov's construction [14] of random groups from expander graph families, leading to a solution [17, Sec. 7] of the Baum-Connes conjecture for groups, with coefficients in commutative C^*-algebras. Here we study Gromov's construction, highlighting the role of the geometry of the metric space on which the group acts. As a result, we isolate key properties of the space acted upon that imply that any isometric action of an appropriate random group has a common fixed point. Using techniques from the theory of metric embeddings in order to establish these properties, we obtain new fixed point results for a variety of spaces that will be described below. This answers in particular a question of Pansu [35] (citing Gromov). In fact, we prove the stronger statement that for every Euclidean building B (see [23]), there exists a torsion-free hyperbolic group for which every isometric action on $\ell_2(B)$ has a common fixed point (this statement extends to appropriate ℓ_2 products of more than one building). Thus, following Ollivier's terminology [31], Gromov's groups are even "wilder" than previously shown.

For $p \geqslant 1$ say that a geodesic metric space (Y, d_Y) is p-uniformly convex if there exists a constant $c > 0$ such that for every $x, y, z \in Y$, every geodesic segment $\gamma : [0,1] \to Y$ with $\gamma(0) = y$, $\gamma(1) = z$, and every $t \in [0,1]$ we have:

$$d_Y(x, \gamma(t))^p \leqslant (1-t) d_Y(x,y)^p + t d_Y(x,z)^p - ct(1-t) d_Y(y,z)^p. \tag{1.1}$$

It is immediate to check that (1.1) can hold only for $p \geqslant 2$. The inequality (1.1) is an obvious extension of the classical notion of p-uniform convexity of Banach spaces (see, e.g., [5]), and when $p = 2$ it is an extension of the CAT(0) property (see, e.g., [9]).

2010 *Mathematics Subject Classification.* 20F65,58E40,46B85.

Key words and phrases. Gromov's random groups, fixed points, Poincaré inequalities.

A.N. is supported in part by NSF grants CCF-0635078 and CCF-0832795, BSF grant 2006009, and the Packard Foundation.

This work was completed when L.S. was an intern at Microsoft Research, Redmond WA, June–August 2004. He wishes to thank Microsoft Research for their hospitality.

We shall say that a metric space (Y, d_Y) admits a sequence of *high girth p-expanders* if there exists $k \in \mathbb{N}$, $\gamma, \eta > 0$, and a sequence of k-regular finite graphs $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} |V_n| = \infty$ such that the length of the shortest non-trivial closed path (the "girth") in G_n is at least $\eta \log |V_n|$, and such that for every $f : V_n \to Y$ we have,

$$\frac{1}{|V_n|^2} \sum_{u,v\in V_n} d_Y(f(u), f(v))^p \leqslant \frac{\gamma}{|E_n|} \sum_{uv\in E_n} d_Y(f(u), f(v))^p. \tag{1.2}$$

When $Y = \mathbb{R}$ it is well-known that inequality (1.2) with $p = 2$ is equivalent to the usual notion of combinatorial expansion (for a survey on expander graphs see [18], especially Section 2). It is less well-known [27] that this is true for all $1 < p < \infty$; we reproduce the proof in Lemma 4.4. It is also worth noting that unless Y consists of a single point, the sequence of graphs considered must necessarily be a sequence of combinatorial expanders.

As we shall see later, a large class of metric spaces of interest consists of spaces that are both p-uniformly convex and admit a sequence of high girth p-expanders. In fact, in all cases that we study, the Poincaré inequality (1.2) holds for *every* sequence of combinatorial expanders. It is an open problem whether the existence of a sequence of bounded degree graphs satisfying (1.2) implies the same conclusion for all combinatorial expanders, but we will not deal with this issue here as the existence statement suffices for our purposes.

Gromov's remarkable construction [14] of random groups is described in detail in Section 6. In order to state our results, we briefly recall it here. Given a (possibly infinite) graph $G = (V, E)$, and integers $j, d \in \mathbb{N}$, a probability distribution over groups Γ associated to G and generated by d elements $s_1, \ldots, s_d$ is defined as follows. Orient the edges of G arbitrarily. For every edge $e \in E$ choose a word w_e of length j in $s_1, \ldots, s_d$ and their inverses uniformly at random from all such $(2d)^j$ words, such that the random variables $\{w_e\}_{e\in E}$ are independent. Each cycle in G induces a random relation obtained by traversing the cycle, and for each edge e of the cycle, multiplying by either w_e or w_e^{-1}, depending on whether e is traversed according to its orientation or not. These relations induce the random group $\Gamma = \Gamma(G, d, j)$.

Our main result is:

Theorem 1.1. *Assume that a geodesic metric space (Y, d_Y) is p-uniformly convex and admits a sequence of high girth p-expanders $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$. Then for all $d \geqslant 2$ and $j \geqslant 1$ with probability tending to 1 as $n \to \infty$, any isometric action of the group $\Gamma(G_n, d, j)$ on Y has a common fixed point.*

It was shown in [14, 32, 3] that for every $d \geqslant 2$, for large enough j (depending only on d and the parameters k, η), the group $\Gamma(G_n, d, j)$ is torsion-free and hyperbolic with probability tending to 1 as $n \to \infty$.

Using a variety of results and techniques from the theory of metric embeddings, we present a list of metric spaces (Y, d_Y) for which the conditions of Theorem 1.1 are satisfied[1]. These spaces include all Lebesgue spaces $L_q(\mu)$ for $1 < q < \infty$, and more generally all Banach lattices which are p-uniformly convex for some $p \in [2, \infty)$. Moreover, they include all (possibly infinite dimensional) Hadamard manifolds (in which case $p = 2, c = 1$), all Euclidean buildings ($p = 2, c = 1$, again), and all p-uniformly convex Gromov hyperbolic metric spaces

[1]Note that our conditions on the metric space (Y, d_Y) in Theorem 1.1 are closed under ℓ_p sums $(\oplus_{s=1}^{N} Y_s)_p$, provided that in (1.2), the same high-girth expander sequence works for all the Y_s. This holds true in all the examples that we present, for which (1.2) is valid for every connected graph, with γ depending only on p, the spectral gap of the graph, and certain intrinsic geometric parameters of Y.

of bounded local geometry. It was asked by Pansu in [35] whether for every symmetric space or Euclidean building an appropriate random group has the fixed point property. Our results imply that this is indeed the case. As a corollary, by a "gluing" construction of [2] (see also [13, Sec. 3.3]) it follows that there exists a torsion-free group that has the fixed point property with respect to *all* the spaces above. This yields one construction of "wild groups". Alternatively, one could follow the original approach of Gromov [14], who considers the group $\Gamma = \Gamma(G, d, j)$, where the graph G is the disjoint union of an appropriate subsequence of the expanders $\{G_n\}_{n=1}^{\infty}$ from Theorem 1.1, which is a torsion-free group with positive probability [14, 32, 3]. For this group Γ, Pansu asked [35] whether it has the fixed point property with respect to *all* symmetric spaces and all buildings of type $\tilde{A}_n$. Our result implies that for every d, j, almost surely Γ will indeed have this fixed point property, and also on all ℓ_2 products of such spaces.

Theorem 1.2. *Let G be the disjoint union of a family of high-girth combinatorial expanders (that is, of a family of graphs for which a single γ applies in* (1.2) *for all $\mathbb{R}$-valued functions). Let $d \geqslant 2$ and $j \geqslant 1$. Then with probability* 1 *the group $\Gamma(G, d, j)$ has the fixed-point property for isometric actions on all p-uniformly convex Banach Lattices, all buildings associated to linear groups, all non-positively curved symmetric spaces, and all p-uniformly convex Gromov hyperbolic spaces. The fixed-point property also holds for an ℓ_p-product of p-uniformly convex spaces, as long as the constant in* (1.2) *is uniformly bounded for these spaces.*

Problems similar to those studied here were also investigated in [20, 19], where criteria were introduced that imply fixed point properties of random groups in Żuk's triangular model [40]. These criteria include a Poincaré-type inequality similar to (1.2), with the additional requirement that the constant γ is small enough (in our normalization, they require $p = 2$ and $\gamma < 2$). Unfortunately, it is not known whether it is possible to establish such a strong Poincaré inequality for the spaces studied here, except for CAT(0) manifolds, trees, and a specific example of an $\tilde{A}_2$ building (see [20, 19]). Our approach is insensitive to the exact value of γ in (1.2). In fact, γ can be allowed to grow to infinity with $|V_n|$; see (4.4) and Theorem 7.6 below.

It was shown in [36] that any cocompact lattice Γ in $\mathrm{Sp}_{n,1}(\mathbb{R})$ admits a fixed-point-free action by linear isometries on L_p for any $p \geqslant 4n + 1$. Also, Γ acts by isometries on the symmetric space of $\mathrm{Sp}_{n,1}(\mathbb{R})$ (which is a Hadamard manifold) without fixed points. Thus, while it is known [14, 37] that Gromov's random groups have property (T), our results do not from follow from property (T) alone. See [12, 4] for a discussion of the relation between property (T) and fixed points of actions on L_p.

We end this introduction by noting that the above gluing-type construction based on Theorem 1.1 yields a non-hyperbolic group. This is necessary, since it was shown in [39] that any hyperbolic group admits a proper (and hence fixed-point free) isometric action on an $L_p(\mu)$ space for p large enough. It remains open whether there exists a *hyperbolic* group with the fixed-point property on all symmetric spaces and Euclidean buildings. Such a group would have no infinite linear images. (This is related to the well known problem of the existence of a hyperbolic group which is not residually finite.)

Overview of the structure of this paper. In Section 2 we recall some background on fixed point properties of groups, and how they are classically proved. The natural approach to finding a fixed point from a bounded orbit by considering the average (or center of mass) of the orbit requires appropriate definitions in general uniformly convex metric spaces; this is discussed in Section 3. But, in our situation orbits are not known to be bounded, so the strategy is to average over certain bounded subsets of an orbit. The hope is that by iterating this averaging procedure we will converge to a fixed point. It turns out that this approach works in the presence of sufficiently good Poincaré inequalities; this is explained in Section 7, a key technical tool being Theorem 3.10 (before reading Section 7, readers should acquaint themselves with the notations and definitions of Section 6, which recalls Gromov's construction of random groups). We prove the desired Poincaré inequalities (in appropriate metric spaces) via a variety of techniques from the theory of metric embeddings; Section 4 and Section 5 are devoted to this topic.

Asymptotic notation. We use $A \lesssim B$, $B \gtrsim A$ to denote the estimate $A \leqslant CB$ for some absolute constant C; if we need C to depend on parameters, we indicate this by subscripts, thus $A \lesssim_p B$ means that $A \leqslant C_p B$ for some C_p depending only on p. We shall also use the notation $A \asymp B$ for $A \lesssim B \ \wedge \ B \lesssim A$.

2. Background on fixed point properties of groups

We start by setting some terminology.

Definition 2.1. Let Γ be a finitely generated group, let (Y, d_Y) be a metric space, and let $\rho\colon \Gamma \to \mathrm{Isom}(Y)$ be an action by isometries. We say that the action satisfies the condition:

(N), if the image $\rho(\Gamma)$ is finite;
(F), if the image $\rho(\Gamma)$ has a common fixed point;
(B), if some (equiv. every) Γ-orbit in Y is bounded.

For a class $\mathcal{C}$ of metric spaces, we say that Γ has property (N$\mathcal{C}$), (F$\mathcal{C}$) or (B$\mathcal{C}$) if every action $\rho\colon \Gamma \to \mathrm{Isom}(Y)$, where $Y \in \mathcal{C}$, satisfies the respective condition.

The Guichardet-Delorme Theorem [16, 10] asserts that if H is Hilbert space then Γ has property (FH) if and only if it has Kazhdan property (T). The reader can take this as the definition of property (T) for the purpose of this paper.

Fixed-point properties can have algebraic implications for the group's structure. For example, finitely generated linear groups have isomorphic embeddings into linear groups over local fields, and these latter groups act by isometries on non-positively curved spaces with well-understood point stabilizers. For completeness and later reference, we include the following simple lemma.

Lemma 2.2 (Strong non-linearity)**.** *Let $\mathcal{S}$ be the class of the symmetric spaces and buildings associated to the groups $\mathrm{GL}_n(F)$, where F is a non-Archimedean local field. Let Γ be a finitely generated group with property* (F$\mathcal{S}$)*. Then any homomorphic image of Γ into a linear group is finite.*

Proof. Let Γ be finitely generated group with property (F$\mathcal{S}$). Let K be a field, and let $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ be a homomorphism. Without loss of generality we can assume K to be the field generated by the matrix elements of the images of the generators of Γ, and then let $A \subset K$ be the set of matrix elements of the images of all elements of Γ. Clearly $\rho(\Gamma)$ is finite iff A is a finite set, and [8, Lem. 2.1] reduces the finiteness of A to showing that the image of A under any embedding of K in a local field F is relatively compact. Hence, let $\iota\colon K \to F$ be such an embedding. This induces a group homomorphism $\mathrm{GL}_n(K) \to \mathrm{GL}_n(F)$

which we also denote ι. Composing with ρ we obtain a homomorphism $\iota \circ \rho \colon \Gamma \to \mathrm{GL}_n(F)$. Now let S be the symmetric space (if F is Archimedean) or Bruhat-Tits building (if F is non-Archimedean) associated to $\mathrm{GL}_n(F)$. Since $\mathrm{GL}_n(F)$ is a group of isometries of S, the image of $\iota \circ \rho$ must fall in the stabilizer in $\mathrm{GL}_n(F)$ of a point of S. Since these stabilizers are compact subgroups of $\mathrm{GL}_n(F)$ we are done. □

Lemma 2.2 implies, via our results as stated in the introduction, that Gromov's wild groups are not isomorphic to linear groups. Alternatively, this fact also follows from the result of [15] that asserts that any linear group admits a coarse embedding into Hilbert space, while it was shown in [14] that Gromov's random group does not admit such an embedding (indeed, this was the original motivation for Gromov's construction). It also follows from Lemma 2.2 that all linear homomorphic images of Gromov's random group are finite. In fact, it was later observed in [13] that the random group has no finite images, and hence also no linear images, since finitely generated linear groups are residually finite.

It is clear that the condition (B) is implied by either condition (N) or (F). When Y is complete and p-uniformly convex the converse holds as well (weaker notions of uniform convexity suffice here). We recall the standard proof of this fact below, since it illustrates a "baby version" of the averaging procedure on uniformly convex spaces that will be used extensively in what follows.

Lemma 2.3 ("Bruhat's Lemma")**.** *Let Y be a uniformly convex geodesic metric space. Then the condition (B) for isometric actions on Y implies condition (F).*

Proof. To any bounded set $A \subset Y$ associate its *radius function* $r_A(y) = \sup_{a \in A} d_Y(y, a)$. For any $a \in A$ and $y_0, y_1 \in Y$ let $y_{1/2}$ be a midpoint of the geodesic segment connecting them. By equation (1.1) we have that $d_Y(a, y_{1/2})^p$ is less than the average of $d_Y(a, y_0)^p$ and $d_Y(a, y_0)^p$ by a positive quantity depending only on $d_Y(y_0, y_1)$ and growing with it. It follows that the diameter of the set $C_\varepsilon \subset Y$ on which r_A exceeds its minimum by no more than ε goes to zero with ε. Since Y is complete it follows that $r_A(y)$ has a unique global minimizer, denoted $\mathrm{c}_\infty(A) \in Y$, and called the *circumcenter* of A. Since its definition involved only the metric on Y, the circumcenter map is equivariant under isometries of Y. It follows that the circumcenter of a bounded orbit for a group action is a fixed point. □

3. Averaging on metric spaces

We saw above how to find a fixed point from a bounded orbit, by forming a kind of "average" (circumcenter) along the orbit. When the orbits are not known to be bounded, it is not possible to form such averages. However, if Γ (generated by $S = S^{-1}$) acts on a p-uniformly convex space Y, it is possible to average over small pieces of the orbit: passing from a point y to an appropriately defined average of the finite set $\{sy\}_{s \in S}$ (the precise notion of averaging is described below). Under suitable conditions this averaging procedure is a contraction on Y, leading to a fixed point. In practice we will need to average over small balls rather than just S itself, but the idea remains the same.

"Averaging" means specifying a function that associates to Borel probability measures σ on Y a point $c(\sigma) \in Y$, in a well-behaved manner. We will not axiomatize the needed properties, instead defining the procedures we will use. We start with a particularly simple example. In what follows all measures are assumed to have finite support—this suffices for our purposes, and the obvious generalizations are standard.

Example 3.1. Let Y be a Banach space, and let σ be a (finitely supported) probability measure on Y. The vector-valued integral

$$\mathrm{c}_{\mathrm{lin}}(\sigma) = \int_Y y d\sigma(y)$$

is called the *linear* center of mass of σ.

This center of mass behaves well under linear maps, but its metric properties are not so clear. Thus even for the purpose of proving fixed-point properties for actions on L_p we use a nonlinear averaging method, related to a metric definition of linear averaging on Hilbert space. This is a standard method in metric geometry (see for example [21, Chapter 3]).

For a metric space (Y, d_Y) we write $\mathcal{M}(Y)$ for the space of probability measures on Y with finite support. Generally it is enough to assume below that the measures have finite pth moment for the appropriate $p \geqslant 2$ but we will not use such measures since our groups are finitely generated.

3.1. **Uniformly convex metric spaces and the geometric center of mass.** We continue with our complete metric space (Y, d_Y). A *geodesic segment* in Y is an isometry $\gamma\colon I \to Y$ where $I \subseteq \mathbb{R}$ is a closed interval, and the metric on I is induced from the standard metric on $\mathbb{R}$. If the endpoints $a < b$ of I are mapped to $y, z \in Y$ respectively, we will say that the segment γ *connects* y to z, and usually denote it by $[y, z]$. Moreover, for any $t \in [0, 1]$ we will use $[y, z]_t$ to denote $\gamma((1-t)a + tb)$. This notation obscures the fact that there may be distinct geodesic segments connecting y to z, but this will not be the case for the spaces we consider (see below).

We now assume that Y is a *geodesic* metric space, i.e., that every two points of Y are connected by a geodesic segment.

Definition 3.2. Let $2 \leqslant p < \infty$. Y is said to be *p-uniformly convex* if there exists a constant $c_Y > 0$ such that for every $x, y, z \in Y$, every geodesic segment $[y, z] \subseteq Y$, and every $t \in [0, 1]$ we have:

$$d_Y(x, [y, z]_t)^p \leqslant (1-t) d_Y(x, y)^p + t d_Y(x, z)^p - c_Y^p t(1-t) d_Y(y, z)^p. \tag{3.1}$$

We say that Y is *uniformly convex* if it is p-uniformly convex for some $p \geqslant 2$.

The above definition is an obvious extension of the notion of p-uniform convexity of Banach spaces (see, e.g., [11, 5]). For concreteness, an $L_p(\mu)$ space is p uniformly convex if $p \in [2, \infty)$ and 2-uniformly convex if $p \in (1, 2]$. In Hilbert space specifically, (3.1) with $p = 2$ and $c_Y = 1$ is an equality, and it follows that the same holds for conclusions such (3.3) below. We also note that it is easy to see that a uniformly convex metric space is uniquely geodesic by examining midpoints.

We now recall the notion of CAT(0) spaces. For $y_1, y_2, y_3 \in Y$, choose $Y_1, Y_2, Y_3 \in \mathbb{R}^2$ such that $\|Y_i - Y_j\|_2 = d_Y(y_i, y_j)$ for any i, j. Such a triplet of *reference points* always exists, and is unique up to a global isometry of $\mathbb{R}^2$. It determines a triangle $\Delta = I_{12} \cup I_{23} \cup I_{13}$ consisting of three segments of lengths $d_Y(y_i, y_j)$. Any choice of three geodesic segments $\gamma_{ij}\colon I_{ij} \to Y$ connecting y_i, y_j gives a *reference map* $R\colon \Delta \to Y$. We say that (Y, d_Y) is a CAT(0) space if for every three points $y_i \in Y$ every associated reference map R does not increase distances. It is a standard fact (see [9]) that (Y, d_Y) is a CAT(0) space iff it is 2-uniformly convex with the constant c_Y in (3.1) equal to 1. CAT(0) spaces are p-uniformly convex for all $p \in [2, \infty)$ since the plane $\mathbb{R}^2$ is p-uniformly convex (it is isometric to a subset of L_p).

Assume that (Y, d_Y) is p-uniformly convex. Let $\sigma \in \mathcal{M}(Y)$. Integrating equation (3.1) we see that for all $y, z \in Y$:

$$c_Y^p t(1-t) d_Y(y,z)^p \leqslant (1-t) d_p(\sigma, y)^p + t d_p(\sigma, z)^p - d_p(\sigma, [y,z]_t)^p, \tag{3.2}$$

where for $w \in Y$ we write

$$d_p(\sigma, w) = \left(\int_Y d_Y(u,w)^p d\sigma(u) \right)^{1/p}.$$

Now let $d = \inf_{y \in Y} d_p(\sigma, y)$, and assume $d_p(\sigma, y), d_p(\sigma, z) \leqslant (d^p + \varepsilon)^{1/p}$. Letting $w \in Y$ denote the midpoint of any geodesic segment connecting y and z we have $d_p(\sigma, w) \geqslant d$ and hence:

$$\frac{c_Y^p}{4} d_Y(y,z)^p \leqslant d^p + \varepsilon - d^p = \varepsilon.$$

In other words, the set of $y \in Y$ such that $d_p(\sigma, y)^p$ is at most $d^p + \varepsilon$ has diameter $\lesssim_{c_Y} \varepsilon^{1/p}$. By the completeness of Y, there exists a unique point $\mathrm{c}_p(\sigma) \in Y$ such that $d_p(\sigma, \mathrm{c}_p(\sigma)) = d$.

To justify the notation $\mathrm{c}_\infty(A)$ introduced in Lemma 2.3 notes that $d_\infty(\sigma, y) = r_A(y)$ where A is the essential support of σ.

Definition 3.3. The point $\mathrm{c}_p(\sigma)$ is called the *geometric* center of mass of σ. We will also use the term p-center of mass when we wish to emphasize the choice of exponent. The point $\mathrm{c}_\infty(A)$ is called the *circumcenter* of A.

Remark 3.4. Consider the special case of the real line with the standard metric, and of $\sigma = t\delta_1 + (1-t)\delta_0$. Then $\mathrm{c}_p(\sigma)$ represents a weighted average of $0, 1 \in \mathbb{R}$. We note that (except for special values of t), the $\mathrm{c}_p(\sigma)$ vary depending on p.

We now apply equation (3.2) where $z = \mathrm{c}_p(\sigma)$. Still using $d_p(\sigma, [y,z]_t) \geqslant d$ we get:

$$c_Y^p t(1-t) d_Y(\mathrm{c}_p(\sigma), y)^p \leqslant (1-t)(d_p(\sigma, y)^p - d^p).$$

Dividing by $1-t$ and letting $t \to 1$ we get the following useful inequality:

$$d_p(\sigma, y)^p \geqslant d_p(\sigma, \mathrm{c}_p(\sigma))^p + c_Y^p d_Y(\mathrm{c}_p(\sigma), y)^p. \tag{3.3}$$

3.2. **Random walks.** Let X be a discrete set. Following Gromov [14] we shall use the following terminology.

Definition 3.5. By a *random walk* (or a Markov chain) on X we shall mean a function $\mu\colon X \to \mathcal{M}(X)$. The space of random walks will be denoted $\mathcal{W}(X)$.

For a random walk μ and $x \in X$ we will denote below the measure $\mu(x)$ by either μ_x or $\mu(x \to \cdot)$. The latter notation emphasizes the view of μ as specifying the transition probabilities of a Markov chain on X. For $\nu \in \mathcal{M}(X)$, $\mu, \mu' \in \mathcal{W}(X)$ we write

$$\nu * \mu \stackrel{\text{def}}{=} \int_X d\nu(x) \mu_x \in \mathcal{M}(X).$$

The map $x \mapsto (\mu' * \mu)_x \stackrel{\text{def}}{=} \mu'_x * \mu$ defines a random walk on X. For $n \in \mathbb{N}$ we define inductively $\mu^{n+1} \stackrel{\text{def}}{=} \mu^n * \mu$.

Let ν be a measure on X. We say that a random walk $\mu \in \mathcal{W}(X)$ is *ν-reversible*, if we have

$$d\nu(x)d\mu(x \to x') = d\nu(x')d\mu(x' \to x), \tag{3.4}$$

as an equality of measures on $X \times X$. If X is finite, we can assume that ν is a probability measure. In general integrating equation (3.4) w.r.t. x', we see that ν is a *stationary measure* for μ, in the sense that $\nu * \mu = \nu$.

Finally, let the discrete group Γ act freely on X. The induced action of Γ on $\mathcal{M}(X)$ preserves $\mathcal{M}(X)$ in this case. The space of Γ-equivariant random walks will be denoted $\mathcal{W}^\Gamma(X)$. Moreover, we have a quotient space $\Gamma\backslash X$. Fixing a probability measure $\bar{\nu}$ on $\Gamma\backslash X$, we will call $\mu \in \mathcal{W}^\Gamma(X)$ $\bar{\nu}$-reversible if it is ν-reversible where ν is the pull-back of $\bar{\nu}$ defined by $\int_X f d\nu = \int_{\Gamma\backslash X} \left(\sum_{\gamma\in\Gamma} f(\gamma x)\right) d\bar{\nu}(x)$ for any $f \in C_c(X)$.

3.3. **Averaging of equivariant functions into uniformly convex spaces.** Continuing with the notation used so far, let $\mu \in \mathcal{W}^\Gamma(X)$ be reversible w.r.t. the probability measure $\bar{\nu}$ on $\Gamma\backslash X$. Let (Y, d_Y) be a p-uniformly convex metric space on which Γ acts by isometries.

Now let $f \colon X \to Y$ be Γ-equivariant. For $x \in X$, the push-forward $f_*\mu_x$ is a probability measure on Y with finite support (the image of the support of μ_x under f). We set:

$$\left|\nabla_\mu(f)\right|_p(x) = \left(\int_X d\mu(x \to x') d_Y\left(f(x), f(x')^p\right)\right)^{1/p}, \tag{3.5}$$

$$\mathcal{E}_\mu^{(p)}(f) = \frac{1}{2}\int_{\Gamma\backslash X} \left(\left|\nabla_\mu(f)\right|_p(x)\right)^p d\bar{\nu}(x), \tag{3.6}$$

$$B(X,Y) = \left\{ f \in C(X,Y)^\Gamma \mid \mathcal{E}_\mu^{(p)}(f) < \infty \right\}. \tag{3.7}$$

For $f, g \in C(X,Y)^\Gamma$, the function $x \mapsto d_Y(f(x), g(x))$ is Γ-invariant, and we can hence set

$$d_p(f,g) \stackrel{\text{def}}{=} \left(\int_{\Gamma\backslash X} d_Y(f(x), g(x))^p d\bar{\nu}(x)\right)^{1/p}.$$

This defines a (possibly infinite) complete metric. The triangle inequality gives:

Lemma 3.6. *We have,*

1. *Let $f, g \in C(X,Y)^\Gamma$. Assume $d_p(f,g) < \infty$. Then $f \in B(X,Y)$ iff $g \in B(X,Y)$.*
2. *Let $f \in B(X,Y)$. Then $\mathcal{E}_{\mu^n}^{(p)}(f) < \infty$ for all $n \geqslant 1$.*

Proof. For all $x, x' \in X$,

$$d_Y(g(x), g(x')) \leqslant d_Y(g(x), f(x)) + d_Y(f(x), f(x')) + d_Y(f(x'), g(x'))$$

and hence

$$3^{1-p} d_Y(g(x), g(x'))^p \leqslant d_Y(g(x), f(x))^p + d_Y(f(x), f(x'))^p + d_Y(f(x'), g(x'))^p. \tag{3.8}$$

Integrating $d\mu(x \to x')$ gives Γ-invariant functions of x which may be integrated $d\bar{\nu}(x)$. Using the stationarity of $d\bar{\nu}$ we then have:

$$\mathcal{E}_\mu^{(p)}(g) \leqslant 3^{p-1}\mathcal{E}_\mu^{(p)}(f) + 3^{p-1} d_p(f,g)^p.$$

Similarly, integrating

$$n^{1-p} d_Y(f(x_0), f(x_n))^p \leqslant \sum_{i=0}^{n-1} d_Y(f(x_i), f(x_{i+1}))^p$$

against $d\bar{\nu}(x_0) \prod_{i=0}^{n-1} d\mu(x_i \to x_{i+1})$ gives $\mathcal{E}^{(p)}_{\mu^n}(f) \leqslant n^{p-1}\mathcal{E}^{(p)}_{\mu}(f)$. ☐

Continuing the analysis of the map $x \mapsto f_*\mu_x$, we note that this is a Γ-equivariant map $X \to \mathcal{M}(Y)$. Since Γ acts by isometries, the map

$$\left(A^{(p)}_{\mu} f\right)(x) \stackrel{\mathrm{def}}{=} \mathrm{c}_p(f_*\mu_x)$$

is also Γ-equivariant; this will be our averaging procedure. If Y is a Hilbert space and $p = 2$, $A^{(p)}_{\mu}$ is the usual linear average with respect to μ. In particular, $A^{(2)}_{\mu_1} A^{(2)}_{\mu_2} = A^{(2)}_{\mu_1 * \mu_2}$. This does not hold in general (for spaces other than Hilbert space, or even in Hilbert space for $p > 2$). In particular, we will later use $A^{(p)}_{\mu^n}$ for large n and not just $\left(A^{(p)}_{\mu}\right)^n$.

We first verify that the averaging procedure remains in the space $B(X, Y)$.

Lemma 3.7. *For $f \in B(X, Y)$ we have*

$$d_p(f, A^{(p)}_{\mu} f) \lesssim_{c_Y} \left(\mathcal{E}^{(p)}_{\mu}(f)\right)^{1/p},$$
$$\mathcal{E}^{(p)}_{\mu}(A^{(p)}_{\mu} f) \lesssim_{p, c_Y} \mathcal{E}^{(p)}_{\mu}(f).$$

Proof. At every $x \in X$ the fundamental estimate (3.3) gives:

$$c_Y^p d_p \left(f(x), A^{(p)}_{\mu} f(x)\right)^p \leqslant d_p(f(x), f_*(\mu_x)^p = \int d_Y(f(x), f(x'))^p d\mu(x \to x').$$

Now both sides are Γ-invariant functions of $x \in X$ and the first claim follows by integrating against $d\bar{\nu}$. For the second claim apply inequality (3.8) from the proof of Lemma 3.6 with $g = A^{(p)}_{\mu}(f)$. ☐

We measure the contractivity of $A^{(p)}_{\mu}$ with respect to the energy $\mathcal{E}^{(p)}_{\mu}$. It is not hard to verify that contraction will imply the existence of fixed points:

Proposition 3.8. *Assume that there exist $n \geqslant 1$ and $c < 1$ such that for all $f \in B(X, Y)$ we have $\mathcal{E}^{(p)}_{\mu}(A^{(p)}_{\mu^n} f) \leqslant c\mathcal{E}^{(p)}_{\mu}(f)$. Suppose that the graph on X given by connecting x, x' if $\mu(x \to x') > 0$ is* connected. *Then, as long as $B(X, Y)$ is non-empty (this is the case, for example, when $\Gamma\backslash X$ is finite), it contains constant maps. In particular, Γ fixes a point in Y.*

Proof. Choose an arbitrary $f_0 \in B(X, Y)$ and let $f_{k+1} = A^{(p)}_{\mu^n} f_k$. By assumption we have $\mathcal{E}^{(p)}_{\mu}(f_k) \leqslant c^k \mathcal{E}^{(p)}_{\mu}(f_0)$. By Lemma 3.6 $\mathcal{E}^{(p)}_{\mu^n}(f_k) \leqslant n^{p-1} c^k \mathcal{E}^{(p)}_{\mu}(f_0)$, and by Lemma 3.7 this means that

$$d_p(f_{k+1}, f_k)^p \lesssim_{p, c_Y, n} c^k \mathcal{E}^{(p)}_{\mu}(f_0).$$

It now follows that f_k are a Cauchy sequence and hence converge to a function $f \in B(X, Y)$. We have $\mathcal{E}^{(p)}_{\mu}(f) = 0$ so $f(x) = f(x')$ whenever $\mu(x \to x') > 0$. By the connectivity assumption this means f is constant on X and its value is the desired fixed point. ☐

We now address the problem of showing that averaging reduces the energy. We prove two technical inequalities:

Proposition 3.9. *(generalization of* [37, B.25]*) We have,*

$$\mathcal{E}_{\mu}^{(p)}\left(A_{\mu^n}^{(p)} f\right) \lesssim_{p,c_Y} \int_{\Gamma\backslash X} d\bar{\nu}(x) \int_X \left[d\mu^{n+1}(x \to x') - d\mu^n(x \to x')\right] d_Y\left(A_{\mu^n}^{(p)} f(x), f(x')\right)^p.$$

$$\int_{\Gamma\backslash X} d\bar{\nu}(x) \int_X d\mu^n(x \to x') d_Y\left(A_{\mu^n}^{(p)} f(x), f(x')\right)^p \lesssim_{p,c_Y} \mathcal{E}_{\mu^n}^{(p)}(f). \tag{3.9}$$

Proof. Recall that $A_{\mu^n}^{(p)} f(x) = \mathrm{c}_p(f_*\mu^n(x \to \cdot))$. The fundamental estimate (3.3) then reads:

$$c_Y^p d_Y\left(y, A_{\mu^n}^{(p)} f(x)\right)^p \\ \leqslant \int_X d_Y(y, f(x'))^p d\mu^n(x \to x') - \int_X d_Y\left(A_{\mu^n}^{(p)} f(x), f(x')\right)^p d\mu^n(x \to x'). \tag{3.10}$$

Setting $y = A_{\mu^n}^{(p)} f(x'')$ integrate (3.10) $d\mu(x'' \to x)$. The resulting function of x'' is Γ-invariant and we integrate it $d\bar{\nu}(x'')$ to get (also using the reversibility),

$$2c_Y^p \mathcal{E}_{\mu}^{(p)}(A_{\mu^n}^{(p)} f) \leqslant \int_{\Gamma\backslash X} d\bar{\nu}(x'') \int_X d\mu^{n+1}(x'' \to x') d_Y(A_{\mu^n}^{(p)} f(x''), f(x'))^p \\ - \int_{\Gamma\backslash X} d\bar{\nu}(x) \int_X d\mu^n(x \to x'') d_Y(A_{\mu^n}^{(p)} f(x''), f(x))^p.$$

Inequality (3.9) follows directly from the triangle inequality and Lemma 3.7. ☐

Theorem 3.10. *Let Γ be a group generated by the symmetric set S of size $2d$, acting by isometries on the p-uniformly convex space Y, let $X = \mathrm{Cay}(\Gamma; S)$ (the Cayley graph of Γ), and let $f \in B(X, Y)$. Let μ be the jth convolution power of the standard random walk on X for an even j. Then*

$$\mathcal{E}_{\mu}^{(p)}\left(A_{\mu^n}^{(p)} f\right) \lesssim_{p,c_Y,d,j} \sqrt{\frac{\log n}{n}} \cdot \mathcal{E}_{\mu^n}^{(p)}(f) + \frac{1}{n} \cdot \mathcal{E}_{\mu}^{(p)}(f).$$

Proof. Pulling back f to a function on the free group on S (acting on Y via the quotient map) we may assume that Γ is the free group and X the $2d$-regular tree. Now, [37, Prop. 2.9] implies that $\mu^{n+1}(x \to x') - \mu^n(x \to x')$ is typically small: given x, except for a set of x' of $(\mu^{n+1} + \mu^n)(x \to \cdot)$-mass $\lesssim_d n^{-\theta}$, the difference is $\lesssim_{d,j,\theta} \sqrt{\frac{\log n}{n}} \mu^n(x \to x')$, where $\theta > 0$ is arbitrary.

Applying this in Proposition 3.9 we find that:

$$\mathcal{E}_{\mu}^{(p)}(A_{\mu^n}^{(p)} f) \lesssim_{p,c_Y,d,j,\theta} \sqrt{\frac{\log n}{n}} \cdot \mathcal{E}_{\mu^n}^{(p)}(f) + n^{-\theta} \max_{\substack{d_X(x,x'') \leqslant j(n+1) \\ 2 | d_X(x,x'')}} d_Y(A_{\mu^n}^{(p)} f(x), f(x''))^p.$$

Now

$$d_Y(A_{\mu^n}^{(p)} f(x), f(x'')^p) \lesssim_{p,c_Y} \max_{\substack{d_X(x,x') \leqslant jn \\ 2 | d_X(x,x')}} d_Y(f(x'), f(x''))^p,$$

and by the triangle inequality

$$d_Y(f(x'), f(x''))^p \leqslant (2n+1)^{p-1} \max_{\substack{d_X(x,x') \leqslant j \\ 2 | d_X(x,x')}} d_Y(f(x), f(x'))^p.$$

Finally, the latter quantity is at most $\lesssim_{j,d} \mathcal{E}_{\mu}^{(p)}(f)$ (one needs that $\mu^j(x \to \cdot)$ is supported on all points x' at even distance from x at most j). Putting it all together we have:

$$\mathcal{E}_{\mu}^{(p)}(A_{\mu^n}^{(p)} f) \lesssim_{p,c_Y,d,j,\theta} \sqrt{\frac{\log n}{n}} \cdot \mathcal{E}_{\mu^n}^{(p)}(f) + n^{p-1-\theta} \mathcal{E}_{\mu}^{(p)}(f),$$

as required. □

It is now clear that (assuming we can arrange n to be large) what is needed is that $\mathcal{E}_{\mu^n}^{(p)}(f)$ is not too large compared to $\mathcal{E}_{\mu}^{(p)}(f)$. This is what we establish in the next two sections.

4. Poincaré inequalities on metric spaces

It turns out that it is hard to show directly that averaging with respect to the generators of the random group reduces the energy (compare [19]). Instead, it is preferable to average with respect to some power of the generators, as in Theorem 3.10, where we gain by making n large. This requires controlling $\mathcal{E}_{\mu^n}^{(p)}(f)$ in terms of $\mathcal{E}_{\mu}^{(p)}(f)$. Such a control takes the form of inequalities involving distances alone rather than centers-of-mass, so that methods from metric embedding theory can be used to prove them. In this section we state the inequalities the we need, and show that they hold for functions from expander graphs to certain target metric spaces (L_p spaces and CAT(0) manifolds). In Section 5 we use metric embeddings to establish these inequalities for additional classes of metric spaces. In section 7 we then show that a strong enough Poincaré inequality for a particular target is enough to control averaging so that the random group has the fixed-point property on that target.

We fix a group Γ, a discrete Γ-space X, a Γ-equivariant random walk $\mu \in \mathcal{W}(X)$, reversible with respect to the Γ-invariant measure ν which gives finite measure to any fundamental domain for $\Gamma \backslash X$.

Definition 4.1. Let Y be a metric space, and $p \geqslant 1$. Let $n > m \geqslant 1$ be integers. We say that a *Poincaré inequality of exponent* p holds if there exists $c > 0$ such that for any $f \in B(X,Y)$,

$$\mathcal{E}_{\mu^n}^{(p)}(f) \leqslant c^p \mathcal{E}_{\mu^m}^{(p)}(f). \tag{4.1}$$

If ν itself is a probability measure, we also say that a Poincaré inequalty holds if exists $\bar{c} > 0$ such that for any f,

$$\int_{X\times X} d\nu(x) d\nu(x') d_Y(f(x), f(x'))^p \leqslant \bar{c}^p \mathcal{E}_{\mu^m}^{(p)}(f). \tag{4.2}$$

Inequality (4.2), when Y is a Hilbert space and $p = 2$ is the classical Poincaré inequality. It is sometimes easier to work with than the inequality (4.1) (for example when proving such results as the extrapolation lemma below). It will be inequality (4.1), however, that will be used for obtaining fixed point properties for the random group. Note that inequality (4.2) can be thought of as the limit as $n \to \infty$ of (4.1).

Lemma 4.2. *Let ν be a probability measure.*

1. *Assume that* (4.2) *holds with the constant $\bar{c}$. Then* (4.1) *holds with $c = 2\bar{c}$ for all $n > m$.*

(2) *Assume that Y is p-uniformly convex, and let $V^{(p)}(f) = \int_X d\nu(x) d_Y^p(f(x), c_p(f_*\nu))$. Then*

$$V^{(p)}(f) \leqslant \int_{X\times X} d\nu(x)d\nu(x')d_Y^p(f(x), f(x')) \leqslant 2^{p-1}V^{(p)}(f).$$

Proof. For any $x, x', x'' \in X$ we raise the triangle inequality to the pth power to obtain:

$$d_Y(f(x), f(x'))^p \leqslant 2^{p-1}d_Y(f(x), f(x''))^p + 2^{p-1}d_Y(f(x'), f(x''))^p.$$

Integrating against $d\nu(x)d\mu^n(x \to x')d\nu(x'')$ and using the stationarity and reversibility of the Markov chain gives:

$$\mathcal{E}^{(p)}_{\mu^n}(f) \leqslant 2^{p-1} \int_{X\times X} d\nu(x)d\nu(x')d_Y(f(x), f(x'))^p,$$

whence the first claim. For the proof of the second claim write $y_0 = c_p(f_*\nu)$, and recall that $\int_X d\nu(x)d_Y(f(x), y_0)^p \leqslant \int_X d\nu(x)d_Y(f(x), y)^p$ holds for all $y \in Y$ by definition of c_p. Setting $y = f(x')$ and averaging w.r.t. x' gives half of the inequality. For the other half use $d_Y(f(x), f(x')) \leqslant d_Y(f(x), y_0) + d_Y(y_0, f(x'))$. □

We study metric inequalities for functions on finite Markov chains (typically, the standard random walks on finite graphs). In the following we use the shorthand (V, μ, ν) for the data of a finite set V ("vertices"), and a Markov chain $\mu \in \mathcal{W}(V)$ reversible with respect to a probability measure $\nu \in \mathcal{M}(V)$. Recall that the Markov chain is *ergodic* if for any $u, v \in V$ there is n such that $\mu^n(u \to v) > 0$. For such a Markov chain the averaging operator $A_\mu^(2)$ acting on $L^2(\nu)$ is the usual nearest-neighbour averaging operator $Af(u) = \int_V f(v)d\mu(u \to v)$. It is well-known that this is a self-adjoint operator with spectrum contained in $[-1, 1]$, with 1 a simple eigenvalue (here we use ergodicity). The *spectral gap* of the chain is then the difference between 1 and the second largest eigenvalue.

Definition 4.3. To the metric space Y we associate its *Poincaré modulus* of exponent p, $p \geqslant 2$. Denoted $\Lambda_Y^{(p)}(\sigma)$, it is the smallest number Λ such that for any finite reversible ergodic Markov chain (V, μ, ν) with spectral gap at least σ and any function $f\colon V \to Y$ we have

$$\int_{V\times V} d\nu(u)d\nu(v)d_Y(f(u), f(v))^p \leqslant \Lambda^p \int_{V\times V} d\nu(u)d\mu(u \to v)d_Y(f(u), f(v))^p. \tag{4.3}$$

Observe that spectrally expanding both sides of (4.3) shows that for Y Hilbert space, $\Lambda_Y^{(2)}(\sigma) = \frac{1}{\sqrt{\sigma}}$.

We also define the *Local Poincaré modulus* of exponent p to be

$$\Lambda_Y^{(p)}(\sigma, N) = \sup\left\{\Lambda_{Y'}^{(p)}(\sigma) \mid Y' \subseteq Y,\, |Y'| \leqslant N\right\}.$$

We say that Y has *small Poincaré moduli* of exponent p if its local Poincaré moduli of that exponent satisfy

$$\Lambda_Y^{(p)}(\sigma, N) \lesssim_{p,\sigma} o\left(\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{2p}}\right). \tag{4.4}$$

Note that a bound of $O(\log N)$ in (4.4) holds true for any metric space, by Bourgain's embedding theorem [7] and (4.6) below.

We shall proceed to bound the Poincaré modulus for non-Hilbertian spaces, i.e., to show that a Poincaré inequality holds for Markov chains on these spaces, with the constant bounded by a function of the spectral gap of the chain. The first case is that of L_p. The proof below is a slight variant of Matoušek's extrapolation lemma for Poincaré inequalities; see [27], and [6, Lem. 5.5]; we include it since it has been previously stated for graphs rather than general Markov chains.

Lemma 4.4 (Matoušek extrapolation)**.** *Let (V,μ,ν) be a reversible Markov chain. Assume the Poincaré inequality* (4.2) *holds with exponent $p \geqslant 1$ and Poincaré modulus Ap for functions $f\colon V \to \mathbb{R}$. Then for any $q \geqslant p$ the inequality* (4.2) *holds for such functions with the exponent q and modulus $4Aq$ and for any $1 < q \leqslant p$ the inequality holds with exponent q and modulus Aq.*

Proof. For $u \in V$ set $g(u) = |f(u)|^{\frac{q}{p}} \operatorname{sgn} f(u)$. Shifting f by a constant does not change the claimed inequalities, and using the intermediate value theorem we may assume $\int g d\nu = 0$. By the convexity of the norm, Hölder's inequality, and the assumed Poincaré inequality, we have:

$$\begin{aligned}\|g\|_{L_p(\nu)} &= \left\| g - \int g d\nu \right\|_{L_p(\nu)} \leqslant \int d\nu(v) \, \|g-g(v)\|_{L_p(\nu)} \leqslant \left(\int d\nu(v) \, \|g-g(v)\|^p_{L_p(\nu)} \right)^{\frac{1}{p}} \\ &= \left(\int d\nu(u) d\nu(v) \, |g(u)-g(v)|^p \right)^{\frac{1}{p}} \leqslant (Ap) \left(\int d\nu(u) d\mu(u \to v) \, |g(u)-g(v)|^p \right)^{\frac{1}{p}}.\end{aligned}$$

We next use the elementary inequality

$$\left| a^{\frac{q}{p}} \pm b^{\frac{q}{p}} \right| \leqslant \frac{q}{p} \, |a \pm b| \left(a^{\frac{q}{p}-1} + b^{\frac{q}{p}-1} \right),$$

to deduce that:

$$\begin{aligned}\|g\|_{L_p(\nu)} &\leqslant (Aq) \left[\int d\nu(u) d\mu(u \to v) \, |f(u)-f(v)|^p \left(|f(u)|^{\frac{q}{p}-1} + |f(v)|^{\frac{q}{p}-1} \right)^p \right]^{\frac{1}{p}} \\ &\leqslant (Aq) \left[\int d\nu(u) d\mu(u \to v) \, |f(u)-f(v)|^q \right]^{\frac{1}{q}} \\ &\quad \cdot \left[\int d\nu(u) d\mu(u \to v) \left(|f(u)|^{\frac{q}{p}-1} + |f(v)|^{\frac{q}{p}-1} \right)^{\frac{qp}{q-p}} \right]^{\frac{q-p}{pq}}, \end{aligned} \tag{4.5}$$

where we used Hölder's inequality.

By the triangle inequality in $L_{qp/(q-p)}$, symmetry and reversibility, the last term in (4.5) is at most:

$$2 \left[\int d\nu(u) \, |f(u)|^{\frac{q-p}{p} \cdot \frac{qp}{q-p}} \right]^{\frac{q-p}{pq}} = 2 \, \|f\|^{\frac{q-p}{p}}_{L_q(\nu)}.$$

Recalling that $|g(u)| = |f(u)|^{\frac{q}{p}}$, this means

$$\|f\|^{\frac{q}{p}}_{L_q(\nu)} \leqslant (2Aq) \left[\int d\nu(u) d\mu(u \to v) \, |f(u)-f(v)|^q \right]^{\frac{1}{q}} \|f\|^{\frac{q}{p}-1}_{L_q(\nu)},$$

and collecting terms finally gives

$$\|f\|_{L_q(\nu)} \leqslant (2Aq) \left[\int d\nu(u) d\mu(u \to v) \, |f(u) - f(v)|^q \right]^{\frac{1}{q}}.$$

To conclude the proof we note that

$$\left[\int d\nu(u) d\nu(v) \, |f(u) - f(v)|^q \right]^{\frac{1}{q}} \leqslant 2 \, \|f\|_{L_q(\nu)}$$

follows by applying the triangle inequality in $L_q(\nu \times \nu)$ to the functions $(u, v) \mapsto f(u)$ and $(u, v) \mapsto -f(u)$. $\square$

Corollary 4.5. *We have $\Lambda_{\mathbb{R}}^{(p)}(\sigma) \leqslant 2p\frac{1}{\sqrt{\sigma}}$. Integrating, this bound also holds for for $\Lambda_{L_p}^{(p)}(\sigma)$.*

Since Hilbert space embeds isometrically into L_p for all $p \geqslant 1$, we see that for $p \geqslant 2$,

$$\Lambda_{L_2}^{(p)}(\sigma) \leqslant \Lambda_{L_p}^{(p)}(\sigma) \leqslant \frac{2p}{\sqrt{\sigma}}. \tag{4.6}$$

Remark 4.6. In [27] it is shown that any N-point metric space embeds in L_p with distortion $\lesssim 1 + \frac{1}{p} \log_N$. It follows that for any metric space Y and any exponent $p \geqslant 2$,

$$\Lambda_Y^{(p)}(\sigma, N) \lesssim \frac{p + \log N}{\sqrt{\sigma}}.$$

Remark 4.7. The argument above was special for L_p spaces. But, using a different method, it was shown in [34] that for any Banach lattice Y that does not contain almost isometric copies of every finite metric space, we have $\Lambda_Y^{(2)}(\sigma) \lesssim_{Y,\sigma} 1$. While this is not stated explicitly in [34], it follows easily from the proof of Lemma A.4 there; this observation is carried out in detail in [30].

We also note for future reference that the Poincaré modulus behaves well under natural operations on metric spaces. The (trivial) proof is omitted.

Proposition 4.8. *Fix a function $L(\sigma, N)$ and let $\mathcal{C}$ be the class of metric spaces Y such that $\Lambda_Y^{(p)} \leqslant L$. Then $\mathcal{C}$ is closed under completion, passing to subspaces, ℓ_p products, and ultralimits. The property of being p-uniformly convex with constant c_Y is also preserved by the same operations, except that that one must pass to convex (i.e. totally geodesic) subspaces.*

In the class of CAT(0) spaces, a further reduction is possible: it is enough to establish the Poincaré inequality for all the *tangent cones* of the space Y. This is essentially an observation from [38, Pf of Thm. 1.1] (see also [20, Lem. 6.2]). It relies on the equivalent formulation from Lemma 4.2. We recall the definition of the *tangent cone* to a metric space Y at the point $y \in Y$. Let $\gamma, \gamma' \colon : [0, \varepsilon] \to Y$ be unit-speed geodesic segments issuing from y. For each $t > 0$ let $\theta_{t,t'}$ be the angle such that

$$d_Y(\gamma(t), \gamma'(t'))^2 = d_Y(y, \gamma(t))^2 + d_Y(y, \gamma'(t'))^2 - 2 d_Y(y, \gamma(t)) d_Y(y, \gamma'(t')) \cos \theta_{t,t'}.$$

The *Alexandroff angle* between γ, γ' is defined as $\theta(\gamma, \gamma') = \limsup_{t,t' \to 0} \theta_{t,t'}$. It is easy to check that this provides a pesudometric on the space of germs of geodesic segments issuing from y. Identifying segments at angle zero gives the *space of directions* $S_y Y$. Now let $T_y Y$ be the infinite cone over $S_y Y$ with the metric $\tilde{d}(a\gamma, b\gamma') = \sqrt{a^2 + b^2 - 2ab \cos \theta(\gamma, \gamma')}$. There is a natural "inverse of the exponential map" $\pi_y\colon Y \to T_yY$ given by mapping $z \in Y$ to $d_Y(y,z)\cdot[y,z]$ where $[y,z]$ is the geodesic segment connecting y to z ($\pi_y(y)$ is the cone point). By definition π_y preserves distances from y, in that $\tilde{d}(\pi_y(y),\pi_y(z)) = d_Y(y,z)$. The key properties for us are that when Y is CAT(0), π_y is 1-Lipschitz (in fact, this is equivalent to the CAT(0) inequality) and that in that case $(T_yY,\tilde{d})$ is a CAT(0) metric space as well. [9, Thm. II.3.19]. It is also important to note that if σ is a probability measure on Y and $y = \mathrm{c}_2(\sigma)$ then $\mathrm{c}_2(\pi_{y*}\sigma) = \pi_y(y)$ (this is since the fact that y minimizes $z \mapsto d_Y(\sigma,z)$ can be stated in terms of distances from y alone; see [20, Prop. 3.5]).

The following proposition was proved in an equivalent form in [38].

Proposition 4.9. *Let Y be a* CAT(0) *space. Then*

$$\Lambda_Y^{(2)}(\sigma,N) \leqslant 2\sup_{y\in Y}\Lambda_{T_yY}^{(2)}(\sigma,N).$$

In particular, $\Lambda_Y^{(2)}(\sigma) \leqslant 2\sup_{y\in Y}\Lambda_{T_yY}^{(2)}(\sigma)$.

Proof. Let (V,μ,ν) be a finite Markov chain as above. For a CAT(0) space Y let $v(Y)$ be minimal such that for all $f\colon V\to Y$ we have

$$V^{(2)}(f) \leqslant v(Y)\mathcal{E}_{\mu^m}^{(2)}(f).$$

Lemma 4.2 shows that the constant c in the Poincaré inequality for functions from V to Y satisfies $v(Y) \leqslant c \leqslant 2v(Y)$. It thus remains to show that $v(Y) \leqslant \sup_{y\in Y} v(T_yY)$. Indeed, let $f\colon X\to Y$, and let $y = \mathrm{c}_2(f_*\nu)$, $\tilde{f} = \pi_y\circ f$. As noted above we have $\mathrm{c}_2(\tilde{f}_*\nu) = \pi_y(y)$, and since distances from y are preserved that $V^{(2)}(f) = V^2(\tilde{f})$. Since π_y is non-expansive, $\mathcal{E}_{\mu^m}^{(2)}(\tilde{f}) \leqslant \mathcal{E}_{\mu^m}^{(2)}(f)$. It follows that $V^{(2)}(f) \leqslant v(T_{\mathrm{c}_2(f_*\nu)})\mathcal{E}_{\mu^m}^{(p)}(f)$ and we are done. ☐

Note that when Y is a Riemannian manifold, the tangent cone constructed above is isometric to the ordinary tangent space at y, equipped with the inner product given by the Riemannian metric at that point. in other words, the tangent cones of a manifold are all isometric to Hilbert spaces. An approximation argument also shows that $\Lambda_Y^{(2)}(\sigma) \geqslant \Lambda_{T_yY}^{(2)}(\sigma)$ for all $y\in Y$.

Corollary 4.10. *Let Y be a Hilbert manifold with a* CAT(0) *Riemannian metric (for example, a finite-dimensional simply connected Riemannian manifold of non-positive sectional curvature). Then $\frac{1}{\sqrt{\sigma}} \leqslant \Lambda_Y^{(2)}(\sigma) \leqslant \frac{2}{\sqrt{\sigma}}$.*

5. Padded decomposability and Nagata dimension

We start by recalling some definitions and results from [26]. Let (X,d_X) be a metric space. Given a partition $\mathscr{P}$ of X and $x\in X$ we denote by $\mathscr{P}(x)$ the unique element of $\mathscr{P}$ containing x. For $\Delta>0$, a distribution Pr over partitions of X is called a Δ-bounded stochastic decomposition if

$$\Pr\left[\forall\ C\in\mathscr{P},\ \mathrm{diam}(C)\leqslant\Delta\right] = 1,$$

i.e., almost surely with respect to Pr partitions of X contain only subsets whose diameter is bounded by Δ. Given $\varepsilon,\delta>0$ we shall say that a Δ-bounded stochastic decomposition Pr is (ε,δ)-padded if for every $x\in X$,

$$\Pr\left[\mathscr{P}(x)\supseteq B_X(x,\varepsilon\Delta)\right]\geqslant\delta.$$

Here, and in what follows, $B_X(x,r) \stackrel{\mathrm{def}}{=} \{y \in X :\ d_X(x,y) \leqslant r\}$ denotes the closed unit ball of radius r centered at x.

Given two metric spaces (Y, d_Y) and (Z, d_Z), and $X \subseteq Y$, we denote by $e(X,Y,Z)$ the infimum over all constant K such that every Lipschitz function $f : X \to Z$ can be extended to a function $\tilde{f} : Y \to Z$ such that $\|\tilde{f}\|_{\mathrm{Lip}} \leqslant K \cdot \|f\|_{\mathrm{Lip}}$. The *absolute Lipschitz extendability* constant of (X, d_X), denoted $\mathrm{ae}(X)$, is defined as

$$\mathrm{ae}(X) \stackrel{\mathrm{def}}{=} \sup \left\{ e(X,Y,Z) :\ Y \supseteq X,\ Z \text{ a Banach space} \right\}.$$

In words, the inequality $\mathrm{ae}(X) < K$ implies that any Banach space valued Lipschitz mapping on X can be extended to any metric space containing X such that the Lipschitz constant of the extension grows by at most a factor of K. This notion was introduced in [26], where several classes of spaces were shown to be absolutely extendable. We note that in the extension theorems we quote below from [26] the role of the target space being a Banach space is very weak, and it can also be, for example, any CAT(0) space; we refer to [26] for a discussion of this issue.

The following theorem was proved in [26].

Theorem 5.1 (Absolute extendability criterion [26])**.** *Fix $\varepsilon, \delta \in (0,1)$ and assume that (X, d_X) admits a 2^k-bounded (ε, δ)-padded stochastic decomposition for every $k \in \mathbb{Z}$. Then*

$$\mathrm{ae}(X) \lesssim \frac{1}{\varepsilon\delta}.$$

In [26] several classes of spaces were shown to satisfy the conditions of Theorem 5.1, including subsets of Riemannian surfaces of bounded genus and doubling metric spaces. For our applications we need to enrich the repertoire of these spaces. We do so by relating the notion of padded decomposability to having finite Nagata dimension, and using results from [24] which bound the Nagata dimension of various classes of spaces (which will be listed shortly).

Let (X, d_X) be a metric space. Following [28, 24], given $\gamma \geqslant 1$ and $d \in \mathbb{N}$ we say that X has Nagata dimension at most d with constant γ if for every $s > 0$ there exists a family of subsets $\mathscr{C} \subseteq 2^X \setminus \{\emptyset\}$ with the following properties.

(1) $\mathscr{C}$ covers X, i.e. $\bigcup_{C \in \mathscr{C}} C = X$.
(2) For every $C \in \mathscr{C}$, $\mathrm{diam}(C) \leqslant \gamma s$.
(3) For every $A \subseteq X$ with $\mathrm{diam}(A) \leqslant s$, we have $|\{C \in \mathscr{C} :\ C \cap A \neq \emptyset\}| \leqslant d+1$.

The infimum over all γ for which X has Nagata dimension at most d with constant γ will be denoted $\gamma_d(X)$. If no such γ exists we set $\gamma_d(X) = \infty$. Finally, the Nagata dimension of X is defined as

$$\dim_N(X) = \inf \left\{ d \geqslant 0 :\ \gamma_d(X) < \infty \right\}.$$

It was proved in [24] that X has finite Nagata dimension if and only if X embeds quasisymmetrically into a product of finitely many trees.

Lemma 5.1 (Bounded Nagata dimension implies padded decomposability)**.** *Let (X, d_X) be a metric space, $\gamma \geqslant 1$ and $d \in \mathbb{N}$. Assume that $\gamma_d(X) < \gamma < \infty$. Then for every $k \in \mathbb{Z}$, X admits a 2^k-bounded $\left(\frac{1}{100\gamma d^2}, \frac{1}{d+1}\right)$-padded stochastic decomposition.*

Proof. It is easy to iterate the definition of Nagata dimension to prove the following fact, which is (part of) Proposition 4.1 in [24] (with explicit, albeit sub-optimal, estimates, that can be easily obtained from an examination of the proof in [24]). Let $r = 50\gamma \cdot d^2$. For every $j \in \mathbb{Z}$ there exists a family of subsets $\mathscr{B} \subseteq 2^X \setminus \{\emptyset\}$ with the following properties.

(1) For every $x \in X$ there exists $B \in \mathscr{B}$ such that $B_X(x, r^j) \subseteq B$.
(2) $\mathscr{B} = \bigcup_{i=0}^d \mathscr{B}_i$, where for every $i \in \{0, \ldots, d\}$ the sets in $\mathscr{B}_i$ are disjoint, and for every $B \in \mathscr{B}_i$, $\mathrm{diam}(B) \leqslant r^{j+1}$.

We now construct a random partition P of X as follows. Let π be a permutation of $\{0, \ldots, d\}$ chosen uniformly at random from all such $(d+1)!$ permutations. Define a family of subsets $\widetilde{\mathscr{B}}_i^\pi \subseteq 2^X \setminus \{\emptyset\}$ inductively as follows: $\widetilde{\mathscr{B}}_0^\pi = \mathscr{B}_{\pi(0)}$, and for $0 \leqslant i < d$,

$$\widetilde{\mathscr{B}}_{i+1}^\pi = \left\{ B \setminus \bigcup_{C \in \bigcup_{\ell=0}^i \widetilde{\mathscr{B}}_{\pi(\ell)}^\pi} C : \; B \in \mathscr{B}_{\pi(i+1)} \right\} \setminus \{\emptyset\}.$$

Finally we set $\mathscr{P}^\pi = \bigcup_{i=0}^d \widetilde{\mathscr{B}}_i^\pi$. Since $\mathscr{B}$ covers X, $\mathscr{P}^\pi$ is a partition of X. Moreover, by construction, for every $C \in \mathscr{P}^\pi$, $\mathrm{diam}(C) \leqslant r^{j+1}$.

Fix $x \in X$. By the first condition above there exists $i \in \{0, \ldots, d\}$ and $B \in \mathscr{B}_i$ such that $B_X(x, r^j) \subseteq B$. If $\pi(0) = i$ then $\mathscr{P}^\pi(x) = B \supseteq B(x, r^j)$. This happens with probability $\frac{1}{d+1}$.

Letting k be the largest integer j such that $r^{j+1} \leqslant 2^k$ we see that $\mathscr{P}^\pi$ is a 2^k-bounded stochastic partition such that for every $x \in X$

$$\Pr\left[\mathscr{P}^\pi(x) \supseteq B\left(x, \frac{2^{k-1}}{r}\right)\right] \geqslant \frac{1}{d+1},$$

as required. □

The following corollary shows that many of the Lipschitz extension theorems proved in [24] are direct conequences of the earlier results of [26]. The cubic dependence on the Nagata dimension is an over-estimate, and can be easily improved. We believe that the true bound should depend linearly on the dimension, but this is irrelevant for the purposes of the present paper.

Corollary 5.2. *For every metric space X and $d \in \mathbb{N}$,*

$$\mathrm{ae}(X) = O\left(\gamma_d(X) d^3\right).$$

Thus, doubling metric spaces, subsets of compact Riemannian surfaces, Gromov hyperbolic spaces of bounded local geometry, Euclidean buildings, symmetric spaces, and homogeneous Hadamard manifolds, all have finite absolute extendability constant.

The list of spaces presented in Corollary 5.2 is a combination of the results of [26] and [24]. In particular the last four classes listed in Corollary 5.2 were shown in [24] to have finite Nagata dimension. It should be remarked here that Lipschitz extension theorems for Gromov hyperbolic spaces of bounded local geometry were previously proved in [29] via different methods.

We will use the following embedding theorem, which follows from the proof of Theorem 5.1 in [25], though it isn't explicitly stated there in full generality. We include the simple proof for the sake of completeness.

Theorem 5.2 (Snowflake embedding)**.** *Fix $\varepsilon,\delta,\theta\in(0,1)$. Let (X,d_X) be a metric space which admits for every $k\in\mathbb{Z}$ a 2^k-bounded (ε,δ)-padded stochastic decomposition. Then the metric space $\left(X,d_X^\theta\right)$ embeds into Hilbert space with bi-Lipschitz distortion $\lesssim\frac{1}{\varepsilon\sqrt{\delta\theta(1-\theta)}}$.*

Proof. For every $k\in\mathbb{Z}$ let Pr_k be an (ε,δ)-padded distribution over 2^k-bounded partitions of X. We also let $\{\sigma_C\}_{C\subseteq X}$ be i.i.d. symmetric ±1 Bernoulli random variables, which are independent of Pr_k. Denote by Ω_k the measure space on which all of these distributions are defined. Let $f_k:X\to L_2(\Omega_k)$ be given by the random variable

$$f_k(x)=\sigma_{\mathscr{P}(x)}\cdot\min\left\{d_X\left(x,X\smallsetminus\mathscr{P}(x)\right),2^k\right\}\quad(\mathscr{P}\text{ is a partition of }X).$$

Finally, define $F:X\to\left(\bigoplus_{k\in\mathbb{Z}}L_2(\Omega_k)\right)\otimes\ell_2$ by

$$F(x)=\sum_{k\in Z}2^{-k(1-\theta)}f_k(x)\otimes e_k.$$

Fix $x,y\in X$ and let $k\in\mathbb{Z}$ be such that $2^k<d_X(x,y)\leqslant 2^{k+1}$. It follows that for every 2^k-bounded partition $\mathscr{P}$ of X, $\mathscr{P}(x)\neq\mathscr{P}(y)$. Thus $\sigma_{\mathscr{P}(x)}$ and $\sigma_{\mathscr{P}(y)}$ are independent random variables, so that

$$\begin{aligned}
&\|F(x)-F(y)\|_2^2\geqslant 2^{-2k(1-\theta)}\left\|f_k(x)-f_k(y)\right\|^2_{L_2(\Omega_k)}\\
&=\ \frac{\mathbb{E}_\sigma\mathbb{E}_{\mathrm{Pr}_k}\left[\sigma_{\mathscr{P}(x)}\cdot\min\left\{d_X\left(x,X\smallsetminus\mathscr{P}(x)\right),2^k\right\}-\sigma_{\mathscr{P}(y)}\cdot\min\left\{d_X\left(y,X\smallsetminus\mathscr{P}(y)\right),2^k\right\}\right]^2}{2^{2k(1-\theta)}}\\
&\overset{(\clubsuit)}{=}\ \frac{\mathbb{E}_{\mathrm{Pr}_k}\left[\min\left\{d_X\left(x,X\smallsetminus\mathscr{P}(x)\right)^2,2^{2k}\right\}\right]+\mathbb{E}_{\mathrm{Pr}_k}\left[\min\left\{d_X\left(y,X\smallsetminus\mathscr{P}(y)\right)^2,2^{2k}\right\}\right]}{2^{2k(1-\theta)}}\\
&\overset{(\spadesuit)}{\geqslant}\ \frac{\delta\left(\varepsilon 2^k\right)^2}{2^{2k(1-\theta)}}\geqslant\frac{\varepsilon^2\delta}{2^{2\theta}}\cdot d_X(x,y)^{2\theta},
\end{aligned}\tag{5.1}$$

where in $(\clubsuit)$ we used the independence of $\sigma_{\mathscr{P}(x)}$ and $\sigma_{\mathscr{P}(y)}$, and in $(\spadesuit)$ we used the (ε,δ)-padded property.

In the reverse direction, for every $j\in\mathbb{Z}$, if $\mathscr{P}$ is a 2^j-bounded partition of X then it is straightforward to check that for all $x,y\in X$ we have the point-wise inequality,

$$\begin{aligned}
&\left|\sigma_{\mathscr{P}(x)}\cdot\min\left\{d_X\left(x,X\smallsetminus\mathscr{P}(x)\right),2^j\right\}-\sigma_{\mathscr{P}(y)}\cdot\min\left\{d_X\left(y,X\smallsetminus\mathscr{P}(y)\right),2^j\right\}\right|\\
&\qquad\leqslant 2\min\left\{d_X(x,y),2^j\right\}.
\end{aligned}\tag{5.2}$$

Indeed, if $d_X(x,y)\geqslant 2^j$ then (5.2) is trivial. If $\mathscr{P}(x)=\mathscr{P}(y)$ then (5.2) follows from the Lipschitz condition $\left|d_X\left(x,X\smallsetminus\mathscr{P}(x)\right)-d_X\left(y,X\smallsetminus\mathscr{P}(x)\right)\right|\leqslant d_X(x,y)$. Finally, if $d_X(x,y)<2^j$ and $\mathscr{P}(x)\neq\mathscr{P}(y)$ then $d_X\left(x,X\smallsetminus\mathscr{P}(x)\right),d_X\left(y,X\smallsetminus\mathscr{P}(y)\right)\leqslant d_X(x,y)<2^j$, implying (5.2) in this case as well.

It follows from (5.2) that

$$\begin{aligned}
\|F(x)-F(y)\|_2^2&\lesssim\sum_{j\in\mathbb{Z}}\frac{\min\left\{d_X(x,y)^2,4^j\right\}}{4^{j(1-\theta)}}\lesssim\sum_{j\leqslant k}4^{j\theta}+d_X(x,y)^2\sum_{j\geqslant k+1}4^{-j(1-\theta)}\\
&\lesssim\frac{4^{k\theta}}{\theta}+d_X(x,y)^2\cdot\frac{4^{-k(1-\theta)}}{1-\theta}\lesssim\frac{d_X(x,y)^{2\theta}}{\theta(1-\theta)}.
\end{aligned}\tag{5.3}$$

Combining (5.1) and (5.3), we get that the bi-Lipschitz distortion of f is $\lesssim\frac{1}{\varepsilon\sqrt{\delta\theta(1-\theta)}}$. $\square$

Corollary 5.3. *Let (Y, d_Y) be a metric space which admits for every $k \in \mathbb{Z}$ a 2^k-bounded (ε, δ)-padded stochastic decomposition (thus, (Y, d_Y) can belong to one of the classes of spaces listed in Corollary 5.2). Then, using the notation of Section 4, for every $p \in [1, \infty)$ we have*

$$\Lambda_Y^{(p)}(\sigma) \lesssim_{\varepsilon,\delta,p,\sigma} 1. \tag{5.4}$$

Proof. By Theorem 5.2 the metric space $\left(Y, \sqrt{d_Y}\right)$ embeds into Hilbert space with distortion $\lesssim_{\varepsilon,\delta} 1$. By (4.6) we know that $\Lambda_{L_2}^{(2p)}(\sigma) \lesssim_p \sigma^{-1/2}$. It follows that $\Lambda_Y^{(p)}(\sigma) \lesssim_{\varepsilon,\delta,p} \sigma^{-1} \lesssim_{\varepsilon,\delta,p,\sigma} 1$, as required. □

Remark 5.4. For our purposes the dependence on σ in (5.4) is irrelevant. Nevertheless, the proof Corollary 5.3 can be optimized as follows. For $\theta \in (0, 1)$, use Theorem 5.2 to embed the metric space (Y, d_Y^θ) into Hilbert space with distortion $\lesssim \frac{1}{\varepsilon\sqrt{\delta\theta(1-\theta)}}$. Since by (4.6) we know that $\Lambda_{L_2}^{(p/\theta)}(\sigma) \lesssim \frac{p}{\theta\sqrt{\sigma}}$. Thus, there exists a universal constant $c > 1$ such that

$$\Lambda_Y^{(p)}(\sigma) \leqslant \left(\frac{p}{\theta\sqrt{\sigma}} \cdot \frac{c}{\varepsilon\sqrt{\theta(1-\theta)}}\right)^{1/\theta}. \tag{5.5}$$

One can then choose θ so as to minimize the right hand side of (5.5). If one cares about the behavior of our bound as $\sigma \to 0$, then the optimal choice is $\theta = 1 - \frac{\log\log(1/\sigma)}{\log(1/\sigma)}$, yielding for $\sigma \in (0, 1/4)$, the estimate

$$\Lambda_Y^{(p)}(\sigma) \lesssim_{\varepsilon,\delta,p} \frac{\log(1/\sigma)}{\sqrt{\sigma}}. \tag{5.6}$$

Using the ideas presented here more carefully, the logarithmic term in (5.6) was subsequently removed in [30] (where the dependence on σ was of importance for certain applications).

6. A brief review of the construction of the random group

We recall here the "graph model" for random groups and the iterative construction of a group from an appropriate sequence of graphs. The construction is due to Gromov [14]; further details may be found in the works of Ollivier [31, 32], or in the more recent work of Arzhantseva-Delzant [3].

Let $G = (V, E)$ be an *undirected* simple graph. The set of edges E then has a natural double cover, the set of *oriented edges* of G

$$\vec{E} = \{(u, v), (v, u) \mid \{u, v\} \in E\}.$$

Now let Γ be a group. A *symmetric* Γ*-labeling* of G is a map $\alpha\colon \vec{E} \to \Gamma$ such that $\alpha(u, v) = \alpha(v, u)^{-1}$ for all $\{u, v\} \in E$. The set of these will be denoted $\mathcal{A}(G, \Gamma)$. More generally an S-labeling is a labeling whose image lies in a (symmetric) subset $S \subseteq \Gamma$. The set of such labels will be denotes $\mathcal{A}(G, S)$.

Let $S \subseteq \Gamma$ be a symmetric subset, $1 \notin S$. The *Cayley graph* $\mathrm{Cay}(\Gamma; S)$ is the graph with vertex set Γ and directed edge set $\{(x, xs) \mid x \in \Gamma, s \in S\}$. This is actually an undirected graph since S is symmetric and carries the natural symmetric labeling $\alpha(x, xs) = s$. The Cayley graph $\mathrm{Cay}(\Gamma; S)$ is connected iff S generates Γ. In that case let $\vec{c}$ be an oriented cycle (that is, a closed path) in that graph, and let $w \in S^*$ be the word in S read along the cycle. It is clear that w is trivial as an element of Γ. Conversely, any relator $w \in S^*$ for Γ induces many closed cycles on $\mathrm{Cay}(\Gamma; S)$: starting at any $x \in \Gamma$ one follows the edges labeled by successive letters in w. Since $w = 1$ in Γ, this path is a closed cycle in the Cayley graph. This observation motivates the following construction.

Given a symmetric Γ-labeling $\alpha \in \mathcal{A}(G, \Gamma)$ and an oriented path $\vec{p} = (\vec{e}_1, \ldots, \vec{e}_r)$ in G, we set $\alpha(\vec{p}) = \alpha(\vec{e}_1) \cdot \ldots \cdot \alpha(\vec{e}_r)$. We write

$$R_\alpha = \{\alpha(\vec{c}) \mid \vec{c} \text{ a cycle in } G\},$$

and will consider groups of the form

$$\Gamma_\alpha = \Gamma / \langle R_\alpha \rangle^{\mathrm{N}}, \tag{6.1}$$

where $\langle R_\alpha \rangle^{\mathrm{N}}$ is the normal closure of $\langle R_\alpha \rangle$. Alternatively, given a presentation $\Gamma = \langle S | R \rangle$ we also have $\Gamma_\alpha = \langle S | R \cup R_\alpha \rangle$ once we write the labels $\alpha(\vec{e})$ as words in S. Given $u \in V(G)$ and $x \in \mathrm{Cay}(\Gamma_\alpha; S)$ we define a map $\alpha_{u \to x} \colon G \to \mathrm{Cay}(\Gamma_\alpha; S)$ as follows. For $v \in V(G)$ choose a path $\vec{p}$ from u to v in G, and define $\alpha_{u \to x}(v) = x\alpha(\vec{p})$. Note that by construction, $\alpha_{u \to x}(v)$ does not depend on the choice of the path $\vec{p}$, and hence $\alpha_{u \to x}$ is well defined.

With a choice of a probability measure Pr on $\mathcal{A}(G, \Gamma)$, the groups Γ_α become "random groups". Note the ad-hoc nature of this construction: it is very useful for proving the existence of groups with desired properties (for example see [33]). However, the groups Γ_α are not "typical" in any sense of the word.

As above, let S be a symmetric set of generators for Γ. For any integer j let $\Pr_j$ on $\mathcal{A}(G, S^j)$ be given by independently assigning a label to each edge, uniformly at random from S^j. Fixing an *orientation* of E (i.e. a section $\iota \colon E \to \vec{E}$ of the covering map $\vec{E} \to E$) shows that that $\mathcal{A}(G, S^j)$ is non-canonically isomorphic to the product space E^{S_j} and identifies $\Pr_j$ with the natural product measure on that space.

Definition 6.1. ([31, Def. 50]) A sequence of finite connected graphs $\{G_i\}_{i=1}^\infty$ is called *good for random quotients* if there exist positive constants C, Δ such that:

1. The maximum degree of G_i satisfies $\Delta(G_i) \leqslant \Delta$.
2. The girth of G_i satisfies $g(G_i) \geqslant C \cdot \mathrm{diam}(G_i)$
3. $|V(G_i)|$ (equivalently, $g(G_i)$) tend to ∞ with i.

Theorem 6.2. *(*[31, Thm. 51], [3, Thm. 6.3]*) Let $\{G_i\}_{i=1}^\infty$ be good for random quotients, let Γ be a non-elementary torsion-free hyperbolic group with property (T), and let $\varepsilon > 0$. Then there exist $A > 0$, an integer $j \geqslant 1$ and a subsequence $\{i_k\}_{k \geqslant 1}$ such that for $G = \bigsqcup_{k \geqslant 1} G_{i_k}$ and α chosen from $\mathcal{A}(G, S^j)$ we have with positive* $\Pr_j$*-probability that:*

1. *For any $K \geqslant 1$ if we set $G_{(K)} = \bigsqcup_{k \leqslant K} G_{i_k}$ and $\alpha_{(K)} = \alpha \restriction_{G_{(K)}}$, then $\Gamma_{(K)} = \Gamma_{\alpha_{(K)}}$ is a torsion-free non-elementary hyperbolic group. In particular, Γ_α is an infinite group.*
2. *For any choice of vertices $u_0, v, w \in V(G_{i_k})$ and $x_0 \in \mathrm{Cay}(\Gamma_\alpha; S)$ the natural map $\alpha_{u_0 \to x_0} \colon G_{i_k} \to X_\alpha \overset{def}{=} \mathrm{Cay}(\Gamma_\alpha; S)$ has*
$$A\left(d_{G_{i_k}}(v, w) - \varepsilon \, \mathrm{diam}(G_{i_k})\right) \leqslant \frac{1}{j} d_{X_\alpha}(\alpha_{u_0 \to x_0}(v), \alpha_{u_0 \to x_0}(w)) \leqslant d_{G_{i_k}}(v, w).$$

When we apply Theorem 6.2 in Section 7, we will take the initial group Γ to be a free group. Even though Γ does not have property (T), Theorem 6.2 still applies if we assume that $\Gamma_{(1)}$, the quotient by the relations on G_{i_1}, satisfies the assumptions of Theorem 6.2. This happens with positive probability if we take i_1 large enough, as explained in the discussion preceding Definition 50 in [31]

7. From Poincaré inequalities to fixed points

Let $\{G_i\}_{i=1}^{\infty}$ be an expander family of graphs, with all vertices of degrees between 3 and d and $g(G_i) \gtrsim \log|V(G_i)|$. For later convenience we assume that the graphs are non-bipartite. Let $G = \bigsqcup_{i\geqslant 1} G_i$ be the disjoint union of the graphs.

Let $\Gamma = \langle S \rangle$ be free on the symmetric set of generators S of size $2k$. We set $X = \mathrm{Cay}(\Gamma; S)$; a $2k$-regular tree. As in Section 6, for $j \geqslant 1$ let $\mathcal{A}(G, S^j)$ denote the space of symmetric maps from the (directed) edges of G to S^j. Given $\alpha \in \mathcal{A}(G, S^j)$ let Γ_α be the quotient of Γ presented by declaring every word read along a cycle in G to be a relator. To every $\alpha \in \mathcal{A}(G, S^j)$ we associate its restrictions α_k to the copy of G_k.

Our model for random groups is obtained by choosing the value of α at each edge independently and uniformly at random. In Section 6 we reviewed the assumptions on G_i needed so that, with high probability, the group Γ_α is infinite. We now show that with probability 1 the quotient group Γ_α has strong fixed-point properties.

We follow below the lines of [37], with the natural changes that are required for handling powers p rather than powers 2, and p-uniformly convex metric spaces rather than CAT(0) spaces. Moreover, the handling of $j > 1$ in [37] was rather awkward. Taking advantage of the fact that we are reproducing much of the analysis of [37], we give a cleaner argument here for the case $j > 1$.

7.1. Simulating random walks and transferring Poincaré inequalities.

Let G be a connected finite graph (one of the G_i). We assume $3 \leqslant \delta(G) \leqslant \Delta(G) \leqslant d$ and let $g = g(G)$, $N = |V(G)|$. We choose $\alpha \in \mathcal{A}(G, S^j)$ uniformly at random. In particular, independently for each edge. Given $u, v \in V(G)$ such that $d_G(u, v) < g/2$, and $x \in X$, let $\beta_{u\to x}(v)$ denote the vertex $x\alpha(\vec{p})$ of X, where $\vec{p}$ is the unique shortest path joining u and v in G. Note that, using the notation of Section 6, $\pi_\alpha(\beta_{u\to x}(v)) = \alpha_{u\to x}(v)$, where $\pi_\alpha : X \to X_\alpha$ is the natural quotient map.

For every $q \in \mathbb{N}$, $q < g/2$, we define the random walk $\mu^q_{G,\alpha}$ on the tree X as follows:

$$\mu^q_{G,\alpha}(x \to \cdot) = \sum_{u\in G} \nu_G(u)\big((\beta_{u\to x})_* \mu^q_G(u \to \cdot)\big), \tag{7.1}$$

where μ_G is the standard random walk on G and ν_G is its stationary measure. Since $\beta_{u\to\gamma x}(v) = \gamma\beta_{u\to x}(v)$, equation (7.1) is a Γ-equivariant random walk on X.

For any fixed $x, x' \in X$, $\mu^q_{G,\alpha}(x \to x')$ is a random variable depending on the choice of α. We denote its expectation by $\bar{\mu}^q_{G,X}(x \to x') \in \mathcal{W}^\Gamma(X)$. It is important to note that while μ^q_G and μ^q_X are indeed q-fold convolutions of the random walks μ_G and μ_X, this is not the case for the other walks we consider such as $\mu^q_{G,\alpha}$.

The walks $\mu^q_{G,\alpha}(x \to x')$ will now be used to "simulate" the walks μ^n_X on X. Indeed, with high (asymptotic) probability the walks $\mu^q_{G,\alpha}(x \to x')$ are close to their expectation values $\bar{\mu}^q_{G,X}(x \to x')$, and these expectation values can be related to walks μ^n_X for appropriate values of n.

Equation (7.1) above furnishes the connection between the averaging notions on X and on G. For computations, however, we rewrite it as:

$$\mu^q_{G,\alpha}(x \to x') = \sum_{|\vec{p}|=q} \nu_G(p_0)\mu^q_G(\vec{p})\mathbb{1}\left(x\alpha(\vec{p}) = x'\right), \tag{7.2}$$

where the sum is over all oriented paths $\vec{p}$ of length q in G starting at p_0, and $\mathbb{1}(x = y)$ is the characteristic function of the diagonal of $X \times X$, so that $\alpha \mapsto \mathbb{1}(x\alpha(\vec{p}) = x')$ is an indicator random variable for the event that $\alpha(\vec{p})$ equals $x^{-1}x'$ as elements of Γ.

We now easily compute the mean walk $\bar{\mu}^q_{G,X}$. We start with the instructive case $q = 1$, where unwinding the definitions of ν_G and μ_G gives:

$$\mu^1_{G,\alpha}(x \to x') = \frac{1}{2\,|E(G)|} \sum_{\vec{e} \in \vec{E}} \mathbb{1}(x\alpha(\vec{e}) = x').$$

Taking expectation we conclude that $\bar{\mu}^1_{G,X}(x \to x')$ equals the probability that following a random word in S^j will lead us from x to x', that is $\mu^j_X(x \to x')$.

A similar calculation for $q > 1$ gives the following.

Lemma 7.1 (generalization of [37, Lem. 2.12])**.** *Let $q < g/2$. We can write $\bar{\mu}^q_{G,X}$ as a convex combination*

$$\bar{\mu}^q_{G,X} = \sum_{l=0}^{q} P^q_G(l)\mu^{jl}_X \tag{7.3}$$

where the weights $P^q_G(l)$ are concentrated on large values of l, in the sense that

$$Q^q_G \overset{def}{=} \sum_{l \leqslant q/6} P^q_G(l) \leqslant e^{-q/18}. \tag{7.4}$$

Also, wherever $\bar{\mu}^q_{G,X}(x \to x')$ is non-zero then it is at least

$$\varepsilon(d,k,j)^q \overset{def}{=} \left(\frac{1}{d(2k)^j}\right)^q. \tag{7.5}$$

Proof. Given a path $\vec{p}$ in G of length $q < g/2$, let $\tilde{p}$ be the shortest path connecting the endpoints of G. Since the ball of radius q in G around the starting vertex p_0 of $\vec{p}$ is a tree, $\tilde{p}$ is unique and can be obtained from $\vec{p}$ by successively cancelling "backtracks" (consecutive steps which traverse a single edge in opposite directions). This $\tilde{p}$ is a simple path, traversing each of its edges exactly once. It follows that the law of the Γ-valued random variable $\alpha \mapsto \alpha(\tilde{p})$ is that of a uniformly chosen element in S^{jl} where $l = |\tilde{p}|$. Moreover, the symmetry of the labelling α shows that the words $\alpha(\vec{p})$ and $\alpha(\tilde{p})$ are equal as elements of the free group Γ. In particular, the expectation of the indicator variable $\mathbb{1}\,(x\alpha(\vec{p}) = x')$ in (7.2) is $\mu^{jl}_X(x, x')$. Equation (7.3) now follows, with

$$P^q_G(l) = \sum_{|\vec{p}|=q, |\tilde{p}|=l} \nu_G(p_0)\mu^q_G(\vec{p}).$$

Note that $P^q_G(l)$ is precisely the probability that q steps of the stationary random walk on G travel a distance l. The bound (7.4) is established in [37, Lem. 2.12].

For the lower bound on $\bar{\mu}^q_{G,X}(x \to x')$ note first that for any path $\vec{p}$ in G of length q, $\mu^q_G(\vec{p}) \geqslant d^{-q}$ since every vertex has degree at most d. Now let $0 \leqslant l \leqslant q$ and assume that l, q have the same parity (if either condition fails then $P^q_G(l) = 0$). Then for any vertex p_0 there exists paths $\vec{p}$ of length q and reduced length l starting at p_0. It follows that $P^q_G(l) \geqslant \sum_{p_0} \nu_G(p_0)d^{-q} \geqslant d^{-q}$ for l as above.

Finally, let $x, x' \in X$ and let their distance be at most jq and have the same parity as jq (otherwise, for every term in (7.3) either $P^q_G(l)$ or $\mu^{jl}_X(x \to x')$ vanishes). Then the same argument shows that $\mu_X^{jq}(x \to x') \geqslant (2k)^{-jq}$. Equation (7.5) now follows from the estimate $\bar{\mu}^q_{G,X}(x \to x') \geqslant P^q_G(q)\mu_X^{jq}(x \to x')$. $\square$

Definition 7.2. We say that $\mu^\bullet_{G,\alpha}$ *effectively simulates* $\mu^\bullet_X$ *up to time* q_0 if for every $1 \leqslant q \leqslant q_0$ and every $x, x' \in X$ we have:

$$\mu^q_{G,\alpha}(x \to x') \geqslant \frac{1}{2}\bar{\mu}^q_{G,X}(x \to x'),$$

and in addition we have for every $x, x' \in X$:

$$\mu^1_{G,\alpha}(x \to x') \leqslant 2\mu^j_X(x \to x').$$

When the walks on G effectively simulate the walks on X we can transfer Poincaré inequalities from G to Γ_α:

Proposition 7.3. *Let $G = (V, E)$ be a finite graph on N vertices, and let σ be the spectral gap of G. Let $\alpha \in \mathcal{A}(G, S^j)$ be such that $\mu^\bullet_{G,\alpha}$ effectively simulates $\mu^\bullet_X$ up to a time $q_0 \gtrsim \log N$. Let Y be a metric space on which Γ_α acts by isometries. Write $B(X,Y)$ for the space of Γ-equivariant functions from X to Y where the free group Γ acts via its quotient Γ_α. Then for every $f \in B(X,Y)$ there exists m comparable to $\log N$ such that*

$$\mathcal{E}^{(p)}_{\mu_X^{jm}}(f) \lesssim \left(\Lambda^{(p)}_Y(\sigma, N)\right)^p \mathcal{E}^{(p)}_{\mu_X^j}(f).$$

Proof. By definition of $\mu^q_{G,\alpha}$, we have for $q < g/2$:

$$\left|\nabla_{\mu^q_{G,\alpha}}(f)\right|^p_p(x) = \sum_{u\in V} \nu_G(u)\left|\nabla_{\mu^q_G}(f \circ \beta_{u\to x})\right|^p_p(u). \tag{7.6}$$

Note that in (7.6) the composition $f \circ \beta_{u\to x}$ is well-defined since $\beta_{u\to x}(v)$ is defined for all $v \in V(G)$ with $d_G(u,v) < g/2$, and $q < g/2$. The same remark applies for the remainder of the computations below, where we treat $\beta_{u\to x}$ as a function even though it is only a partially defined function.

Since the action of Γ on Y factors via Γ_α, the function f can also be viewed as an equivariant function on X_α. Fixing $u_0 \in V$, we use this to set $f_0 = f \circ \alpha_{u_0\to x}$. Then for each $u \in V(G)$ we have

$$\left|\nabla_{\mu^q_G}(f \circ \beta_{u\to x})\right|^p_p(u) = \left|\nabla_{\mu^q_G}(f_0)\right|^p_p(u),$$

by projecting to X_α and translating by the element $\gamma \in \Gamma$ which sends $\alpha_{u_0\to x}(u)$ back to x. It follows that

$$\left|\nabla_{\mu^q_{G,\alpha}}(f)\right|^p_p(x) = 2\mathcal{E}^{(p)}_{\mu^q_G}(f_0). \tag{7.7}$$

Applying the Poincaré inequality (4.1) for maps from G to Y and using (7.7) on both sides we have:

$$\left|\nabla_{\mu^q_{G,\alpha}}(f)\right|^p_p(x) \lesssim \left(\Lambda^{(p)}_Y(\sigma(G), N)\right)^p \left|\nabla_{\mu_{G,\alpha}}(f)\right|^p_p(x). \tag{7.8}$$

If q is small enough then the assumption of effective simulation allows us to replace the random walks in (7.8) by their expectations up to a constant loss. Applying Lemma 7.1 and omitting some (non-negative) terms in the sum in (7.3), we find:

$$\min_{q\geqslant l>q/6}\left|\nabla_{\mu_X^{jl}}(f)\right|^p_p(x) \overset{(7.4)}{\leqslant} \sum_{q\geqslant l>q/6}\frac{P^q_G(l)}{1-Q^q_G}\left|\nabla_{\mu_X^{jl}}(f)\right|^p_p(x) \lesssim \left(\Lambda^{(p)}_Y(\sigma(G), N)\right)^p \left|\nabla_{\mu_X^j}(f)\right|^p_p(x).$$

By assumption we can take $q \asymp \log N$, and the proof is complete. $\square$

Proposition 7.4. *(generalization of* [37, Lem. 2.13]*) Let G be a finite graph with $3 \leqslant \delta(G) \leqslant \Delta(G) \leqslant d$. Let $N = |V(G)|$, and assume $g = g(G) \geqslant C \log N$. Then there exists $C' > 0$ depending on d, k, j, C so that the probability of $\mu^{\bullet}_{G,\alpha}$ failing to effectively simulate $\mu^{\bullet}_X$ up to time $C' \log N$ is $o_{d,k,j}(1)$ as $N \to \infty$.*

Proof. Since Γ acts transitively on X, our measure-valued random variables $\mu^q_{G,\alpha}(x \to \cdot)$ are determined by their value at any particular $x \in X$, which we fix. For each choice of α, the measure $\mu^q_{G,\alpha}(x \to \cdot)$ is supported on the ball $B_X(x, jq)$, so for each q we need to control $|B_X(x, jq)|$ real-valued random variables on $\mathcal{A}(G, S^j)$. Let $\mu^q_{G,\alpha}(x \to x')$ be one such random variable. We give a bound τ_q to its Lipschitz constant as a map from $\mathcal{A}(G, S^j)$ (equipped with the Hamming metric) to $[0, 1]$. For this it suffices to consider a pair of labelings α, α' which agree everywhere except at $e \in E$. We then have (sum over paths which traverse e at some point)

$$\left|\mu^q_{G,\alpha}(x \to x') - \mu^q_{G,\alpha'}(x \to x')\right| \leqslant \sum_{e \in \vec{p}} \nu_G(p_0)\mu^q_G(\vec{p}).$$

There are at most $2qd^{q-1}$ such paths, and each contributes at most $\frac{2d}{3N}3^{-q}$ to the right-hand-side since $\nu_G(u) = d(u)/2|E(G)|$. the vertex degrees allow us to take

$$\tau_q = \frac{4q}{3N}\left(\frac{d}{3}\right)^q.$$

We would like to rule out $\mu^q_{G,\alpha}(x \to x')$ deviating from its non-zero mean $\bar{\mu}^q_{G,X}(x \to x')$ by a factor of at least 2. It enough to bound the probability of deviation by at least $\frac{1}{2}\varepsilon(d, k, j)^q$, where $\frac{1}{2}\varepsilon(d, k, j)$ is as in (7.5)). Azuma's inequality (see, e.g., [1, Thm. 7.2.1]) shows that the probability for this is at most:

$$\exp\left\{-\frac{\varepsilon(d, k, j)^{2q}}{8\,|E(G)|\,\tau_q^2}\right\}.$$

We can choose C' small enough to ensure $\left(\frac{d}{3\varepsilon^2}\right)^{jq}$ is an arbitrary small power of N. Also $|E(G)| \lesssim_d N$, so the probability of deviation is exponentially small in a positive power of N. The number of random variables is polynomial in N (it is at most $(2k)^{qj}$ for each q) so we can take the union bound. A similar analysis shows that probability of some $\mu^1_{G,\alpha}(x \to x')$ being too large also decays. $\square$

7.2. **Fixed points.** Returning to G being the union of finite components G_i, we summarize the result of the previous section:

Theorem 7.5. *Assume that the $\{G_i\}_{i \geqslant 1}$ are connected non-bipartite graphs on N_i vertices with vertex degrees in $[3, d]$, spectral gaps $\sigma(G_i) \geqslant \sigma > 0$ and girths $\gtrsim \log N_i$. Let $G = \bigsqcup_{i \geqslant 1} G_i$ and let Γ_α be constructed at random from $\alpha \in \mathcal{A}(G, S^j)$ with j even. Then almost surely for every metric space Y, and every action of Γ_α on Y by isometries, there exists arbitrarily large N_i such that for any $f \in B(X_\alpha, Y)$ there exist m comparable to $\log N_i$ such that*

$$\mathcal{E}^{(p)}_{\mu_X^{jm}}(f) \lesssim \left(\Lambda^{(p)}_Y(\sigma, N_i)\right)^p \mathcal{E}^{(p)}_{\mu_X^{j}}(f).$$

Theorem 7.6. *Let Γ_α satisfy the conclusion of Theorem 7.5. Let Y be p-uniformly convex, and assume that $\Lambda_Y^{(p)}(\sigma) < \infty$ or, in greater generality, that*

$$\lim_{N\to\infty} \left(\frac{\log\log N}{\log N} \right)^{\frac{1}{2p}} \Lambda_Y^{(p)}(\sigma, N) = 0$$

(in the terminology of Definition 4.3, we are assuming that Y has small Poincaré moduli of exponent p). Then every isometric action of Γ_α on Y fixes a point.

Proof. By Theorems 3.10 and 7.5, there exist arbitrarily large N such that for any equivariant $f \in B(X_\alpha, Y)$ (identified with its pull-back to X) there is some m comparable to $\log N$ such that:

$$\mathcal{E}^{(p)}_{\mu_X^j}\left(A^{(p)}_{\mu_X^{jm}} f\right) \lesssim_{p,c_Y,j,d} \left(Q(N) + \frac{1}{\log N}\right) \mathcal{E}^{(p)}_{\mu_X^j}(f),$$

where $Q(N) \to 0$ as $N \to \infty$. Choosing N large enough, we see can ensure the existence of m such that

$$\mathcal{E}^{(p)}_{\mu_X^j}\left(A^{(p)}_{\mu_X^{jm}} f\right) \leqslant \frac{1}{2} \mathcal{E}^{(p)}_{\mu_X^j}(f).$$

Note that the choice of N was independent of f. Now Proposition 3.8 shows that iterating the averaging (with m depending on f but bounded by N) leads to a sequence converging to a fixed point (here $\Gamma\backslash X$ is a s single point, so $B(X, Y)$ is non-empty).

In more detail, let μ_{X_α} denote the standard random walk on X_α. We have in fact shown the existence of m such that

$$\mathcal{E}^{(p)}_{\mu_{X_\alpha}^j}\left(A^{(p)}_{\mu_{X_\alpha}^{jm}} f\right) \leqslant \frac{1}{2} \mathcal{E}^{(p)}_{\mu_{X_\alpha}^j}(f).$$

In order to apply Proposion 3.8 we further need to verify that a certain graph is connected – specifically the Cayley graph of Γ_α with respect to the set S^j. Since j is even S^j contains S^2 (as sets of elements of Γ_α), so it is enough to verify that S^2 is a set of generators for Γ_α. Indeed, the graphs G_i are non-bipartite and hence contain odd cycles. It follows that some relators in R_α have odd length, so that up to multiplication by a relator, every element of Γ_α can be represented by a word in S of even length. ☐

Remark 7.7. Theorem 7.6 was formulated for the limiting wild group Γ_α, i.e., the group corresponding to the infinite graph G. Arguing identically for the random group corresponding to the relations of each G_i separately, we obtain Theorem 1.1.

Acknowledgements. We are very grateful to the anonymous referee for the careful reading of our manuscript, and for many helpful suggestions.

Added in proof. Francois Dahmani pointed out to us that the answer to one of the questions that we asked in the introduction is known. Specifically, in [22, Sec. 8] it is shown how to construct a hyperbolic group with the fixed-point property on all symmetric spaces and buildings associated to linear groups.

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