ON THE MATHEMATICAL CONTRIBUTIONS OF JORAM LINDENSTRAUSS

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A deep, original, clever and prolific mathematician, Joram Lindenstrauss passed away on April 29, 2012. He was a pillar of modern functional analysis and an enormously influential champion of a tradition of excellence and uncompromising (at times, even harsh) pursuit of the highest possible quality of mathematical research. In addition to proving difficult theorems and solving longstanding open problems, Joram was a conceptual leader who was responsible for the formulation of foundational insights and major research directions that guided and shaped the intensive efforts of many researchers worldwide over several generations, and his legacy will undoubtedly continue to do so in the future.

Joram was our doctoral advisor, and a central and multifaceted source of influence on our mathematical careers. He is deeply missed.

The articles that are contained in this memorial volume are the works of mathematicians who were linked to Joram in various ways: advisees, collaborators, friends, relatives, and others who were influenced by him. The results themselves are, we believe, material that Joram would have enjoyed reading.

Below we describe examples of Joram's research achievements. We make no attempt for a comprehensive discussion, as this would require several monographs. Our goal is to present some highlights that introduce and summarize his outstanding body of work.

Even before his doctoral work, Joram made two influential contributions. His short and elegant proof of Lyapunov's convexity theorem [27] made its way into standard courses in functional analysis (see Rudin's classical book [41]) and also inspired further developments in other areas, including operator algebras [1]. In addition, Joram played an important role in the proof of a famous theorem of his advisor Dvoretzky on almost spherical sections of convex bodies; see [11, page 124]. Dvoretzky's theorem is a cornerstone of modern convex geometry with many powerful ramifications. Joram returned to this topic more than once in the ensuing decades, making important contributions.

Joram's doctoral dissertation dealt with extensions of operators between Banach spaces, a deep and insightful achievement that appeared as the research monograph [25], and led also to the study of preduals of L_1 (see e.g. [23]). Joram's ideas in this context had major influence on subsequent work, and even at the present followup papers are being published on a regular basis.

Another extremely influential contribution that Joram made early on in his career is his seminal work [26] on the nonlinear geometry of Banach spaces. In this work he proved theorems that exhibit remarkable, and arguably unexpected, rigidity phenomena for Banach spaces when they are deformed by uniformly continuous (or Lipschitz) mappings. Joram continued to investigate the quantitative nonlinear geometry of Banach spaces throughout his career, obtaining many deep and important results, and eventually leading to the formulation of a research program that aims to find a dictionary that translates Banach space notions and phenomena to the setting of general metric spaces; see the surveys [4, 38] for an exposition of (part of) this theme. As an example of Joram's achievements in the nonlinear geometry of Banach spaces, consider the Johnson–Lindenstrauss Lipschitz extension theorem [17], which answered a question of Marcus and Pisier [36]. The paper [17] of Johnson and Lindenstrauss discovered a famous dimensionality reduction lemma (by far Joram's most cited result), and asked several influential questions, including a question on Euclidean embeddings of finite metric spaces that was subsequently answered by Bourgain's embedding theorem [7], and a question on the possible validity of a Lipschitz version of Maurey's extension theorem [37], a question that was subsequently answered through the work of Ball [3] and Naor, Peres, Schramm and Sheffield [39].

Additional examples of results of Joram (with several coauthors) on the nonlinear theory of Banach spaces include a rigidity theorem [20] asserting that for any $p \in (1, \infty)$, if a Banach space X is uniformly homeomorphic to ℓ_p then X must be linearly isomorphic to ℓ_p (the case $p = 2$ of this statement was previously proved 27 years earlier by Enflo [12]). His study [5] of rigidity properties of (quantitatively) nonlinear quotient mappings relied on novel quantitative versions of differentiation. Joram's profound work on differentiation of Lipschitz functions culminated in deep and difficult works; see his paper with Preiss [30] and his research monograph with Preiss and Tišer [31]. In this context one must mention Joram's book with Benyamini [6] on geometric nonlinear functional analysis, as a cornerstone of this area. This book formulated unifying phenomena, presented key open questions (many of which have been subsequently solved), and included several results that did not appear elsewhere. The importance of this monumental book is hard to overestimate, as it served a leadership role by crystalizing a fast-growing mathematical discipline.

The local theory of Banach spaces aims to investigate those properties of Banach spaces that are determined by quantitative/numerical invariants of their finite dimensional subspaces. Joram's groundbreaking paper with Pełczyński [29] gave major impetus to this local theory. It starts with a study of Grothendieck's work [16], including a crucial reformulation of Grothendieck's inequality, and proceeds to present several striking applications in addition to the introduction of key concepts such as $\mathcal{L}_p$ spaces. In his paper with Rosenthal [32], Joram continued to pursue the theory of $\mathcal{L}_p$ spaces, and discovered the *principle of local reflexivity*, an important statement that asserts that, for every Banach space X, the finite dimensional subspaces of X^{**} essentially coincide with the finite dimensional subspaces of X.

The local theory of Banach spaces was very convincingly used in Joram's work with Tzafriri [33] that solved the complemented subspace problem, a problem that was at the time one of the oldest and most well known open questions of Banach space theory. They proved that if a Banach space X has the property that there is a bounded projection onto each of its closed subspaces, then X must be isomorphic to a Hilbert space. Finite dimensional invariants as well as Dvoretzky's theorem play a crucial role in the proof of this statement. Obtaining asymptotically sharp estimates in the context of the complemented subspace problem remains an open question, with major progress obtained recently by Kalton [21].

Joram's influential paper [14] with Figiel and Milman revisited Dvoretzky's theorem, discovering improved bounds on the dimension of almost spherical sections under additional geometric assumptions. In a related direction, Joram, in collaboration with Bourgain and Milman [10], studied the geometry of finite dimensional subspaces of L_p spaces. The case $p = 1$ of their investigations amounts to the following natural geometric result about the approximation of zonoids by zonotopes. A zonotope is a Minkowski sum of symmetric segments, i.e., a set of the form $[-x_1, x_1] + \ldots + [-x_m, x_m]$ for some $x_1, \ldots, x_m \in \mathbb{R}^n$, where for $y, z \in \mathbb{R}^n$ we set $[y, z] = \{\lambda y + (1-\lambda)z : \ \lambda \in [0,1]\}$. A zonoid is a limit of zonoids (in the Hausdorff metric). Given $n \in \mathbb{N}$ and $\varepsilon \in (1/n, 1)$, Bourgain, Lindenstrauss and Milman proved via a substantial refinement of a method of [42] that if $K \subseteq \mathbb{R}^n$ is a zonotope then there exists an integer $m \leqslant c(\varepsilon)n(\log n)^3$ and vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ such that if we set $L = [-x_1, x_1] + \ldots + [-x_m, x_m]$ then $L \subseteq K \subseteq (1+\varepsilon)L$. The upper bound on m was subsequently improved by Talagrand [43] to $m \leqslant c(\varepsilon)n \log n$, and it remains an intriguing open problem to determine whether the $\log n$ factor is needed here. For the dependence on ε in this question, see also the work of Joram and Bourgain [9].

Joram, in collaboration with Enflo and Pisier, conducted a profound study of three space problems in [13]. Here one is given a Banach space X and a closed subspace $Y \subseteq X$. One assumes that both Y and X/Y have a certain desirable property, and one wants to deduce from this that X itself also has the same property. Answering a natural question of Palais, corresponding to the case when the property in question is "isomorphic to Hilbert space," for every $n \in \mathbb{N}$ the above authors constructed a normed space X of dimension $n^2 + n$ and subspace $Y \subseteq X$ of dimension n^2 such that both Y and X/Y are isometric to Hilbert space yet the Banach-Mazur distance of X to Hilbert space is at least a constant multiple of $\sqrt{\log n}$. This ingenious construction was subsequently revisited by Kalton and Peck [22], obtaining an improved dependence on n. Conversely, in [13] it was shown that in the above setting the distance to Hilbert space cannot grow faster than a power of $\log n$.

Let X be a Banach space and let $K \subseteq X$ be compact and convex. By the Choquet representation theorem we know that for every $x \in K$ there exists a probability measure μ_x supported on the extreme points of K such that $x = \int y d\mu_x(y)$. If μ_x is unique for every $x \in K$ then K is said to be a simplex. Answering a question of Choquet, in 1961 Poulsen constructed [40] a simplex K in Hilbert space whose extreme points are dense in K. Poulsen's construction of this nonintuitive object is achieved by iterating the following simple operation. Given $\varepsilon \in (0,1)$ and a simplex $S \subseteq \ell_2^n$, let $x_1, \ldots, x_m$ be an ε-dense subset of S. Thinking of ℓ_2^n as the space spanned by the last n coordinates of ℓ_2^{m+n}, and letting $e_1, \ldots, e_{m+n}$ be the standard basis of ℓ_2^{m+n}, consider the convex hull of S and the points $x_1 + \varepsilon e_1, \ldots, x_m + \varepsilon e_m$. One thus obtains a new simplex S' whose extreme points are 2ε-dense in S'. Surprisingly, despite the fact that this construction seems to involve arbitrary choices, Joram, in collaboration with Olsen and Sternfeld, proved in [28] that any two simplices with a dense set of extreme points are affinely homeomorphic. Thus there is only one simplex with dense extreme points, a simplex that is known today as the *Poulsen simplex*. In [28] it is shown that this simplex has beautiful homogeneity properties. The proofs of these facts are ingenious, relying in part on the method of representing matrices that Joram previously introduced with Lazar in [24]. Having been shown to be a canonical and highly symmetric object, the Poulsen simplex plays a role in other areas of mathematics. For example, Glasner and Weiss proved in [15] that a countable discrete group G fails to have Kazhdan's property (T) if and only if the set of all G-invariant measures on $\{0,1\}^G$ is the Poulsen simplex (here, the extreme points are the ergodic measures).

Joram has 126 publications, so the above description of his work is clearly very partial. Hopefully, we succeeded to convey the variety, depth and influence of Joram's body of work. When compared to Joram's own list of selected publications, our choices cover much, but not all of his favorite results. We omitted, for example, Joram's seminal work with Amir [2] on weakly compactly generated Banach spaces. We also omitted Joram's research monograph [8] with Bourgain, Casazza and Tzafriri on uniqueness of unconditional bases, up to permutation.

We already mentioned above several of Joram's books and research monographs [25, 8, 6, 31]. We have yet to mention his very influential books with Tzafriri [35, 34] on classical Banach spaces: these are classics and mandatory reading for anyone who is interested in the geometry of Banach spaces. Joram also edited with Johnson the monumental two volume handbook of the geometry of Banach spaces [18, 19]. Finally, Joram wrote four beautiful textbooks in Hebrew: Advanced Infinitesimal Calculus 1, Advanced Infinitesimal Calculus 2, Introduction to Modern Analysis (coauthored with Pazy and Weiss), and Functional Analysis, Hilbert and Banach Spaces (also coauthored with Pazy and Weiss). These textbooks were the source from which numerous Israeli mathematicians learned the basics of advanced analysis over several decades.

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