A Phase Transition Phenomenon Between the Isometric and Isomorphic Extension Problems for Hölder Functions Between L_p Spaces

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Abstract

We prove that it is possible to extend α Hölder maps from subsets of L_p to L_q, $(1 < p, q \leq 2)$ isometrically if and only if $\alpha \leq p/q^*$ and isomorphically if and only if $\alpha \leq p/2$. We also prove that the set of α's which allow an isomorphic extension for α Hölder maps from subsets of X to Y is monotone when Y is a dual Banach space. Finally, we study the isometric and isomorphic extension problems for Hölder functions between L_p and L_q for general $p, q \geq 1$ and solve a question posed by K. Ball by showing that it is not true that all Lipschitz maps from subsets of Hilbert space into normed spaces extend to the whole of Hilbert space.

1 Introduction

Let (X, d) and (Y, ρ) be metric spaces. Recall that a function $f : X \to Y$ is called α Hölder with constant K if for every $x, y \in X$, $\rho(f(x), f(y)) \leq Kd(x, y)^\alpha$. We denote By $\mathcal{A}(X, Y)$ the set of all $\alpha > 0$ such that for all $D \subset X$ and for all α-Hölder $f : D \to Y$ there is an $\bar{f} : X \to Y$ which is α Hölder with the same constant as that of f and the restriction of $\bar{f}$ to D is f. Such an $\bar{f}$ is called an isometric extension of f. Analogously, we denote by $\mathcal{B}(X, Y)$ the set of all $\alpha > 0$ such that there is a constant $C > 0$ such that for all $D \subset X$ and for any α Hölder function $g : D \to Y$ with constant K there is an α Hölder function with constant CK, $\bar{g} : X \to Y$ which extends g. Such a $\bar{g}$ is called an isomorphic extension of g. Note that $\mathcal{A}(X, Y) \subset \mathcal{B}(X, Y)$.

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Throughout the rest of this paper, for $1 < p < \infty$ we define $p^* = \frac{p}{p-1}$.

In this paper we combine several results to prove the following theorem :

Theorem 1 *For $1 < p, q \le 2$:*

1) $\mathcal{A}(L_p, L_q) = \left(0, \frac{p}{q^*}\right]$.

2) $\mathcal{B}(L_p, L_q) = \left(0, \frac{p}{2}\right]$.

3) For any $\frac{p}{2} < \alpha \le 1$ there is an α Hölder function from a subset of the unit ball of L_p to L_q which cannot be extended to an α Hölder function defined on all of the unit ball of L_p.

Note that for $1 < q < 2$, $q^* > 2$ so that $\frac{p}{2} > \frac{p}{q^*}$. We therefore get a phase transition between the various extension problems.

Part 1) of the theorem for $p = q = 2$ was proved by Kirszbraun [K]. A complete proof of part 1) can be found in [W-W]. We will give here a simplified proof of part 1) which is perhaps of independent interest. The isomorphic extension problem for Lipschitz maps between L_p spaces was studied by Ball in [B]. We will use the main theorem of [B] to prove part 2). We will then use an idea of Lindenstrauss [L] (see also [T]) to prove part 3). In the last section we will state some open problems related to the concepts discussed in this paper. For a thorough discussion of related issues we refer to the book [B-L].

The abstract properties of the sets $\mathcal{A}(X, Y)$ and $\mathcal{B}(X, Y)$ are of independent interest, and we will start this paper with a general theorem which we hope will initiate the study of these sets. Namely, we will prove that if Y is a dual Banach space then $\alpha \in \mathcal{B}(X, Y)$ and $\beta \le \alpha$ implies that $\beta \in \mathcal{B}(X, Y)$. This simple statement seems to be surprisingly non-trivial.

2 Monotonicity of $\mathcal{B}(X, Y)$

Let (X, d) and (Y, ρ) be metric spaces. For every $\alpha > 0$ denote by $K_\alpha(X, Y)$ the infimum of all $K > 0$ such that for all $D \subset X$ and every $f : D \to Y$ which is α Hölder with constant C, there is an extension of f to X which is α Hölder with constant KC. Our goal here is to prove the following :

Theorem 2 *Let (X, d) be a metric space and Y a dual Banach space. If $\alpha \in \mathcal{B}(X, Y)$ and $\beta \le \alpha$ then $\beta \in \mathcal{B}(X, Y)$. Moreover, we have the follow-*

ing estimate:

$$K_\beta(X,Y) \le \frac{30\alpha}{\beta} K_\alpha(X,Y)^2.$$

We will start out with a general lemma :

Lemma 1 *Let (X,d) and (Y,ρ) be metric spaces. Assume $\alpha \in \mathcal{B}(X,Y)$, $K > K_\alpha(X,Y)$ and $\beta \le \alpha$. Take $D \subset X$ and $f : D \to Y$ β Hölder with constant C. Assume also that $g : X \to Y$ is α Hölder with constant a and for every $x \in D$, $\rho(g(x), f(x)) \le b$. Then there is a function $h : X \to Y$ with the following properties :*

1) h is α Hölder with constant $2Ka$.

2) For every $x \in D$,

$$\rho(h(x), f(x)) \le (K+1)C \left(\frac{C}{2a}\right)^{\frac{\beta}{\alpha-\beta}}.$$

3) For every $x \in X$,

$$\rho(h(x), g(x)) \le 2(K+1)b + (K+1)C \left(\frac{C}{2a}\right)^{\frac{\beta}{\alpha-\beta}}.$$

Proof: Put $s = (C/2a)^{1/(\alpha-\beta)}$. Let $\mathcal{N}$ be a s net in D (i.e. a maximal s separated subset of D). Define also :

$$A = \left\{ x \in X; d(x,D) \ge \left(\frac{b}{a}\right)^{1/\alpha} \right\}.$$

We will define $h : \mathcal{N} \bigcup A \to Y$ by :

$$h(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{N} \\ g(x) & \text{if } x \in A \end{cases}$$

We claim that h is α Hölder with constant $2a$. Indeed, if $x, y \in \mathcal{N}$ are distinct then :

$$\rho(h(x), h(y)) = \rho(f(x), f(y)) \le Cd(x,y)^\beta \le \frac{C}{s^{\alpha-\beta}} d(x,y)^\alpha = 2ad(x,y)^\alpha.$$

h is clearly α Hölder with constant $a \le 2a$ on A. Finally, if $x \in \mathcal{N}$ and $y \in A$ then $d(x,y) \ge (b/a)^{1/\alpha}$ so that :

$$\rho(h(x), h(y)) = \rho(f(x), g(y)) \le \rho(f(x), g(x)) + \rho(g(x), g(y)) \le$$

$$\leq b + ad(x,y)^\alpha \leq ad(x,y)^\alpha + ad(x,y)^\alpha = 2ad(x,y)^\alpha.$$

By our assumption, we can extend h to an α Hölder function with constant $2Ka$ defined on X. To prove 2), let $x \in D$ and find $y \in \mathcal{N}$ with $d(x,y) \leq s$. Then :

$$\rho(h(x), f(x)) \leq \rho(h(x), h(y)) + \rho(f(y), f(x)) \leq 2Kad(x,y)^\alpha + Cd(x,y)^\beta \leq$$

$$\leq 2Kas^\alpha + Cs^\beta = (K+1)C\left(\frac{C}{2a}\right)^{\frac{\beta}{\alpha-\beta}}.$$

Part 3) is trivial if $x \in A$. If $x \in D$ then :

$$\rho(h(x), g(x)) \leq \rho(h(x), f(x)) + \rho(f(x), g(x)) \leq$$

$$\leq (K+1)C\left(\frac{C}{2a}\right)^{\frac{\beta}{\alpha-\beta}} + b < 2(K+1)b + (K+1)C\left(\frac{C}{2a}\right)^{\frac{\beta}{\alpha-\beta}}.$$

Finally, if $x \notin A \bigcup D$ then there is $y \in D$ with $d(x,y) \leq (b/a)^{1/\alpha}$. Hence :

$$\rho(h(x), g(x)) \leq \rho(h(x), h(y)) + \rho(h(y), f(y)) + \rho(f(y), g(y)) + \rho(g(y), g(x)) \leq$$

$$\leq 2Kad(x,y)^\alpha + (K+1)C\left(\frac{C}{2a}\right)^{\frac{\beta}{\alpha-\beta}} + b + ad(x,y)^\alpha \leq$$

$$\leq 2Ka\frac{b}{a} + (K+1)C\left(\frac{C}{2a}\right)^{\frac{\beta}{\alpha-\beta}} + b + a\frac{b}{a} =$$

$$= 2(K+1)b + (K+1)C\left(\frac{C}{2a}\right)^{\frac{\beta}{\alpha-\beta}}.$$

■

This lemma is iterated to prove the following extension result.

Proposition 1 *In the above notation assume in addition that Y is complete. Let $D \subset X$ and $f : D \to Y$ be β Hölder with constant C. Then for every $t > 0$ there is a function $\bar{f} : X \to Y$ which extends f and satisfies :*

$$d(x,y) \leq t \Longrightarrow \rho(\bar{f}(x), \bar{f}(y)) \leq \frac{21K^2\alpha}{\beta} Cd(x,y)^\beta.$$

Proof: Put $\lambda = (1/(2K))^{1/(\alpha-\beta)}$. We will define functions $\{g_n\}_{n=0}^\infty$ on X. To define g_0 let $\mathcal{N}$ be a t net in D. If $x, y \in \mathcal{N}$ are distinct then :

$$\rho(f(x), f(y)) \leq Cd(x,y)^\beta \leq \frac{C}{t^{\alpha-\beta}} d(x,y)^\alpha.$$

By our assumption, the restriction of f to $\mathcal{N}$ can be extended to a function $g_0 : X \to Y$ which is α Hölder with constant $(KC)/(t^{\alpha-\beta})$. If $x \in D$ then there is $y \in \mathcal{N}$ with $d(x,y) \le t$. Hence :

$$\rho(f(x), g_0(x)) \le \rho(f(x), f(y)) + \rho(g_0(y), g_0(x)) \le Cd(x,y)^\beta + \frac{KC}{t^{\alpha-\beta}} d(x,y)^\alpha \le$$

$$\le Ct^\beta + \frac{KC}{t^{\alpha-\beta}} t^\alpha = (K+1)Ct^\beta.$$

Assume that we have constructed $g_n : X \to Y$ with the following properties :

a) g_n is α Hölder with constant $\frac{KC}{t^{\alpha-\beta}\lambda^{n(\alpha-\beta)}}$.

b) For all $x \in D$, $\rho(g_n(x), f(x)) \le (K+1)C\lambda^{\beta n} t^\beta$.

By the previous lemma there is $g_{n+1} : X \to Y$ which satisfies:

1) g_{n+1} is α Hölder with constant

$$2K \frac{KC}{t^{\alpha-\beta}\lambda^{n(\alpha-\beta)}} = \frac{KC}{t^{\alpha-\beta}\lambda^{(n+1)(\alpha-\beta)}}.$$

2) For every $x \in D$:

$$\rho(g_{n+1}(x), f(x)) \le (K+1)C \left(\frac{C}{2(KC)/(t^{\alpha-\beta}\lambda^{n(\alpha-\beta)})} \right)^{\frac{\beta}{\alpha-\beta}} =$$

$$= (K+1)C\lambda^{(n+1)\beta} t^\beta.$$

3) For all $x \in X$:

$$\rho(g_{n+1}(x), g_n(x)) \le 2(K+1)^2 C\lambda^{n\beta} t^\beta + (K+1)C\lambda^{(n+1)\beta} t^\beta =$$

$$= (K+1)C\lambda^{n\beta} t^\beta (2K + 2 + \lambda^\beta) \le 10K^2 C t^\beta \lambda^{n\beta}.$$

Hence, if $n > m$,

$$\rho(g_n(x), g_m(x)) \le \sum_{k=m}^{n-1} \rho(g_k(x), g_{k+1}(x)) \le 10K^2 C t^\beta \sum_{k=m}^{n-1} \lambda^{k\beta} =$$

$$= 10K^2 C t^\beta \frac{\lambda^{\beta m} - \lambda^{\beta n}}{1 - \lambda^\beta} \le \frac{10K^2 C t^\beta}{1 - 2^{-\beta/(\alpha-\beta)}} (\lambda^{\beta m} - \lambda^{\beta n}) \le \frac{10K^2 C \alpha t^\beta}{\beta} (\lambda^{\beta m} - \lambda^{\beta n}).$$

In the last inequality we used the fact that for all $\alpha > \beta > 0$, $1 - 2^{-\beta/(\alpha-\beta)} \ge \beta/\alpha$ (to see that this is true put $\gamma = \alpha/\beta$, and use the inequality $2^x \ge 1 + x$ for $x = \frac{1}{\gamma - 1}$).

This shows that for all $x \in X$, $\{g_n(x)\}_{n=0}^{\infty}$ is Cauchy. Since Y is complete there exists $\bar{f}(x) = \lim_{n\to\infty} g_n(x)$. If $x \in D$ then :

$$\rho(\bar{f}(x), f(x)) = \lim_{n\to\infty} \rho(g_n(x), f(x)) \leq \lim_{n\to\infty} (K+1)Ct^\beta \lambda^{\beta n} = 0.$$

So that $\bar{f}$ extends f. Moreover, for every m:

$$\rho(g_m(x), \bar{f}(x)) = \lim_{n\to\infty} \rho(g_m(x), g_n(x)) \leq \lim_{n\to\infty} \frac{10K^2C\alpha t^\beta}{\beta}(\lambda^{\beta m} - \lambda^{\beta n}) =$$

$$= \frac{10K^2C\alpha t^\beta}{\beta}\lambda^{\beta m}.$$

The proof that $\bar{f}$ satisfies the required condition now follows. If $d(x,y) \leq t$ then we can find $n \geq 1$ such that $t\lambda^n \leq d(x,y) \leq t\lambda^{n-1}$. Hence :

$$\rho(\bar{f}(x), \bar{f}(y)) \leq \rho(\bar{f}(x), g_n(x)) + \rho(g_n(x), g_n(y)) + \rho(g_(y), \bar{f}(y)) \leq$$

$$\leq 2\frac{10K^2C\alpha t^\beta}{\beta}\lambda^{\beta n} + \frac{KC}{t^{\alpha-\beta}\lambda^{n(\alpha-\beta)}}d(x,y)^\alpha \leq$$

$$\leq \frac{20K^2C\alpha}{\beta}d(x,y)^\beta + \frac{KC}{d(x,y)^{\alpha-\beta}/\lambda^{\alpha-\beta}}d(x,y)^\alpha \leq \frac{21K^2C\alpha}{\beta}d(x,y)^\beta$$

■

For every $r > 0$ put $D_r = \{x \in X, d(x,D) < r\}$.

Corollary 1 *In the above notation, for every $r > 0$ there is an extension of f to D_r which is β Hölder with constant $\frac{30K^2\alpha C}{\beta}$.*

Proof: Fix $t > r$ and let $\bar{f}$ be the function constructed in the previous proposition. If $x, y \in D_r$ then there are $x', y' \in D$ with $d(x,x') < r$ and $d(y,y') < r$. Now, if $d(x,y) \leq t$ then $\rho(\bar{f}(x), \bar{f}(y)) \leq (21K^2\alpha C/\beta)d(x,y)^\beta$. If $d(x,y) > t$ then :

$$\rho(\bar{f}(x), \bar{f}(y)) \leq \rho(\bar{f}(x), \bar{f}(x')) + \rho(f(x'), f(y')) + \rho(\bar{f}(y'), \bar{f}(y)) \leq$$

$$\leq \frac{21K^2\alpha C}{\beta}(d(x,x')^\beta + d(y,y')^\beta) + Cd(x',y')^\beta \leq$$

$$\leq 2\frac{21K^2\alpha C}{\beta}r^\beta + C(d(x',x) + d(x,y) + d(y,y'))^\beta \leq$$

$$\leq \left(2\frac{21K^2\alpha C}{\beta}\left(\frac{r}{t}\right)^\beta + C\left(1 + \frac{2r}{t}\right)^\beta\right) d(x,y)^\beta.$$

So that $\bar{f}$ is β Hölder with constant $30K^2\alpha C/\beta$ for t large enough. ■

We are now ready for the proof of theorem 2.

Proof of theorem 2: We will use the above notation and assume in addition that Y is a dual Banach space. Fix some $x_0 \in D$. For every n there is $\bar{f}_n : D_n \to Y$ which extends f and is β Hölder with constant $30K^2\alpha C/\beta$. Notice that for every $x \in X$, $\{\bar{f}_n(x)\}_{n=1}^{\infty}$ is bounded. Indeed, $||\bar{f}_n(x) - f(x_0)|| \leq \frac{30K^2\alpha C}{\beta} d(x, x_0)^\beta$. Therefore the w^* compactness of balls in Y, for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we can define :

$$\bar{f}(x) = w^* - \lim_{\mathcal{U}} \bar{f}_n(x).$$

It is clear that $\bar{f}$ extends f. If $x, y \in X$ then for n large enough, $x, y \in D_n$. By our assumptions on $\bar{f}_n$,

$$\bar{f}_n(x) - \bar{f}_n(y) \in B\left(0, \frac{30K^2\alpha C}{\beta} d(x,y)^\beta\right).$$

By the w^* compactness of balls we deduce that :

$$\bar{f}(x) - \bar{f}(y) \in B\left(0, \frac{30K^2\alpha C}{\beta} d(x,y)^\beta\right).$$

In other words, $\bar{f}$ is β Hölder with constant $30K^2\alpha C/\beta$. ■

A simple strengthening of theorem 2 is :

Corollary 2 *If (X, d) is a metric space and Y is a Banach space such that there is a Lipschitz retraction from Y^{**} to Y then $\alpha \in \mathcal{B}(X, Y)$ and $\beta \leq \alpha$ implies that $\beta \in \mathcal{B}(X, Y)$. The same is true if we assume that Y is Lipschitz equivalent to a dual Banach space.*

Remarks:

1) There is a Lipschitz retract from ℓ_∞ to c_0 (see [B-L]), so that the above theorem holds for $Y = c_0$. Note that since c_0 is determined by it's Lipschitz structure, [G-K-L], c_0 isn't Lipschitz equivalent to a dual Banach space. It is an open problem whether for every Banach space Y there is a Lipschitz retraction from Y^{**} to Y (see [B-L]). A positive answer to this problem will therefore prove the monotonicity of $\mathcal{B}(X, Y)$ for every metric space X and Banach space Y.

2) At present we do not have an example of metric spaces X and Y for which $\mathcal{B}(X, Y)$ isn't an interval.

3) We do not know under what conditions $\mathcal{A}(X,Y)$ is monotone. We conjecture that this is true when Y is a Hilbert space, and it seems likely not to be true for a general Banach space Y (although we do not have an example of this).

3 The Isometric Extension Problem

In this section we will prove part 1) of the theorem. As stated above, a complete proof of this result can be found in [W-W]. Some parts of our proof are new, and we give simplifications of other parts.

We will first show that for any $\alpha > \frac{p}{q^*}$, $\alpha \notin \mathcal{A}(L_p, L_q)$. Our approach is based on the following probabilistic lemma.

For every n put $\Omega = \{-1,1\}^n$, and let P be the uniform probability measure on Ω. We denote by $r_1, ..., r_n$ the Rademacher functions on Ω, i.e. $r_j(\epsilon) = \epsilon_j$. Define a random variable λ on Ω by :

$$\lambda(\epsilon) = \frac{|\{i; \epsilon_i = 1\}|}{n}.$$

λ satisfies the following concentration inequality :

$$P\left(\left|\lambda - \frac{1}{2}\right| > t\right) \leq 2e^{-nt^2}.$$

A standard proof of this runs as follows :

$$P\left(\lambda - \frac{1}{2} > t\right) = P\left(\frac{n + \sum_{i=1}^n r_i}{2n} - \frac{1}{2} > t\right) = P\left(\sum_{i=1}^n r_i > 2tn\right) =$$

$$= P\left(e^{t\left(\sum_{i=1}^n r_i - 2tn\right)} > 1\right) \leq \mathbb{E}e^{t\left(\sum_{i=1}^n r_i - 2tn\right)} = e^{-2t^2 n}\left(\mathbb{E}e^{tr_1}\right)^n =$$

$$= e^{-2nt^2}\left(\frac{e^t + e^{-t}}{2}\right)^n \leq e^{-2nt^2}e^{nt^2} = e^{-nt^2}.$$

We can now state the result we need :

Lemma 2 *For all $1 < q < \infty$ there is a constant C which depends only on q such that for any $X \in L_q(\Omega)$*

$$\frac{1}{n}\sum_{j=1}^n \mathbb{E}|X - r_j|^q \geq 1 - C\sqrt{\frac{\log n}{n}}.$$

Proof: We can clearly assume that $|X| \le 1$, since for every $a > 1$, $|a \pm 1| > |1 \pm 1|$ and similarly for $a < -1$. Assuming $|X| \le 1$ we get that for all $t > 0$:

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}|X - r_j|^q = \mathbb{E}\frac{1}{n}\sum_{j=1}^{n} |X - r_j|^q = \mathbb{E}(\lambda|X-1|^q + (1-\lambda)|X+1|^q) \ge$$

$$\ge \mathbb{E}(\lambda|X-1|^q + (1-\lambda)|X+1|^q)\, 1_{\{|\lambda - \frac{1}{2}| \le t\}} =$$

$$= \mathbb{E}\left(\frac{|1-X|^q + |1+X|^q}{2} + \left(\lambda - \frac{1}{2}\right)(|1-X|^q - |X+1|^q)\right) 1_{\{|\lambda-\frac{1}{2}|\le t\}} \ge$$

$$\ge \mathbb{E}\left(1 - \left|\lambda - \frac{1}{2}\right| ||1-X|^q - |1+X|^q|\right) 1_{\{|\lambda-\frac{1}{2}|\le t\}} \ge$$

$$\ge (1 - 2^q t) P(|\lambda - \frac{1}{2}| \le t) \ge (1-2^q t)(1 - 2e^{-nt^2}).$$

Now take $t = \sqrt{\frac{\log n}{n}}$. ■

We can now prove :

Proposition 2 *If $\alpha > \frac{p}{q^*}$ then $\alpha \notin \mathcal{A}(L_p, L_q)$*

Proof: Let $e_1, ..., e_n$ be the unit basis of ℓ_p^n. Put $D = \{e_1, ..., e_n\}$ and define $f : D \to L_q(\Omega)$ by $f(e_j) = r_j$. Then for $i \ne j$:

$$||f(e_i) - f(e_j)||_q = (\mathbb{E}|r_i - r_j|^q)^{1/q} = \left(\frac{1}{2}2^q\right)^{1/q} = 2^{1/q^*} = 2^{1/q^* - \alpha/p}||e_i - e_j||_p^\alpha.$$

So that f is α Hölder with constant $2^{1/q^*-\alpha/p}$. Assume that f has an extension to 0, which is α Hölder with the same constant, and put $X = f(0)$. By the lemma we get that :

$$1 - C\sqrt{\frac{\log n}{n}} \le \frac{1}{n}\sum_{j=1}^{n} ||X - r_j||_q^q \le \frac{1}{n}\sum_{j=1}^{n} 2^{q(1/q^*-\alpha/p)}||0 - e_j||_p^{\alpha q} = 2^{q(1/q^*-\alpha/p)}.$$

Since this is true for all n we deduce that $1/q^* - \alpha/p \ge 0$, or $\alpha \le p/q^*$. ■

It remains to prove that $(0, \frac{p}{q^*}] \subset \mathcal{A}(L_p, L_q)$. We will need some probabilistic results. In order to avoid measurability problems we will assume in what follows that all measure spaces are finite.

Lemma 3 *Let X and Y be identically distributed independent random variables. Then for every $1 \le p \le 2$:*

$$\mathbb{E}|X - Y|^p \le 2\mathbb{E}|X|^p$$

Proof: For every random variable Z put $\phi_Z(t) = \mathbb{E}e^{itZ}$. It is well known that there is a constant C_p such that :

$$\mathbb{E}|Z|^p = C_p Re \int_0^\infty \frac{1-\phi_Z(t)}{t^{p+1}} dt.$$

Hence :

$$\mathbb{E}|X-Y|^p = C_p Re \int_0^\infty \frac{1-\phi_{X-Y}(t)}{t^{p+1}} dt = C_p \int_0^\infty \frac{1-|\phi_X(t)|^2}{t^{p+1}} dt =$$

$$= C_p \int_0^\infty \frac{(1-|\phi(t)|)(1+|\phi(t)|)}{t^{p+1}} dt \leq C_p \int_0^\infty \frac{2(1-Re\phi(t))}{t^{p+1}} dt = 2\mathbb{E}|X|^p.$$

■

Corollary 3 *If X and Y are i.i.d. random vectors in L_p then :*

$$\mathbb{E}||X-Y||_p^p \leq 2\mathbb{E}||X||_p^p$$

Lemma 4 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an anti-symmetric function. If X, Y are i.i.d. random variables then :*

$$2\mathbb{E}_X \left(\mathbb{E}_Y f(X,Y)\right)^2 \leq \mathbb{E}_X \mathbb{E}_Y f(X,Y)^2.$$

Proof: Let Z be an independent copy of X and Y. We have :

$$\mathbb{E}_X \left(\mathbb{E}_Y f(X,Y)\right)^2 = \mathbb{E}_X \mathbb{E}_Y \mathbb{E}_Z f(X,Y) f(X,Z).$$

Dividing the last expectation according to the relative order of $\{X, Y, Z\}$, and relabeling the variables so that $X < Y < Z$ we get, using the anti-symmetry of f :

$$\mathbb{E}_X \mathbb{E}_Y \mathbb{E}_Z f(X,Y) f(X,Z) = \mathbb{E}_X \mathbb{E}_Y \mathbb{E}_Z f(X,Y)^2 1_{\{Z=Y\}} +$$

$$+\mathbb{E}_X \mathbb{E}_Y \mathbb{E}_Z 1_{\{X<Y<Z\}} [f(X,Y)f(X,Z) + f(X,Z)f(X,Y) + f(Y,X)f(Y,Z) +$$

$$+f(Y,Z)f(Y,X) + f(Z,X)f(Z,Y) + f(Z,Y)f(Z,X)] =$$

$$= \mathbb{E}_X \mathbb{E}_Y f(X,Y)^2 P(Z=Y) +$$

$$+\mathbb{E}_{X,Y,Z} 1_{\{X<Y<Z\}} 2 [f(X,Y)f(X,Z) - f(X,Y)f(Y,Z) + f(X,Z)f(Y,Z)] =$$

$$= \mathbb{E}_X \mathbb{E}_Y f(X,Y)^2 P(Z=Y) +$$

$$+\mathbb{E}_X \mathbb{E}_Y \mathbb{E}_Z 1_{\{X<Y<Z\}} \left[f(X,Y)^2 + f(X,Z)^2 + f(Y,Z)^2 - \right.$$

$$- \left(f(X,Y) - f(X,Z) + f(Y,Z)\right)^2 \Big] =$$

$$= \mathbb{E}_X \mathbb{E}_Y 1_{\{X<Y\}} f(X,Y)^2 [P(Z > Y > X) + P(X < Z < Y) + P(Z < X < Y)] +$$

$$+ \mathbb{E}_X \mathbb{E}_Y f(X,Y)^2 P(Z = Y) - \mathbb{E}_{X,Y,Z} [f(X,Y) - f(X,Z) + f(Y,Z)]^2 1_{\{X<Y<Z\}} \leq$$

$$\leq \mathbb{E}_X \mathbb{E}_Y f(X,Y)^2 P(Z = Y) + \mathbb{E}_X \mathbb{E}_Y 1_{\{X<Y\}} f(X,Y)^2 P(Z \neq X \text{ and } Z \neq Y) =$$

$$= \mathbb{E}_X \mathbb{E}_Y f(X,Y)^2 P(Z = Y) + \frac{1}{2} \mathbb{E}_X \mathbb{E}_Y f(X,Y)^2 [P(Z \neq X \text{and} Z \neq Y)] =$$

$$= \frac{1}{2} \mathbb{E}_X \mathbb{E}_Y f(X,Y)^2.$$

■

Lemma 5 *Let $1 < q \leq 2$. If X, Y are i.i.d. random vectors in L_q and $f : L_q \times L_q \to L_q$ is anti-symmetric then :*

$$\mathbb{E}_X \mathbb{E}_Y ||f(X,Y)||_q^{q^*} \geq 2 \mathbb{E}_X ||\mathbb{E}_Y f(X,Y)||_q^{q^*}.$$

Proof: Since we are dealing with finite valued random variables we may assume that X is a random vector in ℓ_p^n. Denote by $L_{q^*}(\mathbb{R}^n \times \mathbb{R}^n, L_q)$ the space of all functions $g : \mathbb{R}^n \times \mathbb{R}^n \to L_q$ equipped with the norm :

$$||g||_{q^*,q} = \left(\mathbb{E}_X \mathbb{E}_Y ||g(X,Y)||_q^{q^*}\right)^{1/q^*}.$$

Define :

$$T(g)(x) = \mathbb{E}_Y \left(\frac{g(x,Y) - g(Y,x)}{2}\right).$$

We think here of $T(g)$ as an element of $L_{q^*}(\mathbb{R}^n, L_q)$, i.e. the space of all $h : \mathbb{R}^n \to L_q$ equipped with the norm :

$$||h||_{q^*,q} = \left(\mathbb{E}_X ||h(X)||_q^{q^*}\right)^{1/q^*}.$$

Integration of the above lemma gives that

$$||T(g)||_{2,2} \leq \frac{1}{\sqrt{2}} \left(\mathbb{E}_X \; E_Y \left\| \frac{g(X,Y) - g(Y,X)}{2} \right\|_2^2 \right)^{1/2} \leq \frac{1}{\sqrt{2}} ||g||_{2,2}.$$

Moreover,

$$||T(g)||_{\infty,1} = \sup_X \left\| E_Y \frac{g(X,Y) - g(Y,X)}{2} \right\|_1 \leq \sup_{X,Y} ||g(X,Y)||_1 = ||g||_{\infty,1}.$$

By a standard theorem from interpolation theory (see [Be-Lö] theorem 5.1.2.) we deduce that :

$$||T(g)||_{q^*,q} \leq \frac{1}{2^{1/q^*}} ||g||_{q^*,q}.$$

And this is the required result for $g = f$. ∎

Lemma 6 *Assume* $1 < q \leq 2$ *and* $\beta \geq q^*$. *If* X *and* Y *are i.i.d. random vectors in* L_q *then :*

$$\mathbb{E}||X - Y||_q^\beta \geq 2\mathbb{E}||X - \mathbb{E}X||_q^\beta$$

Proof: If $\beta = q^*$ use the previous lemma for $f(x, y) = x - y$. If $\beta > q^*$, use q-stable random variables to embed ℓ_q in L_{β^*}. ∎

The geometric property that we will need is the following :

Proposition 3 *Let* I *be some index set,* $\{x_i\}_{i\in I} \subset L_p$, $\{y_i\}_{i\in I} \subset L_q$, $\{r_i\}_{i\in I} \subset (0, \infty)$ *and* $0 < \alpha \leq p/q^*$. *Assume that for all* $i, j \in I$, $||y_i - y_j||_q \leq ||x_i - x_j||_p^\alpha$. *Then :*

$$\bigcap_{i\in I} B(x_i, r_i) \neq \emptyset \Longrightarrow \bigcap_{i\in I} B(y_i, r_i^\alpha) \neq \emptyset.$$

Proof: By a standard weak compactness argument it is enough to prove the proposition for a finite index set I, say $I = \{1, .., n\}$. Fix $x \in \bigcap_{i=1}^n B(x_i, r_i)$ and let F be the linear span of $y_1, .., y_n$. For $\beta = p/\alpha \geq q^*$, define $f_i : F \to \mathbb{R}$ by $f_i(y) = ||y - y_i||_q^\beta - ||x - x_i||_p^p$. By the Min-Max theorem applied to the function $g(\lambda, y) = \sum_{i=1}^n \lambda_i f_i(y)$, $\lambda \in \mathbb{R}^n$, $y \in F$, there are $y \in \text{conv}\{y_1, ..., y_n\}$ and $\lambda_1, ..., \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that for every $z \in \text{conv}\{y_1, ..., y_n\}$

$$\max_{1\leq i\leq n} f_i(y) \leq \sum_{i=1}^n \lambda_i f_i(z).$$

We claim that $y \in \bigcap_{i=1}^n B(y_i, r_i^\alpha)$. To see this define a random vector X by $P(X = x_i) = \lambda_i$ and let X' be an i.i.d. copy of X. We will also write $y_i = \psi(x_i)$. By the previous lemmas:

$$\max_{1\leq i\leq n} f_i(y) \leq \sum_{i=1}^n \lambda_i f_i(\mathbb{E}\psi(X)) = \mathbb{E}\left(||\psi(X) - \mathbb{E}\psi(X)||_q^\beta - ||X - x||_p^p\right) \leq$$

$$\leq \frac{1}{2}\mathbb{E}\left(||\psi(X) - \psi(X')||_q^{p/\alpha} - ||X - X'||_p^p\right) \leq 0,$$

i.e. for every $i = 1, ..., n$, $||y - y_i||_q \leq ||x - x_i||_p^\alpha \leq r_i^\alpha$.

■

We now return to finish the proof of $(0, \frac{p}{q^*}] \subset \mathcal{A}(L_p, L_q)$. If $D \subset L_p$ and $f : D \to L_q$ is α Hölder with constant 1 then for every $y \in L_p \setminus D$ we can use the above proposition to find $z \in \bigcap_{x \in D} B(f(x), ||x-y||_p^\alpha)$ and extend f to y by defining $f(y) = z$. Now a Zorn lemma argument gives the required extension to all of L_p.

4 The Isomorphic Extension Problem

The most powerful method known in the isomorphic extension problem is based on the fundamental results of [B]. In order to proceed we will need to recall the notion of Markov type which was defined there. Let (X, d) be a metric space. A symmetric Markov chain on X is a Markov chain $\{M_k\}_{k=0}^\infty$ on a state space $\{x_1, ...x_m\} \subset X$ with a symmetric transition matrix and such that M_0 is uniformly distributed. In other words, there is a $m \times m$ symmetric stochastic matrix $A = (a_{ij})$ such that for all k, $P(M_{k+1} = x_j | M_k = x_i) = a_{ij}$ and $P(M_0 = x_i) = \frac{1}{m}$. For $0 < p$ we say that X has Markov type p if there is a constant $C > 0$ such that for every symmetric Markov chain on X, $\{M_k\}_{k=1}^\infty$ and for every n : $\mathbb{E}d(M_n, M_0)^p \leq Cn\mathbb{E}d(M_1, M_0)^p$. Notice that C is independent of the state space $\{x_1, ..., x_m\}$. The smallest such C is called the Markov type p constant of X and is denoted by $M_p(X)$. Joining together some of the results from [B] we quote the following theorem :

Theorem 3 *Let $1 < q \leq 2$. If X is a metric space with Markov type 2, $D \subset X$ and $f : D \to L_q$ is a Lipschitz function with constant K, then f can be extended to a Lipschitz function defined on all of X with Lipschitz constant bounded by :*

$$\frac{6M_2(X)}{\sqrt{q-1}}K.$$

The space L_2 has Markov type 2 with constant 1. This was proved in [B]. The fact that L_p, $1 < p \leq 2$ has Markov type p with constant 1 was proved in [B] by using the theory of isometric embeddings into Hilbert space. We will give a direct proof which is based on the following lemma :

Lemma 7 *If $\{M_k\}_{k=1}^\infty$ is a symmetric Markov chain on $\mathbb{R}$ then for every n :*

$$\mathbb{E}\sin^2(M_n - M_0) \leq n\mathbb{E}\sin^2(M_1 - M_0).$$

Proof: We shall use the following identity :

$$\sin^2(a-b) = (\sin^2 a - \sin^2 b)^2 + \frac{1}{4}(\sin 2a - \sin 2b)^2 + \sin^2 a + \sin^2 b -$$

$$- \sin^4 a - \sin^4 b - \frac{1}{4}\sin^2 2a - \frac{1}{4}\sin^2 2b.$$

By using the Markov type 2 property for the Markov chains $\{\sin^2 M_k\}_{k=0}^{\infty}$ and $\{\sin 2M_k\}_{k=0}^{\infty}$ we get that :

$$\mathbb{E}\sin^2(M_n - M_0) \leq n\mathbb{E}\left[(\sin^2 M_1 - \sin^2 M_0)^2 + \frac{1}{4}(\sin 2M_1 - \sin 2M_0)^2\right] +$$

$$+2\mathbb{E}\sin^2 M_0 - 2\mathbb{E}\sin^4 M_0 - \frac{1}{2}\mathbb{E}\sin^2 2M_0 =$$

$$= n\mathbb{E}\sin^2(M_1 - M_0)^2 - (n-1)\mathbb{E}\left(2\sin^2 M_0 - 2\sin^4 M_0 - \frac{1}{2}\sin^2 2M_0\right) =$$

$$= n\mathbb{E}\sin^2(M_1 - M_0)^2 - (n-1)\mathbb{E}\sin^2(M_0 - M_0) = n\mathbb{E}\sin^2(M_1 - M_0)^2.$$

■

Corollary 4 *If $\{M_k\}_{k=0}^{\infty}$ is a symmetric Markov chain on $\mathbb{R}$ and $1 < p < 2$ then :*

$$\mathbb{E}|M_n - M_0|^p \leq n\mathbb{E}|M_1 - M_0|^p.$$

Proof: It is easily seen that there exists a constant $C > 0$ such that for all $t \in \mathbb{R}$:

$$|t|^p = C\int_0^{\infty} \frac{\sin^2 tx}{x^{p+1}} dx.$$

The result now follows by integration from the previous lemma. ■

Using integration again we get :

Corollary 5 *If $1 < p < 2$ and $0 < \alpha \leq p$ then L_p has Markov type α with constant 1.*

Proof: For $\alpha = p$ this is clear from the above. If $\alpha < p$ use p-stable random variables to embed ℓ_p in L_α. ■

Part 2) of the main theorem now follows. Define a metric d on L_p by $d(x,y) = ||x-y||_p^{\alpha}$. We have proved that (L_p, d) has Markov type 2 so that theorem 2 gives the required result.

5 An Unextendable Function

Fix $\alpha > p/2$. Our goal here is to prove part 3) of the main theorem, i.e. to construct an α Hölder function defined on a subset of the unit ball of L_p which cannot be extended to an α Hölder function defined on all of the unit ball of L_p.

For any σ finite measure μ the Mazur map $\phi : L_p(\mu) \to L_2(\mu)$ is defined by :

$$\phi(f) = |f|^{p/2} sign(f).$$

It is well known (see [B-L]) that ϕ is $p/2$ Hölder on the unit ball of $L_p(\mu)$, with constant, say, C. Fix some $\epsilon > 0$ and let $\mathcal{N}_n(\epsilon)$ be an ϵ net in the unit ball of ℓ_p^{2n}. Denote also by $\phi_n : \ell_p^{2n} \to \ell_2^{2n}$ the Mazur map. If $x, y \in \mathcal{N}_n(\epsilon)$ are distinct then :

$$||\phi_n(x) - \phi_n(y)||_2 \leq C||x-y||_p^{p/2} \leq \frac{C}{\epsilon^{\alpha - p/2}}||x-y||_p^{\alpha}.$$

In other words, the restriction of ϕ_n to $\mathcal{N}_n(\epsilon)$ is α Hölder with constant $\frac{C}{\epsilon^{\alpha-p/2}}$. Assume now that the restriction of ϕ_n to $\mathcal{N}_n(\epsilon)$ can be extended to an α Hölder function f with constant K on the unit ball of ℓ_p^{2n}(which we denote by $B(\ell_p^{2n})$). Define $x_1, ..., x_{n+1} \in B(\ell_p^{2n})$ by:

$$x_k = \frac{1}{(2n)^{1/p}} 1_{\{k,\ldots,k+n-1\}}.$$

Following [L] we define for every permutation π of $\{1, ..., 2n\}$ and every $2n$ signs $\theta = (\theta_1, ..., \theta_{2n})$ an isometry of both ℓ_p^{2n} and ℓ_2^{2n} by :

$$U_{\pi,\theta}x(i) = \theta_i x(\pi^{-1}i).$$

We also put :

$$V_{\pi,\theta}f(x) = U_{\pi,\theta}\left(f(U_{\pi,\theta}^{-1}x)\right).$$

Finally, if we put :

$$g(x) = \frac{1}{2^{2n}(2n)!}\sum_{\pi,\theta} V_{\pi,\theta}f(x),$$

then g is α Hölder with constant K and for every π and θ, $U_{\pi,\theta}g(x) = g(U_{\pi,\theta}x)$. This easily implies that the vectors $\{g(x_{k+1}) - g(x_k)\}_{k=1}^n$ are disjointly supported. Thus :

$$||g(x_{n+1}) - g(x_1)||_2 = \left\|\sum_{k=1}^{n}(g(x_{k+1}) - g(x_k))\right\|_2 =$$

$$= \left(\sum_{k=1}^{n} ||g(x_{k+1}) - g(x_k)||_2^2 \right)^{1/2} \leq n^{1/2} K \frac{2^{\alpha/p}}{(2n)^{\alpha/p}} = \frac{K}{n^{\alpha/p-1/2}}.$$

Since ϕ_n is invariant under permutations and signs, for every x in the domain of f :

$$||\phi_n(x) - g(x)||_2 \leq ||\phi_n - f(x)||_2.$$

Now, we can find $y \in \mathcal{N}_n(\epsilon)$ with $||x-y||_p \leq \epsilon$. Hence, since f extends ϕ_n on $\mathcal{N}_n(\epsilon)$:

$$||f(x) - \phi_n(x)||_2 \leq ||f(x) - f(y)||_2 + ||\phi_n(y) - \phi_n(x)||_2 \leq K\epsilon^\alpha + C\epsilon^{p/2}.$$

So,

$$2K\epsilon^\alpha + 2C\epsilon^{p/2} \geq ||g(x_{n+1}) - \phi_n(x_{n+1})||_2 + ||g(x_1) - \phi_n(x_1)||_2 \geq$$

$$\geq ||(\phi_n(x_{n+1}) - \phi_n(x_1)) - (g(x_{n+1}) - g(x_1))||_2 \geq$$

$$\geq ||\phi_n(x_{n+1}) - \phi_n(x_1)||_2 - ||g(x_{n+1}) - g(x_1)||_2 \geq 1 - \frac{K}{n^{\alpha/p-1/2}}.$$

Choose $\epsilon = \frac{1}{n^{1/p-1/2\alpha}}$. Then :

$$3K \geq n^{\alpha/p-1/2} \left(1 - \frac{2C}{n^{1/2-p/(4\alpha)}}\right).$$

Since $1/2 - p/(4\alpha) > 0$, the right hand side is greater than $cn^{\alpha/p-1/2}$ for n large enough, so that we have proved the following:

Proposition 4 *There is a universal constant $c > 0$ such that if $\alpha > p/2$ and $f : B(\ell_p^{2n}) \to \ell_2^{2n}$ is α Hölder with constant K which coincides with ϕ_n on $\mathcal{N}_n\left(\frac{1}{n^{1/p-1/2\alpha}}\right)$ then $K \geq cn^{\alpha/p-1/2}$.*

Corollary 6 *Let $1 < p, q \leq 2$. For every n there is a subset $A_n \subset B(\ell_p)$ and an α Hölder function $f_n : A_n \to L_q$ with constant 1 such that if $g : B(\ell_p) \to L_q$ is α Hölder with constant K, and coincides with f on A_n, then $K \geq n$. Moreover, $0 \in A_n$ and $f_n(0) = 0$.*

Proof: It is well known, that L_q contains a complemented copy of ℓ_2. Fix such a copy and let $P : L_q \to \ell_2$ be a bounded projection. We also denote by Q_k the orthogonal projection from ℓ_2 to ℓ_2^k. Fix some integer k and put $A = \mathcal{N}_n\left(\frac{1}{k^{1/p-1/2\alpha}}\right)$. If a is the smallest possible α Hölder constant of the restriction of ϕ_k to A then put $f = \frac{1}{a}\phi_k$. Note that we have proved above that $a \leq Ck^{(1/p-1/(2\alpha))(\alpha-p/2)}$. If $g : B(\ell_p) \to L_q$ extends f and is α Hölder with constant K, then $aQ_{2k}Pg : B(\ell_p^{2k}) \to \ell_2^{2k}$ extends ϕ_k on A and is α Hölder with constant $||P||KCk^{(1/p-1/(2\alpha))(\alpha-p/2)}$. Hence :

$$||P||KCk^{(1/p-1/(2\alpha))(\alpha-p/2)} \geq ck^{\alpha/p-1/2}.$$

So that $K \geq c'k^{1/2-p/(4\alpha)}$. Since $1/2 - p/(4\alpha) > 0$ we are done. ■

To finish the construction note that it is clear that we may replace L_p in part 3) of theorem 1 by ℓ_p. Fix A_n and f_n as above. Let $\{e_n\}_{n=1}^{\infty}$ be the unit basis of ℓ_p. Define :

$$D = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{1/p}} e_n + \frac{1}{4} A_n \right).$$

The sets in the above union are disjoint. Indeed if $k \neq n$ and $x \in A_k$, $y \in A_n$ then :

$$\left\| \left(\frac{1}{2^{1/p}} e_k + \frac{1}{4} x \right) - \left(\frac{1}{2^{1/p}} e_n + \frac{1}{4} y \right) \right\| \geq 1 - \frac{1}{4}(||x|| + ||y||) \geq \frac{1}{2}.$$

We now define $f : D \to L_q$ by putting for $x \in A_n$, $f(\frac{1}{2^{1/p}} e_n + \frac{1}{4}x) = f_n(x)$. f is clearly α Hölder with constant 4^α on $\frac{1}{2^{1/p}} e_n + \frac{1}{4}A_n$. Fix $n \neq k$, $x \in A_k$ and $y \in A_n$. Since $f_n(0) = f_k(0) = 0$, $||f_k(x) - f_n(y)|| \leq 2$ so that:

$$\left\| f\left(\frac{1}{2^{1/p}} e_k + \frac{1}{4} x \right) - f\left(\frac{1}{2^{1/p}} e_n + \frac{1}{4} y \right) \right\| \leq$$

$$\leq 2^{1+\alpha} \left\| \left(\frac{1}{2^{1/p}} e_k + \frac{1}{4} x \right) - \left(\frac{1}{2^{1/p}} e_n + \frac{1}{4} y \right) \right\|.$$

This shows that f is α Hölder with constant 4.

If $g : B(\ell_p) \to L_q$ is α Hölder with constant K and extends f, then for every n, the function $h(x) = g(1/2^{1/p}e_n + 1/4x)$ extends f_n, and is α Hölder with constant $4K$. Hence $4K \geq n$, which is a contradiction.

6 Remarks and Open Problems

In [W-W] it was proved that :

$$\mathcal{A}(L_p, L_q) = \begin{cases} (0, p/q^*] & \text{if } 1 < p, q \leq 2 \\ (0, p/q] & \text{if } 1 < p \leq 2 \leq q < \infty \\ (0, p^*/q] & \text{if } 2 \leq p, q < \infty \\ (0, p^*/q^*] & \text{if } 2 \leq p < \infty \text{ and } 1 < q \leq 2 \end{cases}$$

It is possible to prove of all these equalities along the same lines as the proof in section 2. Here is an indication of the required changes. The analogue of lemma 2 is :

Lemma 8 *For all $1 < q < \infty$ there is a constant C which depends only on q such that for all $x \in \ell_q^n$:*

$$\frac{1}{n}\sum_{j=1}^{n} ||x - e_j||_q^q \geq 1 - \frac{C}{n^{q^*-1}}.$$

Proof: As in the proof of lemma 2, we may assume that for all i, $0 \leq x_i \leq 1$. Define also $e = (1, ..., 1) \in \ell_q^n$. Then :

$$\frac{1}{n}\sum_{j=1}^{n} ||x - e_j||_q^q = \frac{1}{n}\sum_{j=1}^{n} \left[||x||_q^q + (|x_j - 1|^q - |x_j|^q)\right] =$$

$$= \left(1 - \frac{1}{n}\right)||x||_q^q + \frac{1}{n}||e - x||_q^q \geq \left(1 - \frac{1}{n}\right)||x||_q^q + \frac{1}{n}\left(n^{1/q} - ||x||_q\right)^q.$$

The lemma now follows from calculation of the minimum of $f(t) = (n-1)t^q + (1-t)^q$, $0 \leq t \leq 1$. ∎

The corresponding analogues of proposition 2 follow when we take $f(e_j) = e_j$ when $1 < p \leq 2 \leq q < \infty$; $f(r_j) = e_j$ when $p, q \geq 2$ and $f(r_j) = r_j$ when $1 < q \leq 2 \leq p$.

In order to prove the reverse inclusion of the above results one proves the required analogues of proposition 3, by using the following probabilistic inequalities in the obvious places :

Lemma 9 *If $p \geq 2$ and X, Y are i.i.d. random vectors in L_p then :*
a) $\mathbb{E}||X - Y||_p^{p^*} \leq 2\mathbb{E}||X||_p^{p^*}$.
b) $\mathbb{E}||X - Y||_p^p \geq 2\mathbb{E}||X - \mathbb{E}X||_p^p$.

Proof: Assume that X is a random variable on the measure space (Ω, P). To prove part a) for every $f : \Omega \to L_p$ define $T(f) : \Omega \times \Omega \to L_p$ by $T(f)(\omega, \tau) = f(\omega) - f(\tau)$. It is easy to verify that :

$$||T(f)||_{L_2(\Omega\times\Omega, L_2)}^2 = 2||f||_{L_2(\Omega, L_2)}^2.$$

And

$$||T(f)||_{L_1(\Omega\times\Omega, L_\infty)} \leq ||f||_{L_1(\Omega, \infty)}.$$

Now interpolate to get the necessary bound on T as an operator from $L_{p^*}(\Omega, L_p)$ to $L_{p^*}(\Omega \times \Omega, L_p)$.

To prove part b) repeat the proof of lemma 5, but now interpolate between $L_2(L_2)$ and $L_\infty(L_\infty)$. ■

For the set $\mathcal{B}(L_p, L_q)$ we have the following inclusion :

Proposition 5

$$\mathcal{B}(L_p, L_q) \subset \begin{cases} (0, p/2] & \textit{if } 1 < p, q \leq 2 \\ (0, p/q] & \textit{if } 1 < p \leq 2 \leq q < \infty \\ (0, 2/q] & \textit{if } 2 \leq p, q < \infty \\ (0, 1] & \textit{if } 2 \leq p < \infty \textit{ and } 1 < q \leq 2 \end{cases}$$

Proof: The first inclusion was already proved and the last inclusion in trivial. The second inclusion is proved by repeating the arguments of section 4, and replacing in all places the Mazur map from $L_p(\mu)$ to $L_2(\mu)$ by the Mazur map from $L_p(\mu)$ to $L_q(\mu)$. The third inclusion now follows since L_p contains a copy of L_2. ■

Combining these results we see that the phase transition phenomenon that appeared in theorem 1 does not always happen :

Theorem 4 *If* $1 < p \leq 2 \leq q < \infty$ *then :*

$$\mathcal{A}(L_p, L_q) = \mathcal{B}(L_p, L_q) = \left(0, \frac{p}{q}\right].$$

In particular, we get that if $q > 2$ then $1 \notin \mathcal{B}(L_2, L_q)$, so that it is not true that all Lipschitz maps from subsets of Hilbert space into normed spaces extend to the whole of Hilbert space. This answers a question posed by Keith Ball in [B].

In the paper [B] the notion of Markov cotype 2 was defined. Analogously we say that a normed space X has Markov cotype q if there is a constant K such that for every $n \times n$ symmetric stochastic matrix A, $0 < \alpha < 1$ and $x_1, ..., x_n \in X$:

$$\alpha \sum a_{i,j} \left\| \sum c_{ir} x_r - \sum c_{j,s} x_s \right\|^q \leq K^q (1 - \alpha) \sum c_{ij} ||x_i - x_j||^q.$$

Where $C = (1 - \alpha)(I - \alpha A)^{-1}$. The least K for which this holds is denoted by $N_q(X)$.

The obvious modifications of theorem 1.7. in [B] give :

Theorem 5 *Let (X, d) be a metric space with Markov type q and Y a reflexive Banach space with Markov cotype q. Then $1 \in \mathcal{B}(X, Y)$ and:*

$$K_1(X, Y) \leq 3M_q(X)N_q(Y).$$

Again, simple modifications of theorem 4.1. of [B] show that if $q \geq 2$ then L_q has Markov cotype q.

In [B] it was conjectured that for $p \geq 2$, L_p has Markov type 2. This conjecture leads to the following:

Conjecture Assume that $p \geq 2$. Then if $1 < q \leq 2$, $\mathcal{B}(L_p, L_q) = (0, 1]$. If $q \geq 2$ then $\mathcal{B}(L_p, L_q) = (0, 2/q]$.

Assuming that L_p has Markov type 2 for $P \geq 2$ we can prove this conjecture. In the first case $1 \in \mathcal{B}(L_p, L_q)$ because of the above theorem. In the second case use the above theorem for $X = L_p$, equipped with the metric $d(x, y) = ||x-y||_p^{2/q}$ and deduce that $2/q \in \mathcal{B}(L_p, L_q)$. The result now follows from theorem 2 and proposition 5.

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