CLASS NOTES ON LIPSCHITZ EXTENSION FROM FINITE SETS

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Our goal here is to give a self-contained proof of the following theorem, which was originally proved in [LN05]. The proof below is based on the same ideas as in [LN05], but some steps and constructions are different, leading to simplifications. The previously best-known bound on this problem was due to [JLS86].

Theorem 0.1. *Suppose that (X, d_X) is a metric space and $(Z, \|\cdot\|_Z)$ is a Banach space. Fix an integer $n \geqslant 3$ and $A \subseteq X$ with $|A| = n$. Then for every Lipschitz function $f : A \to Z$ there exists a function $F : X \to Z$ that extends f and*

$$\|F\|_{\mathrm{Lip}} \lesssim \frac{\log n}{\log\log n}\|f\|_{\mathrm{Lip}}. \tag{1}$$

By normalization, we may assume from now on that $\|f\|_{\mathrm{Lip}} = 1$. Write $A = \{a_1, \ldots, a_n\}$. For $r \in [0, \infty)$ let A_r denote the r-neighborhood of A in X, i.e.,

$$A_r \stackrel{\mathrm{def}}{=} \bigcup_{j=1}^{n} B_X(a_j, r),$$

where for $x \in X$ and $r \geqslant 0$ we denote $B_X(x, r) \stackrel{\mathrm{def}}{=} \{y \in X : \ d_X(x, y) \leqslant r\}$. Given a permutation $\pi \in S_n$ and $r \in [0, \infty)$, for every $x \in A_r$ let $j_r^\pi(x) \in \{1, \ldots, n\}$ be the smallest $j \in \{1, \ldots, n\}$ for which $d_X(a_{\pi(j)}, x) \leqslant r$. Such a j must exist since $x \in A_r$. Define $\mathfrak{a}_r^\pi : X \to A$ by

$$\forall\, x \in X, \qquad \mathfrak{a}_r^\pi(x) \stackrel{\mathrm{def}}{=} \begin{cases} x & \text{if } x \in A, \\ a_{j_r^\pi(x)} & \text{if } x \in A_r \smallsetminus A, \\ a_1 & \text{if } x \in X \smallsetminus A_r. \end{cases} \tag{2}$$

We record the following lemma for future use; compare to inequality (3) in [MN07].

Lemma 0.2. *Suppose that $r > 0$ and that $x, y \in A_r$ satisfy $d_X(x, y) \leqslant r$. Then*

$$\frac{|\{\pi \in S_n : \ \mathfrak{a}_r^\pi(x) \neq \mathfrak{a}_r^\pi(y)\}|}{n!} \leqslant 1 - \frac{|A \cap B_X(x, r - d_X(x, y))|}{|A \cap B_X(x, r + d_X(x, y))|}.$$

Proof. Suppose that $\pi \in S_n$ is such that the minimal $j \in \{1, \ldots, n\}$ for which $a_{\pi(j)} \in B_X(x, r + d_X(x, y))$ actually satisfies $a_{\pi(j)} \in B_X(x, r - d_X(x, y))$. Hence $j_r^\pi(x) = j$ and therefore $a_{\pi(j)} = \mathfrak{a}_r^\pi(x)$. Also, $d_X(a_{\pi(j)}, y) \leqslant d_X(a_{\pi(j)}, x) + d_X(x, y) \leqslant r$, so $j_r^\pi(y) \leqslant j$. But $d_X(x, a_{j_r^\pi(y)}) \leqslant d_X(y, a_{j_r^\pi(y)}) + d_X(x, y) \leqslant r + d_X(x, y)$, so by the definition of j we must have $j_r^\pi(y) \geqslant j$. Thus $j_r^\pi(y) = j$, so that $\mathfrak{a}_r^\pi(y) = a_{\pi(j)} = \mathfrak{a}_r^\pi(x)$. We have shown that if in the random order that π induces on A the first element that falls in the ball $B_X(x, r + d_X(x, y))$ actually falls in the smaller ball $B_X(x, r - d_X(x, y))$, then $\mathfrak{a}_r^\pi(y) = \mathfrak{a}_r^\pi(x)$. If π is chosen uniformly at random from S_n then the probability of this event equals $|A \cap B_X(x, r - d_X(x, y))|/|A \cap B_X(x, r + d_X(x, y))|$. Hence,

$$\frac{|\{\pi \in S_n : \ \mathfrak{a}_r^\pi(x) = \mathfrak{a}_r^\pi(y)\}|}{n!} \geqslant \frac{|A \cap B_X(x, r - d_X(x, y))|}{|A \cap B_X(x, r + d_X(x, y))|}. \qquad \square$$

Corollary 0.3. *Suppose that $0 \leqslant u \leqslant v$ and $x, y \in A_u$ satisfy $d_X(x, y) \leqslant \min\{u - d_X(x, A), v - u/2\}$. Then*

$$\int_u^v \left(\frac{1}{t}\int_t^{2t} \frac{|\{\pi \in S_n : \ \mathfrak{a}_r^\pi(x) \neq \mathfrak{a}_r^\pi(y)\}|}{n!} dr\right) dt \lesssim d_X(x, y)\log n.$$

Proof. Denote $d \stackrel{\mathrm{def}}{=} d_X(x, y)$ and

$$\forall\, r \geqslant 0, \qquad g(r) \stackrel{\mathrm{def}}{=} \frac{|\{\pi \in S_n : \ \mathfrak{a}_r^\pi(x) \neq \mathfrak{a}_r^\pi(y)\}|}{n!} \qquad \text{and} \qquad h(r) \stackrel{\mathrm{def}}{=} \log\left(|A \cap B_X(x, r)|\right). \tag{3}$$

Since $x,y\in A_u$ and for $r\geqslant u$ we have $A_u\subseteq A_r$ and $d_X(x,y)\leqslant u\leqslant r$, Lemma 0.2 implies that
$$\forall\, r\geqslant u,\qquad g(r)\leqslant 1-e^{h(r-d)-h(r+d)}\leqslant h(r+d)-h(r-d), \tag{4}$$
where in the last step of (4) we used the elementary inequality $1-e^{-\alpha}\leqslant \alpha$, which holds for every $\alpha\in\mathbb{R}$.

Note that by Fubini we have
$$\int_u^v\left(\frac{1}{t}\int_t^{2t}g(r)dr\right)dt=\int_u^{2v}\left(\int_{\max\{u,r/2\}}^{\min\{v,r\}}\frac{g(r)}{t}dt\right)dr=\int_u^{2v}g(r)\log\left(\frac{\min\{v,r\}}{\max\{u,r/2\}}\right)dr. \tag{5}$$
Since $\min\{v,r\}/\max\{u,r/2\}\leqslant 2$, it follows that
$$\int_u^v\left(\frac{1}{t}\int_t^{2t}g(r)dr\right)dt\overset{(5)}{\lesssim}\int_u^{2v}g(r)dr\overset{(4)}{\leqslant}\int_{u+d}^{2v+d}h(s)ds-\int_{u-d}^{2v-d}h(s)ds=\int_{2v-d}^{2v+d}h(s)ds-\int_{u-d}^{u+d}h(s)ds, \tag{6}$$
where the last step of (6) is valid because $2v-d\geqslant u+d$, due to our assumption that $d\leqslant v-u/2$.

Since h is nondecreasing, for every $s\in[2v-d,2v+d]$ we have $h(s)\leqslant h(2v+d)$, and for every $s\in[u-d,u+d]$ we have $h(s)\geqslant h(u-d)$. It therefore follows from (6) that
$$\int_u^v\left(\frac{1}{t}\int_t^{2t}g(r)dr\right)dt\lesssim d\big(h(2v+d)-h(u-d)\big)\overset{(3)}{=}d\log\left(\frac{|A\cap B_X(x,2v+d)|}{|A\cap B_X(x,u-d)|}\right)\lesssim d\log n, \tag{7}$$
where in the last step of (7) we used the fact that $|A\cap B_X(x,2v+d)|\leqslant |A|=n$, and, due to our assumption $d\leqslant u-d_X(x,A)$ ($\iff d_X(x,A)\leqslant u-d$), that $A\cap B_X(x,u-d)\neq\emptyset$, so that $|A\cap B_X(x,u-d)|\geqslant 1$. $\square$

Returning to the proof of Theorem 0.1, fix $\varepsilon\in(0,1/2)$. Fix also any $(2/\varepsilon)$-Lipschitz function $\phi_\varepsilon:\mathbb{R}\to[0,1]$ that vanishes outside $[\varepsilon/2,1+\varepsilon/2]$ and $\phi_\varepsilon(s)=1$ for every $s\in[\varepsilon,1]$. Note that
$$\log(1/\varepsilon)=\int_\varepsilon^1\frac{ds}{s}\leqslant\int_0^\infty\frac{\phi_\varepsilon(s)}{s}ds\leqslant\int_{\varepsilon/2}^{1+\varepsilon/2}\frac{ds}{s}\leqslant\log(3/\varepsilon).$$
Hence, if we define $c(\varepsilon)\in(0,\infty)$ by
$$\frac{1}{c(\varepsilon)}\overset{\mathrm{def}}{=}\int_0^\infty\frac{\phi_\varepsilon(s)}{s}ds, \tag{8}$$
then
$$\frac{1}{\log(3/\varepsilon)}\leqslant c(\varepsilon)\leqslant\frac{1}{\log(1/\varepsilon)}. \tag{9}$$
Define $F:X\to Z$ by setting $F(x)=f(x)$ for $x\in A$ and
$$\forall\, x\in X\smallsetminus A,\qquad F(x)\overset{\mathrm{def}}{=}\frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\int_0^\infty\frac{1}{t^2}\phi_\varepsilon\left(\frac{2}{t}d_X(x,A)\right)\left(\int_t^{2t}f\left(\mathfrak{a}_r^\pi(x)\right)dr\right)dt. \tag{10}$$
By definition, F extends f. Next, suppose that $x\in X$ and $y\in X\smallsetminus A$. Fix any $z\in A$ that satisfies $d_X(x,z)=d_X(x,A)$ (thus if $x\in A$ then $z=x$). We have the following identity.
$$F(y)-F(x)$$
$$=\frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}}{t^2}\left(f(a)-f(z)\right)drdt. \tag{11}$$
To verify the validity of (11), note that for every $w\in X\smallsetminus A$ we have
$$\frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\phi_\varepsilon\left(\frac{2d_X(w,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(w)=a\}}}{t^2}f(z)drdt=\frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\int_0^\infty\int_t^{2t}\frac{\phi_\varepsilon\left(\frac{2d_X(w,A)}{t}\right)}{t^2}f(z)drdt$$
$$=c(\varepsilon)\left(\int_0^\infty\frac{1}{t}\phi_\varepsilon\left(\frac{2}{t}d_X(w,A)\right)dt\right)f(z)\overset{(*)}{=}c(\varepsilon)\left(\int_0^\infty\frac{\phi_\varepsilon(s)}{s}ds\right)f(z)\overset{(8)}{=}f(z), \tag{12}$$
where in $(*)$ we made the change of variable $s=2d_X(w,A)/t$, which is allowed since $d_X(w,A)>0$. Due to (12), if $x,y\in X\smallsetminus A$ then (11) is a consequence of the definition (10). If $x\in A$ (recall that $y\in X\smallsetminus A$) then $z=x$ and $\phi_\varepsilon(2d_X(x,A)/t)=0$ for all $t>0$. So, in this case (11) follows once more from (12) and (10).

By (11) we have

$$
\begin{aligned}
&\|F(x)-F(y)\|_Z\\
&\leqslant \frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t^2}\|f(a)-f(z)\|_Z drdt\\
&\leqslant \frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t^2}d_X(a,z)drdt \qquad (13)\\
&\leqslant \frac{2c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t^2}d_X(x,a)drdt, \qquad (14)
\end{aligned}
$$

where in (13) we used the fact that $\|f\|_{\mathrm{Lip}}=1$ and in (14) we used the fact that for every $a\in A$ we have $d_X(a,z)\leqslant d_X(a,x)+d_X(x,z)\leqslant 2d_X(a,x)$, due to the choice of z as the point in A that is closest to x.

To estimate (14), fix $t>0$ and $r\in[t,2t]$. If $\phi_\varepsilon(2d_X(y,A)/t)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}\neq\phi_\varepsilon(2d_X(x,A)/t)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}$ then either $a=\mathfrak{a}_r^\pi(x)$ and $2d_X(x,A)/t\in\mathrm{supp}(\phi_\varepsilon)$ or $a=\mathfrak{a}_r^\pi(y)$ and $2d_X(y,A)/t\in\mathrm{supp}(\phi_\varepsilon)$. Recalling that $\mathrm{supp}(\phi_\varepsilon)\subseteq[\varepsilon/2,1+\varepsilon/2]$, it follows that either $a=\mathfrak{a}_r^\pi(x)$ and $d_X(x,A)<t$ or $a=\mathfrak{a}_r^\pi(y)$ and $d_X(y,A)<t$. If $a=\mathfrak{a}_r^\pi(x)$ and $d_X(x,A)<t$ then since $t\leqslant r$ it follows that $x\in A_r$, and so the definition of $\mathfrak{a}_r^\pi(x)$ implies that $d_X(x,a)=d_X(\mathfrak{a}_r^\pi(x),x)\leqslant r$. On the other hand, if $a=\mathfrak{a}_r^\pi(y)$ and $d_X(y,A)<t$ then as before we have $d_X(y,a)=d_X(\mathfrak{a}_r^\pi(y),y)\leqslant r$, and therefore $d_X(x,a)\leqslant d_X(x,y)+d_X(y,a)\leqslant d_X(x,y)+r$. We have thus checked that $d_X(x,a)\leqslant d_X(x,y)+r\leqslant d_X(x,y)+2t$ whenever the integrand in (14) is nonzero. Consequently,

$$
\begin{aligned}
&\|F(x)-F(y)\|_Z\\
&\overset{(14)}{\leqslant} \frac{2c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}+\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}}{t^2}d_X(x,y)drdt\\
&\qquad+\frac{4c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t}drdt.\\
&\overset{(12)}{=} 4d_X(x,y)+\frac{4c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t}drdt.
\end{aligned}
$$

Therefore, in order to establish the validity of (1) it suffice to show that we can choose $\varepsilon\in(0,1/2)$ so that

$$
\frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t}drdt\lesssim\frac{\log n}{\log\log n}d_X(x,y). \qquad (15)
$$

We shall prove below that for every $\varepsilon\in(0,1/2]$ we have

$$
\begin{aligned}
&\frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t}drdt\\
&\qquad\qquad\lesssim\left(\frac{1}{\varepsilon}+\log n\right)\frac{d_X(x,y)}{\log(1/\varepsilon)}. \qquad (16)
\end{aligned}
$$

Once proved, (16) would imply (15), and hence also Theorem 0.1, if we choose $\varepsilon\asymp 1/\log n$.

Fix $t>0$ and $r\in[t,2t]$ and note that if $\phi_\varepsilon(2d_X(y,A)/t)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}\neq\phi_\varepsilon(2d_X(x,A)/t)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}$ then $\{2d_X(x,A),2d_X(y,A)/t\}\cap\mathrm{supp}(\phi_\varepsilon)\neq\emptyset$, implying that $\max\{d_X(x,A),d_X(y,A)\}\geqslant\varepsilon t/4$. Hence, since

$$
\forall\,\pi\in S_n,\qquad \sum_{a\in A}\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|\leqslant 2,
$$

we have

$$\begin{aligned}\frac{1}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}&\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t}drdt\\
&\leqslant 2\int_0^{\frac{4}{\varepsilon}\max\{d_X(x,A),d_X(y,A)\}}\int_t^{2t}\frac{drdt}{t}\\
&\lesssim\frac{\max\{d_X(x,A),d_X(y,A)\}}{\varepsilon}\\
&\lesssim\frac{d_X(x,y)+\min\{d_X(x,A),d_X(y,A)\}}{\varepsilon}.\end{aligned}$$

In combination with the upper bound on $c(\varepsilon)$ in (9), we therefore have the following corollary (the constant $\frac{5}{3}$ that appears in it isn't crucial; it was chosen only to simplify some of the ensuing expressions).

Corollary 0.4. *If* $\min\{d_X(x,A),d_X(y,A)\}\leqslant\frac{5}{3}d_X(x,y)$ *then*

$$\frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t}drdt\lesssim\frac{d_X(x,y)}{\varepsilon\log(1/\varepsilon)}.$$

Corollary 0.4 implies that (16) holds true when $\min\{d_X(x,A),d_X(y,A)\}\leqslant 5d_X(x,y)/3$. We shall therefore assume from now on that the assumption of Corollary 0.4 fails, i.e., that

$$d_X(x,y)<\frac{3}{5}\min\{d_X(x,A),d_X(y,A)\}. \tag{17}$$

Define

$$U_\varepsilon(x,y)\stackrel{\mathrm{def}}{=}\left\{t\in(0,\infty):\ \left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\right|>0\right\}, \tag{18}$$

and

$$V_\varepsilon(x,y)\stackrel{\mathrm{def}}{=}\left\{t\in(0,\infty):\ \phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)+\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)>0\right\}. \tag{19}$$

Then, for every $\pi\in S_n$, $t>0$ and $r\in[t,2t]$ we have

$$\begin{aligned}&\sum_{a\in A}\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|\\
&=\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\right|\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=\mathfrak{a}_r^\pi(y)\}}\\
&\quad+\left(\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)+\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)\neq\mathfrak{a}_r^\pi(y)\}}\\
&\lesssim\frac{d_X(x,y)}{\varepsilon t}\mathbf{1}_{U_\varepsilon(x,y)}+\mathbf{1}_{V_\varepsilon(x,y)}\cdot\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)\neq\mathfrak{a}_r^\pi(y)\}}.\end{aligned} \tag{20}$$

Where in (20) we used the fact that ϕ_ε is $(2/\varepsilon)$-Lipschitz and that $|d_X(x,A)-d_X(y,A)|\leqslant d_X(x,y)$. Consequently, in combination with the upper bound on $c(\varepsilon)$ in (9), it follows from (20) that

$$\begin{aligned}\frac{c(\varepsilon)}{n!}\sum_{\pi\in S_n}\sum_{a\in A}\int_0^\infty\int_t^{2t}&\frac{\left|\phi_\varepsilon\left(\frac{2d_X(y,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(y)=a\}}-\phi_\varepsilon\left(\frac{2d_X(x,A)}{t}\right)\mathbf{1}_{\{\mathfrak{a}_r^\pi(x)=a\}}\right|}{t}drdt\\
&\lesssim\frac{d_X(x,y)}{\varepsilon\log(1/\varepsilon)}\int_{U_\varepsilon(x,y)}\frac{dt}{t}+\frac{1}{\log(1/\varepsilon)}\int_{V_\varepsilon(x,y)}\left(\frac{1}{t}\int_t^{2t}\frac{|\{\pi\in S_n:\ \mathfrak{a}_r^\pi(x)\neq\mathfrak{a}_r^\pi(y)\}|}{n!}dr\right)dt.\end{aligned} \tag{21}$$

To bound the first term in (21), denote

$$m(x,y)\stackrel{\mathrm{def}}{=}\min\{d_X(x,A),d_X(y,A)\}\qquad\text{and}\qquad M(x,y)\stackrel{\mathrm{def}}{=}\max\{d_X(x,A),d_X(y,A)\}.$$

If $t\in[0,\infty)$ satisfies $t<2m(x,y)/(1+\varepsilon/2)$ then $\min\{2d_X(x,A)/t,2d_X(y,A)/t\}>1+\varepsilon/2$, and therefore by the definition of ϕ_ε we have $\phi_\varepsilon(2d_X(x,A)/t)=\phi_\varepsilon(2d_X(y,A)/t)=0$. Similarly, if $t\in[0,\infty)$

satisfies $t > 4M(x,y)/\varepsilon$ then $\max\{2d_X(x,A)/t, 2d_X(y,A)/t\} < \varepsilon/2$, and therefore by the definition of ϕ_ε we also have $\phi_\varepsilon(2d_X(x,A)/t) = \phi_\varepsilon(2d_X(y,A)/t) = 0$. Finally, if $2M(x,y) \leqslant t \leqslant 2m(x,y)/\varepsilon$ then $2d_X(x,A)/t, 2d_X(y,A)/t \in [\varepsilon, 1]$, so by the definition of ϕ_ε we have $\phi_\varepsilon(2d_X(x,A)/t) = \phi_\varepsilon(2d_X(y,A)/t) = 1$. By the definition of $U_\varepsilon(x,y)$ in (18), we have thus shown that

$$U_\varepsilon(x,y) \subseteq \left[\frac{2m(x,y)}{1+\varepsilon/2}, 2M(x,y)\right] \cup \left[\frac{2m(x,y)}{\varepsilon}, \frac{4M(x,y)}{\varepsilon}\right].$$

Consequently,

$$\int_{U_\varepsilon(x,y)} \frac{dt}{t} \leqslant \int_{\frac{2m(x,y)}{1+\varepsilon/2}}^{2M(x,y)} \frac{dt}{t} + \int_{\frac{2m(x,y)}{\varepsilon}}^{\frac{4M(x,y)}{\varepsilon}} \frac{dt}{t} \lesssim \log\left(\frac{2M(x,y)}{m(x,y)}\right) \lesssim 1, \qquad (22)$$

where the last step of (22) holds true because, due to the triangle inequality and (17), we have

$$M(x,y) \leqslant d_X(x,y) + m(x,y) < \frac{3}{5}m(x,y) + m(x,y) \lesssim m(x,y).$$

To bound the second term in (21), note that by the definition of $V_\varepsilon(x,y)$ in (19) and the choice of ϕ_ε,

$$t \in V_\varepsilon(x,y) \implies \left\{\frac{2d_X(x,A)}{t}, \frac{2d_X(y,A)}{t}\right\} \cap \left[\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}\right] \neq \emptyset. \qquad (23)$$

Hence,

$$V_\varepsilon(x,y) \subseteq \left[\frac{2d_X(x,A)}{1+\varepsilon/2}, \frac{4d_X(x,A)}{\varepsilon}\right] \cup \left[\frac{2d_X(y,A)}{1+\varepsilon/2}, \frac{4d_X(y,A)}{\varepsilon}\right], \qquad (24)$$

and therefore, using the notation for $g : [0,\infty) \to [0,1]$ that was introduced in (3),

$$\int_{V_\varepsilon(x,y)} \left(\frac{1}{t}\int_t^{2t} g(r)dr\right) dt \leqslant \int_{\frac{2d_X(x,A)}{1+\varepsilon/2}}^{\frac{4d_X(x,A)}{\varepsilon}} \left(\frac{1}{t}\int_t^{2t} g(r)dr\right) dt + \int_{\frac{2d_X(y,A)}{1+\varepsilon/2}}^{\frac{4d_X(y,A)}{\varepsilon}} \left(\frac{1}{t}\int_t^{2t} g(r)dr\right) dt. \qquad (25)$$

We wish to use Corollary 0.3 to estimate the two integrals that appear in the right hand side of (25). To this end we need to first check that the assumptions of Corollary 0.3 are satisfied. Denote $u_x = 2d_X(x,A)/(1+\varepsilon/2)$ and $u_y = 2d_X(y,A)/(1+\varepsilon/2)$. Since $u_x \geqslant d_X(x,A)$ we have $x \in A_{u_x}$. Analogously, $y \in A_{u_y}$. Also,

$$d_X(y,A) \leqslant d_X(x,y) + d_X(x,A) \overset{(17)}{\leqslant} \frac{3}{5}d_X(x,A) + d_X(x,A) = \frac{4+2\varepsilon}{5}u_x \leqslant u_x, \qquad (26)$$

where the last step of (26) is valid because $\varepsilon \leqslant 1/2$. From (26) we see that $y \in A_{u_x}$, and the symmetric argument shows that $x \in A_{u_y}$. It also follows from (26) that $d_X(x,y) \leqslant u_x - d_X(x,A)$, and by symmetry also $d_X(x,y) \leqslant u_y - d_X(y,A)$. Next, denote $v_x = 4d_X(x,A)/\varepsilon$ and $v_x = 4d_X(y,A)/\varepsilon$. In order to verify the assumptions of Corollary 0.3, it remains to check that $d_X(x,y) \leqslant \min\{v_x - u_x/2, v_y - u_y/2\}$. Indeed,

$$\frac{d_X(x,y)}{v_x - u_x/2} \overset{(17)}{<} \frac{3d_X(x,A)/5}{v_x - u_x/2} = \frac{\frac{3}{5}}{\frac{4}{\varepsilon} - \frac{1}{1+\varepsilon/2}} = \frac{3\varepsilon(1+\varepsilon/2)}{5(4+\varepsilon)} < 1,$$

and the symmetric argument shows that also $d_X(x,y) < v_y - u_y/2$. Having checked that the assumptions of Corollary 0.3 hold true, it follows from (25) and Corollary 0.3 that

$$\int_{V_\varepsilon(x,y)} \left(\frac{1}{t}\int_t^{2t} \frac{|\{\pi \in S_n :\ \mathfrak{a}_r^\pi(x) \neq \mathfrak{a}_r^\pi(y)\}|}{n!} dr\right) dt \lesssim d_X(x,y)\log n. \qquad (27)$$

The desired estimate (16) now follows from a substitution of (22) and (27) into (21). □

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