

CAYLEY GRAPHS THAT HAVE A QUANTUM ERGODIC EIGENBASIS

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ABSTRACT. We investigate which finite Cayley graphs admit a quantum ergodic eigenbasis, proving that this holds for any Cayley graph on a group of size n for which the sum of the dimensions of its irreducible representations is $o(n)$, yet there exist Cayley graphs that do not have any quantum ergodic eigenbasis.

1. INTRODUCTION

We will prove here the following theorem; see Theorem 3 for a companion impossibility result.

Theorem 1. *There exists an absolute constant $c > 0$ with the following property. Suppose that $\varepsilon > 0$ and let G be a finite group whose irreducible representations have total dimension at most $c\varepsilon^2 |G|$, i.e.,*

$$\sum_{\sigma \in \widehat{G}} d_\sigma \leq c\varepsilon^2 |G|. \quad (1)$$

Then, any Cayley graph on G has an orthonormal eigenbasis \mathcal{B} consisting of functions $\phi: G \rightarrow \mathbb{C}$ satisfying

$$\forall f: G \rightarrow \mathbb{C}, \quad \mathbb{E}_{\phi \in \mathcal{B}} \left[\left| \mathbb{E}_{x \in G} [f(x)|\phi(x)|^2] - \mathbb{E}f \right| \right] \leq \varepsilon \|f\|_2. \quad (2)$$

In the statement of Theorem 1, \widehat{G} is the set of irreducible unitary representations of a finite group G and the dimension of each $\sigma \in \widehat{G}$ is denoted d_σ . The representation theory of finite groups that we will use below is rudimentary; see e.g. [16, 28]. A Cayley graph on G is a graph whose vertex set is G such that there is a symmetric subset $\mathfrak{S} \subseteq G$ that generates G and $\{g, h\} \subseteq G$ forms an edge if and only if $gh^{-1} \in \mathfrak{S}$.

In Theorem 1 and throughout what follows, we will adhere to the convention that a finite set X is only equipped with the uniform probability measure; thus, all expectations, scalar products and L_p norms of functions from X to \mathbb{C} will be with respect to this measure, i.e., for every $f, g: X \rightarrow \mathbb{C}$ and $1 \leq p \leq \infty$,

$$\mathbb{E}f = \mathbb{E}_{x \in X} [f(x)] = \frac{1}{|X|} \sum_{x \in X} f(x) \quad \text{and} \quad \langle f, g \rangle = \mathbb{E}_{x \in X} [\overline{f(x)}g(x)] \quad \text{and} \quad \|f\|_p = \left(\mathbb{E}[|f|^p] \right)^{\frac{1}{p}}. \quad (3)$$

So, a set of functions $\phi_1, \dots, \phi_{|X|}: X \rightarrow \mathbb{C}$ is an orthonormal basis if $\|\phi_j\|_2 = 1$ and $\langle \phi_j, \phi_k \rangle = 0$ for every distinct $j, k \in \{1, \dots, |X|\}$. If X is a graph, then we say that $\mathcal{B} = \{\phi_1, \dots, \phi_{|X|}\}$ is an orthonormal eigenbasis of X if it is an orthonormal basis consisting of eigenfunctions of the adjacency matrix of X .

Theorem 1 is a finitary statement in the spirit of quantum ergodicity on manifolds, e.g. Šnirel'man's classical theorem [30, 9, 31]. Investigations along these lines include notably [4], and we refer also to [2, 5] and the survey [3] for background and motivation. From these works, we extract the following definition.

Definition 2 (quantum ergodic basis). Given a finite set X and $\varepsilon > 0$, we say that an orthonormal basis \mathcal{B} of functions $\phi: X \rightarrow \mathbb{C}$ is ε -quantum ergodic if

$$\forall f: X \rightarrow \mathbb{C}, \quad \mathbb{E}_{\phi \in \mathcal{B}} \left[\left| \mathbb{E}_{x \in X} [f(x)|\phi(x)|^2] - \mathbb{E}f \right| \right] \leq \varepsilon \|f\|_\infty. \quad (4)$$

The only difference between the conclusion (2) of Theorem 1 and the requirement (4) of Definition 2 is that the quantity $\|f\|_2$ in the right hand side of (2) is replaced in the right hand side of (4) by the larger quantity $\|f\|_\infty$. Therefore, Theorem 1 implies that any Cayley graph of a finite group whose irreducible representations have total dimension at most $c\varepsilon^2 |G|$ has an ε -quantum ergodic eigenbasis. The stronger

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conclusion (2) of Theorem 1 can be significantly stronger when e.g. in (2) we take f to be the indicator of a small nonempty subset S of G , as in this case $\|f\|_\infty = 1$ while $\|f\|_2 = \sqrt{|S|/|G|}$.

The reason why we formulated Definition 2 using the L_∞ norm of f rather than its L_2 norm, is first and foremost because this is how the subject is treated in the literature, but also because the following impossibility result rules out even the weaker requirement (4).

Theorem 3. *There exist a universal constant $c > 0$ and arbitrarily large Cayley graphs that do not admit any c -quantum ergodic orthonormal eigenbasis.*

The groups that we will construct in the proof of Theorem 3 will be direct products of cyclic groups with an appropriately chosen fixed group (specifically, a group that was constructed in [27]); the resulting Cayley graphs will have bounded girth as the size of the graph tends to ∞ , which is one of the reasons why their existence does not contradict the literature on quantum ergodicity.

Problem 4. For a finite group G let $\varepsilon(G)$ be the infimum over $\varepsilon > 0$ such that every Cayley graph on G has an ε -quantum ergodic orthonormal eigenbasis. Characterize those sequences $\{G_n\}_{n=1}^\infty$ of groups for which $\lim_{n \rightarrow \infty} \varepsilon(G_n) = 0$. More ambitiously, how can one estimate $\varepsilon(G)$ up to universal constant factors?

Any Abelian group G satisfies $\varepsilon(G) = 0$, as seen by considering the eigenbasis \mathcal{B} of Fourier characters: each $\phi \in \mathcal{B}$ takes values among the roots of unity, so the left-hand side of (2) vanishes for every $f : G \rightarrow \mathbb{C}$. Theorem 1 furnishes many more examples of sequences $\{G_n\}_{n=1}^\infty$ of groups with $\lim_{n \rightarrow \infty} \varepsilon(G_n) = 0$.

If $\eta > 0$ and G is a group with at most $\eta|G|$ conjugacy classes (e.g. by [20, Theorem 2] this holds with $\eta = 2^{n-1}/|G|$ if G is any subgroup of the permutation group S_n), then every Cayley graph on G has a $O(\sqrt[\eta]{\eta})$ -quantum ergodic orthonormal eigenbasis. Indeed,

$$\sum_{\sigma \in \widehat{G}} d_\sigma \leq |\widehat{G}|^{\frac{1}{2}} \left(\sum_{\sigma \in \widehat{G}} d_\sigma^2 \right)^{\frac{1}{2}} \leq \sqrt{\eta} |G|,$$

where the first step uses Cauchy—Schwarz and the second step uses the fact that $|\widehat{G}|$ equals the number of conjugacy classes of G , so it is at most $\eta|G|$, as well as $\sum_{\sigma \in \widehat{G}} d_\sigma^2 = |G|$. Hence, (1) holds with $\varepsilon = \sqrt[\eta]{\eta}/\sqrt{c}$.

The above example includes the groups that are quasirandom in the sense of Gowers [14]. Recall that given $D \in \mathbb{N}$, a finite group G is said to be D -quasirandom in the sense of Gowers if every nontrivial unitary representation of G has dimension at least D . If G is nontrivial and D -quasirandom in the sense of Gowers, then G has at most $2|G|/(D^2 + 1)$ conjugacy classes, and hence every Cayley graph on G has a $O(1/\sqrt{D})$ -quantum ergodic orthonormal eigenbasis. Indeed,

$$|G| = \sum_{\sigma \in \widehat{G}} d_\sigma^2 = 1 + \sum_{\sigma \in \widehat{G} \setminus \{\text{triv}\}} d_\sigma^2 \geq 1 + D^2(|\widehat{G}| - 1). \quad (5)$$

Thus, $|\widehat{G}| \leq 1 + (|G| - 1)/D^2 \leq 2|G|/(D^2 + 1)$, where the last step holds as $|G| > 1$ and therefore the second sum in (5) is nonempty, so in fact $|G| \geq D^2 + 1$. By an inspection of the tables on pages 769–770 of [10] and the classification of finite simple groups, if G is a non-cyclic simple group, then we can take D to be at least a universal constant multiple of $(\log|G|)/\log\log|G|$; for most simple groups a much better lower bound on D is available, and many more examples appear in the literature (see e.g. [29, Chapter 1, §1.3]).

At the same time, Theorem 3 demonstrates that some assumption on $\{G_n\}_{n=1}^\infty$ must be imposed to ensure that $\lim_{n \rightarrow \infty} \varepsilon(G_n) = 0$. Thus, Problem 4 remains an intriguing open question.

Remark. See [21] for incomparable results in a different but related setting of quantum unique ergodicity, rather than quantum ergodicity, in Cayley graphs. In yet another different but related setting, our forthcoming work [24] treats the question of existence of delocalized eigenbases for Cayley graphs, as a special case of a much more general phenomenon; the present work is an outgrowth of our investigations in [24], which were motivated by unrelated analytic and combinatorial issues that arose in [25, 27], though delocalization statements are commonly precursors to results on quantum ergodicity (e.g. [8]).

2. PROOF OF THEOREM 1

The Haar probability measure on a compact topological group Γ will be denoted h_Γ . Given $d \in \mathbb{N}$, the standard coordinate basis of \mathbb{C}^d will be denoted e_1, \dots, e_d and the unitary group of $d \times d$ matrices will be denoted $\mathbb{U}(d)$. The Hilbert–Schmidt norm of a $d \times d$ matrix $A = (a_{jk}) \in M_d(\mathbb{C})$ will be denoted

$$\|A\|_{\text{HS}} = \left(\sum_{j=1}^d \sum_{k=1}^d |a_{jk}|^2 \right)^{\frac{1}{2}}.$$

Our construction of the basis \mathcal{B} of Theorem 1 will be randomized; its main probabilistic input is the following lemma whose proof appears in Section 2.1 below. This lemma provides a deviation inequality for averages of absolute values of tuples of quadratic forms on products of unitary groups. Suppose that we are given a collection of traceless square matrices $\{A_s \in M_{d_s}(\mathbb{C})\}_{s \in S}$, where S is some finite index set and for each $s \in S$ the dimension of A_s is $d_s \in \mathbb{N}$. For each $s \in S$ and $k \in \{1, \dots, d_s\}$ consider the quadratic function that assigns the scalar $e_k^* U_s^* A_s U_s e_k \in \mathbb{C}$ to a unitary matrix $U_s \in \mathbb{U}(d_s)$. The total number of these quadratic functions is $\sum_{s \in S} d_s$. The following lemma bounds from above the probability, with respect to the product of the Haar measures on the Cartesian product of $\{\mathbb{U}(d_s)\}_{s \in S}$, that the average of the absolute values of these quadratic functions is large, which turns out to be controlled using the Hilbert–Schmidt norms of the matrices $\{A_s\}_{s \in S}$ through the parameter $\alpha \geq 0$ that appears in equation (6) below.

Lemma 5. *There exists a universal constant $0 < \eta < 1$ with the following property. Let S be a finite set. For every $s \in S$ fix an integer $d_s \in \mathbb{N}$ and a $d_s \times d_s$ matrix $A_s \in M_{d_s}(\mathbb{C})$ whose trace satisfies $\text{Tr}(A_s) = 0$. Denote*

$$\alpha = \left(\frac{\sum_{s \in S} \frac{1}{d_s} \|A_s\|_{\text{HS}}^2}{\sum_{s \in S} d_s} \right)^{\frac{1}{2}} \quad \text{and} \quad T = \bigcup_{s \in S} (\{s\} \times \{1, \dots, d_s\}) = \{(s, k) : s \in S \wedge k \in \{1, \dots, d_s\}\}. \quad (6)$$

Consider the direct product $\Gamma = \prod_{s \in S} \mathbb{U}(d_s)$ of the unitary groups $\{\mathbb{U}(d_s)\}_{s \in S}$. Then, for every $\beta \geq 2$ we have

$$h_\Gamma \left[\left\{ U = (U_s)_{s \in S} \in \Gamma : \mathbb{E}_{(s,k) \in T} [|e_k^* U_s^* A_s U_s e_k|] \geq \beta \alpha \right\} \right] \leq e^{-\eta \beta^2 \sum_{s \in S} d_s}. \quad (7)$$

Fix a finite group G and fix also a symmetric subset $\mathfrak{S} \subseteq G$ that generates G . Let $n = |G|$. The adjacency matrix $A(G; \mathfrak{S}) \in M_n(\{0, 1\})$ of the Cayley graph that is induced by \mathfrak{S} on G acts on a function $f : G \rightarrow \mathbb{C}$ by $A(G; \mathfrak{S})f(x) = \sum_{\sigma \in \mathfrak{S}} f(\sigma x)$ for every $x \in G$.

We will apply Lemma 5 with the index set S

$$S = \bigcup_{\rho \in \widehat{G}} (\{\rho\} \times \{1, \dots, d_\rho\}) = \{(\rho, j) : \rho \in \widehat{G} \wedge j \in \{1, \dots, d_\rho\}\}.$$

and $d_s = d_\rho$ for every $s = (\rho, j) \in S$. For this S , the set T in (6) becomes

$$T = \{(\rho, j, k) : \rho \in \widehat{G} \wedge (j, k) \in \{1, \dots, d_\rho\}^2\}.$$

Henceforth, $\Gamma = \prod_{(\rho, j) \in S} \mathbb{U}(d_\rho) \cong \prod_{\rho \in \widehat{G}} \mathbb{U}(d_\rho)^{d_\rho}$ will be the group from Lemma 5.

Suppose that for each $\rho \in \widehat{G}$ and $j, k \in \{1, \dots, d_\rho\}$ we are given $a_{\rho, j, k} \in \mathbb{C}^{d_\rho}$ and $b_{\rho, j} \in \mathbb{C}^{d_\rho}$ such that

$$\forall j, j', k, k' \in \{1, \dots, d_\rho\}, \quad a_{\rho, j, k}^* a_{\rho, j, k'} = \mathbf{1}_{\{k=k'\}} \quad \text{and} \quad b_{\rho, j}^* b_{\rho, j'} = \mathbf{1}_{\{j=j'\}}. \quad (8)$$

This is an orthonormality requirement¹ with respect to the standard (not normalized) scalar product on \mathbb{C}^{d_ρ} . The statement of Schur orthogonality is that whenever (8) holds the following collection of functions from G to \mathbb{C} (indexed by T) is orthonormal with respect to the scalar product in (3); as $|T| = \sum_{\rho \in \widehat{G}} d_\rho^2 = n$, it is an orthonormal basis of G :

$$\left\{ (x \in G) \mapsto d_\rho^{\frac{1}{2}} a_{\rho, j, k}^* \rho(x)^* b_{\rho, j} \right\}_{(\rho, j, k) \in T}. \quad (9)$$

¹To be consistent with our normalization convention in (3), for every $d \in \mathbb{N}$ we will use matrix notation as in (8) when treating the standard scalar product on \mathbb{C}^d .

These expressions are also natural through the lens of non-Abelian Fourier analysis. It is mechanical to check that (9) consists of eigenfunctions of the adjacency matrix $A(G; \mathfrak{S})$ if for each $\rho \in \widehat{G}$ we choose $b_{\rho,1}, \dots, b_{\rho,d_\rho} \in \mathbb{C}^{d_\rho}$ to be eigenvectors of the (Hermitian, as \mathfrak{S} is symmetric) matrix

$$\widehat{\mathbf{I}}_{\mathfrak{S}}(\rho) = \mathbb{E}_{\sigma \in \mathfrak{S}} [\rho(\sigma)^*] \in M_{d_\rho}(\mathbb{C}).$$

So, we will henceforth assume that $\{b_{\rho,j}\}_{j=1}^{d_\rho}$ are eigenvectors of $\widehat{\mathbf{I}}_{\mathfrak{S}}(\rho)$ and satisfy (8) for each $\rho \in \widehat{G}$.

We will prove Theorem 1 by choosing the rest of the datum in (9) uniformly at random. Namely, vectors $\{a_{\rho,j,k}\}_{(\rho,j,k) \in T}$ as above can be parameterized by taking $U = (U_{\rho,j})_{(\rho,j) \in S} \in \Gamma$ and letting $a_{\rho,j,k} = U_{\rho,j} e_k$ for every $(\rho, j, k) \in T$. Using this notation, the orthonormal eigenbasis of G in (9) becomes

$$\mathcal{B}_U = \left\{ (x \in G) \mapsto d_\rho^{\frac{1}{2}} e_k^* U_{\rho,j}^* \rho(x)^* b_{\rho,j} \right\}_{(\rho,j,k) \in T}.$$

We will show that if (1) holds and $U \in \Gamma$ is distributed according to the Haar probability measure h_Γ , then \mathcal{B}_U satisfies the conclusion of Theorem 1 with probability at least $1 - e^{-n}$.

We will see that the following lemma is an instantiation of Lemma 5.

Lemma 6. *Let $\eta > 0$ be the universal constant of Lemma 5. For every $\beta \geq 2$ and $f : G \rightarrow \mathbb{C}$ we have*

$$h_\Gamma \left[\left\{ U \in \Gamma : \mathbb{E}_{\phi \in \mathcal{B}_U} \left[\left| \mathbb{E}_{x \in G} [f(x)|\phi(x)|^2] - \mathbb{E}f \right| \right] \geq \beta \left(\frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \right)^{\frac{1}{2}} \|f\|_2 \right\} \right] \leq e^{-\eta \beta^2 n}. \quad (10)$$

Prior to proving Lemma 6, we will explain how it implies Theorem 1.

Deduction of Theorem 1 from Lemma 6. There are $f_1, \dots, f_{5^{2n}} : G \rightarrow \mathbb{C}$ with $\|f_1\|_2, \dots, \|f_{5^{2n}}\|_2 \leq 1$ such that every $f : G \rightarrow \mathbb{C}$ with $\|f\|_2 \leq 1$ is in the convex hull of $\{2f_1, \dots, 2f_{5^{2n}}\}$.² As for fixed $U \in \Gamma$ the mapping

$$(f : G \rightarrow \mathbb{C}) \mapsto \mathbb{E}_{\phi \in \mathcal{B}_U} \left[\left| \mathbb{E}_{x \in G} [f(x)|\phi(x)|^2] - \mathbb{E}f \right| \right]$$

is convex (in the variable f), it follows that

$$\sup_{\substack{f: G \rightarrow \mathbb{C} \\ \|f\|_2=1}} \mathbb{E}_{\phi \in \mathcal{B}_U} \left[\left| \mathbb{E}_{x \in G} [f(x)|\phi(x)|^2] - \mathbb{E}f \right| \right] \leq 2 \max_{\ell \in \{1, \dots, 5^{2n}\}} \mathbb{E}_{\phi \in \mathcal{B}_U} \left[\left| \mathbb{E}_{x \in G} [f_\ell(x)|\phi(x)|^2] - \mathbb{E}f_\ell \right| \right].$$

Consequently, if η is the universal constant in (10), then

$$\begin{aligned} h_\Gamma \left[\left\{ U \in \Gamma : \forall f : G \rightarrow \mathbb{C}, \quad \mathbb{E}_{\phi \in \mathcal{B}_U} \left[\left| \mathbb{E}_{x \in G} [f(x)|\phi(x)|^2] - \mathbb{E}f \right| \right] \leq \frac{5}{\sqrt{\eta}} \left(\frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \right)^{\frac{1}{2}} \|f\|_2 \right\} \right] \\ \geq 1 - \sum_{\ell=1}^{5^{2n}} h_\Gamma \left[\left\{ U \in \Gamma : \mathbb{E}_{\phi \in \mathcal{B}_U} \left[\left| \mathbb{E}_{x \in G} [f_\ell(x)|\phi(x)|^2] - \mathbb{E}f_\ell \right| \right] \geq \frac{5}{2\sqrt{\eta}} \left(\frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \right)^{\frac{1}{2}} \right\} \right] \\ \geq 1 - 5^{2n} \cdot e^{-5n} \geq 1 - e^{-n} > 0. \end{aligned}$$

Hence, there is $U \in \Gamma$ such that if (1) holds with $c = \eta/25$, then the orthonormal eigenbasis \mathcal{B}_U satisfies

$$\forall f : G \rightarrow \mathbb{C}, \quad \mathbb{E}_{\phi \in \mathcal{B}_U} \left[\left| \mathbb{E}_{x \in G} [f(x)|\phi(x)|^2] - \mathbb{E}f \right| \right] \leq \frac{5}{\sqrt{\eta}} \left(\frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \right)^{\frac{1}{2}} \|f\|_2 \leq \frac{5\sqrt{c}}{\sqrt{\eta}} \varepsilon \|f\|_2 = \varepsilon \|f\|_2. \quad \square$$

We will next prove Lemma 6 assuming Lemma 5, after which we will pass (in Section 2.1) to the proof of Lemma 5, thus completing the proof of Theorem 1.

²Using e.g. [12, Lemma 2.4] there are $f_1, \dots, f_{5^{2n}} : G \rightarrow \mathbb{C}$ with $\|f_1\|_2, \dots, \|f_{5^{2n}}\|_2 \leq 1$ such that $\min_{\varphi \in \mathcal{N}} \|f - \varphi\|_2 \leq 1/2$ for every $f : G \rightarrow \mathbb{C}$ with $\|f\|_2 \leq 1$, where $\mathcal{N} = \{f_1, \dots, f_{5^{2n}}\}$. Take $\varphi_0 \in \mathcal{N}$ with $\|f - \varphi_0\|_2 \leq 1/2$, then take $\varphi_1 \in \mathcal{N}$ with $\|2(f - \varphi_0) - \varphi_1\|_2 \leq 1/2$, then take $\varphi_2 \in \mathcal{N}$ with $\|2(2(f - \varphi_0) - \varphi_1) - \varphi_2\|_2 \leq 1/2$, and so forth. We have $\|f - \varphi_0 - \varphi_1/2 - \varphi_2/4 - \dots - \varphi_m/2^m\|_2 \leq 1/2^{m+1}$ for every $m \in \mathbb{N}$. Therefore, $f = \varphi_0 + \varphi_1/2 + \varphi_2/4 + \dots \in \text{conv}(2\mathcal{N})$. Better upper bounds on the number of vertices of such polytopal approximation of balls can be found in [7, 6, 26], but they only affect the constant c in Theorem 1.

Deduction of Lemma 6 from Lemma 5. As $\|f - \mathbb{E}f\|_2 \leq \|f\|_2$, it suffices to prove (10) under the additional assumptions $\mathbb{E}f = 0$ and $\|f\|_2 = 1$. Observe that for every $(\rho, j, k) \in T$ and $U \in \Gamma$ we have

$$\mathbb{E}_{x \in G} \left[|f(x)| d_\rho^{\frac{1}{2}} e_k^* U_{\rho,j}^* \rho(x)^* b_{\rho,j} \right]^2 = e_k^* U_{\rho,j}^* A_{\rho,j}^f U_{\rho,j} e_k,$$

where we introduce the notation

$$A_{\rho,j}^f = d_\rho \mathbb{E}_{x \in G} [f(x) \rho(x)^* b_{\rho,j} b_{\rho,j}^* \rho(x)] \in M_{d_\rho}(\mathbb{C}).$$

For every $(\rho, j) \in S$,

$$\mathrm{Tr}[A_{\rho,j}^f] = d_\rho \mathbb{E} \left[f(x) \mathrm{Tr}[\rho(x)^* b_{\rho,j} b_{\rho,j}^* \rho(x)] \right] = d_\rho (\mathbb{E}f) \mathrm{Tr}[b_{\rho,j} b_{\rho,j}^*] = 0,$$

where we used the cyclicity of the trace and that $\rho(x)$ is unitary for every $x \in G$. Also,

$$\|A_{\rho,j}^f\|_{\mathrm{HS}}^2 = \mathrm{Tr}[(A_{\rho,j}^f)^* A_{\rho,j}^f] = d_\rho^2 \mathbb{E}_{(x,y) \in G \times G} \left[\overline{f(x)} f(y) \mathrm{Tr}[\rho(x)^* b_{\rho,j} b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j} b_{\rho,j}^* \rho(y)] \right]. \quad (11)$$

Using the cyclicity of the trace once more, for every $x, y \in G$ we have

$$\mathrm{Tr}[\rho(x)^* b_{\rho,j} b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j} b_{\rho,j}^* \rho(y)] = |b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j}|^2.$$

In combination with (11), this gives that

$$\begin{aligned} \|A_{\rho,j}^f\|_{\mathrm{HS}}^2 &= d_\rho^2 \mathbb{E}_{(x,y) \in G \times G} \left[\overline{f(x) b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j}} \left(f(y) b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j} \right) \right] \\ &\leq d_\rho^2 \mathbb{E}_{(x,y) \in G \times G} \left[|f(x)|^2 |b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j}|^2 \right] = d_\rho \mathbb{E}_{x \in G} \left[|f(x)|^2 d_\rho \mathbb{E}_{y \in G} \left[|b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j}|^2 \right] \right], \end{aligned}$$

where the penultimate step uses Cauchy–Schwarz. By Schur orthogonality, for every $x \in G$ we have

$$d_\rho \mathbb{E}_{y \in G} \left[|b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j}|^2 \right] = ((\rho(x)^* b_{\rho,j})^* \rho(x)^* b_{\rho,j}) (b_{\rho,j}^* b_{\rho,j}) = (b_{\rho,j}^* b_{\rho,j})^2 = 1.$$

Therefore, $\|A_s^f\|_{\mathrm{HS}}^2 \leq d_\rho \|f\|_2^2 \leq d_\rho$ for every $s \in S$. The desired estimate (10) now follows from (7) because

$$\sum_{s \in S} d_s = \sum_{\rho \in \hat{G}} d_\rho^2 = n \quad \text{and} \quad \sum_{s \in S} \frac{1}{d_s} \|A_s^f\|_{\mathrm{HS}}^2 \leq |S| = \sum_{\rho \in \hat{G}} d_\rho. \quad \square$$

2.1. Concentration. Given $d \in \mathbb{N}$, let g_d be the standard Riemannian metric on $\mathbb{U}(d)$, namely the geodesic distance that is induced by taking the Hilbert–Schmidt metric on all of the tangent spaces.

The following theorem is a concatenation of known results that we formulate for ease of later reference. Its quick justification below uses fundamental properties of logarithmic Sobolev inequalities [15] on metric probability spaces; good expositions of what we need can be found in the monographs [19, 22].

Theorem 7 (concentration of measure on Pythagorean products of rescaled unitary groups). *Let S be a finite set and $\{d_s\}_{s \in S} \in \mathbb{N}$. Denote $\Gamma = \prod_{s \in S} \mathbb{U}(d_s)$. Suppose that $K > 0$ and that $f : \Gamma \rightarrow \mathbb{R}$ satisfies*

$$\forall U = (U_s)_{s \in S}, V = (V_s)_{s \in S} \in \Gamma, \quad |f(U) - f(V)| \leq K \left(\sum_{s \in S} d_s g_{d_s}(U_s, V_s)^2 \right)^{\frac{1}{2}}. \quad (12)$$

In other words, (12) is the requirement that f is K -Lipschitz with respect on the Pythagorean product of the metric spaces $\{\mathbb{U}(d_s), \sqrt{d_s} g_{d_s}\}_{s \in S}$. Then, for every $\varepsilon > 0$ we have

$$h_\Gamma \left[f \geq \varepsilon + \int_\Gamma f d h_\Gamma \right] \leq \exp \left(-\frac{\varepsilon^2}{3\pi^2 K^2} \right). \quad (13)$$

Proof. By the paragraph after Theorem 15 in [23], for every $d \in \mathbb{N}$ the logarithmic Sobolev constant of the metric probability space $(\mathbb{U}(d), g_d, h_{\mathbb{U}(d)})$ is at most $3\pi^2/(2d)$. As the logarithmic Sobolev constant scales quadratically with rescaling of the metric, it follows that the metric probability space $(\mathbb{U}(d), \sqrt{d} g_d, h_{\mathbb{U}(d)})$

has logarithmic Sobolev constant at most $3\pi^2/2$. By the tensorization property of the logarithmic Sobolev constant under Pythagorean products (see [19, Corollary 5.7]), if we define

$$\forall U = (U_s)_{s \in S}, V = (V_s)_{s \in S} \in \Omega, \quad \rho(U, V) = \left(\sum_{s \in S} d_s g_{d_s}(U_s, V_s)^2 \right)^{\frac{1}{2}},$$

then the logarithmic Sobolev constant of the metric probability space (Γ, ρ, h_Γ) is at most $3\pi^2/2$. The desired conclusion (13) follows by the classical Herbst argument [11, 1, 18] (see [19, Theorem 5.3]). \square

It is worthwhile to formulate separately the following quick corollary of Theorem 7.

Corollary 8. *Continuing with the notation of Theorem 7, suppose that $\{K_s\}_{s \in S} \subseteq (0, \infty)$ and that for each $s \in S$ we are given a function $f_s : \mathbb{U}(d_s) \rightarrow \mathbb{R}$ that is K_s -Lipschitz with respect to the geodesic metric g_{d_s} , i.e., $|f_s(U) - f_s(V)| \leq K_s g_{d_s}(U, V)$ for every $U, V \in \mathbb{U}(d_s)$. Then, for every $\varepsilon > 0$ we have*

$$h_\Gamma \left[\left\{ U = (U_s)_{s \in S} \in \Gamma : \mathbb{E}_{s \in S} [f_s(U_s)] \geq \mathbb{E}_{s \in S} \left[\int_{\mathbb{U}(d_s)} f_s d h_{\mathbb{U}(d_s)} \right] + \varepsilon \right\} \right] \leq \exp \left(- \frac{\varepsilon^2 |S|^2}{3\pi^2 \sum_{s \in S} \frac{1}{d_s} K_s^2} \right).$$

Proof. Define $f : \Gamma \rightarrow \mathbb{R}$ by setting $f(U) = \mathbb{E}_{s \in S} [f_s(U_s)]$ for $U = (U_s)_{s \in S} \in \Gamma$. If $U, V \in \Gamma$, then

$$|f(U) - f(V)| \leq \mathbb{E}_{s \in S} [|f_s(U_s) - f_s(V_s)|] \leq \frac{1}{|S|} \sum_{s \in S} K_s g_s(U_s, V_s) \leq \frac{1}{|S|} \left(\sum_{s \in S} \frac{1}{d_s} K_s^2 \right)^{\frac{1}{2}} \left(\sum_{s \in S} d_s g_{d_s}(U_s, V_s)^2 \right)^{\frac{1}{2}},$$

where the final step is Cauchy–Schwarz. Now apply Theorem 7. \square

The following lemma connects the above general discussion to Lemma 5.

Lemma 9. *Suppose that $\varphi_1, \dots, \varphi_d : \mathbb{C} \rightarrow \mathbb{C}$ are 1-Lipschitz and $A \in M_n(\mathbb{C})$. Define $f : \mathbb{U}(d) \rightarrow \mathbb{C}$ by setting*

$$\forall U \in \mathbb{U}(d), \quad f(U) = \sum_{k=1}^d \varphi_k(e_k^* U^* A U e_k).$$

Then, the Lipschitz constant of f with respect to the geodesic distance g_d is at most $2\|A\|_{\text{HS}}$, i.e.,

$$\forall U, V \in \mathbb{U}(d), \quad |f(U) - f(V)| \leq 2\|A\|_{\text{HS}} g(U, V).$$

Proof. Fix $U, V \in \mathbb{U}(d)$. By the definition of $g = g_d(U, V)$, there is a smooth curve (unit-speed geodesic) $\gamma : [0, g] \rightarrow \mathbb{U}(d)$ that satisfies $\gamma(0) = U$, $\gamma(g) = V$, and such that $\|\gamma'(t)\|_{\text{HS}} = 1$ for every $t \in [0, g]$. Then,

$$\begin{aligned} |f(U) - f(V)| &\leq \sum_{k=1}^d |\varphi_k(e_k^* U^* A U e_k) - \varphi_k(e_k^* V^* A V e_k)| \leq \sum_{k=1}^d |e_k^* U^* A U e_k - e_k^* V^* A V e_k| \\ &= \sum_{k=1}^d \left| \int_0^g \frac{d}{dt} (e_k^* \gamma(t)^* A \gamma(t) e_k) dt \right| \leq \int_0^g \left(\sum_{k=1}^d \left| \frac{d}{dt} (e_k^* \gamma(t)^* A \gamma(t) e_k) \right| \right) dt. \end{aligned}$$

It therefore suffices to prove the following point-wise estimate:

$$\forall t \in [0, g], \quad \sum_{k=1}^d \left| \frac{d}{dt} (e_k^* \gamma(t)^* A \gamma(t) e_k) \right| \leq 2\|A\|_{\text{HS}}. \quad (14)$$

This indeed holds because by Cauchy–Schwarz for every $t \in [0, g]$ and $k \in \{1, \dots, d\}$,

$$\begin{aligned} \left| \frac{d}{dt} (e_k^* \gamma(t)^* A \gamma(t) e_k) \right| &= e_k^* \gamma'(t)^* A \gamma(t) e_k + e_k^* \gamma(t)^* A \gamma'(t) e_k \\ &\leq (e_k^* \gamma'(t)^* \gamma'(t) e_k)^{\frac{1}{2}} (e_k^* \gamma(t)^* A \gamma(t) e_k)^{\frac{1}{2}} + (e_k^* \gamma(t)^* A A^* \gamma(t) e_k)^{\frac{1}{2}} (e_k^* \gamma'(t)^* \gamma'(t) e_k)^{\frac{1}{2}}. \end{aligned}$$

By summing this over $k \in \{1, \dots, d\}$ and using Cauchy–Schwarz, we conclude the proof of (14) as follows.

$$\begin{aligned} \sum_{k=1}^d \left| \frac{d}{dt} (e_k^* \gamma(t)^* A \gamma(t) e_k) \right| &\leq \left(\sum_{k=1}^d e_k^* \gamma'(t)^* \gamma'(t) e_k \right)^{\frac{1}{2}} \left(\left(\sum_{k=1}^d e_k^* \gamma(t)^* A^* A \gamma(t) e_k \right)^{\frac{1}{2}} + \left(\sum_{k=1}^d e_k^* \gamma(t)^* A A^* \gamma(t) e_k \right)^{\frac{1}{2}} \right) \\ &= \left(\text{Tr}[\gamma'(t)^* \gamma'(t)] \right)^{\frac{1}{2}} \left(\left(\text{Tr}[\gamma(t)^* A^* A \gamma(t)] \right)^{\frac{1}{2}} + \left(\text{Tr}[\gamma(t)^* A A^* \gamma(t)] \right)^{\frac{1}{2}} \right) \\ &= \left(\text{Tr}[\gamma'(t)^* \gamma'(t)] \right)^{\frac{1}{2}} \left(\left(\text{Tr}[A^* A] \right)^{\frac{1}{2}} + \left(\text{Tr}[A A^*] \right)^{\frac{1}{2}} \right) \\ &= 2 \|\gamma'(t)\|_{\text{HS}} \|A\|_{\text{HS}} = 2 \|A\|_{\text{HS}}, \end{aligned}$$

where in the penultimate step we used the fact that $\gamma(t) \in \mathbb{U}(d)$ and the cyclicity of the trace. \square

We can now prove Lemma 5, thus completing the proof of Theorem 1.

Proof of Lemma 5. For every $d \in \mathbb{N}$ and $k \in \{1, \dots, d\}$ we have

$$\forall A \in M_d(\mathbb{C}), \quad \int_{\mathbb{U}(d)} |e_k^* U^* A U e_k|^2 d h_{\mathbb{U}(d)}(U) = \frac{\|A\|_{\text{HS}}^2 + |\text{Tr}(A)|^2}{d(d+1)}. \quad (15)$$

One checks (15) by noting that if U is distributed according to the Haar probability measure on $\mathbb{U}(d)$, then $U e_k$ is distributed according to the normalized surface area measure on $\{z \in \mathbb{C}^d : |z_1|^2 + \dots + |z_d|^2 = 1\}$, expanding the squares and substituting the resulting spherical integrals that are computed in e.g. [13].

Returning to the setting and notation of Lemma 5, for every $s \in S$ and $U \in \mathbb{U}(d_s)$ define

$$f_s(U) = \sum_{k=1}^{d_s} |e_k^* U^* A_s U e_k|.$$

By Lemma 9, the assumption of Corollary 8 holds with $K_s = 2 \|A_s\|_{\text{HS}}$. Because $\text{Tr}(A_s) = 0$, by using (15) and Cauchy–Schwarz we get that

$$\int_{\mathbb{U}(d_s)} f_s d h_{\mathbb{U}(d_s)} = \sum_{k=1}^{d_s} \int_{\mathbb{U}(d_s)} |e_k^* U^* A_s U e_k| d h_{\mathbb{U}(d_s)}(U) \leq \sum_{k=1}^{d_s} \left(\int_{\mathbb{U}(d_s)} |e_k^* U^* A_s U e_k|^2 d h_{\mathbb{U}(d_s)}(U) \right)^{\frac{1}{2}} \leq \|A_s\|_{\text{HS}}.$$

Using Cauchy–Schwarz and recalling the definition of α in (6), we therefore have

$$\mathbb{E}_{s \in S} \left[\int_{\mathbb{U}(d_s)} f_s d h_{\mathbb{U}(d_s)} \right] \leq \mathbb{E}_{s \in S} [\|A_s\|_{\text{HS}}] \leq \frac{1}{|S|} \left(\sum_{s \in S} d_s \right)^{\frac{1}{2}} \left(\sum_{s \in S} \frac{1}{d_s} \|A_s\|_{\text{HS}}^2 \right)^{\frac{1}{2}} = \frac{\sum_{s \in S} d_s}{|S|} \alpha.$$

Corollary 8 therefore implies the following estimate for every $\beta \geq 2$:

$$h_{\Gamma} \left[\left\{ U = (U_s)_{s \in S} \in \Gamma : \mathbb{E}_{s \in S} [f_s(U_s)] \geq \frac{\sum_{s \in S} d_s}{|S|} \beta \alpha \right\} \right] \leq \exp \left(- \frac{(\beta - 1)^2}{3\pi^2} \sum_{s \in S} d_s \right) \leq \exp \left(- \frac{\beta^2}{12\pi^2} \sum_{s \in S} d_s \right).$$

This coincides with the desired estimate (7) with $\eta = 1/(12\pi^2)$. \square

3. PROOF OF THEOREM 3

For the statement of the following proposition, observe that if H is a finite group and \mathfrak{S} a symmetric generating subset of H , then $\mathfrak{S} \times \{-1, 1\}$ generates $H \times (\mathbb{Z}/m\mathbb{Z})$ for any odd integer $m \in 1 + 2\mathbb{N}$. Indeed, if $(h, k) \in H \times (\mathbb{Z}/m\mathbb{Z})$, then take $a \in \mathbb{N}$ and $\sigma_1, \dots, \sigma_a \in \mathfrak{S}$ such that $h = \sigma_1 \cdots \sigma_a$. Since m is odd, there exists $b \in \mathbb{N}$ such that $a + 2b \equiv k \pmod{m}$. We then have $(h, k) = (\sigma_1, 1) \cdots (\sigma_a, 1) (\sigma_1, 1)^b (\sigma_1^{-1}, 1)^b$.

Proposition 10 (from quantum ergodicity to existence of delocalized eigenfunctions). *Let H be a finite group and fix a symmetric generating subset \mathfrak{S} of H . There is $\ell = \ell(H, \mathfrak{S}) \in \mathbb{N}$ with the following property. Let $p > 3$ be a prime that does not divide ℓ . Consider the direct product $G = H \times (\mathbb{Z}/p\mathbb{Z})$. Suppose that the Cayley graph that is induced on G by the generating set $\mathfrak{S} \times \{-1, 1\}$ has an ε -quantum ergodic eigenbasis for some $\varepsilon > 0$. Then, for every nonzero eigenvalue λ of the Cayley graph that is induced on H by \mathfrak{S} there exists an eigenfunction $\psi : H \rightarrow \mathbb{C}$ whose eigenvalue is λ and $0 < \|\psi\|_{\infty} \leq \sqrt{2(1 + 2|H|^3\varepsilon)} \|\psi\|_2$.*

Prior to proving Proposition 10, we will explain how it implies Theorem 3. Since for every fixed finite Cayley graph the conclusion of Proposition 10 is an upper bound on $\|\psi\|_\infty$ that does not depend on p even though p can tend to ∞ , if Theorem 3 failed, then by taking $p \rightarrow \infty$ and $\varepsilon \rightarrow 0$ it would follow that for every nonzero eigenvalue of any finite Cayley admits a normalized eigenfunction that is $O(1)$ bounded point-wise. This conclusion is at odds with (a very small part of) the following theorem from [27]:

Theorem 11. *There is a universal constant $\kappa > 0$ with the following property. For arbitrarily large $N \in \mathbb{N}$ there exists a group H with $|H| = N$ and a symmetric generating subset \mathfrak{S} of H such that the adjacency matrix $A(H; \mathfrak{S})$ has a nonzero eigenvalue λ with the property that $\|\psi\|_\infty / \|\psi\|_2 \geq \kappa \sqrt{\log N} / \log \log N$ for every nonzero eigenfunction ψ of $A(H; \mathfrak{S})$ whose eigenvalue is λ .*

The statement of Theorem 1.2 in [27] coincides with Theorem 11, except that it does not include the assertion that the eigenvalue is nonzero, but this is stated in the proof of [27, Theorem 1.2].

Deduction of Theorem 3 from Proposition 10. If Theorem 3 does not hold, then by Proposition 10 for any nonzero eigenvalue λ of any finite Cayley graph there is an eigenfunction ψ of that Cayley graph whose eigenvalue is λ and $\|\psi\|_\infty \leq \sqrt{2} \|\psi\|_2$. This contradicts Theorem 11. \square

Our proof of Proposition 10 uses the following basic lemma about algebraic numbers; the rudimentary facts from Galois theory and cyclotomic fields that appear in its proof can be found in e.g. [17].

Lemma 12. *Let \mathbb{K} be a finite degree number field. There exists $\ell = \ell(\mathbb{K}) \in \mathbb{N}$ such that if $p > 3$ is a prime that does not divide ℓ , then $\cos(2\pi j/p) / \cos(2\pi k/p) \notin \mathbb{K}$ for all distinct $j, k \in \{0, 1, \dots, (p-1)/2\}$.*

Proof. Denote $\mathbb{Q}^{\text{cyc}} = \mathbb{Q}(\{\exp(2\pi i/k)\}_{k=1}^\infty)$. Let $\mathbb{K}' = \mathbb{K} \cap \mathbb{Q}^{\text{cyc}} \subseteq \mathbb{K}$. By the primitive element theorem, there exists $\alpha \in \mathbb{K}'$ such that $\mathbb{K}' = \mathbb{Q}(\alpha)$. Since $\alpha \in \mathbb{Q}^{\text{cyc}}$, there exists $\ell \in \mathbb{N}$ such that $\alpha \in \mathbb{Q}(\exp(2\pi i/\ell))$. Therefore, $\mathbb{K} \cap \mathbb{Q}^{\text{cyc}} \subseteq \mathbb{Q}(\exp(2\pi i/\ell))$. Observe that $\mathbb{Q}(\exp(2\pi i/\ell)) \cap \mathbb{Q}(\exp(2\pi i/p)) = \mathbb{Q}$ for any prime p that does not divide ℓ (as the field generated by $\mathbb{Q}(\exp(2\pi i/\ell))$ and $\mathbb{Q}(\exp(2\pi i/p))$ is $\mathbb{Q}(\exp(2\pi i/(\ell p)))$, and its degree is $\varphi(\ell p) = \varphi(\ell)\varphi(p)$, where $\varphi(\cdot)$ is Euler's totient function, while the degrees of $\mathbb{Q}(\exp(2\pi i/\ell))$ and $\mathbb{Q}(\exp(2\pi i/p))$ are, respectively, $\varphi(\ell)$ and $\varphi(p)$). Therefore

$$\mathbb{K} \cap \mathbb{Q}\left(e^{\frac{2\pi i}{p}}\right) = (\mathbb{K} \cap \mathbb{Q}^{\text{cyc}}) \cap \mathbb{Q}\left(e^{\frac{2\pi i}{p}}\right) \subseteq \mathbb{Q}\left(e^{\frac{2\pi i}{\ell}}\right) \cap \mathbb{Q}\left(e^{\frac{2\pi i}{p}}\right) = \mathbb{Q}. \quad (16)$$

Denoting $\zeta = \exp(2\pi i/p)$, it follows from (16) that if $\cos(2\pi j/p) / \cos(2\pi k/p) = (\zeta^j + \zeta^{-j}) / (\zeta^k + \zeta^{-k}) \in \mathbb{K}$ for some distinct $j, k \in \{0, 1, \dots, (p-1)/2\}$, then actually $(\zeta^j + \zeta^{-j}) / (\zeta^k + \zeta^{-k}) \in \mathbb{Q}$. This cannot happen for the following reason. Suppose that there are $a, b \in \mathbb{Z} \setminus \{0\}$ for which $a(\zeta^j + \zeta^{-j}) - b(\zeta^k + \zeta^{-k}) = 0$. Given $r \in (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$, we can apply the automorphism of $\mathbb{Q}(\zeta)$ which maps ζ to ζ^r . Since $p > 3$, we can choose r so that $jr, kr \not\equiv (p-1)/2 \pmod{p}$. We therefore deduce that $a(\zeta^{jr} + \zeta^{-jr}) - b(\zeta^{kr} + \zeta^{-kr}) = 0$ for some distinct integers $0 \leq u, v < (p-1)/2$. Without loss of generality, $u < v$. Then $a(\zeta^{u+v} + \zeta^{v-u}) - b(\zeta^{2v} + 1) = 0$. We have thus found a nonzero polynomial with integer coefficients of degree $2v < p-1$ that vanishes at ζ , contradicting the fact that the minimal polynomial of ζ is $P(t) = t^{p-1} + \dots + t + 1$. \square

We can now prove Proposition 10, thus completing the proof of Theorem 3.

Proof of Proposition 10. Denote the distinct nonzero eigenvalues of the adjacency matrix $A(H; \mathfrak{S})$ by $\lambda_1, \dots, \lambda_s \in \mathbb{R} \setminus \{0\}$, and for each $j \in \{1, \dots, s\}$ let $\Lambda_j \subseteq \mathbb{C}^H$ be the eigenspace of $A(H; \mathfrak{S})$ that corresponds to the eigenvalue λ_j . Also, let $\Lambda_0 \subseteq \mathbb{C}^H$ be the kernel of $A(H; \mathfrak{S})$. Define

$$M = \max \left\{ \inf_{\psi \in \Lambda_1 \setminus \{0\}} \frac{\|\psi\|_\infty}{\|\psi\|_2}, \dots, \inf_{\psi \in \Lambda_s \setminus \{0\}} \frac{\|\psi\|_\infty}{\|\psi\|_2} \right\}.$$

The desired conclusion of Proposition 10 is the same as requiring that $M \leq \sqrt{2(1+2|H|^3\varepsilon)}$. If $M \leq \sqrt{2}$, then there is nothing to prove, so suppose from now on that $M > \sqrt{2}$.

Let ℓ be as in Lemma 12 applied to the field $\mathbb{K} = \mathbb{Q}(\lambda_1, \dots, \lambda_s)$. Fix a prime $p > 3$ that does not divide ℓ and let $G = H \times (\mathbb{Z}/p\mathbb{Z})$ be as in the statement of Proposition 10. For $k \in \mathbb{Z}$ denote $\mu_k = 2 \cos(2\pi k/p)$. As p is odd, $\mu_k \neq 0$. Write $\chi_k(x) = \exp(2\pi i kx/p)$ for $x \in \mathbb{Z}/p\mathbb{Z}$ and let E_k be the span of χ_k and χ_{-k} in $\mathbb{C}^{\mathbb{Z}/p\mathbb{Z}}$.

Then, $\dim(E_0) = 1$ and $\dim(E_k) = 2$ for $k \in \{1, \dots, (p-1)/2\}$, and E_k is the eigenspace of $A(\mathbb{Z}/p\mathbb{Z}; \{-1, 1\})$ whose eigenvalue is μ_k . As p is odd, the eigenspace decomposition of $A(\mathbb{Z}/p\mathbb{Z}; \{-1, 1\})$ is

$$\mathbb{C}^{\mathbb{Z}/p\mathbb{Z}} = \bigoplus_{k=0}^{\frac{p-1}{2}} E_k.$$

The nonzero eigenvalues of $A(G, \mathfrak{S} \times \{-1, 1\})$ are $\{\lambda_j \mu_k : (j, k) \in \{1, \dots, s\} \times \{0, \dots, (p-1)/2\}\}$; we claim that these numbers are distinct, so that the eigenspace decomposition of $A(G, \mathfrak{S} \times \{-1, 1\})$ is

$$\mathbb{C}^G \cong \mathbb{C}^H \otimes \mathbb{C}^{\mathbb{Z}/p\mathbb{Z}} = \left(\Lambda_0 \otimes \mathbb{C}^{\mathbb{Z}/p\mathbb{Z}} \right) \bigoplus \left(\bigoplus_{j=1}^s \bigoplus_{k=0}^{\frac{p-1}{2}} \Lambda_j \otimes E_k \right).$$

Indeed, if $j, j' \in \{1, \dots, s\}$ and $k, k' \in \{0, \dots, (p-1)/2\}$ are such that $\lambda_j \mu_k = \lambda_{j'} \mu_{k'}$, then $\mu_k / \mu_{k'} = \lambda_{j'} / \lambda_j \in \mathbb{K}$, so $k = k'$ by Lemma 12 and therefore also $j = j'$.

Fix $j \in \{1, \dots, s\}$ at which M is attained, i.e., $\|\psi\|_\infty \geq M \|\psi\|_2$ for every $\psi \in \Lambda_j$. Let $\phi : G \rightarrow \mathbb{C}$ be an eigenfunction of $A(G, \mathfrak{S} \times \{-1, 1\})$ whose eigenvalue is $\lambda_j \mu_k$ for some $k \in \{0, \dots, (p-1)/2\}$. So, $\phi \in \Lambda_j \otimes E_k$ and therefore there exist $\psi_+, \psi_- \in \Lambda_j$ with $\|\psi_+\|_2^2 + \|\psi_-\|_2^2 = \|\phi\|_2^2$ such that $\phi = \psi_+ \otimes \chi_k + \psi_- \otimes \chi_{-k}$. There is $\psi \in \{\psi_+, \psi_-\}$ with $\|\psi\|_2^2 \geq \|\phi\|_2^2/2$. Fix $h_\phi \in H$ for which $|\psi(h_\phi)| = \|\psi\|_\infty$. Then,

$$\begin{aligned} \mathbb{E}_{x \in \mathbb{Z}/p\mathbb{Z}} [|\phi(h_\phi, x)|^2] &= \mathbb{E}_{x \in \mathbb{Z}/p\mathbb{Z}} \left[\left| \psi_+(h_\phi) e^{\frac{2\pi i k x}{p}} + \psi_-(h_\phi) e^{-\frac{2\pi i k x}{p}} \right|^2 \right] \\ &= |\psi_+(h_\phi)|^2 + |\psi_-(h_\phi)|^2 \geq |\psi(h_\phi)|^2 = \|\psi\|_\infty^2 \geq M^2 \|\psi\|_2^2 \geq \frac{M^2}{2} \|\phi\|_2^2. \end{aligned}$$

If $\mathcal{B} \subseteq \mathbb{C}^G$ is an orthonormal eigenbasis of $A(G, \mathfrak{S} \times \{-1, 1\})$, then let $\mathcal{B}_j \subseteq \mathcal{B}$ be the elements of \mathcal{B} whose eigenvalue is $\lambda_j \mu_k$ for some $k \in \{0, \dots, (p-1)/2\}$. Thus, $|\mathcal{B}_j| = \dim(\Lambda_j) p \geq p$. By the pigeonhole principle there are $\mathcal{B}'_j \subseteq \mathcal{B}_j$ and $h \in H$ such that $|\mathcal{B}'_j| \geq |\mathcal{B}_j|/|H| \geq p/|H|$ and $h_\phi = h$ for every $\phi \in \mathcal{B}'_j$. Consequently,

$$\begin{aligned} \mathbb{E}_{\phi \in \mathcal{B}} \left[\left| \mathbb{E}_{x \in G} [\mathbf{1}_{\{h\} \times \mathbb{Z}/p\mathbb{Z}}(x) |\phi(x)|^2] - \mathbb{E}[\mathbf{1}_{\{h\} \times \mathbb{Z}/p\mathbb{Z}}] \right| \right] &= \frac{1}{p|H|^2} \sum_{\phi \in \mathcal{B}} \left| \mathbb{E}_{x \in \mathbb{Z}/p\mathbb{Z}} [|\phi(h, x)|^2] - 1 \right| \\ &\geq \frac{1}{p|H|^2} \sum_{\phi \in \mathcal{B}'_j} \left| \mathbb{E}_{x \in \mathbb{Z}/p\mathbb{Z}} [|\phi(h_\phi, x)|^2] - 1 \right| \geq \frac{|\mathcal{B}'_j|}{p|H|^2} \left(\frac{M^2}{2} - 1 \right) \geq \frac{1}{|H|^3} \left(\frac{M^2}{2} - 1 \right). \end{aligned} \quad (17)$$

If \mathcal{B} is ε -quantum ergodic, then the first term in (17) is at most ε , and therefore $M \leq \sqrt{2(1 + |H|^3 \varepsilon)}$. \square

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