NONPOSITIVE CURVATURE IS NOT COARSELY UNIVERSAL

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ABSTRACT. We prove that not every metric space embeds coarsely into an Alexandrov space of nonpositive curvature. This answers a question of Gromov (1993) and is in contrast to the fact that any metric space embeds coarsely into an Alexandrov space of nonnegative curvature, as shown by Andoni, Naor and Neiman (2015). We establish this statement by proving that a metric space which is q-barycentric for some $q \in [1, \infty)$ has metric cotype q with sharp scaling parameter. Our proof utilizes nonlinear (metric space-valued) martingale inequalities and yields sharp bounds even for some classical Banach spaces. This allows us to evaluate the bi-Lipschitz distortion of the ℓ_∞ grid $[m]_\infty^n = (\{1, \ldots, m\}^n, \|\cdot\|_\infty)$ into ℓ_q for all $q \in (2, \infty)$, from which we deduce the following discrete converse to the fact that ℓ_∞^n embeds with distortion $O(1)$ into ℓ_q for $q = O(\log n)$. A rigidity theorem of Ribe (1976) implies that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that if $[m]_\infty^n$ embeds into ℓ_q with distortion $O(1)$, then q is necessarily at least a universal constant multiple of $\log n$. Ribe's theorem does not give an explicit upper bound on this m, but by the work of Bourgain (1987) it suffices to take $m = n$, and this was the previously best-known estimate for m. We show that the above discretization statement actually holds when m is a universal constant.

1. INTRODUCTION

A complete geodesic metric space (X, d_X) is an Alexandrov space of nonpositive curvature if for any quadruple of points $x, y, z, w \in X$ such that w is a metric midpoint of x and y, i.e., $d_X(w, x) = d_X(w, y) = \frac{1}{2} d_X(x, y)$, we have

$$d_X(z, w)^2 + \frac{1}{4} d_X(x, y)^2 \leqslant \frac{1}{2} d_X(z, x)^2 + \frac{1}{2} d_X(z, y)^2. \tag{1}$$

If the reverse inequality to (1) holds true for any such quadruple $x, y, z, w \in X$, then X is an Alexandrov space of nonnegative curvature.[1] See e.g. [30, 146, 27, 167, 29, 169, 73] for more on these fundamental notions.

A metric space (Y, d_Y) is said to embed coarsely into a metric space (X, d_X) if there exist two nondecreasing moduli $\omega, \Omega : [0, \infty) \to [0, \infty)$ satisfying $\omega \leqslant \Omega$ pointwise and $\lim_{t \to \infty} \omega(t) = \infty$, and a mapping $f : Y \to X$ such that

$$\forall x, y \in Y, \qquad \omega(d_Y(x, y)) \leqslant d_X(f(x), f(y)) \leqslant \Omega(d_Y(x, y)). \tag{2}$$

A mapping $f : Y \to X$ that satisfies (2) is called a coarse embedding (with moduli ω, Ω).

The notion of a coarse embedding was introduced by Gromov in [67, §4], where such an embedding was called a "placement," and further studied by him in [69, Section 7.E], where such an embedding was called a "uniform embedding." The subsequent change to the currently commonly used term "coarse embedding" is due to the need to avoid conflict with prior terminology in the functional analysis literature; see e.g. the explanation in [19, 159, 140, 143].

As a special case of a more general result that will be described later, we will prove here the following theorem.

Theorem 1. *There is a metric space Y that does not embed coarsely into any nonpositively curved Alexandrov space X.*

Theorem 1 is the first time that the mere existence of such a metric space Y is established (thus addressing a long-standing question of Gromov [69]; see below), but we will actually see that one could take here $Y = \ell_p$ for any $p > 2$.

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[1]Strictly speaking, the above definition is of Alexandrov spaces of *global* nonpositive curvature, also known as CAT(0) spaces or Hadamard spaces. See [27] for the local counterpart of this definition; we will not treat it here, and therefore it will be convenient to drop the term "global" throughout the present text, because the ensuing results are vacuously false under the weaker local assumption (since the 1-dimensional simplicial complex that is associated to any connected combinatorial graph is an Alexandrov space of local nonpositive curvature). For nonnegative curvature, the local and global notions coincide due to the (metric version of the) Alexandrov–Toponogov theorem; see e.g. [152].

It follows from [4] that the statement of Theorem 1 is false if one replaces in it the term "nonpositively curved" by "nonnegatively curved." Namely, by [4] every metric space embeds coarsely into some nonnegatively curved Alexandrov space. This difference between the coarse implications of the "sign" of curvature is discussed further in Section 1.4 below, where we also explain how our work answers other open questions that were posed in [4]. A conceptual contribution that underlies Theorem 1 is to specify an invariant which is preserved under embeddings that may incur very large distortion, such that this invariant holds when the curvature is nonpositive yet it does not follow from nonnegative curvature. Prior to this, nonnegative curvature was observed to be "better behaved" than nonpositive curvature, in the sense that all such invariants that were previously computed for Alexandrov spaces held/failed equally well in the presence of either nonpositive or nonnegative curvature (for different reasons), or they held for spaces of nonnegative curvature and not necessarily for spaces of nonpositive curvature; see Section 1.5 below.

The geometric faithfulness that definition (2) of a coarse embedding imposes is weak (much more so than, say, that of a bi-Lipschitz embedding), but it nevertheless has strong implications in topology, K-theory and group theory; see e.g. [69, 52, 181, 74, 183, 140, 143]. The fact that the requirement (2) is not stringent implies that large classes of metric spaces admit a coarse embedding into "nice" metric spaces (see [64, 34, 28, 74, 182] for examples of theorems of this type), and it raises the question of finding invariants that serve as obstructions to the existence of coarse embeddings.

The above (open-ended) question was first raised in [69, page 218; Remark (b)], where it is stated that

"There is no known geometric obstruction for uniform embeddings into infinite dimensional spaces. In particular, it is unclear whether every separable metric space can be uniformly embedded into the Hilbert space $\mathbb{R}^\infty$."

<div align="right">M. Gromov, 1993.</div>

Here, the term "infinite dimensional" alludes to the fact that Gromov's investigations in [67, 69] dealt with coarse embeddings into nonpositively curved spaces which exhibit a certain (appropriately defined) finite dimensionality. In that setting, he indeed found examples of the sought-after obstructions (e.g. relying on a coarse notion of dimension, or proving that the rank of a symmetric space serves as an obstruction for coarse embeddings of one symmetric space into another). The particular case of embedding into a Hilbert space that is mentioned separately in the above quote is clearly the natural place to start, but it is also important since coarse embeddings into a Hilbert space have profound topological and K-theoretical implications, as conjectured later by Gromov [52] and proven by Yu [181]. Theorem 1 answers the above question in the setting in which it was originally posed, namely ruling out the existence of a coarse embedding into an arbitrary Alexandrov space of nonpositive curvature.

Gromov's question influenced the development of several approaches which resolve its Hilbertian case (and variants for embeddings into certain Banach spaces, partially motivated by an application in [92]), starting with the initial solution by Dranishnikov, Gong, Lafforgue, and Yu [42], who were the first to prove that there exists a metric space which does not embed coarsely into a Hilbert space. The main threads in this line of research are

(1) Dranishnikov, Gong, Lafforgue, and Yu [42] were the first to answer the Hilbertian case of Gromov's question, by adapting a classical argument of Enflo [44] based on an invariant which he called *generalized roundness*.

(2) Gromov himself found [70, 72] a different solution, showing that a metric space which contains any sequence of arbitrarily large bounded degree *expander graphs* fails to admit a coarse embedding into a Hilbert space.

(3) Pestov [147] proved that the universal Urysohn metric space $\mathbb{U}$ does not admit a coarse embedding into any uniformly convex Banach space, relying on works of Hrushovski [78], Solecki [166] and Vershik [177].

(4) A Fourier-analytic argument of Khot and Naor [94] shows that for every $n \in \mathbb{N}$ one can choose a lattice $\Lambda_n \subseteq \mathbb{R}^n$ such that any metric space that contains the flat tori $\{\mathbb{R}^n/\Lambda_n\}_{n=1}^\infty$ does not embed coarsely into a Hilbert space.

(5) Johnson and Randrianarivony [87] proved that if $p > 2$, then ℓ_p does not embed coarsely into a Hilbert space, by building on a method of Aharoni, Maurey and Mityagin [1]. Randrianarivony [154] proceeded to use this approach to characterize the Banach spaces that admit a coarse embedding into a Hilbert space as those that are linearly isomorphic to a closed subspace of an $L_0(\mu)$ space.

(6) Mendel and Naor introduced [121] an invariant called *metric cotype* and showed that it yields an obstruction to coarse embeddings provided that an auxiliary quantity called the "scaling parameter" has a sharp asymptotic behavior; all of the relevant terminology will be recalled in Section 1.2 below, since this is the strategy of our proof of Theorem 1. In [121], such a sharp metric cotype inequality was established for K-convex Banach spaces via a vector-valued Fourier-analytic argument; here we take a different route in lieu of Fourier analysis, due to its unavailability for functions that take values in metric spaces that are not Banach spaces.

(7) Kalton investigated coarse embeddings in [88] where, building in part on classical work of Raynaud [155], he introduced an invariant called *Kalton's Property $\mathcal{Q}$*, and used it to show that certain Banach spaces (including

notably the space c_0 of null sequences, the James quasi-reflexive space [82, 83], and non-reflexive uniformly non-octahedral spaces [84, 86, 85, 151]) do not admit a coarse embedding into any reflexive Banach space. Property $\mathcal{Q}$ can also be used to rule out the existence of coarse embeddings of certain Banach spaces into the Schatten–von Neumann trace class S_1; see [60, page 172]. In the same work [88], Kalton considered a sequence $\{K_r(\mathbb{N})\}_{r=1}^\infty$ of infinite connected combinatorial graphs (equipped with their shortest-path metric), which are called *Kalton's interlacing graphs*, and proved that any metric space which contains $\{K_r(\mathbb{N})\}_{r=1}^\infty$ does not admit a coarse embedding into any *stable* metric space, hence a fortiori into a Hilbert space (here, the notion of stability of a metric space is in the sense of Garling's definition [59] which builds on that of Krivine and Maurey [97]; see the survey [15], and we will return to this matter in Section 2 below). This circle of ideas has been substantially developed in several directions [91, 90, 17, 104, 16], leading in particular to coarse non-embeddability results also into some non-classical Banach spaces, including the Tsirelson space [173], the James quasi-reflexive space, and spaces that are reflexive and asymptotically c_0 [117].

(8) Lafforgue [100, 101] and Mendel and Naor [124, 125] constructed a special type of (a sequence of) expander graphs (namely, expanders relative to certain metric spaces, and even *super-expanders*), which, via a straightforward generalization of the aforementioned argument of Gromov [70, 72], do not admit a coarse embedding into families of metric spaces that include Hilbert spaces but are much richer; we will describe this approach later as it relates to important open problems that pertain to a potential strengthening of Theorem 1.

(9) Arzhantseva and Tessera [7] introduced the notion of *relative expander graphs*, which is a weakening of the classical notion of expander graph, and showed that a metric space that contains any sequence of relative expanders does not admit a coarse embedding into a Hilbert space. They exhibited examples of such spaces into which no sequence of expander graph embeds coarsely. We will elaborate on this topic in Section 2 below.

(10) Naor and Schechtman [136] introduced an invariant called a *metric* KS *inequality*, evaluated it for some spaces (including ℓ_2), and showed that it is an obstruction to coarse embeddings of powers of hypercubes.

Section 2 below returns to the above list of previous results/methods, explaining why they do not yield Theorem 1. Briefly, in some cases this is so because one can construct an Alexandrov space of nonpositive curvature for which the given approach fails. Some of the other methods in the above list rely so heavily on linear properties of the underlying Banach space that we do not see how to interpret those approaches when the target space is not a Banach space. And, for a couple of items in the above list, their possible applicability to the setting of Theorem 1 requires the (positive) solution of an open question that is of interest beyond its potential use as a different route to Theorem 1.

In the special case of embeddings into a simply connected Riemannian manifold (M, d_M) of nonpositive sectional curvature, Gromov's question is (implicitly) answered by an argument of Wang [180], which implies that a metric space which contains any sequence of arbitrarily large bounded degree expander graphs fails to admit a coarse embedding into (M, d_M); this also follows from the work of Izeki and Nayatani [81], and an explicit exposition of the proof appears in [138, pages 1159-1160]. The assumption that (M, d_M) is a Riemannian manifold can be relaxed to a technical requirement which restricts its possible singularities; in [72, Remark 1.2.C(b)] Gromov calls such Alexandrov spaces of nonpositive curvature "CAT(0) spaces with bounded singularities." Following this approach, several authors studied [81, 171, 138, 57, 129, 172] special cases of spaces X for which the conclusion Theorem 1 holds true; to indicate just one of several such examples that are available in the literature, Fujiwara and Toyoda [57, Corollary 1.9] proved that this is so when X is an arbitrary CAT(0) cube complex. However, in [71, page 187] and [72, page 117] Gromov himself proposed a construction of Alexandrov spaces of nonpositive curvature for which his approach fails, and this was carried out by Kondo [96]. By [125], there are metric spaces (namely, those that contain a specially-constructed sequence of bounded degree expander graphs) that do not embed coarsely into most such "Gromov–Kondo spaces," but the proof of this statement in [125] relies on particular properties of the specific construction.

1.1. Barycentric metric spaces.
For a set Ω, let $\mathcal{P}_\Omega^{<\infty}$ be the space of all finitely supported probability measures on Ω. A function $\mathfrak{B} : \mathcal{P}_\Omega^{<\infty} \to \Omega$ is said to be a barycenter map if $\mathfrak{B}(\delta_x) = x$ for every $x \in \Omega$, where δ_x is the point mass at x.

Following the terminology of [123], if $q \in [1, \infty)$, then a metric space (X, d_X) is said to be q-barycentric with constant $\beta \in (0, \infty)$ if there exists a barycenter map $\mathfrak{B} : \mathcal{P}_X^{<\infty} \to X$ such that every $x \in X$ and $\mu \in \mathcal{P}_X^{<\infty}$ satisfy

$$d_X\big(\mathfrak{B}(\mu), x\big)^q + \frac{1}{\beta^q} \int_X d_X\big(\mathfrak{B}(\mu), y\big)^q \, \mathrm{d}\mu(y) \leqslant \int_X d_X(x, y)^q \, \mathrm{d}\mu(y). \tag{3}$$

A metric space is said to be barycentric if it is q-barycentric for some $q \in [1, \infty)$. We imposed the restriction $q \geqslant 1$ above because it is mandated by the barycentric requirement (3) (unless X is a singleton). Moreover, if (X, d_X) contains a

geodesic segment, then (3) implies that $q \geqslant 2$, though barycentric metric spaces need not necessarily contain any geodesic segment. See Section 5 below for a quick justification of these facts and further discussion.

It is well known (see e.g. [105, Lemma 4.1] or [169, Theorem 6.3]) that any Alexandrov space of nonpositive curvature is 2-barycentric with constant $\beta = 1$. Hence, Theorem 1 is a special case of the following theorem.

Theorem 2. *There exists a metric space Z that does not embed coarsely into any barycentric metric space.*

In fact, we will establish the following more refined version of Theorem 2.

Theorem 3. *If $p, q \in [1, \infty]$ satisfy $p > q$, then ℓ_p does not embed coarsely into any q-barycentric metric space.*

1.2. **Sharp metric cotype.** Following [121], a metric space (X, d_X) is said to have metric cotype $q \in (0, \infty)$ with constant $\Gamma \in (0, \infty)$ if for every $n \in \mathbb{N}$ there exists some $m = m(n, q, X) \in \mathbb{N}$ such that every function $f : \mathbb{Z}_{2m}^n \to X$ satisfies

$$\left(\sum_{i=1}^{n} \sum_{x \in \mathbb{Z}_{2m}^n} d_X \big(f(x + me_i), f(x) \big)^q \right)^{\frac{1}{q}} \leqslant \Gamma m \left(\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{2m}^n} d_X \big(f(x + \varepsilon), f(x) \big)^q \right)^{\frac{1}{q}}. \tag{4}$$

In (4) and in what follows, $\mathbb{Z}_{2m} = \mathbb{Z}/(2m\mathbb{Z})$ and the additions that appear in the arguments of f are modulo $2m$. Also, $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ is the standard basis of $\mathbb{Z}_{2m}^n$ (or, with a subsequent slight abuse of notation, of $\mathbb{R}^n$).

See [121] for an explanation of this terminology, as well as its motivation within an extensive long-term research program called the *Ribe program*. Explaining this larger context is beyond the scope of the present article, but accessible introductory surveys are available [89, 128, 12, 143, 63, 132]. It suffices to say here that our work is yet another example of an application of the Ribe program in which insights that are inspired by Banach space theory are used to answer a geometric question about objects which a priori have nothing to do with Banach spaces. As we shall see below, in addition to the aforementioned application, revisiting these matters in the fully nonlinear setting of Alexandrov spaces forced us to find a different approach that led to new results even for Banach spaces such as ℓ_q.

Remark 4. The above definition of metric cotype q is not identical to the definition of [121]. The difference is that in [121] the average in the right hand side of (4) is over the 3^n points $\varepsilon \in \{-1, 0, 1\}^n$ rather than over the 2^n points $\varepsilon \in \{-1, 1\}^n$. In the context of [121] it was more natural to average over the ℓ_∞^n edges $\varepsilon \in \{-1, 0, 1\}^n$, while in the present context an average over the sign vectors $\varepsilon \in \{-1, 1\}^n$ arises naturally. This nuance is irrelevant for the application to coarse embeddings. Nevertheless, in Section 4 below we will prove that the above variant of the definition of metric cotype q in fact coincides with the original definition of [121].

As formulated above (and in the literature), the notion of metric cotype q suppresses the value of the so-called "scaling parameter" $m = m(n, q, X)$. Nevertheless, it was shown in [121] that obtaining a good upper bound on m is important for certain applications, including as an obstruction to coarse embeddings. As explained in [121, Lemma 2.3], if (X, d_X) is any non-singleton metric space that satisfies (4), then necessarily $m \geqslant \frac{1}{\Gamma} n^{1/q}$. So, say that (X, d_X) has *sharp metric cotype q* if (X, d_X) has metric cotype q with m, Γ in (4) satisfying[2] the bounds $m \lesssim_{q,X} n^{1/q}$ and $\Gamma \lesssim_{q,X} 1$.

Prior to the present work, the literature contained only one theorem which establishes that a certain class of metric spaces has sharp metric cotype q. Namely, by [121] this is so for K-convex Banach spaces of Rademacher cotype q (see e.g. [116] for the definitions of the relevant Banach space concepts; we will not use them in the ensuing proofs). The question whether every Banach space of Rademacher cotype q has sharp metric cotype q remains a fundamental open problem [121]. On the other hand, in [176, Theorem 1.5] it was shown that for some $q \in [1, \infty)$, some classes of metric spaces (including ultrametrics) have metric cotype q but do not have sharp metric cotype q; another such example appears in Remark 16 below. The following theorem yields a new setting in which sharp metric cotype holds.

Theorem 5 (q-barycentric implies sharp metric cotype q)**.** *Fix $q, \beta \in [1, \infty)$ and let (X, d_X) be a q-barycentric metric space with constant β. Then, for every $n \in \mathbb{N}$ and $m \in 2\mathbb{N}$, every function $f : \mathbb{Z}_{2m}^n \to X$ satisfies*

$$\left(\sum_{i=1}^{n} \sum_{x \in \mathbb{Z}_{2m}^n} d_X \big(f(x + me_i), f(x) \big)^q \right)^{\frac{1}{q}} \leqslant \left(4n^{\frac{1}{q}} + \beta m \right) \left(\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{2m}^n} d_X \big(f(x + \varepsilon), f(x) \big)^q \right)^{\frac{1}{q}}. \tag{5}$$

In particular, if $m \geqslant \frac{1}{\beta} n^{1/q}$, then (X, d_X) satisfies the metric cotype q inequality (4) with constant $\Gamma \lesssim \beta$.

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[2]In addition to the usual "$O(\cdot), o(\cdot)$" asymptotic notation, it will be convenient to use throughout this article the following (also standard) asymptotic notation. Given two quantities $Q, Q' > 0$, the notations $Q \lesssim Q'$ and $Q' \gtrsim Q$ mean that $Q \leqslant CQ'$ for some universal constant $C > 0$. The notation $Q \asymp Q'$ stands for $(Q \lesssim Q') \wedge (Q' \lesssim Q)$. If we need to allow for dependence on parameters, we indicate this by subscripts. For example, in the presence of auxiliary objects (e.g. numbers or spaces) $\phi, \mathfrak{Z}$, the notation $Q \lesssim_{\phi, \mathfrak{Z}} Q'$ means that $Q \leqslant C(\phi, \mathfrak{Z}) Q'$, where $C(\phi, \mathfrak{Z}) > 0$ is allowed to depend only on $\phi, \mathfrak{Z}$; similarly for the notations $Q \gtrsim_{\phi, \mathfrak{Z}} Q'$ and $Q \asymp_{\phi, \mathfrak{Z}} Q'$.

Remark 6. Beyond metric cotype, dimension-dependent scaling parameters occur (for conceptually distinct reasons) in other metric inequalities that arise in the Ribe program. Determining their asymptotically sharp values is a major difficulty that pertains to important open problems; see [120, 61, 176, 60, 135, 130, 137, 48]. The currently best-known general bound [60] for the metric cotype q scaling parameter of Banach spaces of Rademacher cotype q is $m \lesssim n^{1+1/q}$.

The deduction of Theorem 3 (hence also its special cases Theorem 2 and Theorem 1) from Theorem 5 follows from an argument of [121]; for completeness, we will provide this derivation in Section 1.3.1 below. Beyond this application, it turns out that Theorem 5 sometimes yields new information even when the underlying metric space is a Banach space. To explain this, fix $q \in [2, \infty)$ and $\mathcal{K} \in [1, \infty)$. Following the terminology of [13] (itself inspired by [148, 54]; see also the treatment in [150]), a Banach space $(X, \|\cdot\|_X)$ is said to be q-uniformly convex with constant $\mathcal{K} \in [1, \infty)$ if

$$\forall\, x, y \in X, \qquad 2\|x\|_X^q + \frac{2}{\mathcal{K}^q}\|y\|_X^q \leq \|x+y\|_X^q + \|x-y\|_X^q. \tag{6}$$

The minimum $\mathcal{K}$ for which (6) holds is denoted $\mathcal{K}_q(X)$. As shown in [11, Lemma 3.1] (for $q = 2$) and [124, Lemma 6.5] (for general $q \geq 2$), every q-uniformly convex Banach space is also q-barycentric with constant $\beta \leq 2\mathcal{K}_q(X)$. Combining this fact with Theorem 5, we deduce the following statement

Corollary 7. *Suppose that $q \in [2, \infty)$ and let $(X, \|\cdot\|_X)$ be a q-uniformly convex Banach space. Then, for every $n \in \mathbb{N}$ and $m \in 2\mathbb{N}$, every function $f : \mathbb{Z}_{2m}^n \to X$ satisfies*

$$\left(\sum_{i=1}^n \sum_{x \in \mathbb{Z}_{2m}^n} \|f(x+me_i) - f(x)\|_X^q \right)^{\frac{1}{q}} \lesssim \left(n^{\frac{1}{q}} + \mathcal{K}_q(X)m \right)\left(\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{2m}^n} \|f(x+\varepsilon) - f(x)\|_X^q \right)^{\frac{1}{q}}. \tag{7}$$

In [121, Section 4] a bound that is similar to (7) was obtained under the assumption that X is a K-convex Banach space of Rademacher cotype q, in which case in the right hand side of (7) the quantity $\mathcal{K}_q(X)$ is replaced in [121] by the product of the operator norm of the Rademacher projection on $L_q(\{-1,1\}^n; X)$ and the Rademacher cotype q constant of X. These two results are incomparable, in the sense that there are Banach spaces X for which (7) is stronger than the bound of [121], and vice versa. To examine a concrete example, by the Clarkson inequality [36], if $q \in [2, \infty)$ and $X = \ell_q$, then we have $\mathcal{K}_q(X) = 1$, so the first term in the right hand side of (7) becomes $n^{1/q} + m$, which makes (7) sharp in this case, up to the implicit absolute constant factor. In contrast, the Rademacher cotype q constant of ℓ_q is equal to 1 and, for sufficiently large $n = n(q) \in \mathbb{N}$, the norm of the Rademacher projection on $L_q(\{-1,1\}^n; \ell_q)$ is at least a universal constant multiple of $\sqrt{q}$ (for a justification of the latter statement, consider e.g. [79, Lemma 7.4.11] with $N \asymp p$). Hence the corresponding term in the bound of [121] is $n^{1/q} + \sqrt{q}m$, which is significantly weaker than (7) if q is large; we will describe in Section 1.3.2 below a geometric consequence of (7) that relies on its behavior in the large q regime and does not follow from its counterpart in [121]. On a more conceptual level, the fact that the Rademacher projection appears in the bound of [121] reflects the Fourier-analytic nature of its proof in [121]. In the present setting, we need an argument that works for functions that take values in barycentric metric spaces rather than Banach spaces, in which case we do not know how to interpret the considerations of [121]. The new route that we take here leads to the aforementioned better dependence on q as $q \to \infty$ when $X = \ell_q$, though, as we already mentioned above, it is neither stronger nor weaker than the bound of [121] for general Banach spaces.

1.3. **Non-embeddability.** Here we will derive some geometric consequences of Theorem 5, including Theorem 1.

1.3.1. *Coarse, uniform and quasisymmetric embeddings.* A metric space (Y, d_Y) is said to embed uniformly into a metric space (X, d_X) if there exists a one-to-one mapping $f : Y \to X$ such that both f and $f^{-1} : f(Y) \to Y$ are uniformly continuous. (Y, d_Y) is said to embed quasisymmetrically into (X, d_X) if there exists a one-to-one mapping $f : Y \to X$ and an increasing modulus $\eta : (0, \infty) \to (0, \infty)$ with $\lim_{t\to 0}\eta(t) = 0$ such that for every distinct $x, y, z \in Y$ we have

$$\frac{d_Y\big(f(x), f(y)\big)}{d_Y\big(f(x), f(z)\big)} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)} \right).$$

See [80, 174, 19, 75, 143] and the references therein for (parts of) the large literature on these topics.

The proof of the following proposition is a straightforward abstraction of the arguments in [121] (for coarse and uniform embeddings) and [127] (for quasisymmetric embeddings).

Proposition 8. *Suppose that $p, q \in [2, \infty]$ satisfy $p > q$. Then ℓ_p does not admit a coarse, uniform or quasisymmetric embedding into a metric space (X, d_X) that has sharp metric cotype q. More generally, if a Banach space $(Y, \|\cdot\|_Y)$ admits a coarse, uniform or quasisymmetric embedding into such (X, d_X), then Y has Rademacher cotype $q + \varepsilon$ for any $\varepsilon > 0$.*

For completeness, we shall now quickly prove Proposition 8 in the case of coarse embeddings, thus establishing that Theorem 5 implies Theorem 3.

Proof of Proposition 8 in the case of coarse embeddings. Let $\omega, \Omega : [0, \infty) \to [0, \infty)$ satisfy $\omega \le \Omega$ and $\lim_{t \to \infty} \omega(t) = \infty$. Suppose that (X, d_X) is a non-singleton metric space that satisfies the sharp metric cotype q condition (4) for every $n \in \mathbb{N}$, some $\Gamma = \Gamma(q, X) > 0$ and $m = m(n, q, X) \in \mathbb{N}$ obeying $m \le C n^{1/q}$ for some $C = C(q, X) > 0$. By [121, Lemma 2.3] we have $m \ge \frac{1}{\Gamma} n^{1/q}$. Suppose also that $\phi : \ell_p^n \to X$ satisfies $\omega(d_X(x, y)) \le \|\phi(x) - \phi(y)\|_p \le \Omega(d_X(x, y))$ for all $x, y \in \ell_p^n$.

Consider the function $f : \mathbb{Z}_{2m}^n \to X$ that is given by

$$\forall x = (x_1, \ldots, x_n) \in \mathbb{Z}_{2m}^n, \qquad f(x) \stackrel{\text{def}}{=} \phi\left(n^{-\frac{1}{p}} m e^{\frac{\pi i}{m} x_1}, \ldots, n^{-\frac{1}{p}} m e^{\frac{\pi i}{m} x_n}\right).$$

Then, for every $x \in \mathbb{Z}_{2m}^n$, every $j \in \{1, \ldots, n\}$ and every $\varepsilon \in \{-1, 1\}^n$ we have

$$d_X\big(f(x + m e_j), f(x)\big) \ge \omega\left(n^{-\frac{1}{p}} m |e^{\pi i} - 1|\right) \ge \omega\left(\frac{2}{\Gamma} n^{\frac{1}{q} - \frac{1}{p}}\right) \qquad \text{and} \qquad d_X\big(f(x + \varepsilon), f(x)\big) \le \Omega\left(m |e^{\frac{\pi i}{m}} - 1|\right) \le \Omega(\pi).$$

Due to these bounds, the assumed sharp metric cotype q inequality yields the estimate

$$\omega\left(\frac{2}{\Gamma} n^{\frac{1}{q} - \frac{1}{p}}\right) n^{\frac{1}{q}} \le \Gamma m \Omega(\pi) \le C \Omega(\pi) n^{\frac{1}{q}} \implies \omega\left(\frac{2}{\Gamma} n^{\frac{1}{q} - \frac{1}{p}}\right) \le C \Omega(\pi). \tag{8}$$

Since $q < p$, the validity of (8) for every $n \in \mathbb{N}$ contradicts the assumption that $\lim_{t \to \infty} \omega(t) = \infty$. The final statement of Proposition 8 follows by combining this conclusion with a (deep) classical theorem of Maurey and Pisier [118]. $\square$

To state one concrete example of a locally finite metric space Y for which the conclusion of Theorem 1 holds true, by using the above proof of Proposition 8 with the (arbitrarily chosen) value $p = 3$, while recalling that by Theorem 5, when X is an Alexandrov space of nonpositive curvature one can work with $m = \sqrt{n}$, shows that one can take

$$(Y, d_Y) = \coprod_{m=1}^{\infty} \left(\sqrt[3]{m} \left\{e^{\frac{\pi i}{m}}, e^{\frac{2\pi i}{m}}, e^{\frac{3\pi i}{m}}, \ldots, e^{2\pi i}\right\}^{m^2}, \|\cdot\|_{\ell_3^{m^2}}\right). \tag{9}$$

Here (and throughout), we use the standard convention that the disjoint union of bounded metric spaces is their set-theoretic disjoint union, equipped with the metric that coincides with the given metric within each "cluster," and the distance between points that belong to two different clusters is the maximum of the diameters of these two clusters.

The metric space in (9) is locally finite, but not of bounded geometry (see [143, Definition 1.66]). We therefore ask

Question 9. Does there exist a metric space of bounded geometry that does not embed coarsely into any Alexandrov space of nonpositive curvature? Does this hold true for coarse embedding into any barycentric space?

Given that Theorem 1 has now been established, it seems plausible that the answer to Question 9 is positive. However, it isn't clear whether the expander-based approach [70, 72] that was previously used to address this matter for embeddings into a Hilbert space could be be applied here (certainly, by [96], one cannot use arbitrary expanders, as was done before. A tailor-made expander would be needed); see Section 2 for more on this very interesting issue.

1.3.2. *Bi-Lipschitz distortion.* Suppose that (U, d_U) and (V, d_V) are non-singleton metric spaces, and that $f : U \to V$ is a one-to-one mapping. The (bi-Lipschitz) distortion of f, denoted $\text{dist}(f) \in [1, \infty]$, is the quantity

$$\text{dist}(f) \stackrel{\text{def}}{=} \sup_{\substack{x, y \in U \\ x \ne y}} \frac{d_V\big(f(x), f(y)\big)}{d_U(x, y)} \cdot \sup_{\substack{a, b \in U \\ a \ne b}} \frac{d_U(a, b)}{d_V\big(f(a), f(b)\big)}.$$

We shall also use the convention that $\text{dist}(f) = \infty$ if f is not one-to-one. The distortion of U in V, denoted $\mathsf{c}_V(U)$, is the infimum of $\text{dist}(f)$ over all possible $f : U \to V$. In particular, by re-scaling we see that the distortion of a metric space (M, d_M) in a Banach space $(X, \|\cdot\|_X)$ is the infimum over those $D \in [1, \infty]$ for which there is $f : M \to X$ that satisfies

$$\forall x, y \in M, \qquad \frac{d_M(x, y)}{D} \le \|f(x) - f(y)\|_X \le d_M(x, y). \tag{10}$$

For $p \in [1, \infty]$ and a finite metric space (Φ, d_Φ), we will use below the simpler notation $\mathsf{c}_{\ell_p}(\Phi) = \mathsf{c}_p(\Phi)$.

Given $m, n \in \mathbb{N}$, let $[m]_\infty^n \subseteq \ell_\infty^n$ denote the finite grid $\{1, \ldots, m\}^n \subseteq \mathbb{R}^n$, equipped with the ℓ_∞ metric on $\mathbb{R}^n$. For each $q \in [1, \infty]$ we consider two "trivial" embeddings of $[m]_\infty^n$ into ℓ_q. The first is the restriction of the identity mapping from ℓ_∞^n to $\ell_q^n \subseteq \ell_q$, which we denote below by $\text{Id}_{[m]_\infty^n \to \ell_q}$. The distortion of $\text{Id}_{[m]_\infty^n \to \ell_q}$ equals $n^{1/q}$. For the second embedding, fix an arbitrary enumeration $\{x_1, \ldots, x_{m^n}\}$ of the m^n elements of $[m]_\infty^n$, and let $\text{Forget}_{[m]_\infty^n \to \ell_q}$ be the mapping

$$\forall j \in \{1, \ldots, m^n\}, \qquad \text{Forget}_{[m]_\infty^n \to \ell_q}(x_j) = e_j \in \ell_q^{m^n}.$$

The reason for the above notation/terminology is that this mapping "forgets" the metric structure of the grid $[m]_\infty^n$ altogether, as it is an arbitrary bijection of $[m]_\infty^n$ and the vertices of a simplex of m^n vertices, on which the ℓ_q metric is trivial (equilateral). Since the diameter of $[m]_\infty^n$ equals m, the distortion of the embedding $\mathsf{Forget}_{[m]_\infty^n \to \ell_q}$ equals m.

The following consequence of Corollary 7 shows that when $q \geqslant 2$, the better of the two trivial embeddings $\mathsf{Id}_{[m]_\infty^n \to \ell_q}$ and $\mathsf{Forget}_{[m]_\infty^n \to \ell_q}$ yields the smallest-possible (up to universal constant factors) distortion of the grid $[m]_\infty^n$ into ℓ_q.

Corollary 10. *For every $m, n \in \mathbb{N}$ and $q \in [2, \infty)$ we have*

$$\mathsf{c}_q\big([m]_\infty^n\big) \asymp \min\Big\{\mathrm{dist}\big(\mathsf{Id}_{[m]_\infty^n \to \ell_q}\big), \mathrm{dist}\big(\mathsf{Forget}_{[m]_\infty^n \to \ell_q}\big)\Big\} = \min\Big\{n^{\frac{1}{q}}, m\Big\}. \tag{11}$$

Proof. It will be notationally convenient to show that $\mathsf{c}_q([m+1]_\infty^{2n}) \gtrsim \min\{n^{1/q}, m\}$, which is of course equivalent to the assertion of Corollary 10, up to a possible modification of the implicit universal constant factor in (11). So, suppose that $f : [m+1]_\infty^{2n} \to \ell_q$ satisfies (10) with $M = [m+1]_\infty^{2n}$ and $X = \ell_q$. The task is to deduce that $D \gtrsim \min\{n^{1/q}, m\}$.

Let $d_{\mathbb{Z}_{2m}^n} : \mathbb{Z}_{2m}^n \times \mathbb{Z}_{2m}^n \to \mathbb{N} \cup \{0\}$ be the shortest-path metric on the Cayley graph of $\mathbb{Z}_{2m}^n$ induced by $\{-1, 0, 1\}^n \subseteq \mathbb{Z}_{2m}^n$. By [121, Lemma 6.12], there is an embedding $\psi : \mathbb{Z}_{2m}^n \to \{1, \dots, m+1\}^{2n}$ whose distortion as a mapping from $(\mathbb{Z}_{2m}^n, d_{\mathbb{Z}_{2m}^n})$ to $[m+1]_\infty^{2n}$ is $O(1)$. So, there are $\alpha, \beta > 0$ with $\beta/\alpha \lesssim 1$ such that $\alpha d_{\mathbb{Z}_{2m}^n}(x, y) \leqslant \|\psi(x) - \psi(y)\|_\infty \leqslant \beta d_{\mathbb{Z}_{2m}^n}(x, y)$ for every $x, y \in \mathbb{Z}_{2m}^n$. Consider the function $h = f \circ \psi : \mathbb{Z}_{2m}^n \to \ell_q$. Then $\|h(x + me_i) - h(x)\|_q \geqslant \alpha d_{\mathbb{Z}_{2m}^n}(x + me_i, x)/D = \alpha m/D$ and $\|h(x+\varepsilon) - h(x)\|_q \leqslant \beta d_{\mathbb{Z}_{2m}^n}(x+\varepsilon, x) \leqslant \beta$ for all $x \in \mathbb{Z}_{2m}^n$, $i \in \{1, \dots, n\}$ and $\varepsilon \in \{-1, 1\}^n$. Therefore, recalling that $\mathcal{K}_q(\ell_q) = 1$, an application of Corollary 7 to h gives the estimate $\alpha n^{1/q} m/D \lesssim \beta(n^{1/q} + m)$. Since $\beta/\alpha \lesssim 1$, this means that

$$D \gtrsim \frac{n^{\frac{1}{q}} m}{n^{\frac{1}{q}} + m} \asymp \min\Big\{n^{\frac{1}{q}}, m\Big\}. \qquad \square$$

In the setting of Corollary 10, the previous lower bound on $\mathsf{c}_q([m]_\infty^n)$ is due to [121], and is smaller than the sharp estimate (11) by a factor of $\sqrt{q}$. We will next see applications for which large values of q are needed and using the bound of [121] leads to asymptotically suboptimal results. Note that one could analogously fix $p \in (q, \infty)$ and investigate the asymptotic behavior of $\mathsf{c}_q([m]_p^n)$, where $[m]_p^n$ denotes the grid $\{1, \dots, m\}^n \subseteq \mathbb{R}^n$ equipped with the ℓ_p metric on $\mathbb{R}^n$. In this more general setting, using Corollary 10 we again obtain a lower bound that is better than that of [121] by a factor of $\sqrt{q}$, however we no longer know that it is sharp up to universal constant factors as in (11) (the best-known upper bound on $\mathsf{c}_q([m]_p^n)$ follows from an argument of [119]). This is due to a subtlety that relates to a longstanding open question in metric embeddings; see the discussion of this intriguing issue in [135, Remark 1.13] and [49].

By a (special case of a) theorem of Ribe [157], for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that if $\mathsf{c}_p([m]_\infty^n) = O(1)$ for some $p \in [2, \infty)$, then necessarily $p \gtrsim \log n$.[3] The fact that Ribe's theorem does not provide any estimate on m was addressed by Bourgain [25], who found a different proof which yields an explicit estimate (Bourgain's discretization theorem), albeit it gives here the weak bound $m = \exp(\exp(O(n)))$. However, since we are dealing with embeddings of ℓ_∞^n rather than a general n-dimensional normed space, in the same paper he showed that it suffices to take $m = n$ in our setting (specifically, this follows from of [25, Proposition 5]); note that since $[m]_\infty^n$ contains an isometric copy of $[m]_\infty^m$ for every $m \in \{1, \dots, n\}$, this implies formally that one can also take m to be at most any fixed positive power of n, by adjusting the implicit constant in the conclusion $p \gtrsim \log n$. Different proofs of this fact were found in [121, 62], but it remained the best-known bound on m to date. Corollary 10 implies that actually it suffices to take $m = O(1)$.

To justify the latter statement, observe that if $\mathsf{c}_p([m]_\infty^n) = O(1)$, then by (11) we have $\min\{n^{1/p}, m\} = O(1)$. Hence, provided that m is bigger than an appropriate universal constant (specifically, it needs to be bigger than the implicit constant in the $O(1)$ notation), it follows that $n^{1/p} = O(1)$, thus implying the desired lower bound $p \gtrsim \log n$. One should note the importance of having the sharp bound (11) at our disposal here, because the aforementioned weaker bound $\mathsf{c}_p([m]_\infty^n) \gtrsim \min\{n^{1/p}/\sqrt{p}, m\}$ of [121] does not imply the lower bound $p \gtrsim \log n$ for any $m \in \mathbb{N}$ whatsoever.

Another perspective on the above reasoning arises by examining a parameter $\mathsf{p}(X) \in [2, \infty)$ that was defined for a finite metric space (X, d_X) in [129, Section 1.1] as the infimum over those $p \in [2, \infty)$ for which $\mathsf{c}_p(X) < 10$. The value 10 in this definition was chosen arbitrarily in [129] for notational simplicity, and clearly given $\alpha > 1$ one could consider an analogous quantity $\mathsf{p}_\alpha(X)$ by defining it to be the infimum over those $p \in [2, \infty)$ for which $\mathsf{c}_p(X) < \alpha$. In addition to its intrinsic interest, the study of the quantity $\mathsf{p}(X)$ is motivated by its algorithmic significance to approximate nearest neighbor search; see [129, Remark 4.12] as well as [133, 14] and a more recent improvement in [5]. Due to Corollary 10, we can now evaluate these parameters up to universal constant factors for $X = [m]_\infty^n$.

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[3] A shorter proof of this fact, using an ultrapower and differentiation argument, follows form [76]. Because for this particular case of Ribe's theorem the target space is ℓ_q, a further simplification of the differentiation step is possible; see also [19, 111]

Corollary 11. *There exists $m_0 \in \mathbb{N}$ such that $p([m]_\infty^n) \asymp \log n$ for every $m, n \in \mathbb{N}$ with $m \geqslant m_0$. More generally, there exists a universal constant $C \geqslant 1$ such that $p_\alpha([m]_\infty^n) \asymp (\log n)/\log \alpha$ for every $\alpha \geqslant 2$ and $m, n \in \mathbb{N}$ with $m \geqslant C\alpha$.*

Proof. Fix $p, \alpha \geqslant 2$ and $m, n \in \mathbb{N}$. Since $\mathsf{c}_p([m]_\infty^n) \leqslant \mathrm{dist}(\mathrm{Id}_{[m]_\infty^n \to \ell_p}) = n^{1/p}$, we trivially have $p_\alpha([m]_\infty^n) \leqslant (\log n)/\log \alpha$. In the reverse direction, suppose that $\mathsf{c}_p(X) < \alpha$. Therefore $\min\{n^{1/p}, m\} \lesssim \alpha$, by Corollary 10. Provided that $m \geqslant C\alpha$ for a sufficiently large universal constant $C \geqslant 1$, it follows from this that $n^{1/p} \lesssim \alpha$, and hence $p \gtrsim (\log n)/\log \alpha$. $\qquad \square$

Using the bound of [121] in place of Corollary 10, one gets $p([m]_\infty^n) \gtrsim (\log n)/\log\log n$ for $m \geqslant m_0$. So, Corollary 11 improves over the available bound by the modest term $\log\log n$, but the resulting estimate is now sharp up to a universal constant factor. Having established this fact, one could hope for the following even more precise understanding. Fixing $\alpha \in \{2, 3, \ldots\}$ (e.g. focus here on $\alpha = 10$ as in [129]), since $[m]_\infty^n$ has distortion m from an m^n-simplex, we have $p_\alpha([m]_\infty^n) = 2$ if $m < \alpha$. At the same time, Corollary 11 asserts that $p_\alpha([m]_\infty^n) \asymp_\alpha \log n$ if $m \geqslant C\alpha$, so the asymptotic behavior of $p_\alpha([m]_\infty^n)$ changes markedly as m ranges over the interval $[\alpha - 1, C\alpha]$. This might occur abruptly rather than obeying intermediate asymptotics within the bounded interval; the following natural question seems accessible.

Question 12. Fix $\alpha \in \{2, 3, \ldots\}$. Do there exist $m_0 = m_0(\alpha) \in \mathbb{N}$ and $n_0 = n_0(\alpha) \in \mathbb{N}$ such that for every $n \in \{n_0, n_0 + 1, \ldots\}$ we have $p_\alpha([m]_\infty^n) = 2$ if $m \in \{1, \ldots, m_0 - 1\}$, yet $p_\alpha([m]_\infty^n) \gtrsim_\alpha \log n$ for $m \in \{m_0, m_0 + 1, \ldots\}$? If so, perhaps $m_0 = \alpha$?

1.4. **Quadratic metric inequalities.** Here we will briefly explain how the present work answers further questions that were posed in the article [4] of Andoni, Naor and Neiman, which investigates issues related to the fundamental (still wide open) problem of obtaining an intrinsic characterization of those metric spaces that admit a bi-Lipschitz embedding into some Alexandrov space of nonpositive or nonnegative curvature. This topic falls under the intriguing general question that Gromov calls the "curvature problem" [73, Section 1.19_+] (see also his earlier formulation[4] in [71, §15], the overall discussion of this topic in [4, 18], and [167, 56, 20, 162, 106, 21] for progress on the isometric setting).

Following [4], given $n \in \mathbb{N}$ and two n-by-n matrices with nonnegative entries $\mathsf{A} = (a_{ij}), \mathsf{B} = (b_{ij}) \in \mathsf{M}_n([0, \infty))$, a metric space (X, d_X) is said to satisfy the (A, B)-quadratic metric inequality if

$$\forall x_1, \ldots, x_n \in X, \qquad \sum_{i=1}^n \sum_{j=1}^n a_{ij} d_X(x_i, x_j)^2 \leqslant \sum_{i=1}^n \sum_{j=1}^n b_{ij} d_X(x_i, x_j)^2. \tag{12}$$

As explained in [4, Section 4], there exist two collections of pairs of matrices with nonnegative entries

$$\mathcal{A}^{\leqslant 0}, \mathcal{A}^{\geqslant 0} \subseteq \bigcup_{n=1}^\infty \Big(\mathsf{M}_n([0, \infty)) \times \mathsf{M}_n([0, \infty)) \Big),$$

such that bi-Lipschitz embeddability into some Alexandrov space of nonpositive or nonnegative curvature is characterized by the quadratic metric inequalities that are associated to $\mathcal{A}^{\leqslant 0}$ and $\mathcal{A}^{\geqslant 0}$, respectively. Namely, a metric space (Y, d_Y) admits an Alexandrov space of nonpositive (respectively, nonnegative) curvature $(X, d_X) = (X(Y), d_{X(Y)})$ for which $\mathsf{c}_X(Y) = O(1)$ if and only if (Y, d_Y) satisfies the (A, B)-quadratic metric inequality for every $(\mathsf{A}, \mathsf{B}) \in \mathcal{A}^{\leqslant 0}$ (respectively, for every $(\mathsf{A}, \mathsf{B}) \in \mathcal{A}^{\geqslant 0}$). See [4, Proposition 3] for a more refined formulation of this fact that spells out the dependence on the implicit constant in the $O(1)$ notation. One possible (and desirable) way to obtain an intrinsic characterization of those metric spaces that admit a bi-Lipschitz embedding into some Alexandrov space of nonpositive or nonnegative curvature would be to specify concrete families $\mathcal{A}^{\leqslant 0}, \mathcal{A}^{\geqslant 0}$ as above. Note that one could describe such $\mathcal{A}^{\leqslant 0}, \mathcal{A}^{\geqslant 0}$ by considering *all the possible* quadratic metric inequalities that *every* such Alexandrov spaces satisfies, so this question necessarily has some vagueness built into it, depending on what is considered to be "concrete" here.

A concrete candidate for $\mathcal{A}^{\geqslant 0}$ has not yet been proposed, and it would be very interesting to investigate this further. In [4, Section 5] a (quite complicated) candidate for $\mathcal{A}^{\leqslant 0}$ was derived, and the question whether or not it satisfies the desired property remains an interesting open problem that is perhaps tractable using currently available methods.

It was shown in [4] that whatever $\mathcal{A}^{\geqslant 0}$ may be, the corresponding family of inequalities "trivializes" if the distances are not squared, i.e., if $\big(\mathsf{A} = (a_{ij}), \mathsf{B} = (b_{ij})\big) \in \big(\mathsf{M}_n(\mathbb{R}) \times \mathsf{M}_n(\mathbb{R})\big) \cap \mathcal{A}^{\geqslant 0}$, then for *every* metric space (M, d_M) we have

$$\forall x_1, \ldots, x_n \in M, \qquad \sum_{i=1}^n \sum_{j=1}^n a_{ij} d_M(x_i, x_j) \lesssim \sum_{i=1}^n \sum_{j=1}^n b_{ij} d_M(x_i, x_j).$$

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[4]Specifically, in [71, §15(b)] Gromov wrote "The geodesic property is one logical level up from concentration inequalities as it involves the *existential quantifier*. It is unclear if there is a simple $\exists$-free description of (nongeodesic!) *subspaces* in CAT(κ)-spaces." The term "concentration inequalities" is defined in [71, §15(a)] to be the same inequalities as the quadratic metric inequalities that we consider in (12), except that in [71] they are allowed to involve arbitrary powers of the pairwise distances. However, due to [4] it suffices to consider only quadratic inequalities for the purpose of the simple intrinsic description that Gromov hopes to obtain (though, as he indicates, it may not exist).

In light of this result, [4, Section 1.4.1] naturally raised the question whether the same "trivialization" property holds for $\mathcal{A}^{\leqslant 0}$. Theorem 5 resolves this question, as exhibited by the sharp metric cotype 2 inequality itself, which is a quadratic metric inequality that we now know holds in any Alexandrov space of nonpositive curvature, but if one raises all the distances that occur in it to power 1 rather than squaring them, then it is straightforward to verify that the resulting distance inequality fails for the metric space $(\mathbb{Z}_{2m}^n, d_{\mathbb{Z}_{2m}^n})$, even allowing for any loss of a constant factor.

An inspection of the proof of Theorem 5 that appears below reveals that the sharp metric cotype 2 inequality for Alexandrov spaces of nonpositive curvature belongs to the family of quadratic metric inequalities that were derived in [4, Section 5.2]. Checking this assertion is somewhat tedious but entirely mechanical. Specifically, our proof of Theorem 5 consists only of (many) applications of the triangle inequality and (many) applications of an appropriate variant of Pisier's martingale inequality for barycentric metric spaces (see Proposition 26 below). The latter inequality is due to [123], where its proof is an iterative application of the barycentric condition (3). As such, the ensuing derivation of the sharp metric cotype 2 inequality for Alexandrov spaces of nonpositive curvature can be recast as falling into the hierarchical framework of [4, Section 5.2]. Hence, the above lack of "trivialization" of $\mathcal{A}^{\leqslant 0}$ when the squares are removed already occurs within this hierarchy, thus answering another question that was raised in [4].

1.4.1. *Wasserstein spaces.* Given $p \geqslant 1$ and a separable complete metric space (X, d_X), let $\mathcal{P}_p(X)$ denote the space of all Borel probability measures on X of finite p'th moment, equipped with the Wasserstein-p metric; see e.g. [179] for all of the relevant terminology and background (which will not be used in any of the ensuing proofs).

It was proved in [4] that if $p > 1$ and $\theta \in (0, 1/p]$, then for every finite metric space (M, d_M), its θ-snowflake, i.e., the metric space (M, d_M^θ), embeds with distortion arbitrarily close to 1 into $\mathcal{P}_p(\mathbb{R}^3)$. Hence, by basic facts (see e.g. [95, Section 2.4]) about ultrapowers of metric spaces, (Y, d_Y^θ) embeds isometrically into an ultrapower of $\mathcal{P}_p(\mathbb{R}^3)$ for every (not necessarily finite) metric space (Y, d_Y). In particular, since by [144, 170, 114] an ultrapower of $\mathcal{P}_2(\mathbb{R}^3)$ is an Alexandrov space of nonnegative curvature, it follows that any metric space embeds via an embedding which is simultaneously coarse, uniform and quasisymmetric into some Alexandrov space of nonnegative curvature.

Initially, namely prior to the present work, the validity of the above universality result led some researchers to suspect that the conclusion of Theorem 1 might actually be false. Indeed, a main open question of [4] was whether for every metric space (Y, d_Y) the metric space $(Y, \sqrt{d_Y})$ admits a bi-Lipschitz embedding into some Alexandrov space of nonpositive curvature; Theorem 1 resolves this question.

Since the proof of Theorem 2 relies only on the sharp metric cotype of the target space, which involves only its finite subsets, the following corollary is a consequence of a combination of the aforementioned result of [4] and Theorem 2.

Corollary 13. *If $p > 1$, then $\mathcal{P}_p(\mathbb{R}^3)$ does not admit a coarse, uniform or quasisymmetric embedding into any barycentric metric space, and hence, in particular, it admits no such embedding into any Alexandrov space of nonpositive curvature.*

Because $\mathcal{P}_2(\mathbb{R}^3)$ is an Alexandrov space of nonnegative curvature, the following corollary is nothing more than a special case of Corollary 13, but it seems worthwhile to state separately.

Corollary 14. *There exists an Alexandrov space of nonnegative curvature that does not embed coarsely into any Alexandrov space of nonpositive curvature.*

The potential validity of Corollary 13 (and the underlying universality phenomenon that was used to deduce it) for $p = 1$ remains a very interesting open question; see [4] for a thorough discussion of this matter, which is closely related to an old open question of Bourgain [23]. Also, the conclusion of Corollary 13 with $\mathcal{P}_p(\mathbb{R}^3)$ replaced by $\mathcal{P}_p(\mathbb{R}^2)$ remains intriguingly open; see [8] for partial information in this direction. In relation to the latter question, observe that due to Proposition 8, a positive answer to the following question would imply that the snowflake universality phenomenon that was established in [4] for $\mathcal{P}_2(\mathbb{R}^3)$ does not hold true for $\mathcal{P}_2(\mathbb{R}^2)$.

Question 15. Does $\mathcal{P}_2(\mathbb{R}^2)$ have sharp metric cotype q for any $q \in [2, \infty)$?

Remark 16. Answering a question that was posed in [4], our forthcoming work [48] establishes that any Alexandrov space of nonnegative curvature has metric cotype 2. In particular, both $\mathcal{P}_2(\mathbb{R}^2)$ and $\mathcal{P}_2(\mathbb{R}^3)$ have metric cotype 2, but by combining the aforementioned snowflake universality of $\mathcal{P}_2(\mathbb{R}^3)$ that was proved in [4] with Proposition 8, we see that $\mathcal{P}_2(\mathbb{R}^3)$ fails to have metric cotype q with sharp scaling parameter for any $q \in [2, \infty)$. This yields another example of a metric space whose metric cotype is not sharp, which is yet another indication that the phenomenon of Theorem 5 is a rare occurrence. This also shows that the answer to Question 15 with $\mathcal{P}_2(\mathbb{R}^2)$ replaced by $\mathcal{P}_2(\mathbb{R}^3)$ is negative.

The following question on understanding the nonnegative-curvature counterpart of Theorem 5 seems accessible.

Question 17. For $n \in \mathbb{N}$, what is the asymptotic behavior of the smallest $m(n) \in \mathbb{N}$ such that any Alexandrov space of nonnegative curvature has metric cotype 2 with scaling parameter $m = m(n)$, namely (4) holds for $q = 2$ and $\Gamma = O(1)$?

As stated in Remark 16, the fact that in Question 17 the desired $m(n) \in \mathbb{N}$ exists is due to our forthcoming work [48], but the bound that follows from the proof of [48] is probably far from being asymptotically sharp.

1.5. **Implications of the sign of curvature under large deformations.** The present work shows that any Alexandrov space of nonpositive curvature has metric cotype 2, and [48] shows that also any Alexandrov space of nonnegative curvature has metric cotype 2. So, on a qualitative level both "signs of curvature" imply the same (best possible, because of the requirement that the space is geodesic) metric cotype. In terms of this specific quadratic metric inequality (recall (12) for the general form of a quadratic metric inequality), the difference between the two possible signs of curvature turns out to be more nuanced, namely it manifests itself in the asymptotic behavior of the scaling parameter m in (4) (with $q = 2$) as $n \to \infty$. Specifically, Theorem 3 shows that $m \leqslant \sqrt{n}$ suffices when (X, d_X) is an Alexandrov space of nonpositive curvature, while by [4] we know that $m/\sqrt{n}$ must tend to ∞ as $n \to \infty$ if (X, d_X) is the Alexandrov space of nonnegative curvature $\mathcal{P}_2(\mathbb{R}^3)$ (recall Question 17).[5] If one puts aside this deeper (and important) subtlety, Alexandrov spaces of nonnegative curvature are actually "better behaved" than Alexandrov spaces of nonpositive curvature in terms of those quadratic invariants that have been computed thus far; see Table 1.

TABLE 1. Influence of the sign of curvature of Alexandrov spaces on quadratic bi-Lipschitz invariants.

sign of curvature	Enflo type 2	Markov type 2	metric cotype 2	Markov convexity 2	nonlinear spectral gap
$\geqslant 0$	yes	yes	yes	yes	no
$\leqslant 0$	yes	no	yes	no	no

In addition to metric cotype, the Ribe program produced several quadratic metric invariants that have a variety of applications (at the same time, questions on how to formulate certain other invariants that should capture in a metrical way key Banach space properties remain important "missing steps" of the Ribe program; see e.g. [128]). Those include Enflo type [45, 46] (which, in its quadratic special case, coincides with invariants that were considered by Gromov [66] and Bourgain, Milman and Wolfson [26]), K. Ball's Markov type [11], and Markov convexity [108]. Table 1 explains how the sign of the curvature of an Alexandrov space influences the validity of such invariants; e.g. its second column indicates that Enflo type 2 is implied by both nonnegative and nonpositive curvature (this follows by a variant of an inductive argument that essentially goes back to Enflo; see e.g. [141]), whereas its third column says that, due to [141], any Alexandrov space of nonnegative curvature has Markov type 2, while, due to [96], there exist Alexandrov spaces of nonpositive curvature that do not have Markov type 2 (even any nontrivial Markov type). The fourth column of Table 1 was already discussed above, and the top entry of its fifth column is due to [8] (see also [110, 48] for variants of the argument in [8]), while the bottom entry of the fifth column is an immediate consequence of the fact that (the 1-dimensional simplicial complex of) a tree is an Alexandrov spaces of nonpositive curvature.

The last column of Table 1 deals with the validity of the nonlinear spectral gap inequality (13) below for *every* classical expander $\{\mathsf{G}(n)\}_{n=1}^{\infty}$. While, as we discussed earlier, this phenomenon holds for a variety of spaces, Table 1 indicates that it fails for Alexandrov spaces of either nonpositive or nonnegative curvature, due to [96] and [4], respectively.[6]

Table 1 is not exhaustive. For example, our forthcoming work [48] formulates a new metric invariant (inspired by [107]) called *diamond convexity*, and proves that Alexandrov spaces of either nonpositive or nonnegative curvature have diamond convexity 2. Other metric invariants that arose from the Ribe program [136, 135, 137] have not yet been computed for Alexandrov spaces, and this is so also for invariants that arose in other (related) contexts [94, 9, 102, 139].

While the literature contains several geometric inequalities that distinguish between the sign of curvature of Alexandrov spaces (quadrilateral inequality [20], tripod inequality [106], Ptolemy inequality [56] etc.), these had initially been

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[5]This demonstrates that the role of the scaling parameter in the definition of metric cotype is not as subsidiary as it may seem from [121], where it did not have a crucial role in the metric characterization of Rademacher cotype or the nonlinear Maurey–Pisier theorem. While sharp metric cotype was shown in [121] to have implications to coarse and uniform embeddings, there is a definite possibility that metric cotype q and sharp metric cotype q coincide for Banach spaces (though this is a major open problem); here we see that this is markedly not so in the setting of Alexandrov geometry, leading to formidable qualitative differences between the coarse implications of the sign of curvature.

[6]In terms of bi-Lipschitz distortion of arbitrary expanders, Table 1 does not fully indicate the extent to which nonnegative curvature behaves better than nonpositive curvature. The fact [141] that an Alexandrov space of nonnegative curvature has Markov type 2 implies (by examining the standard random walk on the graph) that if $\mathscr{G} = \{\mathsf{G}(n)\}_{n=1}^{\infty}$ is an expander, then $\mathsf{c}_X(\mathsf{G}(n)) \gtrsim_{\mathscr{G}} \sqrt{\log |\mathsf{G}(n)|}$ for any nonnegatively curved Alexandrov space X. In contrast, there is [96] an Alexandrov space of nonpositive curvature which contains an expander with $O(1)$ distortion.

relevant only for the isometric (or almost-isometric) regime, in the sense that if one considers the same inequalities up to a sufficiently large universal constant factor (for the above examples, factor 4 suffices), then *any* metric space satisfies the resulting invariant. So, these initial investigations did not rule out embeddings of very large distortion into Alexandrov spaces. In recent years, some invariants that do retain their non-triviality after arbitrarily large deformations were found, but the crux of the above comparison is that they turned out to always behave either the same or better in the setting of nonnegative curvature than in the setting of nonpositive curvature. This is so even for metric cotype 2, but the present work establishes that the asymptotic behavior of the scaling parameter in the cotype 2 inequality provides an invariant with respect to which nonpositive curvature is in fact better behaved than nonnegative curvature. The search for such an invariant was one of the motivations of the present investigation, but there clearly remains much to be done in order to understand the coarse/large distortion implications of the sign of curvature in Alexandrov geometry (this general theme is in the spirit of the Riemannian counterpart that is presented in [68]).

2. Limitations of previous approaches and further open questions

The main purpose of this section is to explain why Theorem 1 does not follow from the previously available results (which we listed in the Introduction) on obstructions to coarse embeddings. The ensuing synthesis of the literature and examination of the conceptual limitations of existing tools to address Gromov's question also naturally leads us to state several open problems; we believe that these are rich, fertile and important directions for future research.

While the present section is helpful for understanding the significance of our results as well as the challenges that remain, it will not be used in the subsequent proofs and therefore it could be skipped on first reading.

We will next elaborate on all of the pointers to the literature that were listed at the start of the Introduction, but it will be instructive to do so thematically rather than following the chronological order that we used previously.

2.1. **Metric cotype.** The direct precursor (and inspiration) of the present work is [121], where the validity of sharp metric cotype q is demonstrated for certain Banach spaces. The proof of [121] uses the underlying linear structure through an appeal to the boundedness of an oscillatory convolution operator (the Rademacher projection), which relies on cancellations. We do not see how to interpret that strategy of [121] for targets that are not Banach spaces.

2.2. **Negative definite kernels.** The works of Johnson and Randrianarivony [87] and Randrianarivony [154] proceed via a reduction to the linear theory through an influential approach of Aharoni, Maurey and Mitaygin [1], which uses negative definite kernels and was invented to treat uniform embeddings. This strategy relies heavily on the underlying linear (even Hilbertian) structure and we doubt that it could be interpreted for targets that are not Banach spaces. If such an interpretation were possible, then it would be a very interesting achievement, likely of value elsewhere.

2.3. **The Urysohn space.** Parts of Pestov's proof [147] that the Urysohn space $\mathbb{U}$ does not admit a coarse embedding into any uniformly convex Banach space $(X, \|\cdot\|_X)$ rely on linear considerations (e.g. using the Hahn–Banach theorem), so at the outset its relevance to the fully nonlinear setting of Theorem 1 is questionable. But, even putting this matter aside, there is the following conceptual reason why the approach of [147] is ill-suited to treating nonpositively curved targets. The crux of the argument in [147] is the recursive use of a strong self-similarity property [78, 166, 177] of $\mathbb{U}$ to demonstrate that if $\mathbb{U}$ embeds coarsely into X, then an ultrapower of $\ell_2(X)$ contains arbitrarily large complete binary trees in which all pairs of vertices that are either siblings or form an edge are distorted by at most a $O(1)$-factor (see [147, Section 6] for a precise formulation of this statement). By a classical result in Banach space theory (see e.g. [50, Chapter 9]), this conclusion contradicts the premise that X is uniformly convex. However, (the 1-dimensional simplicial complex of) a connected tree is an Alexandrov space of nonpositive curvature, so the above approach detects a geometric structure that is compatible with nonpositive curvature. Note that, in hindsight, due to universality properties of $\mathbb{U}$, Theorem 1 implies formally that $\mathbb{U}$ does not embed coarsely into any Alexandrov space of nonpositive curvature, but the reason that we obtain here for this fact is entirely different from the strategy of [147].

2.4. **Arbitrary expanders.** Throughout what follows, graphs are tacitly assumed to be finite, connected and regular. The vertex set of a graph G is denoted V_{G} and its edge set is denoted E_{G}. The shorted-path metric that G induces on V_{G} is denoted d_{G}. When we consider a graph G as a metric space, it is always understood to be $(V_{\mathsf{G}}, d_{\mathsf{G}})$.

A sequence of graphs $\{\mathsf{G}(n)\}_{n=1}^{\infty}$ is an expander with respect to a metric space (X, d_X) if $\lim_{n\to\infty} |V_{\mathsf{G}(n)}| = \infty$, the degree of $\mathsf{G}(n)$ is bounded above independently of $n \in \mathbb{N}$, and there exists $\gamma \in (0, \infty)$ such that for any $n \in \mathbb{N}$ we have

$$\forall \{x_u\}_{u \in V_{\mathsf{G}(n)}} \subseteq X, \qquad \frac{1}{|V_{\mathsf{G}(n)}|^2} \sum_{(u,v) \in V_{\mathsf{G}(n)} \times V_{\mathsf{G}(n)}} d_X(x_u, x_v)^2 \leqslant \frac{\gamma}{|E_{\mathsf{G}(n)}|} \sum_{\{u,v\} \in E_{\mathsf{G}(n)}} d_X(x_u, x_v)^2. \tag{13}$$

An extensive discussion of expanders with respect to metric spaces can be found in [122, 124, 125]. In the special case when $\{\mathsf{G}(n)\}_{n=1}^\infty$ is an expander with respect to a Hilbert space (equivalently with respect to $\mathbb{R}$, or even, by an application of Cheeger's inequality [32, 53, 40, 3], with respect to $\{0,1\}$), the common simpler terminology is that $\{\mathsf{G}(n)\}_{n=1}^\infty$ is an expander (in the classical sense); detailed background on this important notion appears in e.g. [77].

As noted in [70], by a short argument (that is also implicit in works of Linial, London and Rabinovich [113] and Matoušek [115], both of which considered bi-Lipschitz embeddings), one shows that if $\{\mathsf{G}(n)\}_{n=1}^\infty$ is an expander with respect to (X, d_X), then any metric space (Y, d_Y) that contains an isometric copy of each of the finite metric spaces $\{(V(\mathsf{G}_n), d_{\mathsf{G}(n)})\}_{n=1}^\infty$ (e.g. their disjoint union or their Pythagorean product) does not embed coarsely into (X, d_X). In particular, since (classical) expanders are known to exist (by now, via a wealth of different constructions; see [77]), we thus obtain Gromov's proof [70, 72] that there exists a metric space that does not embed coarsely into a Hilbert space.

As indicated by Gromov [71, 72] and proven in detail by Kondo [96], there exists an Alexandrov space of nonpositive curvature that contains with $O(1)$ distortion a classical expander; for an even more "wild" example of such an Alexandrov space (with respect to which random regular graphs are almost surely *not* expanders), see [125]. In particular, it follows that some Alexandrov spaces of nonpositive curvature do not admit a coarse embedding into a Hilbert space. It also follows that the aforementioned important realization of [70, 72] that the presence of *any expander whatsoever* implies coarse non-embeddability into a Hilbert space is irrelevant for proving Theorem 5.

2.5. **Arbitrary relative expanders.** The notion of a *relative expander*, or an expander relative to a partition, was introduced by Arzhantseva and Tessera [7] for studying aspects of coarse embeddings into a Hilbert space; it was also used implicitly in the earlier work [2], and was studied for algorithmic purposes (clustering) in [10]. We will now present the obvious generalization of the definition of [7] to a notion of a relative expander with respect to a metric space.

Say that a sequence of graphs $\{\mathsf{G}(n)\}_{n=1}^\infty$ is a relative expander with respect to a metric space (X, d_X) if the degree of $\mathsf{G}(n)$ is bounded above independently of $n \in \mathbb{N}$, and there exists a constant $\rho > 0$ such that for every $n \in \mathbb{N}$ there is a partition $\mathscr{P}(n) = \{C_1(n), C_2(n), \dots, C_{k(n)}(n)\}$ of $V_{\mathsf{G}(n)}$ for which $\lim_{n\to\infty} \min_{j\in\{1,\dots,k(n)\}} |C_j(n)| = \infty$, and

$$\forall \{x_u\}_{u \in V_{\mathsf{G}(n)}} \subseteq X, \qquad \frac{1}{|V_{\mathsf{G}(n)}|} \sum_{j=1}^{k(n)} \frac{1}{|C_j(n)|} \sum_{u,v \in C_j(n)} d_X(x_u, x_v)^2 \leqslant \frac{\rho}{|E_{\mathsf{G}(n)}|} \sum_{\{u,v\} \in E_{\mathsf{G}(n)}} d_X(x_u, x_v)^2. \tag{14}$$

When (X, d_X) is a Hilbert space one simply says that $\{\mathsf{G}(n)\}_{n=1}^\infty$ is a relative expander. This is a weakening of the definition 13 of an expander with respect to (X, d_X), which corresponds to the special case when the partition $\mathscr{P}(n)$ is the trivial partition $\{V_{\mathsf{G}(n)}\}$ for every $n \in \mathbb{N}$. It is, in fact, a very substantial weakening, because by [7] there is a relative expander $\{\mathsf{G}(n)\}_{n=1}^\infty$ such that $\bigsqcup_{n=1}^\infty \mathsf{H}(n)$ does not embed coarsely into $\bigsqcup_{n=1}^\infty \mathsf{G}(n)$ for *any* expander $\{\mathsf{H}(n)\}_{n=1}^\infty$. Despite this, the deduction that (13) implies that $\bigsqcup_{n=1}^\infty \mathsf{G}(n)$ does not embed coarsely into (X, d_X) carries over effortlessly[7] to show that this remains valid under the weaker hypothesis that $\{\mathsf{G}(n)\}_{n=1}^\infty$ is a relative expander with respect to (X, d_X).

Because, as we discussed above, one cannot use arbitrary expanders to prove Theorem 1, also arbitrary relative expanders cannot be used for this purpose, since the latter is an even larger family of (sequences of) graphs. When in Theorem 1 the target space X is a simply connected Riemannian manifold of nonpositive sectional curvature rather than a general (potentially singular) Alexandrov space of nonpositive curvature, by [180, 81] one can take Y to be $\bigsqcup_{n=1}^\infty \mathsf{G}(n)$ for any expander $\{\mathsf{G}(n)\}_{n=1}^\infty$. So, arbitrary expanders suffice for targets that are not singular (or have "bounded singularities" [72, 171]), but the proof of this in [180, 81, 138] does not carry over to relative expanders.

Question 18. Does there exist a relative expander $\{\mathsf{G}(n)\}_{n=1}^\infty$ and a simply connected Riemannian manifold (M, d_M) of nonpositive sectional curvature such that $\bigsqcup_{n=1}^\infty \mathsf{G}(n)$ embeds coarsely into (M, d_M)?

Despite the fact that Question 18 is inherently not a route towards a different proof of Theorem 1, it is a natural (and perhaps quite accessible) question in Riemannian geometry that arises from the present considerations.

2.6. **Specially-crafted expanders.** In order to apply the idea that we recalled in Section 2.4 to show that there exists a metric space that does not embed coarsely into some non-Hilbertian metric space (X, d_X), one only needs to show that (X, d_X) admits *some* expander rather than to show that *any* (classical) expander whatsoever is also an expander with respect to (X, d_X). See [115, 145, 81, 171, 138, 149, 129, 126, 33, 172, 131] for theorems that provide a variety of spaces (X, d_X) that do satisfy the latter stronger requirement for any possible expander. In general, however, it could be that no expander with respect to (X, d_X) exists (certainly not with respect to, say, ℓ_∞). This is so even for (seemingly) "nice" spaces; e.g. by [4] there is an Alexandrov space of nonnegative curvature, namely $\mathcal{P}_2(\mathbb{R}^3)$, with respect to which

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[7]The only additional observation that is needed for this is that, because the degree of $\mathsf{G}(n)$ is $O(1)$, a quick and standard counting argument (see e.g. the justification of equation (36) in [134]) shows that $d_{\mathsf{G}(n)}(u, v) \gtrsim \log|C_j(n)| \to \infty$ for a constant fraction of $(u, v) \in C_j(n) \times C_j(n)$.

no sequence of graphs is an expander. Obtaining a useful/workable intrinsic characterization of those metric spaces with respect to which some expander exists is an important (likely difficult, perhaps intractable) open question.

This leaves the possibility, as a potential alternative route towards a proof Theorem 1, that one could somehow come up with a special sequence of graphs $\{G(n)\}_{n=1}^{\infty}$ which is simultaneously an expander with respect to *every* Alexandrov space of nonpositive curvature. If this were indeed possible, then it would require constructing a specially-crafted expander, and finding a way to prove the quadratic distance inequality (13) that relies on geometric considerations (based solely on nonpositive curvature) rather than the straightforward linear algebra/spectral consideration that underlies its Euclidean counterpart (recall that by [4] this cannot be accomplished for nonnegative curvature).

Question 19. Is there a sequence of graphs $\{G(n)\}_{n=1}^{\infty}$ which is an expander with respect to every Alexandrov space of nonpositive curvature X? More modestly, is there such an expander that does not embed coarsely into any such X?

Question 19 does not originate in the present work, and is in fact a well-known open problem that has been broached several times (see e.g. [81, 171, 138, 57, 96, 124, 7, 125, 4, 172]); we restated it above due to its relevance to the present discussion (and importance). Note that if Question 19 had a positive answer, then, since expanders have bounded degree, the disjoint union of the resulting expander $\{G(n)\}_{n=1}^{\infty}$ would be a space of bounded geometry that fails to admit a coarse embedding into any Alexandrov space of nonpositive curvature; thus answering Question 9.

V. Lafforgue devised [100, 101] a brilliant method to prove that certain special graph sequences are expanders with respect to every Banach space of nontrivial Rademacher type; see also [112, 38, 37] for further perspectives on such *super-expanders*. Lafforgue's approach relies heavily on the linear structure of the underlying Banach space through the use of Fourier-analytic considerations, including appeal to the substantial work [22]. As such, we do not see how these methods could be relevant to Question 19, though it would be very interesting if they could be implemented for metric spaces that are not Banach spaces. Further examples of such expanders [163, 39, 55] were more recently obtained using Roe's warped cones [160] and a discretization procedure of Vigolo [178], but these take Lafforgue's expanders as input and therefore they suffer from the same deficiency vis-à-vis Question 19.

An entirely different strategy to construct super-expanders was found by Mendel and Naor [124], using the zigzag graph product of Reingold, Vadhan and Wigderson [156]. This strategy does apply to targets that are not Banach spaces (see e.g. its implementation in [125]), but its potential applicability to Question 19 remains unclear because it relies on an iterative construction and at present we lack a "base graph" to start the induction. Due to [123], finding such a base graph is the *only* issue that remains to be overcome in order to answer Question 19 (positively) via the zigzag strategy of [124, 125]. This seems to be a difficult question; see Section 2.7 for a candidate base graph.

A more ambitious strengthening of Question 9 (which is also a well-known problem) is the following question.

Question 20. Does there exist a finitely generated group (equipped with a word metric associated to any finite symmetric set of generators) that does not embed coarsely into any Alexandrov space of nonpositive curvature?

As explained in [138], by applying the graphical random group construction of [72] (see also [6, 142]), a positive answer to Question 21 below would imply a positive answer to Question 20. By [138], this would also yield a group G such that any isometric action of G on an Alexandrov space of nonpositive curvature has a fixed point; the existence of such a group is itself a major open problem, in addition to Question 20 and Question 21.

Question 21. Does there exist a sequence of graphs $\{G(n)\}_{n=1}^{\infty}$ which is an expander with respect to every Alexandrov space of nonpositive curvature, and also for every $n \in \mathbb{N}$ the girth of $G(n)$ is at least $c \log|V_{G(n)}|$ for some $c \in (0,\infty)$?

Unfortunately, the approaches of [100, 101, 124] for constructing super-expanders seem inherently ill-suited for producing the high-girth graphs that Question 21 aims to find.

2.7. **Quotients of the Hamming cube.** Following [124], a candidate for the base graph in the inductive construction of [124] arises from the work [94] of Khot and Naor, where it is shown that for every $n \in \mathbb{N}$ there is a linear subspace $V_n \subseteq \mathbb{F}_2^n = \{0,1\}^n$, namely the polar of an "asymptotically good linear code," such that for every $f : \mathbb{F}_2^n/V_n \to \ell_2$ satisfies

$$\frac{1}{4^n} \sum_{(x,y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n} \|f(x+V_n) - f(y+V_n)\|_2^2 \lesssim \frac{1}{n2^n} \sum_{j=1}^{n} \sum_{x \in \mathbb{F}_2^n} \|f(x+e_j+V_n) - f(x+V_n)\|_2^2. \tag{15}$$

The estimate (15) is close to the assertion that the quotient Hamming graphs $\{\mathbb{F}_2^n/V_n\}_{n=1}^{\infty}$ (namely, the Cayley graphs of the Abelian groups $\mathbb{F}_2^n/V_n$ with respect to the generators $\{e_1,\ldots,e_n\}$) form an expander, except that they do not have bounded degrees. Nonetheless, it follows from (15), similarly to the aforementioned application of (13), that any metric space that contains these quotient Hamming graphs does not embed coarsely into a Hilbert space (for this, one

only needs to estimate the average distance in these graphs, as done in [94]). In the same vein, a positive answer to Question 22 below would yield a different proof of Theorem 1. Much more significantly, by [123, 124, 125] this would yield the aforementioned desired base graph so as to answer Question 19 positively; see [124, Section 7] for a variant of this approach using the heat semigroup on $\mathbb{F}_2^n$ which would also yield such a base graph, as well as a closely related harmonic-analytic question in [124, Section 5] that remains open even for uniformly convex Banach spaces.

Question 22. Let $V_n \subseteq \mathbb{F}_2^n$ be linear subspaces (namely, polars of asymptotically good codes) as in [94] and above. Is it true that for every Alexandrov space of nonpositive curvature (X, d_X), for every $f : \mathbb{F}_2^n / V_n \to X$ we have

$$\frac{1}{4^n} \sum_{(x,y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n} d_X\big(f(x+V_n), f(y+V_n)\big)^2 \lesssim \frac{1}{n2^n} \sum_{j=1}^n \sum_{x \in \mathbb{F}_2^n} d_X\big(f(x+e_j+V_n), f(x+V_n)\big)^2. \tag{16}$$

Despite the fact that the proof of (15) in [94] is Fourier-analytic (this is so also for the variants in [124]), we believe that establishing its analogue (16) for Alexandrov spaces of nonpositive curvature is currently the most viable approach towards Question 19. Nevertheless, it seems that a substantial new idea is required here.

It should be noted that while [94] shows that quotient Hamming graphs $\{\mathbb{F}_2^n / V_n\}_{n=1}^\infty$ yield a solution of the Hilbertian case of Gromov's question that we quoted earlier, we do not know if this inherently does not follow from a reduction to the case of classical expanders, as formulated in the following natural geometric question.

Question 23. Continuing with the above notation, does there exist a (bounded degree, classical) expander $\{\mathsf{G}(n)\}_{n=1}^\infty$ such that $\bigsqcup_{n=1}^\infty \mathsf{G}(n)$ embeds coarsely into $\bigsqcup_{n=1}^\infty (\mathbb{F}_2^n / V_n)$?

2.8. **Generalized roundness.** The first proof that there exists a metric space which does not embed coarsely into a Hilbert space was obtained by Dranishnikov, Gong, Lafforgue, and Yu [42]. They showed that this is so for coarse embeddings into any metric space that satisfies a classical condition, called generalized roundness $p \in (0, \infty)$, which was introduced and used by Enflo [44] to answer an old question of Smirnov [65] whether every metric space embeds uniformly into a Hilbert space (strictly speaking, [42] considered only the case $p = 2$, but the argument works mutatis mutandis for any $p > 0$). A unified treatment of these approaches, so as to rule out coarse and uniform embeddings simultaneously, appears in [93]. As noted by several authors [158, 103, 153, 93], a "vanilla" application of the approach of [42] fails to apply to targets that are general Alexandrov spaces of nonpositive curvature, i.e., to prove Theorem 1. Indeed, a combination of the characterization of generalized roundness in [109] with the work [51] shows that there are Alexandrov spaces of nonpositive curvature that do not have positive generalized roundness; specifically, this is so for the quaternionic hyperbolic space (and not so for the real and complex hyperbolic spaces [58, 51]). However, the quaternionic hyperbolic space embeds coarsely into a Hilbert space (e.g. by [165]), so the conclusion of Theorem 1 does hold for it by a direct reduction to the case of Hilbertian targets. In other words, the correct setting of the method of [42] is those spaces that embed coarsely into some metric space of positive generalized roundness rather than those spaces that have positive generalized roundness themselves. A decisive demonstration that the approach of [42] is inherently ill-suited for proving Theorem 1 is that, by combining [42] with the classical Schoenberg embedding criterion [164], one sees that if a metric space has positive generalized roundness, then it embeds coarsely into a Hilbert space. Hence the aforementioned Gromov–Kondo spaces are Alexandrov spaces of nonpositive curvature that do not embed coarsely into any metric space of positive generalized roundness. (Incidently, this answers questions that were posed in [103, page 155] and [153, page 10], but it seems to have gone unnoticed that [96] resolves them.)

2.9. **Flat tori.** For every $n \in \mathbb{N}$ let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the standard scalar product on $\mathbb{R}^n$, which induces the Hilbert space ℓ_2^n. For each $n \in \mathbb{N}$ fix a lattice $\Lambda_n \subseteq \mathbb{R}^n$ for which the dual lattice $\Lambda_n^* = \bigcap_{y \in \Lambda_n} \{x \in \mathbb{R}^n : \langle x, y \rangle \in \mathbb{Z}\}$ is a $O(\sqrt{n})$-net of ℓ_2^n, namely $\|x - y\|_2 \gtrsim \sqrt{n}$ for all distinct $x, y \in \Lambda_n^*$, and for every $z \in \mathbb{R}^n$ there exists $w \in \Lambda_n^*$ such that $\|z - w\|_2 \lesssim \sqrt{n}$. The fact that such lattices exist is a classical theorem of Rogers [161]; see also [31, 24] for other proofs.

The reasoning in [94, Section 4] implies that if we endow the Abelian group $\mathbb{R}^n / \Lambda_n$ with the (flat torus) quotient Riemannian metric $d_{\mathbb{R}^n / \Lambda_n} : (\mathbb{R}^n / \Lambda_n) \times (\mathbb{R}^n / \Lambda_n) \to [0, \infty)$ that is induced by the ℓ_2^n metric, then the Pythagorean product

$$\mathfrak{T} \overset{\text{def}}{=} \left(\bigoplus_{n=1}^\infty \big(\mathbb{R}^n / \Lambda_n, d_{\mathbb{R}^n / \Lambda_n}\big) \right)_2 \tag{17}$$

does not embed coarsely into $L_1(\mathbb{R})$, and hence also it does not embed coarsely into a Hilbert space. This is so because by (a change of variable in) [94, Lemma 9], if we let μ_n denote the normalized Riemannian volume measure on the torus $\mathbb{R}^n / \Lambda_n$ and γ_n denote the standard Gaussian measure on $\mathbb{R}^n$, then every measurable $\phi : \mathbb{R}^n / \Lambda_n \to L_1(\mathbb{R})$ satisfies

$$\iint_{(\mathbb{R}^n / \Lambda_n) \times (\mathbb{R}^n / \Lambda_n)} \|\phi(u) - \phi(v)\|_1 \, \mathrm{d}\mu_n(u) \, \mathrm{d}\mu_n(v) \lesssim \iint_{(\mathbb{R}^n / \Lambda_n) \times \mathbb{R}^n} \left\| \phi\Big(w + \frac{z}{\sqrt{n}} + \Lambda_n\Big) - \phi(w) \right\|_1 \, \mathrm{d}\mu_n(w) \, \mathrm{d}\gamma_n(z). \tag{18}$$

If in addition we have $\omega(d_{\mathbb{R}^n/\Lambda_n}(u,v)) \leq \|\phi(u) - \phi(v)\|_1 \leq \Omega(d_{\mathbb{R}^n/\Lambda_n}(u,v))$ for all $u, v \in \mathbb{R}^n/\Lambda_n$ and some nondecreasing moduli $\omega, \Omega : [0,\infty) \to [0,\infty)$, then it follows from (18) that $\omega(c\sqrt{n}) \lesssim \Omega(1)$, where $c > 0$ is a universal constant. For this computation, use the proof of [94, Lemma 10] to deduce that $d_{\mathbb{R}^n/\Lambda_n}(u,v) \gtrsim \sqrt{n}$ for a constant proportion (with respect to $\mu_n \times \mu_n$) of $(u,v) \in (\mathbb{R}^n/\Lambda_n) \times (\mathbb{R}^n/\Lambda_n)$, combined with the fact that $\|\phi(a) - \phi(b)\|_1 \leq \Omega(1)(d_{\mathbb{R}^n/\Lambda_n}(a,b) + 1)$ for every $(a,b) \in (\mathbb{R}^n/\Lambda_n) \times (\mathbb{R}^n/\Lambda_n)$, which follows from a standard application of the triangle inequality.

The proof of (18) in [94] is Fourier-analytic, and as such it does not apply when $L_1(\mathbb{R})$ is replaced by a target which is not a Banach space. It would be very interesting if one could find a way to reason analogously about embeddings of $\mathfrak{T}$ into an Alexandrov space of nonpositive curvature. Investigating the geometry of $\mathfrak{T}$ is interesting in its own right, and some aspects of this were discussed in [129, 4]. It follows immediately from (17) that $\mathfrak{T}$ is an Alexandrov space of nonnegative curvature, and we do not know if every (classical) expander is also an expander with respect to $\mathfrak{T}$, or even if $\mathfrak{T}$ admits some expander. Determining whether $\mathfrak{T}$ has sharp metric cotype q for any $q \geq 2$ would also be worthwhile.

2.10. **Metric Kwapień–Schütt.**

Metric KS inequalities are metric invariants that were introduced in [136] (the nomenclature refers to the works [98, 99] of Kwapień and Schütt). The reasoning of [136, Section 3] shows that if a metric space satisfies such an inequality, then it does not admit a coarse embedding of $\ell_2(\ell_p)$ for any $p \in [1,2)$. Thus, $\ell_2(\ell_p)$ fails to satisfy any metric KS inequality despite the fact that its 2-barycentric constant tends to 1 as $p \to 2^-$. Also, it was shown in [136, Section 3] that if $p \in (2,\infty)$, then ℓ_p fails to satisfy any metric KS inequality even though it is p-barycentric with constant 1. We do not know how to interpret for spaces that are not Banach spaces the Fourier-analytic proof of [136, Theorem 2.1] that a Hilbert space satisfies the quadratic metric KS inequality. In particular, we do not know if Alexandrov spaces of nonpositive curvature satisfy the quadratic metric KS inequality; the above examples show that in the (in our opinion unlikely) event that this were true, then its proof must use the fact that such spaces are 2-barycentric, and moreover that their 2-barycentric constant equals 1, i.e., for any $\varepsilon > 0$ it fails for spaces that are either 2-barycentric with constant $1 + \varepsilon$, or are $(2 + \varepsilon)$-barycentric with constant 1.

2.11. **Stable metrics and interlacing graphs.**

Following Garling's definition [59], which extends the fundamental contribution of Krivine and Maurey [97] in the setting of Banach spaces, a metric space (X, d_X) is said to be stable if

$$\lim_{m\to\infty}\lim_{n\to\infty} d_X(x_m, y_n) = \lim_{n\to\infty}\lim_{m\to\infty} d_X(x_m, y_n) \tag{19}$$

for any bounded sequences $\{x_m\}_{m=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq X$ for which both iterated limits in (19) exist. It is simple to check that locally compact metric spaces and Hilbert spaces are stable; see the survey [15] for more examples and non-examples.

In [88], Kalton investigated the use of stability of metrics to rule out coarse embeddings, profoundly building on (and inspired by) a classical work of Raynaud [155] which studied stable metrics in the context of uniform embeddings. Among the results of [88], Kalton considered for each $r \in \mathbb{N}$ the following infinite graph, which we denote by $\mathsf{K}_r(\mathbb{N})$. The vertices of $\mathsf{K}_r(\mathbb{N})$ are the subsets of $\mathbb{N}$ of cardinality r. Two vertices σ, τ of $\mathsf{K}_r(\mathbb{N})$ are declared to form an edge if they interlace, i.e., if one could write $\sigma = \{m_1, \dots, m_r\}$ and $\tau = \{n_1, \dots, n_r\}$ such that either $m_1 \leq n_1 \leq m_2 \leq n_2 \leq \dots \leq m_r \leq n_r$ or $n_1 \leq m_1 \leq n_2 \leq m_2 \leq \dots \leq n_r \leq m_r$. Below, $\mathsf{K}_r(\mathbb{N})$ will always be understood to be equipped with the shortest-path metric that this graph structure induces. By a short iterative application of the definition of stability of a metric space (see [19, Lemma 9.19]), it was shown in [88] that $\bigsqcup_{r=1}^\infty \mathsf{K}_r(\mathbb{N})$ does not embed coarsely into any stable metric space.

Question 24. Does every Alexandrov space of nonpositive curvature embed coarsely into some stable metric space?

By the aforementioned result from [88], a positive answer to Question 24 would imply that $\bigsqcup_{r=1}^\infty \mathsf{K}_r(\mathbb{N})$ does not embed coarsely into any Alexandrov space of nonpositive curvature. This would yield a way to prove Theorem 1 that is entirely different from how we proceed here, though note that if $r \in \{2, 3, \dots\}$, then $\mathsf{K}_r(\mathbb{N})$ is not locally finite, and this is an inherent attribute of this approach because locally compact metric spaces are stable. Beyond its mere applicability to potentially proving Theorem 1, a positive answer to Question 24 would be important in its own right, as a nontrivial structural consequence of nonpositive curvature (which, by [4], fails for nonnegative curvature). In our opinion, if true, then this would be a fundamental result that will likely have applications elsewhere, though we suspect that the answer to Question 24 is negative, and that proving this might be quite accessible. Regardless, it would be interesting to determine if $\bigsqcup_{r=1}^\infty \mathsf{K}_r(\mathbb{N})$ can embed coarsely into some Alexandrov space of nonpositive curvature.

The above idea of [88] (partially building on [155]) inspired a series of investigations [88, 91, 90, 17, 104, 16] over recent years that led to major coarse non-embeddability results for certain Banach spaces, starting with Kalton's incorporation [88] of Ramsey-theoretic reasoning which led to (among other things) his proof in [88] that c_0 does not embed coarsely into any reflexive Banach space. We will not survey these ideas here, and only state that they rely on the linear theory in multiple ways, so their relevance to the setting of Theorem 1 is questionable.

3. Proof of Theorem 5

Prior to carrying out the proof of Theorem 5, we will quickly present a probabilistic tool on which it relies.

3.1. Nonlinear martingales.

We will next describe basic facts about martingales that don't necessarily take values in a Banach space. There are multiple ways to extend the linear theory of martingales, but we will only discuss one such approach, namely following that of [123, Section 2], on which we will rely in the proof of Theorem 5 . There is substantial literature on martingales in metric spaces (including [41, 43, 47, 168, 35]), but because our sole purpose here is to use nonlinear martingales to prove a purely geometric result rather than the (independently interesting) foundational probabilistic perspective, we will not delve into the general theory and comparison of different approaches.

Let Ω be a finite set and let $\mathcal{F} \subseteq 2^{\Omega}$ be a σ-algebra. For every $\omega \in \Omega$, define $\mathcal{F}(\omega) \subseteq \Omega$ to be the unique atom of $\mathcal{F}$ for which $\omega \in \mathcal{F}(\omega)$. Suppose that $\mu : 2^{\Omega} \to [0,1]$ is a probability measure of full support, i.e., $\mu(\{\omega\}) > 0$ for every $\omega \in \Omega$. Let X be a set equipped with a barycenter map $\mathfrak{B} : \mathcal{P}_X^{<\infty} \to X$. For a function $Z : \Omega \to X$, define its μ-conditional barycenter $\mathfrak{B}_\mu(Z|\mathcal{F}) : \Omega \to X$ by setting

$$\forall \omega \in \Omega, \qquad \mathfrak{B}_\mu\big(Z|\mathcal{F}\big)(\omega) = \mathfrak{B}\Big(\frac{1}{\mu\big(\mathcal{F}(\omega)\big)} \sum_{a \in \mathcal{F}(\omega)} \mu(a)\delta_{Z(a)}\Big). \qquad (20)$$

Fix $n \in \mathbb{N}$ and σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n \subseteq 2^{\Omega}$ (a filtration). Say that a sequence of mappings $\{Z_i : \Omega \to X\}_{i=0}^n$ (X-valued random variables) is a μ-martingale with respect to the filtration $\{\mathcal{F}_i\}_{i=0}^n$ if

$$\forall i \in \{1, \ldots, n\}, \qquad \mathfrak{B}_\mu(Z_i|\mathcal{F}_{i-1}) = Z_{i-1}. \qquad (21)$$

We will use below the following simple monotonicity property for martingales in barycentric metric space.

Lemma 25. *Fix $q \geq 1$ and $n \in \mathbb{N}$. Let (X, d_X) be a q-barycentric metric space. Suppose that μ is a probability measure of full support on a finite set Ω and that $\{Z_i : \Omega \to X\}_{i=0}^n$ is a μ-martingale with respect to a filtration $\{\mathcal{F}_i\}_{i=0}^n$. Then,*

$$\forall x \in X, \qquad \int_\Omega d_X(Z_0, x)^q \, d\mu \leq \int_\Omega d_X(Z_1, x)^q \, d\mu \leq \ldots \leq \int_\Omega d_X(Z_n, x)^q \, d\mu. \qquad (22)$$

Proof. Fix $x \in X$ and $i \in \{0, 1, \ldots, n-1\}$. Suppose that $\{A_j\}_{j=1}^k$ are the atoms of $\mathcal{F}_i$. For each $j \in \{1, \ldots, k\}$ and $\omega \in A_j$,

$$d_X(Z_i(\omega), x)^q \overset{(20) \wedge (21)}{=} d_X\Big(\mathfrak{B}\Big(\frac{1}{\mu(A_j)} \sum_{a \in A_j} \mu(a)\delta_{Z_{i+1}(a)}\Big), x\Big)^q \overset{(3)}{\leq} \frac{1}{\mu(A_j)} \sum_{a \in A_j} \mu(a) d_X(Z_{i+1}(a), x)^q.$$

Consequently,

$$\forall j \in \{1, \ldots, k\}, \qquad \int_{A_j} d_X(Z_i, x)^q \, d\mu \leq \sum_{a \in A_j} \mu(a) d_X(Z_{i+1}(a), x)^q.$$

As $\{A_j\}_{j=1}^k$ is a partition of Ω, by summing this over $j \in \{1, \ldots, k\}$ we get that $\int_\Omega d_X(Z_i, x)^q \, d\mu \leq \int_\Omega d_X(Z_{i+1}, x)^q \, d\mu$. $\square$

A key ingredient of the proof of Theorem 5 is the following proposition, which is due to [123, Lemma 2.1]; it is analogous to an important inequality of Pisier [148] for martingales with values in uniformly convex Banach spaces.

Proposition 26 (Pisier's martingale cotype inequality in q-barycentric metric spaces). *Fix $q > 0$ and $n \in \mathbb{N}$. Let (X, d_X) be a q-barycentric metric space with constant $\beta > 0$. Suppose that μ is a probability measure of full support on a finite set Ω and that $\{Z_i : \Omega \to X\}_{i=0}^n$ is a μ-martingale with respect to a filtration $\{\varnothing, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n \subseteq 2^{\Omega}$. Then,*

$$\forall x \in X, \qquad \int_\Omega d_X(Z_0, x)^q \, d\mu + \frac{1}{\beta^p} \sum_{i=1}^n \int_\Omega d_X(Z_i, Z_{i-1})^q \, d\mu \leq \int_\Omega d_X(Z_n, x)^q \, d\mu. \qquad (23)$$

3.2. Cotype.

Fix $n \in \mathbb{N}$. Denote the uniform probability measure on $\{-1, 1\}^n$ by μ. For $i \in \{1, \ldots, n\}$ let $\mathcal{F}_i \subseteq 2^{\{-1,1\}^n}$ be the σ-algebra that is generated by the coordinate functions $\varepsilon_1, \ldots, \varepsilon_i : \{-1, 1\}^n \to \{-1, 1\}$. Write also $\mathcal{F}_0 = \{\varnothing, \{-1, 1\}^n\}$.

Suppose that (X, d_X) is a metric space equipped with a barycenter map $\mathfrak{B} : \mathcal{P}_X^{<\infty} \to X$. For each $h : \{-1, 1\}^n \to X$ we recursively construct a sequence of functions $\{\mathsf{E}_i h : \{-1, 1\}^n \to X\}_{i=0}^n$ by setting $\mathsf{E}_n h = h$ and for every $i \in \{0, 1, \ldots, n-1\}$,

$$\forall \varepsilon \in \{-1, 1\}^n, \qquad (\mathsf{E}_i h)(\varepsilon) \overset{\text{def}}{=} \mathfrak{B}_\mu\big(\mathsf{E}_{i+1} h | \mathcal{F}_i\big)(\varepsilon). \qquad (24)$$

By definition, $\{\mathsf{E}_i h\}_{i=0}^n$ is a μ-martingale with respect to the filtration $\{\mathcal{F}_i\}_{i=0}^n$. In particular, for each $\varepsilon \in \{-1, 1\}^n$ the value $(\mathsf{E}_i h)(\varepsilon)$ depends only on $\varepsilon_1, \ldots, \varepsilon_i$. We will therefore sometimes write $(\mathsf{E}_i h)(\varepsilon_1, \ldots, \varepsilon_i)$ instead of $(\mathsf{E}_i h)(\varepsilon)$.

Fix $m, n \in \mathbb{N}$ and $x \in \mathbb{Z}_m^n$. For every function $f : \mathbb{Z}_m^n \to X$, denote by $f_x : \{-1, 1\}^n \to X$ the function that is given by

$$\forall \, \varepsilon \in \{-1, 1\}^n, \qquad f_x(\varepsilon) = f(x + \varepsilon). \tag{25}$$

We record here the following simple identity for ease of later reference.

Lemma 27. *Fix $m, n \in \mathbb{N}$, $x \in \mathbb{Z}_m^n$ and a fuction $f : \mathbb{Z}_m^n \to X$. For every $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, i\}$ and $\varepsilon \in \{-1, 1\}^n$, we have*

$$\left(\mathsf{E}_i f_{x - 2\varepsilon_j e_j} \right)(\varepsilon) = \left(\mathsf{E}_i f_x \right)(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_i). \tag{26}$$

Proof. The proof proceeds by reverse induction on $i \in \{0, \ldots, n\}$. For $i = n$, we have

$$\left(\mathsf{E}_n f_{x - 2\varepsilon_j e_j} \right)(\varepsilon) = f_{x - 2\varepsilon_j e_j}(\varepsilon) = f(x - 2\varepsilon_j e_j + \varepsilon) = f_x(-2\varepsilon_j e_j + \varepsilon) = \left(\mathsf{E}_n f_x \right)(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n).$$

For the inductive step, fix $i \in \{1, \ldots, n\}$ and notice that for every $j \in \{1, \ldots, i-1\}$, we have

$$\left(\mathsf{E}_{i-1} f_{x - 2\varepsilon_j e_j} \right)(\varepsilon_1, \ldots, \varepsilon_{i-1}) = \mathfrak{B} \left(\frac{1}{2} \delta_{(\mathsf{E}_i f_{x - 2\varepsilon_j e_j})(\varepsilon_1, \ldots, \varepsilon_{i-1}, -1)} + \frac{1}{2} \delta_{(\mathsf{E}_i f_{x - 2\varepsilon_j e_j})(\varepsilon_1, \ldots, \varepsilon_{i-1}, 1)} \right)$$

$$= \mathfrak{B} \left(\frac{1}{2} \delta_{(\mathsf{E}_i f_x)(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{i-1}, -1)} + \frac{1}{2} \delta_{(\mathsf{E}_i f_x)(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{i-1}, 1)} \right) = \left(\mathsf{E}_{i-1} f_x \right)(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{i-1}),$$

where the first and last equalities use the definition (24) and the middle equality uses the inductive hypothesis. $\square$

Proof of Theorem 5. Fix $m, n \in \mathbb{N}$ and a function $f : \mathbb{Z}_{4m}^n \to X$. It will be notationally convenient to prove the desired estimate (5) with m replaced by $2m$, i.e., our goal is now to show that the following inequality holds true.

$$\left(\sum_{i=1}^n \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f(x + 2m e_i), f(x) \big)^q \right)^{\frac{1}{q}} \le \left(4 n^{\frac{1}{q}} + 2 \beta m \right) \left(\frac{1}{2^n} \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f(x + \varepsilon), f(x) \big)^q \right)^{\frac{1}{q}}. \tag{27}$$

Recalling the notation (25) and using translation invariance on $\mathbb{Z}_{4m}^n$, for every $i \in \{1, \ldots, n\}$ and $\varepsilon \in \{-1, 1\}^n$ we have

$$\sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f_{x + 2m\varepsilon_i e_i}(\varepsilon), f_x(\varepsilon) \big)^q = \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f_{x + 2m e_i}(\varepsilon), f_x(\varepsilon) \big)^q = \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f(x + 2m e_i), f(x) \big)^q.$$

Hence,

$$\left(\sum_{i=1}^n \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f(x + 2m e_i), f(x) \big)^q \right)^{\frac{1}{q}} = \left(\frac{1}{2^n} \sum_{i=1}^n \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f_{x + 2m\varepsilon_i e_i}(\varepsilon), f_x(\varepsilon) \big)^q \right)^{\frac{1}{q}}. \tag{28}$$

By the triangle inequality in (X, d_X), each of the summands in the right hand side of (28) can be bounded as follows.

$$d_X \big(f_{x + 2m\varepsilon_i e_i}(\varepsilon), f_x(\varepsilon) \big) \le d_X \big(f_{x + 2m\varepsilon_i e_i}(\varepsilon), (\mathsf{E}_i f_{x + 2m\varepsilon_i e_i})(\varepsilon) \big) + d_X \big((\mathsf{E}_i f_{x + 2m\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big) + d_X \big(f_x(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big). \tag{29}$$

A substitution of (29) into (28) in combination with the triangle inequality in L_q gives the bound

$$\left(\sum_{i=1}^n \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f(x + 2m e_i), f(x) \big)^q \right)^{\frac{1}{q}}$$

$$\le 2 \left(\frac{1}{2^n} \sum_{i=1}^n \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f_x(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}} + \left(\frac{1}{2^n} \sum_{i=1}^n \sum_{\varepsilon \in \{-1, 1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_{x + 2m\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}}, \tag{30}$$

where we used the fact that, by translation invariance on $\mathbb{Z}_{4m}^n$ once more, we have

$$\sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f_{x + 2m\varepsilon_i e_i}(\varepsilon), (\mathsf{E}_i f_{x + 2m\varepsilon_i e_i})(\varepsilon) \big)^q = \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f_x(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q.$$

To bound the first term on the right hand side of (30), use the triangle inequality in (X, d_X) to deduce that for every $i \in \{1, \ldots, n\}$, every $x \in \mathbb{Z}_{4m}^n$ and every $\varepsilon \in \{-1, 1\}^n$, we have

$$d_X \big(f_x(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big) \le d_X \big(f(x + \varepsilon), f(x) \big) + d_X \big((\mathsf{E}_i f_x)(\varepsilon), f(x) \big).$$

Hence, using the triangle inequality in L_q we see that

$$\left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f_x(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}}$$

$$\leqslant \left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f(x+\varepsilon), f(x) \big)^q \right)^{\frac{1}{q}} + \left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_x)(\varepsilon), f(x) \big)^q \right)^{\frac{1}{q}} \tag{31}$$

$$= n^{\frac{1}{q}} \left(\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f(x+\varepsilon), f(x) \big)^q \right)^{\frac{1}{q}} + \left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_x)(\varepsilon), f(x) \big)^q \right)^{\frac{1}{q}}.$$

By Lemma 25, for each fixed $i \in \{1, \dots, n\}$ and $x \in \mathbb{Z}_{4m}^n$, we have

$$\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} d_X \big((\mathsf{E}_i f_x)(\varepsilon), f(x) \big)^q \leqslant \frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} d_X \big((\mathsf{E}_n f_x)(\varepsilon), f(x) \big)^q = \frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} d_X \big(f(x+\varepsilon), f(x) \big)^q. \tag{32}$$

By combining (31) and (32) we therefore conclude that

$$\left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f_x(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}} \leqslant 2 n^{\frac{1}{q}} \left(\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big(f(x+\varepsilon), f(x) \big)^q \right)^{\frac{1}{q}}. \tag{33}$$

To bound the second term on the right hand side of (30), use the triangle inequality in (X, d_X) to deduce that for every $i \in \{1, \dots, n\}$, every $\varepsilon \in \{-1,1\}^n$ and every $x \in \mathbb{Z}_{4m}^n$, we have

$$d_X \big((\mathsf{E}_i f_{x+2m\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big) \leqslant \sum_{s=1}^{m} d_X \big((\mathsf{E}_i f_{x-2s\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_{x-2(s-1)\varepsilon_i e_i})(\varepsilon) \big).$$

By combining this with the triangle inequality in L_q, we see that

$$\left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_{x+2m\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}}$$

$$\leqslant \sum_{s=1}^{m} \left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_{x-2s\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_{x-2(s-1)\varepsilon_i e_i})(\varepsilon) \big)^q \right)^{\frac{1}{q}} \tag{34}$$

$$= m \left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_{x-2\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}}.$$

where in the last step we used translation invariance on $\mathbb{Z}_{4m}^n$. Due to Lemma 27, the estimate (34) is the same as

$$\left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_{x+2m\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}}$$

$$\leqslant m \left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_x)(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}}. \tag{35}$$

Each summand in the right hand side of (35) can be bounded using the triangle inequality in (X, d_X) as follows.

$$d_X \big((\mathsf{E}_i f_x)(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i), (\mathsf{E}_i f_x)(\varepsilon) \big) \leqslant d_X \big((\mathsf{E}_i f_x)(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i), (\mathsf{E}_{i-1} f_x)(\varepsilon) \big) + d_X \big((\mathsf{E}_{i-1} f_x)(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)$$

$$= d_X \big((\mathsf{E}_i f_x)(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i), (\mathsf{E}_{i-1} f_x)(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i) \big) + d_X \big((\mathsf{E}_{i-1} f_x)(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big), \tag{36}$$

where the final step of (36) holds because $(\mathsf{E}_{i-1} f_x)(\varepsilon)$ depends only on the variables $\varepsilon_1, \dots, \varepsilon_{i-1}$. A substitution of (36) into (35) together with an application of the triangle inequality in L_q gives that

$$\left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_i f_{x+2m\varepsilon_i e_i})(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}} \leqslant 2m \left(\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X \big((\mathsf{E}_{i-1} f_x)(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \right)^{\frac{1}{q}}. \tag{37}$$

For a fixed $x \in \mathbb{Z}_{4m}^n$, Proposition 26 applied to the martingale $\{\mathsf{E}_i f_x\}_{i=0}^{n}$ (on $\{-1,1\}^n$) gives the estimate

$$\frac{1}{2^n} \sum_{i=1}^{n} \sum_{\varepsilon \in \{-1,1\}^n} d_X \big((\mathsf{E}_{i-1} f_x)(\varepsilon), (\mathsf{E}_i f_x)(\varepsilon) \big)^q \leqslant \frac{\beta^q}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} d_X \big(f(x+\varepsilon), f(x) \big)^q.$$

Due to (37), we therefore have

$$\left(\frac{1}{2^n}\sum_{i=1}^n\sum_{\varepsilon\in\{-1,1\}^n}\sum_{x\in\mathbb{Z}_{4m}^n}d_X\big((\mathsf{E}_if_{x+2m\varepsilon_ie_i})(\varepsilon),(\mathsf{E}_if_x)(\varepsilon)\big)^q\right)^{\frac{1}{q}}\leqslant 2\beta m\left(\frac{1}{2^n}\sum_{\varepsilon\in\{-1,1\}^n}\sum_{x\in\mathbb{Z}_{4m}^n}d_X\big(f(x+\varepsilon),f(x)\big)^q\right)^{\frac{1}{q}}. \tag{38}$$

The desired estimate (27) now follows by substituting (33) and (38) into (30). $\qquad\square$

4. The definitions of metric cotype with ℓ_∞ edges and sign edges coincide

Let (X,d_X) be an arbitrary metric space. For every $q\in[1,\infty)$ and $m,n\in\mathbb{N}$ denote by $\mathscr{C}_{q,m,n}^{\{-1,0,1\}}(X,d_X)$ the infimum over those $\mathscr{C}\in(0,\infty)$ such that for every $f:\mathbb{Z}_{2m}^n\to X$ we have

$$\left(\sum_{i=1}^n\sum_{x\in\mathbb{Z}_{2m}^n}d_X\big(f(x+me_i),f(x)\big)^q\right)^{\frac{1}{q}}\leqslant\mathscr{C}m\left(\frac{1}{3^n}\sum_{\varepsilon\in\{-1,0,1\}^n}\sum_{x\in\mathbb{Z}_{2m}^n}d_X\big(f(x+\varepsilon),f(x)\big)^q\right)^{\frac{1}{q}}. \tag{39}$$

Analogously, denote by $\mathscr{C}_{q,m,n}^{\{-1,1\}}(X,d_X)$ the infimum over those $\mathscr{C}_*\in(0,\infty)$ such that for every $f:\mathbb{Z}_{2m}^n\to X$ we have

$$\left(\sum_{i=1}^n\sum_{x\in\mathbb{Z}_{2m}^n}d_X\big(f(x+me_i),f(x)\big)^q\right)^{\frac{1}{q}}\leqslant\mathscr{C}_*m\left(\frac{1}{2^n}\sum_{\varepsilon\in\{-1,1\}^n}\sum_{x\in\mathbb{Z}_{2m}^n}d_X\big(f(x+\varepsilon),f(x)\big)^q\right)^{\frac{1}{q}}. \tag{40}$$

The estimate (39) corresponds to the original definition of metric cotype in [121], while here we considered the variant (40), in which the averaging in the right hand side is over the 2^n possible sign vectors in $\{-1,1\}^n$ rather than over the 3^n possible ℓ_∞-edges in $\{-1,0,1\}^n$. We will now show that these definitions are essentially equivalent, up to universal constant factors, by establishing the two estimates in (41) below, which hold for every metric space (X,d_X), every $q\in[1,\infty)$ and every $m,n\in\mathbb{N}$. This confirms a prediction of [135, Section 5.2], where a special case was treated. Despite this fact, one should note that (39) is to some extent a more natural inequality than (40), because the vectors $\{-1,0,1\}^n$ generate $\mathbb{Z}_{2m}^n$ while the vectors $\{-1,1\}^n$ do not. Therefore, unlike the right hand side of (40), the right hand side of (39) corresponds to an "L_q metric gradient" on a connected (Cayley) graph. We shall next establish that

$$\mathscr{C}_{q,m,n}^{\{-1,0,1\}}(X,d_X)\leqslant 6^{\frac{1}{q}}\max_{k\in\{1,\dots,n\}}\mathscr{C}_{q,m,k}^{\{-1,1\}}(X,d_X)\qquad\text{and}\qquad\mathscr{C}_{q,2m,n}^{\{-1,1\}}(X,d_X)\leqslant 2\mathscr{C}_{q,m,n}^{\{-1,0,1\}}(X,d_X). \tag{41}$$

Thus, using the assumptions and notation of Theorem 5, its conclusion implies that also $\mathscr{C}_{q,2m,n}^{\{-1,0,1\}}(X,d_X)\lesssim n^{1/q}+\beta m$.

The proof of the first inequality in (41) is via the following simple reasoning; see also [121, Lemma 2.7]. Denote

$$\mathscr{C}_*\stackrel{\text{def}}{=}\max_{k\in\{1,\dots,n\}}\mathscr{C}_{q,m,k}^{\{-1,1\}}(X,d_X). \tag{42}$$

If $A\subseteq\{1,\dots,n\}$, then consider $\mathbb{Z}_{2m}^A$ to be a subset of $\mathbb{Z}_{2m}^n$, by identifying $y\in\mathbb{Z}_{2m}^A$ with $\sum_{i\in A}y_ie_i\in\mathbb{Z}_{2m}^n$. For $A\subseteq\{1,\dots,n\}$ and $w\in\mathbb{Z}_{2m}^{\{1,\dots,n\}\smallsetminus A}$, define an auxiliary function $f_{A,w}:\mathbb{Z}_{2m}^A\to X$ by setting $f_{A,w}(y)=f(y+w)$ for every $y\in\mathbb{Z}_{2m}^A$. Recalling (42), an application of (40) to $f_{A,w}$ (with n replaced by $|A|$) gives the following estimate.

$$\sum_{i\in A}\sum_{y\in\mathbb{Z}_{2m}^A}d_X\big(f(y+w+me_i),f(y+w)\big)^q=\sum_{i\in A}\sum_{y\in\mathbb{Z}_{2m}^A}d_X\big(f_{A,w}(y+me_i),f_{A,w}(y)\big)^q$$

$$\leqslant\frac{\mathscr{C}_*^q}{2^{|A|}}m^q\sum_{\delta\in\{-1,1\}^A}\sum_{y\in\mathbb{Z}_{2m}^A}d_X\big(f_{A,w}(y+\delta),f_{A,w}(y)\big)^q=\frac{\mathscr{C}_*^q}{2^{|A|}}m^q\sum_{\delta\in\{-1,1\}^A}\sum_{y\in\mathbb{Z}_{2m}^A}d_X\big(f(y+w+\delta),f(y+w)\big)^q. \tag{43}$$

It remains to observe that

$$\sum_{A\subseteq\{1,\dots,n\}}\sum_{w\in\mathbb{Z}_{2m}^{\{1,\dots,n\}\smallsetminus A}}2^{|A|}\sum_{i\in A}\sum_{y\in\mathbb{Z}_{2m}^A}d_X\big(f(y+w+me_i),f(y+w)\big)^q=\sum_{i=1}^n\left(\sum_{\substack{A\subseteq\{1,\dots,n\}\\i\in A}}2^{|A|}\right)\sum_{x\in\mathbb{Z}_{2m}^n}d_X\big(f(x+me_i),f(x)\big)^q$$

$$=\sum_{i=1}^n\left(\sum_{k=1}^n\binom{n-1}{k-1}2^k\right)\sum_{x\in\mathbb{Z}_{2m}^n}d_X\big(f(x+me_i),f(x)\big)^q=\frac{3^{n-1}}{2}\sum_{i=1}^n\sum_{x\in\mathbb{Z}_{2m}^n}d_X\big(f(x+me_i),f(x)\big)^q, \tag{44}$$

and

$$\sum_{A\subseteq\{1,\dots,n\}}\sum_{w\in\mathbb{Z}_{2m}^{\{1,\dots,n\}\smallsetminus A}}\sum_{\delta\in\{-1,1\}^A}\sum_{y\in\mathbb{Z}_{2m}^A}d_X\big(f(y+w+\delta),f(y+w)\big)^q=\sum_{\varepsilon\in\{-1,0,1\}^n}\sum_{x\in\mathbb{Z}_{2m}^n}d_X\big(f(x+\varepsilon),f(x)\big)^q. \tag{45}$$

By multiplying (43) by $2^{|A|}$, summing the resulting bound over all $A \subseteq \{1, \ldots, n\}$ and $w \in \mathbb{Z}_{2m}^{\{1,\ldots,n\} \smallsetminus A}$, and using the identities (44) and (45), we thus obtain the first inequality in (41). Note that this deduction was entirely combinatorial and did not use the triangle inequality, but the proof of the second inequality in (41) does use (a modicum of) geometry.

For the second inequality in (41), write $\mathscr{C} \stackrel{\text{def}}{=} \mathscr{C}_{q,m,n}^{\{-1,0,1\}}(X, d_X)$. Fix $f : \mathbb{Z}_{4m}^n \to X$. For each $\eta \in \{0,1\}^n$ define an auxiliary function $\phi_\eta : \mathbb{Z}_{2m}^n \to X$ by setting $\phi_\eta(y) = f(2y + \eta)$ for every $y \in \mathbb{Z}_{2m}^n$ (for this, one should note that the mapping $y \mapsto 2y$ is well-defined as a mapping from $\mathbb{Z}_{2m}^n$ to $\mathbb{Z}_{4m}^n$). An application of (39) to each of the 2^n functions $\{\phi_\eta\}_{\eta \in \{0,1\}}$ gives

$$
\sum_{i=1}^n \sum_{y \in \mathbb{Z}_{2m}^n} d_X\big(f(2y + \eta + 2me_i), f(2y + \eta)\big)^q = \sum_{i=1}^n \sum_{y \in \mathbb{Z}_{2m}^n} d_X\big(\phi_\eta(y + me_i), \phi_\eta(y)\big)^q
$$
$$
\leqslant \frac{\mathscr{C}^q}{3^n} m^q \sum_{\varepsilon \in \{-1,0,1\}^n} \sum_{y \in \mathbb{Z}_{2m}^n} d_X\big(\phi_\eta(y + \varepsilon), \phi_\eta(y)\big)^q = \frac{\mathscr{C}^q}{3^n} m^q \sum_{\varepsilon \in \{-1,0,1\}^n} \sum_{y \in \mathbb{Z}_{2m}^n} d_X\big(f(2y + \eta + 2\varepsilon), f(2y + \eta)\big)^q.
$$
(46)

The mapping $(y, \eta) \mapsto 2y + \eta$ is a bijection between $\mathbb{Z}_{2m}^n \times \{0,1\}^n$ and $\mathbb{Z}_{4m}^n$. So, by summing (46) over $\eta \in \{0,1\}^n$ we get

$$
\sum_{i=1}^n \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + 2me_i), f(x)\big)^q \leqslant \frac{\mathscr{C}^q}{3^n} \sum_{\varepsilon \in \{-1,0,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + 2\varepsilon), f(x)\big)^q.
$$
(47)

For every $A \subseteq \{1, \ldots, n\}$ and $v \in \mathbb{Z}_{2m}^n$, denote its restriction to $\mathbb{Z}_{2m}^A$ by $v_A = \sum_{i \in A} v_i e_i$. Observe that

$$
\sum_{\varepsilon \in \{-1,0,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + 2\varepsilon), f(x)\big)^q = \sum_{A \subseteq \{1,\ldots,n\}} \frac{1}{2^{n-|A|}} \sum_{\delta \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + 2\delta_A), f(x)\big)^q.
$$
(48)

For every fixed $(x, \delta) \in \mathbb{Z}_{4m}^n \times \{-1,1\}^n$ and every fixed $A \subseteq \{1, \ldots, n\}$, the triangle inequality in (X, d_X) combined with the convexity of the mapping $t \mapsto t^q$ on $[0, \infty)$ gives that

$$
d_X\big(f(x + 2\delta_A), f(x)\big)^q \leqslant 2^{q-1} d_X\big(f(x + 2\delta_A), f(x + \delta)\big)^q + 2^{q-1} d_X\big(f(x + \delta), f(x)\big)^q
$$
$$
= 2^{q-1} d_X\big(f(x + \delta), f(x + \delta + \delta_A - \delta_{\{1,\ldots,n\} \smallsetminus A})\big)^q + 2^{q-1} d_X\big(f(x + \delta), f(x)\big)^q.
$$
(49)

Summing (49) over $x \in \mathbb{Z}_{4m}^n$, while keeping $\delta \in \{-1,1\}^n$ and $A \subseteq \{1, \ldots, n\}$ fixed and using translation invariance, gives

$$
\sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + 2\delta_A), f(x)\big)^q \leqslant 2^{q-1} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x), f(x + \delta_A - \delta_{\{1,\ldots,n\} \smallsetminus A})\big)^q + 2^{q-1} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + \delta), f(x)\big)^q.
$$
(50)

Since for a fixed $A \subseteq \{1, \ldots, n\}$, if $\delta \in \{-1,1\}^n$ is equi-distributed over $\{-1,1\}^n$, then so is $\delta_A - \delta_{\{1,\ldots,n\} \smallsetminus A}$, by (50) we have

$$
\sum_{\delta \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + 2\delta_A), f(x)\big)^q \leqslant 2^q \sum_{\delta \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + \delta), f(x)\big)^q.
$$

A substitution of this bound into (48) gives that

$$
\sum_{\varepsilon \in \{-1,0,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + 2\varepsilon), f(x)\big)^q \leqslant 2^q \left(\sum_{k=1}^n \frac{\binom{n}{k}}{2^{n-k}} \right) \sum_{\delta \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + \delta), f(x)\big)^q
$$
$$
= \frac{2^q 3^n}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \sum_{x \in \mathbb{Z}_{4m}^n} d_X\big(f(x + \varepsilon), f(x)\big)^q.
$$
(51)

By substituting (51) into (47), we thus obtain the second inequality in (41).

5. Remarks on the barycentric condition

Fix $q, \beta > 0$ and suppose that (X, d_X) is a q-barycentric metric space with constant β, with respect to the barycenter map $\mathfrak{B} : \mathscr{P}_X^{<\infty} \to X$. If $a, b \in X$ are distinct points, then two applications of the definition (3) to the probability measure $\mu_{a,b} = (\delta_a + \delta_b)/2$ shows that $d_X(\mathfrak{B}(\mu_{a,b}), a)^q, d_X(\mathfrak{B}(\mu_{a,b}), b)^q \leqslant d_X(a, b)^q/2$; for this, we do not need the second term in the left hand side of (3). Hence, $d_X(a, b) \leqslant d_X(\mathfrak{B}(\mu_{a,b}), a) + d_X(\mathfrak{B}(\mu_{a,b}), b) \leqslant 2^{1-1/q} d_X(a, b)$, and therefore $q \geqslant 1$. This is the reason why we assume $q \geqslant 1$ throughout, thus permitting the use of convexity in various steps.

We do not know if there exists a q-barycentric metric space for some $q \in [1, 2)$, but we have the following statement.

Proposition 28. *Suppose that* $q \in [1, 2)$ *and* (X, d_X) *is a non-singleton q-barycentric metric space. Then there is a metric* $\rho : X \times X \to [0, \infty)$ *and* $\theta \in (0, 1)$ *such that the metric spaces* (X, d_X) *and* (X, ρ^θ) *are bi-Lipschitz equivalent to each other.*

It follows in particular that a metric space (X, d_X) as in Proposition 28 cannot contain any rectifiable curve. Thus, any q-barycentric metric space which contains a geodesic segment must satisfy $q \geqslant 2$ (this is so, in particular, for geodesic metric spaces such as Alexandrov spaces). Note that there do exist barycentric metric spaces that do not contain any rectifiable curve, as exhibited by, say, the real line equipped with the metric given by $\sqrt{|s-t|}$ for all $s, t \in \mathbb{R}$, which is 4-barycentric (since the real line itself is 2-barycentric).

Proof of Proposition 28. By a "de-snowflaking" result of Laakso [175, Theorem 7.2], if for no metric $\rho : X \times X \to [0, \infty)$ and $\theta \in (0, 1)$ the desired conclusion holds, then for each $\varepsilon \in (0, 1)$ there would be a metric d_ε on the disjoint union $X \sqcup [-1, 1]$ and (a scaling factor) $\lambda_\varepsilon \in (0, \infty)$ with the following properties. Firstly, $d_\varepsilon(s, t) = |s - t|$ for all $s, t \in [-1, 1]$. Secondly, $d_\varepsilon(x, y) = \lambda_\varepsilon d_X(x, y)$ for all $x, y \in X$. Finally, for all $s \in [-1, 1]$ there exists $\sigma(s, \varepsilon) \in X$ such that $d_\varepsilon(\sigma(s, \varepsilon), s) \leqslant \varepsilon$.

Suppose that (X, d_X) is q-barycentric with constant β with respect to the barycenter map $\mathfrak{B} : \mathscr{P}_X^{<\infty} \to X$. Denote

$$\forall\, s \in [\varepsilon, 1], \qquad \mu_\varepsilon^s = \frac{1}{2}\delta_{\sigma(-s,\varepsilon)} + \frac{1}{2}\delta_{\sigma(s,\varepsilon)} \in \mathscr{P}_X^{<\infty}.$$

Since $q \geqslant 1$, we can use convexity to bound from below the second term in the left hand side of (3) when $\mu = \mu_\varepsilon^s$ by

$$\left(\int_X d_X\big(\mathfrak{B}(\mu_\varepsilon^s), y\big)^q \,\mathrm{d}\mu_\varepsilon^s(y) \right)^{\frac{1}{q}} \geqslant \frac{d_X\big(\mathfrak{B}(\mu_\varepsilon^s), \sigma(-s, \varepsilon)\big) + d_X\big(\mathfrak{B}(\mu_\varepsilon^s), \sigma(s, \varepsilon)\big)}{2} \geqslant \frac{d_X\big(\sigma(s, \varepsilon), \sigma(-s, \varepsilon)\big)}{2}$$

$$= \frac{d_\varepsilon\big(\sigma(s, \varepsilon), \sigma(-s, \varepsilon)\big)}{2\lambda_\varepsilon} \geqslant \frac{d_\varepsilon(-s, s) - d_\varepsilon\big(-s, \sigma(-s, \varepsilon)\big) - d_\varepsilon\big(s, \sigma(s, \varepsilon)\big)}{2\lambda_\varepsilon} \geqslant \frac{s - \varepsilon}{\lambda_\varepsilon}. \tag{52}$$

A substitution of (52) into two applications of (3) gives the estimates

$$d_X\big(\mathfrak{B}(\mu_\varepsilon^s), \sigma(-1, \varepsilon)\big)^q + \frac{(s - \varepsilon)^q}{(\beta\lambda_\varepsilon)^q} \leqslant \frac{d_X\big(\sigma(-1, \varepsilon), \sigma(-s, \varepsilon)\big)^q + d_X\big(\sigma(-1, \varepsilon), \sigma(s, \varepsilon)\big)^q}{2}, \tag{53}$$

and

$$d_X\big(\mathfrak{B}(\mu_\varepsilon^s), \sigma(1, \varepsilon)\big)^q + \frac{(s - \varepsilon)^q}{(\beta\lambda_\varepsilon)^q} \leqslant \frac{d_X\big(\sigma(1, \varepsilon), \sigma(-s, \varepsilon)\big)^q + d_X\big(\sigma(1, \varepsilon), \sigma(s, \varepsilon)\big)^q}{2}. \tag{54}$$

By averaging (53) and (54) and using convexity ($q \geqslant 1$) followed by the triangle inequality, we see that

$$\frac{d_X\big(\sigma(-1, \varepsilon), \sigma(-s, \varepsilon)\big)^q + d_X\big(\sigma(-1, \varepsilon), \sigma(s, \varepsilon)\big)^q + d_X\big(\sigma(1, \varepsilon), \sigma(-s, \varepsilon)\big)^q + d_X\big(\sigma(1, \varepsilon), \sigma(s, \varepsilon)\big)^q}{4}$$

$$\geqslant \left(\frac{d_X\big(\mathfrak{B}(\mu_\varepsilon^s), \sigma(-1, \varepsilon)\big) + d_X\big(\mathfrak{B}(\mu_\varepsilon^s), \sigma(1, \varepsilon)\big)}{2} \right)^q + \frac{(s - \varepsilon)^q}{(\beta\lambda_\varepsilon)^q} \geqslant \frac{d_X\big(\sigma(-1, \varepsilon), \sigma(1, \varepsilon)\big)^q}{2^q} + \frac{(s - \varepsilon)^q}{(\beta\lambda_\varepsilon)^q}. \tag{55}$$

Next, we have

$$d_X\big(\sigma(-1, \varepsilon), \sigma(1, \varepsilon)\big) = \frac{d_\varepsilon\big(\sigma(-1, \varepsilon), \sigma(1, \varepsilon)\big)}{\lambda_\varepsilon} \geqslant \frac{d_\varepsilon(-1, 1) - d_\varepsilon\big(-1, \sigma(-1, \varepsilon)\big) - d_\varepsilon\big(1, \sigma(1, \varepsilon)\big)}{\lambda_\varepsilon} \geqslant \frac{2(1 - \varepsilon)}{\lambda_\varepsilon}.$$

Also,

$$d_X\big(\sigma(-1, \varepsilon), \sigma(-s, \varepsilon)\big) = \frac{d_\varepsilon\big(\sigma(-1, \varepsilon), \sigma(-s, \varepsilon)\big)}{\lambda_\varepsilon} \leqslant \frac{d_\varepsilon(-1, -s) + d_\varepsilon\big(-1, \sigma(-1, \varepsilon)\big) + d_\varepsilon\big(-s, \sigma(-s, \varepsilon)\big)}{\lambda_\varepsilon} \leqslant \frac{1 - s + 2\varepsilon}{\lambda_\varepsilon}.$$

Analogously, $d_X\big(\sigma(-1, \varepsilon), \sigma(s, \varepsilon)\big), d_X\big(\sigma(1, \varepsilon), \sigma(-s, \varepsilon)\big) \leqslant (1 + s + 2\varepsilon)/\lambda_\varepsilon$ and $d_X\big(\sigma(1, \varepsilon), \sigma(s, \varepsilon)\big) \leqslant (1 - s + 2\varepsilon)/\lambda_\varepsilon$. A substitution of these estimates into (55) yields the bound $2(1 - \varepsilon)^q + 2(s - \varepsilon)^q/\beta^q \leqslant (1 - s + 2\varepsilon)^q + (1 + s + 2\varepsilon)^q$, which holds for every $0 < \varepsilon \leqslant s \leqslant 1$. By taking $\varepsilon \to 0$, we see that $2s^q/\beta^q \leqslant (1 + s)^q + (1 - s)^q - 2 \lesssim s^2$. Hence necessarily $q \geqslant 2$. $\qquad\square$

References

[1] I. Aharoni, B. Maurey, and B. S. Mityagin. Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces. *Israel J. Math.*, 52(3):251–265, 1985.

[2] N. Alon, A. Lubotzky, and A. Wigderson. Semi-direct product in groups and zig-zag product in graphs: connections and applications (extended abstract). In *42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001)*, pages 630–637. IEEE Computer Soc., Los Alamitos, CA, 2001.

[3] N. Alon and V. D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory Ser. B*, 38(1):73–88, 1985.

[4] A. Andoni, A. Naor, and O. Neiman. Snowflake universality of Wasserstein spaces. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3):657–700, 2018.

[5] A. Andoni, A. Naor, A. Nikolov, I. Razenshteyn, and E. Waingarten. Locality sensitive hashing via nonlinear spectral gaps. In *STOC'18— Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 787–800. ACM, New York, 2018.

[6] G. Arzhantseva and T. Delzant. Examples of random groups. To appear in *J. Topol.*, preprint available at `www.mat.univie.ac.at/~arjantseva/Abs/random.pdf`, 2011.

[7] G. Arzhantseva and R. Tessera. Relative expanders. *Geom. Funct. Anal.*, 25(2):317–341, 2015.

[8] T. Austin and A. Naor. On the bi-Lipschitz structure of Wasserstein spaces. Preprint, 2017.

[9] T. Austin, A. Naor, and R. Tessera. Sharp quantitative nonembeddability of the Heisenberg group into superreflexive Banach spaces. *Groups Geom. Dyn.*, 7(3):497–522, 2013.

[10] P. Awasthi, M. Charikar, R. Krishnaswamy, and A. K. Sinop. Spectral embedding of k-cliques, graph partitioning and k-means. In *ITCS'16—Proceedings of the 2016 ACM Conference on Innovations in Theoretical Computer Science*, pages 301–310. ACM, New York, 2016.

[11] K. Ball. Markov chains, Riesz transforms and Lipschitz maps. *Geom. Funct. Anal.*, 2(2):137–172, 1992.

[12] K. Ball. The Ribe programme. *Astérisque*, (352):Exp. No. 1047, viii, 147–159, 2013. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.

[13] K. Ball, E. A. Carlen, and E. H. Lieb. Sharp uniform convexity and smoothness inequalities for trace norms. *Invent. Math.*, 115(3):463–482, 1994.

[14] Y. Bartal and L. Gottlieb. Approximate nearest neighbor search for ℓ_p-spaces $(2 < p < \infty)$ via embeddings. In *Proceedings of LATIN 2018*, volume 10807 of *Lecture Notes in Computer Science*, pages 120–133. Springer, 2018.

[15] F. Baudier. On the metric geometry of stable metric spaces. Preprint available at `https://arxiv.org/abs/1409.7738`, 2014.

[16] F. Baudier, G. Lancien, P. Motakis, and Th. Schlumprecht. A new coarsely rigid class of Banach spaces. Preprint available at `https://arxiv.org/abs/1806.00702`, 2018.

[17] F. Baudier, G. Lancien, and Th. Schlumprecht. The coarse geometry of Tsirelson's space and applications. *J. Amer. Math. Soc.*, 31(3):699–717, 2018.

[18] M. Bačác. Old and new challenges in Hadamard spaces. Preprint available at `https://arxiv.org/abs/1807.01355`, 2018.

[19] Y. Benyamini and J. Lindenstrauss. *Geometric nonlinear functional analysis. Vol. 1*, volume 48 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2000.

[20] I. D. Berg and I. G. Nikolaev. Quasilinearization and curvature of Aleksandrov spaces. *Geom. Dedicata*, 133:195–218, 2008.

[21] I. D. Berg and I. G. Nikolaev. Characterization of Aleksandrov spaces of curvature bounded above by means of the metric Cauchy-Schwarz inequality. *Michigan Math. J.*, 67(2):289–332, 2018.

[22] J. Bourgain. A Hausdorff-Young inequality for B-convex Banach spaces. *Pacific J. Math.*, 101(2):255–262, 1982.

[23] J. Bourgain. The metrical interpretation of superreflexivity in Banach spaces. *Israel J. Math.*, 56(2):222–230, 1986.

[24] J. Bourgain. On lattice packing of convex symmetric sets in $\mathbb{R}^n$. In *Geometrical aspects of functional analysis (1985/86)*, volume 1267 of *Lecture Notes in Math.*, pages 5–12. Springer, Berlin, 1987.

[25] J. Bourgain. Remarks on the extension of Lipschitz maps defined on discrete sets and uniform homeomorphisms. In *Geometrical aspects of functional analysis (1985/86)*, volume 1267 of *Lecture Notes in Math.*, pages 157–167. Springer, Berlin, 1987.

[26] J. Bourgain, V. Milman, and H. Wolfson. On type of metric spaces. *Trans. Amer. Math. Soc.*, 294(1):295–317, 1986.

[27] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[28] N. Brown and E. Guentner. Uniform embeddings of bounded geometry spaces into reflexive Banach space. *Proc. Amer. Math. Soc.*, 133(7):2045–2050, 2005.

[29] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[30] Y. Burago, M. Gromov, and G. Perelman. A. D. Aleksandrov spaces with curvatures bounded below. *Uspekhi Mat. Nauk*, 47(2(284)):3–51, 222, 1992.

[31] G. J. Butler. Simultaneous packing and covering in euclidean space. *Proc. London Math. Soc. (3)*, 25:721–735, 1972.

[32] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. Princeton Univ. Press, Princeton, N. J., 1970.

[33] Q. Cheng. Sphere equivalence, property H, and Banach expanders. *Studia Math.*, 233(1):67–83, 2016.

[34] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette. *Groups with the Haagerup property*. Modern Birkhäuser Classics. Birkhäuser/Springer, Basel, 2001. Gromov's a-T-menability, Paperback reprint of the 2001 edition [MR1852148].

[35] T. Christiansen and K. T. Sturm. Expectations and martingales in metric spaces. *Stochastics*, 80(1):1–17, 2008.

[36] J. A. Clarkson. Uniformly convex spaces. *Trans. Amer. Math. Soc.*, 40(3):396–414, 1936.

[37] M. de la Salle. Towards strong Banach property (T) for SL$(3, \mathbb{R})$. *Israel J. Math.*, 211(1):105–145, 2016.

[38] T. de Laat and M. de la Salle. Strong property (T) for higher-rank simple Lie groups. *Proc. Lond. Math. Soc. (3)*, 111(4):936–966, 2015.

[39] T. de Laat and F. Vigolo. Superexpanders from group actions on compact manifolds. To appear in *Geom. Dedicata*. Preprint available at `https://arxiv.org/abs/1707.01399`, 2017.

[40] J. Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

[41] S. Doss. Moyennes conditionnelles et martingales dans un espace métrique. *C. R. Acad. Sci. Paris*, 254:3630–3632, 1962.

[42] A. N. Dranishnikov, G. Gong, V. Lafforgue, and G. Yu. Uniform embeddings into Hilbert space and a question of Gromov. *Canad. Math. Bull.*, 45(1):60–70, 2002.

[43] M. Émery. *Stochastic calculus in manifolds*. Universitext. Springer-Verlag, Berlin, 1989. With an appendix by P.-A. Meyer.

[44] P. Enflo. On a problem of Smirnov. *Ark. Mat.*, 8:107–109, 1969.

[45] P. Enflo. On the nonexistence of uniform homeomorphisms between L_p-spaces. *Ark. Mat.*, 8:103–105, 1969.

[46] P. Enflo. On infinite-dimensional topological groups. In *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, pages Exp. No. 10–11, 11. École Polytech., Palaiseau, 1978.

[47] A. Es-Sahib and H. Heinich. Barycentre canonique pour un espace métrique à courbure négative. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 355–370. Springer, Berlin, 1999.

[48] A. Eskenazis, M. Mendel, and A. Naor. Diamond convexity: a bifurcation in the Ribe program. Preprint, 2018.

[49] A. Eskenazis and A. Naor. On coarse and uniform embeddings into L_p. Preprint, 2018.

[50] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucí a, J. Pelant, and V. Zizler. *Functional analysis and infinite-dimensional geometry*, volume 8 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, 2001.

[51] J. Faraut and K. Harzallah. Distances hilbertiennes invariantes sur un espace homogène. *Ann. Inst. Fourier (Grenoble)*, 24(3):171–217, 1974.

[52] S. C. Ferry, A. Ranicki, and J. Rosenberg. A history and survey of the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 7–66. Cambridge Univ. Press, Cambridge, 1995.

[53] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Math. J.*, 23(98):298–305, 1973.

[54] T. Figiel. On the moduli of convexity and smoothness. *Studia Math.*, 56(2):121–155, 1976.

[55] D. Fisher, T. Nguyen, and W. van Limbeek. Rigidity of warped cones and coarse geometry of expanders. *Adv. Math.*, 346:665–718, 2019.

[56] T. Foertsch, A. Lytchak, and V. Schroeder. Nonpositive curvature and the Ptolemy inequality. *Int. Math. Res. Not. IMRN*, (22):Art. ID rnm100, 15, 2007.

[57] K. Fujiwara and T. Toyoda. Random groups have fixed points on CAT(0) cube complexes. *Proc. Amer. Math. Soc.*, 140(3):1023–1031, 2012.

[58] R. Gangolli. Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 3:121–226, 1967.

[59] D. J. H. Garling. Stable Banach spaces, random measures and Orlicz function spaces. In *Probability measures on groups (Oberwolfach, 1981)*, volume 928 of *Lecture Notes in Math.*, pages 121–175. Springer, Berlin-New York, 1982.

[60] O. Giladi, M. Mendel, and A. Naor. Improved bounds in the metric cotype inequality for Banach spaces. *J. Funct. Anal.*, 260(1):164–194, 2011.

[61] O. Giladi and A. Naor. Improved bounds in the scaled Enflo type inequality for Banach spaces. *Extracta Math.*, 25(2):151–164, 2010.

[62] O. Giladi, A. Naor, and G. Schechtman. Bourgain's discretization theorem. *Ann. Fac. Sci. Toulouse Math. (6)*, 21(4):817–837, 2012.

[63] G. Godefroy. De Grothendieck à Naor: une promenade dans l'analyse métrique des espaces de Banach. *Gaz. Math.*, (151):13–24, 2017.

[64] G. Gong and G. Yu. Volume growth and positive scalar curvature. *Geom. Funct. Anal.*, 10(4):821–828, 2000.

[65] E. A. Gorin. On uniformly topological imbedding of metric spaces in Euclidean and in Hilbert space. *Uspehi Mat. Nauk*, 14(5 (89)):129–134, 1959.

[66] M. Gromov. Filling Riemannian manifolds. *J. Differential Geom.*, 18(1):1–147, 1983.

[67] M. Gromov. Rigid transformations groups. In *Géométrie différentielle (Paris, 1986)*, volume 33 of *Travaux en Cours*, pages 65–139. Hermann, Paris, 1988.

[68] M. Gromov. Sign and geometric meaning of curvature. *Rend. Sem. Mat. Fis. Milano*, 61:9–123 (1994), 1991.

[69] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[70] M. Gromov. Spaces and questions. *Geom. Funct. Anal.*, (Special Volume, Part I):118–161, 2000. GAFA 2000 (Tel Aviv, 1999).

[71] M. Gromov. CAT(κ)-spaces: construction and concentration. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 280(Geom. i Topol. 7):100–140, 299–300, 2001.

[72] M. Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003.

[73] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, english edition, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[74] E. Guentner, N. Higson, and S. Weinberger. The Novikov conjecture for linear groups. *Publ. Math. Inst. Hautes Études Sci.*, (101):243–268, 2005.

[75] J. Heinonen. *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001.

[76] S. Heinrich and P. Mankiewicz. Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces. *Studia Math.*, 73(3):225–251, 1982.

[77] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561, 2006.

[78] E. Hrushovski. Extending partial isomorphisms of graphs. *Combinatorica*, 12(4):411–416, 1992.

[79] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach spaces. Vol. II*, volume 67 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2017. Probabilistic methods and operator theory.

[80] J. R. Isbell. *Uniform spaces*. Mathematical Surveys, No. 12. American Mathematical Society, Providence, R.I., 1964.

[81] H. Izeki and S. Nayatani. Combinatorial harmonic maps and discrete-group actions on Hadamard spaces. *Geom. Dedicata*, 114:147–188, 2005.

[82] R. C. James. Bases and reflexivity of Banach spaces. *Ann. of Math. (2)*, 52:518–527, 1950.

[83] R. C. James. A non-reflexive Banach space isometric with its second conjugate space. *Proc. Nat. Acad. Sci. U. S. A.*, 37:174–177, 1951.

[84] R. C. James. A nonreflexive Banach space that is uniformly nonoctahedral. *Israel J. Math.*, 18:145–155, 1974.

[85] R. C. James. Nonreflexive spaces of type 2. *Israel J. Math.*, 30(1-2):1–13, 1978.

[86] R. C. James and J. Lindenstrauss. The octahedral problem for Banach spaces. pages 100–120. Various Publ. Ser., No. 24, 1975.

[87] W. B. Johnson and N. L. Randrianarivony. l_p ($p > 2$) does not coarsely embed into a Hilbert space. *Proc. Amer. Math. Soc.*, 134(4):1045–1050, 2006.

[88] N. J. Kalton. Coarse and uniform embeddings into reflexive spaces. *Q. J. Math.*, 58(3):393–414, 2007.

[89] N. J. Kalton. The nonlinear geometry of Banach spaces. *Rev. Mat. Complut.*, 21(1):7–60, 2008.

[90] N. J. Kalton. Lipschitz and uniform embeddings into ℓ_∞. *Fund. Math.*, 212(1):53–69, 2011.

[91] N. J. Kalton and N. L. Randrianarivony. The coarse Lipschitz geometry of $l_p \oplus l_q$. *Math. Ann.*, 341(1):223–237, 2008.

[92] G. Kasparov and G. Yu. The coarse geometric Novikov conjecture and uniform convexity. *Adv. Math.*, 206(1):1–56, 2006.

[93] C. Kelleher, D. Miller, T. Osborn, and A. Weston. Strongly non-embeddable metric spaces. *Topology Appl.*, 159(3):749–755, 2012.

[94] S. Khot and A. Naor. Nonembeddability theorems via Fourier analysis. *Math. Ann.*, 334(4):821–852, 2006.

[95] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, (86):115–197 (1998), 1997.

[96] T. Kondo. CAT(0) spaces and expanders. *Math. Z.*, 271(1-2):343–355, 2012.

[97] J.-L. Krivine and B. Maurey. Espaces de Banach stables. *Israel J. Math.*, 39(4):273–295, 1981.

[98] S. Kwapień and C. Schütt. Some combinatorial and probabilistic inequalities and their application to Banach space theory. *Studia Math.*, 82(1):91–106, 1985.

[99] S. Kwapień and C. Schütt. Some combinatorial and probabilistic inequalities and their application to Banach space theory. II. *Studia Math.*, 95(2):141–154, 1989.

[100] V. Lafforgue. Un renforcement de la propriété (T). *Duke Math. J.*, 143(3):559–602, 2008.

[101] V. Lafforgue. Propriété (T) renforcée banachique et transformation de Fourier rapide. *J. Topol. Anal.*, 1(3):191–206, 2009.

[102] V. Lafforgue and A. Naor. Vertical versus horizontal Poincaré inequalities on the Heisenberg group. *Israel J. Math.*, 203(1):309–339, 2014.

[103] J.-F. Lafont and S. Prassidis. Roundness properties of groups. *Geom. Dedicata*, 117:137–160, 2006.

[104] G. Lancien, C. Petitjean, and A. Procházka. On the coarse geometry of the James space. Preprint available at https://arxiv.org/abs/1805.05171, 2018.

[105] U. Lang, B. Pavlović, and V. Schroeder. Extensions of Lipschitz maps into Hadamard spaces. *Geom. Funct. Anal.*, 10(6):1527–1553, 2000.

[106] N. Lebedeva and A. Petrunin. Curvature bounded below: a definition a la Berg-Nikolaev. *Electron. Res. Announc. Math. Sci.*, 17:122–124, 2010.

[107] J. R. Lee and A. Naor. Embedding the diamond graph in L_p and dimension reduction in L_1. *Geom. Funct. Anal.*, 14(4):745–747, 2004.

[108] J. R. Lee, A. Naor, and Y. Peres. Trees and Markov convexity. *Geom. Funct. Anal.*, 18(5):1609–1659, 2009.

[109] C. J. Lennard, A. M. Tonge, and A. Weston. Generalized roundness and negative type. *Michigan Math. J.*, 44(1):37–45, 1997.

[110] S. Li. Markov convexity and nonembeddability of the Heisenberg group. *Ann. Inst. Fourier (Grenoble)*, 66(4):1615–1651, 2016.

[111] S. Li and A. Naor. Discretization and affine approximation in high dimensions. *Israel J. Math.*, 197(1):107–129, 2013.

[112] B. Liao. Strong Banach property (T) for simple algebraic groups of higher rank. *J. Topol. Anal.*, 6(1):75–105, 2014.

[113] N. Linial, E. London, and Y. Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.

[114] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[115] J. Matoušek. On embedding expanders into l_p spaces. *Israel J. Math.*, 102:189–197, 1997.

[116] B. Maurey. Type, cotype and K-convexity. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1299–1332. North-Holland, Amsterdam, 2003.

[117] B. Maurey, V. D. Milman, and N. Tomczak-Jaegermann. Asymptotic infinite-dimensional theory of Banach spaces. In *Geometric aspects of functional analysis (Israel, 1992–1994)*, volume 77 of *Oper. Theory Adv. Appl.*, pages 149–175. Birkhäuser, Basel, 1995.

[118] B. Maurey and G. Pisier. Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. *Studia Math.*, 58(1):45–90, 1976.

[119] M. Mendel and A. Naor. Some applications of Ball's extension theorem. *Proc. Amer. Math. Soc.*, 134(9):2577–2584, 2006.

[120] M. Mendel and A. Naor. Scaled Enflo type is equivalent to Rademacher type. *Bull. Lond. Math. Soc.*, 39(3):493–498, 2007.

[121] M. Mendel and A. Naor. Metric cotype. *Ann. of Math. (2)*, 168(1):247–298, 2008.

[122] M. Mendel and A. Naor. Towards a calculus for non-linear spectral gaps. In *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2010, Austin, Texas, USA, January 17-19, 2010*, pages 236–255, 2010.

[123] M. Mendel and A. Naor. Spectral calculus and Lipschitz extension for barycentric metric spaces. *Anal. Geom. Metr. Spaces*, 1:163–199, 2013.

[124] M. Mendel and A. Naor. Nonlinear spectral calculus and super-expanders. *Publ. Math. Inst. Hautes Études Sci.*, 119:1–95, 2014.

[125] M. Mendel and A. Naor. Expanders with respect to Hadamard spaces and random graphs. *Duke Math. J.*, 164(8):1471–1548, 2015.

[126] M. Mimura. Sphere equivalence, Banach expanders, and extrapolation. *Int. Math. Res. Not. IMRN*, (12):4372–4391, 2015.

[127] A. Naor. An application of metric cotype to quasisymmetric embeddings. In *Metric and differential geometry*, volume 297 of *Progr. Math.*, pages 175–178. Birkhäuser/Springer, Basel, 2012.

[128] A. Naor. An introduction to the Ribe program. *Jpn. J. Math.*, 7(2):167–233, 2012.

[129] A. Naor. Comparison of metric spectral gaps. *Anal. Geom. Metr. Spaces*, 2:1–52, 2014.

[130] A. Naor. Discrete Riesz transforms and sharp metric X_p inequalities. *Ann. of Math. (2)*, 184(3):991–1016, 2016.

[131] A. Naor. A spectral gap precludes low-dimensional embeddings. In *33rd International Symposium on Computational Geometry*, volume 77 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 50, 16. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2017.

[132] A. Naor. Metric dimension reduction: A snapshot of the Ribe program. In *Proceedings of the 2018 International Congress of Mathematicians, Rio de Janeiro, Volume I*, pages 759–838, 2018.

[133] A. Naor and Y. Rabani. On approximate nearest neighbor search in ℓ_p, $p > 2$. Unpublished manuscript, available on request, 2006.

[134] A. Naor and Y. Rabani. On Lipschitz extension from finite subsets. *Israel J. Math.*, 219(1):115–161, 2017.

[135] A. Naor and G. Schechtman. Metric X_p inequalities. *Forum Math. Pi*, 4:e3, 81, 2016.

[136] A. Naor and G. Schechtman. Pythagorean powers of hypercubes. *Ann. Inst. Fourier (Grenoble)*, 66(3):1093–1116, 2016.

[137] A. Naor and G. Schechtman. Obstructions to metric embeddings of Schatten classes. Preprint, 2018.

[138] A. Naor and L. Silberman. Poincaré inequalities, embeddings, and wild groups. *Compos. Math.*, 147(5):1546–1572, 2011.

[139] A. Naor and R. Young. Vertical perimeter versus horizontal perimeter. *Ann. of Math. (2)*, 188(1):171–279, 2018.

[140] P. W. Nowak and G. Yu. *Large scale geometry*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[141] S.-I. Ohta. Markov type of Alexandrov spaces of non-negative curvature. *Mathematika*, 55(1-2):177–189, 2009.

[142] D. Osajda. Small cancellation labellings of some infinite graphs and applications. Preprint available at `https://arxiv.org/abs/1406.5015`, 2014.

[143] M. I. Ostrovskii. *Metric embeddings*, volume 49 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2013. Bilipschitz and coarse embeddings into Banach spaces.

[144] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[145] N. Ozawa. A note on non-amenability of $\mathscr{B}(l_p)$ for $p = 1, 2$. *Internat. J. Math.*, 15(6):557–565, 2004.

[146] G. Perelman. Spaces with curvature bounded below. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 517–525. Birkhäuser, Basel, 1995.

[147] V. G. Pestov. A theorem of Hrushovski-Solecki-Vershik applied to uniform and coarse embeddings of the Urysohn metric space. *Topology Appl.*, 155(14):1561–1575, 2008.

[148] G. Pisier. Martingales with values in uniformly convex spaces. *Israel J. Math.*, 20(3-4):326–350, 1975.

[149] G. Pisier. Complex interpolation between Hilbert, Banach and operator spaces. *Mem. Amer. Math. Soc.*, 208(978):vi+78, 2010.

[150] G. Pisier. *Martingales in Banach spaces*, volume 155 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[151] G. Pisier and Q. H. Xu. Random series in the real interpolation spaces between the spaces v_p. In *Geometrical aspects of functional analysis (1985/86)*, volume 1267 of *Lecture Notes in Math.*, pages 185–209. Springer, Berlin, 1987.

[152] C. Plaut. Metric spaces of curvature $\geqslant k$. In *Handbook of geometric topology*, pages 819–898. North-Holland, Amsterdam, 2002.

[153] S. Prassidis and A. Weston. Manifestations of nonlinear roundness in analysis, discrete geometry and topology. In *Limits of graphs in group theory and computer science*, pages 141–170. EPFL Press, Lausanne, 2009.

[154] N. L. Randrianarivony. Characterization of quasi-Banach spaces which coarsely embed into a Hilbert space. *Proc. Amer. Math. Soc.*, 134(5):1315–1317, 2006.

[155] Y. Raynaud. Espaces de Banach superstables, distances stables et homéomorphismes uniformes. *Israel J. Math.*, 44(1):33–52, 1983.

[156] O. Reingold, S. Vadhan, and A. Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Ann. of Math. (2)*, 155(1):157–187, 2002.

[157] M. Ribe. On uniformly homeomorphic normed spaces. *Ark. Mat.*, 14(2):237–244, 1976.

[158] G. Robertson. Crofton formulae and geodesic distance in hyperbolic spaces. *J. Lie Theory*, 8(1):163–172, 1998.

[159] J. Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[160] J. Roe. Warped cones and property A. *Geom. Topol.*, 9:163–178, 2005.

[161] C. A. Rogers. A note on coverings and packings. *J. London Math. Soc.*, 25:327–331, 1950.

[162] T. Sato. An alternative proof of Berg and Nikolaev's characterization of CAT(0)-spaces via quadrilateral inequality. *Arch. Math. (Basel)*, 93(5):487–490, 2009.

[163] D. Sawicki. Super-expanders and warped cones. Preprint available at `https://arxiv.org/abs/1704.03865`, 2017.

[164] I. J. Schoenberg. Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.*, 44(3):522–536, 1938.

[165] Z. Sela. Uniform embeddings of hyperbolic groups in Hilbert spaces. *Israel J. Math.*, 80(1-2):171–181, 1992.

[166] S. Solecki. Extending partial isometries. *Israel J. Math.*, 150:315–331, 2005.

[167] K. T. Sturm. Metric spaces of lower bounded curvature. *Exposition. Math.*, 17(1):35–47, 1999.

[168] K.-T. Sturm. Nonlinear martingale theory for processes with values in metric spaces of nonpositive curvature. *Ann. Probab.*, 30(3):1195–1222, 2002.

[169] K.-T. Sturm. Probability measures on metric spaces of nonpositive curvature. In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, volume 338 of *Contemp. Math.*, pages 357–390. Amer. Math. Soc., Providence, RI, 2003.

[170] K.-T. Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.

[171] T. Toyoda. CAT(0) spaces on which a certain type of singularity is bounded. *Kodai Math. J.*, 33(3):398–415, 2010.

[172] T. Toyoda. Fixed point property for a CAT(0) space which admits a proper cocompact group action. *Kodai Math. J.*, 39(1):129–153, 2016.

[173] B. S. Tsirelson. It is impossible to imbed ℓ_p or c_0 into an arbitrary Banach space. *Funkcional. Anal. i Priložen.*, 8(2):57–60, 1974.

[174] P. Tukia and J. Väisälä. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):97–114, 1980.

[175] J. T. Tyson and J.-M. Wu. Characterizations of snowflake metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 30(2):313–336, 2005.

[176] E. Veomett and K. Wildrick. Spaces of small metric cotype. *J. Topol. Anal.*, 2(4):581–597, 2010.

[177] A. M. Vershik. Globalization of the partial isometries of metric spaces and local approximation of the group of isometries of Urysohn space. *Topology Appl.*, 155(14):1618–1626, 2008.

[178] F. Vigolo. Measure expanding actions, expanders and warped cones. *Trans. Amer. Math. Soc.*, 371(3):1951–1979, 2019.

[179] C. Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

[180] M.-T. Wang. A fixed point theorem of discrete group actions on Riemannian manifolds. *J. Differential Geom.*, 50(2):249–267, 1998.

[181] G. Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.

[182] G. Yu. Hyperbolic groups admit proper affine isometric actions on l^p-spaces. *Geom. Funct. Anal.*, 15(5):1144–1151, 2005.

[183] G. Yu. Higher index theory of elliptic operators and geometry of groups. In *International Congress of Mathematicians. Vol. II*, pages 1623–1639. Eur. Math. Soc., Zürich, 2006.