An Eulerian-Lagrangian approach to the Navier-Stokes equations.

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Abstract

We present a formulation of the incompressible viscous Navier-Stokes equation based on a generalization of the inviscid Weber formula, in terms of a diffusive "back-to-labels" map and a virtual velocity. We derive a generalization of the inviscid Cauchy formula and obtain certain bounds for the objects introduced.

1 Introduction

This work presents an Eulerian-Lagrangian approach to the Navier-Stokes equation. An Eulerian-Lagrangian description of the Euler equations has been used in ([4], [5]) for local existence results and constraints on blow-up. Eulerian coordinates (fixed Euclidean coordinates) are natural for both analysis and laboratory experiment. Lagrangian variables have a certain theoretical appeal. In this work I present an approach to the Navier-Stokes equations that is phrased in unbiased Eulerian coordinates, yet describes objects that have Lagrangian significance: particle paths, their dispersion and diffusion. The commutator between Lagrangian and Eulerian derivatives plays an important role in the Navier-Stokes equations: it contributes a singular perturbation to the Euler equations, in addition to the Laplacian. The Navier-Stokes equations are shown to be equivalent to the system

$$\Gamma v = 2\nu C\nabla v$$

where C are the coefficients of the commutator between Eulerian and Lagrangian derivatives, and Γ is the operator of material derivative and viscous diffusion. The physical pressure is not explicitly present in this formulation. The Eulerian velocity u is related to v in a non-local fashion, and one may recover the physical pressure dynamically from the evolution of the gradient part of v. When one sets $\nu = 0$ the commutator coefficients C do not enter the equation, and then v is a passive rearrangement of its initial value. When $\nu \neq 0$ the perturbation involves the curvature of the particle paths, and the gradients of v: a singular perturbation. Fortunately, the coefficients C start from zero, and, as long as they remain small v does not grow too much.

A different but not unrelated approach ([17], ([21]) is based on a variable w that has the same curl as the Eulerian velocity u. The velocity is recovered then from w by applying the Leray-Hodge projection $\mathbf{P}$ on divergence-free functions. The evolution equation for w

$$\Gamma w + (\nabla u)^* w = 0,$$

conserves local helicity and circulation (when $\nu = 0$). We will refer to this equation informally as "the cotangent equation" because it is the equation obeyed by the Eulerian gradient of any scalar ϕ that solves $\Gamma\phi = 0$. There exists a large literature on various aspects of the Euler equations in this and related formulations ([22], [25], [1], [10], [14], [18], [3], [23]).

The variable w is related to v:

$$w = (\nabla A)^* v$$

where A is the "back-to-labels" map that corresponds when $\nu = 0$ to the inverse of the Lagrangian path map. When $\nu = 0$,

$$u = \mathbf{P}\left((\nabla A)^* v\right)$$

is the Weber formula ([24]). $A(x,t)$ is an active vector obeying

$$\Gamma A = 0,$$

$A(x,0) = x$. Both v and A have a Lagrangian meaning when $\nu = 0$, but the dynamical development of w is the product of two processes, the growth of the deformation tensor (given by the evolution of ∇A) and the rearrangement of a fixed function, given by the evolution of v. In the presence of viscosity,

v's evolution is not by rearrangement only. It is therefore useful to study separately the growth of ∇A and the shift of v.

Recently certain model equations have been proposed ([2], [15]) as modifications of the Euler and Navier-Stokes equations. They can be obtained in the context described above simply by smoothing u in the cotangent equation. Smoothing means that one approximates the linear zero-order nonlocal operator $u = \mathbf{P}w$ that relates u to w in the cotangent equation by applying a smoothing approximation of the identity J_δ, thus $u = J_\delta \mathbf{P}w$. When $\nu = 0$ the models have a Kelvin circulation theorem.

In this paper we consider the Navier-Stokes equations and obtain rigorous bounds for the particle paths and for the virtual velocity v. The main bounds concern the Lagrangian displacement, its first and second spatial derivatives, are obtained under general conditions and require no assumptions. Higher derivatives can be bounded also under certain natural quantitative smoothness assumptions. We define the virtual vorticity as the Eulerian-Lagrangian curl of the virtual velocity, and derive a Cauchy formula that generalizes the classical formula to the viscous situation. In it, the Eulerian vorticity is obtained from the gradient of the back to labels map and the virtual vorticity. In the absence of viscosity, the virtual vorticity is transported passively on fluid paths. In the presence of viscosity the evolution of the virtual vorticity is given by a dissipative equation in which the commutator coefficients enter multiplied by viscosity.

2 Velocity and displacement

The Eulerian velocity $u(x,t)$ has three components u^i, $i = 1, 2, 3$ and is a function of three Eulerian space coordinates x and time t. We decompose the Eulerian velocity $u(x,t)$:

$$u^i(x,t) = \frac{\partial A^m(x,t)}{\partial x_i} v_m(x,t) - \frac{\partial n(x,t)}{\partial x_i}. \tag{1}$$

Repeated indices are summed. There are three objects that appear in this formula. The first one, $A(x,t)$, has a Lagrangian interpretation. In the absence of viscosity, A is the "back-to-labels" map, the inverse of the particle trajectory map $a \mapsto x = X(a,t)$. In the presence of viscosity we require this map to obey a diffusive equation, departing thus from its conventional interpretation as inverse of particle trajectories. The vector

$$\ell(x,t) = A(x,t) - x \tag{2}$$

will be called the "Eulerian-Lagrangian displacement vector", or simply "displacement". ℓ joins the current Eulerian position x to the original Lagrangian position $a = A(x,t)$. $A(x,t)$ and $\ell(x,t)$ have dimensions of length, ∇A is non-dimensional. The second object in (1), $v(x,t)$, has dimensions of velocity and, in the absence of viscosity, is just the initial velocity composed with the back-to-labels map; in this case (1) is the Weber formula ([24]) that has been used in numerical and theoretical studies ([11], [12], [16]).

We refer to v as the "virtual velocity". Its evolution marks the difference between the Euler and Navier-Stokes equations most clearly. The third object in (1) is a scalar function $n(x,t)$ that will be referred to as "the Eulerian-Lagrangian potential". It plays a mathematical role akin to that played by the physical pressure but has dimensions of length squared per time, like the kinematic viscosity. If $A(x,t)$ is known, then there are four functions entering the decomposition of u, three v-s and one n. If the velocity is divergence-free

$$\nabla \cdot u = 0,$$

then there is one relationship between the four unknown functions.

3 Eulerian-Lagrangian derivatives and commutators

When one considers the map $x \mapsto A(x,t)$ as a change of variables one can pull back the Lagrangian differentiation with respect to particle position and write it in Eulerian coordinates using the chain rule. Let us call this pull-back of Lagrangian derivatives the Eulerian-Lagrangian derivative,

$$\nabla_A = Q^* \nabla_E. \tag{3}$$

Here

$$Q(x,t) = \left(\nabla A(x,t)\right)^{-1}, \tag{4}$$

and the notation Q^* refers to the transpose of the matrix Q. The expression of ∇_A on components is

$$\nabla_A^i = Q_{ji}\partial_j \tag{5}$$

where we wrote ∂_i for differentiation in the i-th Eulerian Cartesian coordinate direction,

$$\partial_i = \nabla_E^i.$$

The Eulerian spatial derivatives can be expressed in terms of the Eulerian-Lagrangian derivatives via

$$\nabla_E^i = (\partial_i A_m)\, \nabla_A^m \tag{6}$$

The commutation relations

$$[\nabla_E^i, \nabla_E^k] = 0, \quad \left[\nabla_A^i, \nabla_A^k\right] = 0$$

hold. The commutators between Eulerian-Lagrangian and Eulerian derivatives do not vanish, in general:

$$\left[\nabla_A^i, \nabla_E^k\right] = C_{m,k;i} \nabla_A^m. \tag{7}$$

The coefficients $C_{m,k;i}$ are given by

$$C_{m,k;i} = \{\nabla_A^i(\partial_k \ell_m)\}. \tag{8}$$

Note that

$$C_{m,k;i} = Q_{ji}\partial_j\partial_k A_m = \nabla_A^i(\nabla_E^k A_m) = [\nabla_A^i, \nabla_E^k]A_m.$$

The commutator coefficents C are related to the Christoffel coefficients Γ_{ij}^m of the trivial flat connection in $\mathbf{R}^3$ computed at $a = A(x,t)$ by the formula

$$\Gamma_{ij}^m = -Q_{kj} C_{m,k;i}.$$

A straight Eulerian line $x(s) = x_0 + sm$ is transformed in the Lagrangian label curve $a(s) = A(x(s),t)$. The geodesic equation $\frac{d^2 a^i}{ds^2} + \Gamma_{jk}^i \frac{da^j}{ds}\frac{da^k}{ds} = 0$ is equivalent to the equation $\frac{d^2 a^i}{ds^2} + C_{i,j;k}\frac{dx^j}{ds}\frac{da^k}{ds} = 0$. But the interest here is not in the geometry of $\mathbf{R}^3$: the commutator coefficients play an important role in dynamics.

4 The evolution of A

We associate to a given divergence-free velocity $u(x,t)$ the operator

$$\partial_t + u\cdot\nabla - \nu\Delta = \Gamma_\nu(u,\nabla). \tag{9}$$

We write ∂_t for time derivative. We write Γ for $\Gamma_\nu(u, \nabla)$ when the u we use is clear from the context. The coefficient $\nu > 0$ is the kinematic viscosity of the fluid. When applied to a vector or a matrix, Γ acts as a diagonal operator, i.e. on each component separately. The operator Γ obeys a maximum principle: If a function q solves

$$\Gamma q = S$$

and the function q has homogeneous Dirichlet or periodic boundary conditions, then the sup-norm $\|q\|_{L^\infty(dx)}$ satisfies

$$\|q(\cdot, t)\|_{L^\infty(dx)} \le \|q(\cdot, t_0)\|_{L^\infty(dx)} + \int_{t_0}^{t} \|S(\cdot, s)\|_{L^\infty(dx)} ds$$

for any $t_0 \le t$. The operator $\Gamma_\nu(u, \nabla)$ is not a derivation (that means an operator that satisfies the product rule); Γ satisfies a product rule that is similar to that of a derivation:

$$\Gamma(fg) = (\Gamma f)g + f(\Gamma g) - 2\nu(\partial_k f)(\partial_k g). \tag{10}$$

We require the back-to-labels map A to obey

$$\Gamma A = 0. \tag{11}$$

By (11) we express therefore the advection *and diffusion* of A. We will use sometimes the equation obeyed by ℓ

$$(\partial_t + u \cdot \nabla - \nu\Delta)\, \ell + u = 0 \tag{12}$$

which is obviously equivalent to (11). We will discuss periodic boundary conditions

$$\ell(x + Le_j, t) = \ell(x, t),$$

where e_j is the unit vector in the j-th direction. Some of our inequalities will hold also for the physical boundary condition that require $\ell(x, t) = 0$ at the boundary.

It is important to note that the initial data for the displacement is zero:

$$\ell(x, 0) = 0. \tag{13}$$

The matrix $\nabla A(x, t)$ is invertible as long as the evolution is smooth. This is obvious when $\nu = 0$ because the determinant of this matrix equals 1 for all time, but in the viscous case the statement needs proof. We differentiate (11) in order to obtain the equation obeyed by ∇A

$$\Gamma(\nabla A) + (\nabla A)(\nabla u) = 0. \tag{14}$$

The product $(\nabla A)(\nabla u)$ is matrix product in the order indicated. We consider

$$\Gamma Q = (\nabla u)Q + 2\nu Q\partial_k(\nabla A)\partial_k Q. \tag{15}$$

It is clear that the solutions of both (14) and (15) are smooth as long as the advecting velocity u is sufficiently smooth. It is easy to verify using (10) that the matrix $Z = (\nabla A)Q - I$ obeys the equation

$$\Gamma Z = 2\nu Z\partial_k(\nabla A)\partial_k Q$$

with initial datum $Z(x, 0) = 0$. Thus, as long as u is smooth, $Z(x, t) = 0$ and it follows that the solution Q of (15) is the inverse of ∇A.

The commutator coefficients $C_{m,k;i}$ enter the important commutation relation between the Eulerian-Lagrangian label derivative and Γ:

$$\left[\Gamma, \nabla_A^i\right] = 2\nu C_{m,k;i}\nabla_E^k\nabla_A^m \tag{16}$$

The proof of this formula can be found in Appendix B.

The evolution of the coefficients $C_{m,k;i}$ defined in (8) can be computed using (14) and (16):

$$\Gamma\left(C_{m,k;i}\right) = -(\partial_l A_m)\nabla_A^i(\partial_k(u_l))$$

$$-(\partial_k(u_l))C_{m,l;i} + 2\nu C_{j,l;i}\cdot\partial_l\left(C_{m,k;j}\right). \tag{17}$$

The calculation leading to (17) is presented in Appendix B.

5 The evolution of v

We require the virtual velocity to obey

$$\Gamma v = 2\nu C\nabla v + Q^* f. \tag{18}$$

This equation is, on components

$$\Gamma_\nu(u,\nabla)v_i = 2\nu C_{m,k;i}\partial_k v_m + Q_{ji}f_j. \tag{19}$$

The vector $f = f(x,t)$ represents the body forces. The boundary conditions are periodic

$$v(x + Le_j, t) = v(x,t)$$

and the initial data are, for instance

$$v(x,0) = u_0(x). \tag{20}$$

The reason for requiring the equation (18) is

Proposition 1. *Assume that u is given by the expression (1) above and that the displacement ℓ and the virtual velocity v obey the equations (12) and respectively (18). Then the velocity u satisfies the Navier-Stokes equation*

$$\partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = f$$

with pressure p determined from the Eulerian-Lagrangian potential by

$$\Gamma_\nu(u,\nabla)n + \frac{|u|^2}{2} + c = p$$

where c is a free constant.

Proof. We denote for convenience

$$D_t = \partial_t + u\cdot\nabla. \tag{21}$$

We apply D_t to the velocity representation (1) and use the commutation relation

$$[D_t, \partial_k]\, g = -(\nabla u)^*\nabla g. \tag{22}$$

We obtain

$$D_t(u^i) = (\partial_i(D_t A^m))\, v_m + (\partial_i A^m)D_t v_m - \partial_i\left(\frac{|u|^2}{2} + D_t n\right).$$

We substitute the equations for A (12) and for v (18):

$$D_t(u^i) = -\partial_i\left(\frac{|u|^2}{2} + D_t n\right) + (\partial_i(\nu\Delta A^m))\, v_m +$$

$$(\partial_i A^m)\left\{\nu\Delta v_m + Q^*_{mj}\left(2\nu\partial_k(\nabla\ell)^*_{jl}\partial_k v_l + f_j\right)\right\}.$$

Now we use the facts that

$$(\partial_i A^m)Q^*_{mj} = \delta_{ij}$$

(Kronecker's delta), and

$$\partial_k(\nabla\ell)^*_{il} = \partial_k(\nabla A)^*_{il} = \partial_k(\partial_i A^l)$$

to deduce

$$D_t(u^i) = -\partial_i\left(\frac{|u|^2}{2} + D_t n\right) + f_i$$

$$+\nu(\Delta\partial_i A^m)v_m + \nu(\partial_i A^m)\Delta v_m + 2\nu\partial_k(\partial_i A^l)\partial_k v_l$$

and so, changing the dummy summation index l to m in the last expression

$$D_t(u^i) = -\partial_i\left(\frac{|u|^2}{2} + D_t n\right) + \nu\Delta((\partial_i A^m)v_m) + f_i.$$

Using (1) we obtained

$$D_t(u^i) = \nu\Delta u_i - \partial_i\left(\frac{|u|^2}{2} - \nu\Delta n + D_t n\right) + f_i$$

and that concludes the proof.

Observation The incompressibility of velocity has not yet been used. This is why no restriction on the potential $n(x,t)$ was needed. The incompressibility

$$\nabla\cdot u = 0 \tag{23}$$

can be imposed in two ways. The first approach is static: one considers the ansatz (1) and one requires that n maintains the incompressibility at each instance of time. This results in the equation

$$\Delta n = \nabla\cdot(\nabla A)^* v). \tag{24}$$

In this way n is computed from A in a time independent manner and the basic formula (1) can be understood as

$$u = \mathbf{P}\left((\nabla A)^* v\right) \tag{25}$$

where $\mathbf{P}$ is the Leray-Hodge projector on divergence-free functions. The second approach is dynamic: one computes the physical Navier-Stokes pressure

$$p = R_i R_j (u^i u^j) + c \tag{26}$$

where c is a free constant and $R_i = (-\Delta)^{-\frac{1}{2}} \partial_i$ is the Riesz transform for periodic boundary conditions. The formula for p follows by taking the divergence of the Navier-Stokes equation and using (23). Substituting (26) in the expression for the pressure in Proposition 1 one obtains the evolution equation

$$\Gamma n = R_i R_j (u^i u^j) - \frac{|u|^2}{2} + c \tag{27}$$

for n. Incompressibility can be enforced either by solving at each time the static equation (24) or by evolving n according to (27).

Proposition 2. *Let u be given by (1) and assume that the displacement solves (12) and that the virtual velocity solves (18). Assume in addition that the potential obeys (24) (respectively (27)). Then u obeys the incompressible Navier-Stokes equations,*

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \quad \nabla \cdot u = 0,$$

the pressure p satisfies (26) and the potential obeys also (27) (respectively (24)).

The same results hold for the case of the whole $\mathbf{R}^3$ with boundary conditions requiring u and ℓ to vanish at infinity. In the presence of boundaries, if the boundary conditions for u are homogeneous Dirichlet ($u = 0$) then the boundary conditions for v are Dirichlet, but not homogeneous. In that case one needs to solve either one of the equations (24),(27) for n (with Dirichlet or other physical boundary condition) and the v equation (18) with

$$v = \nabla_A n$$

at the boundary.

Proposition 3. *Let u be an arbitrary spatially periodic smooth function and assume that a displacement ℓ solves the equation (12) and a virtual velocity*

v obeys the equation (18) with periodic boundary conditions and with C computed using $A = x + \ell$. Then w defined by

$$w_i = (\partial_i A^m) v_m \tag{28}$$

obeys the cotangent equation

$$\Gamma w + (\nabla u)^* w = f. \tag{29}$$

Proof. The proof is a straightforward calculation. One uses (10) to write

$$\Gamma w_i = (\partial_i A^m) \Gamma v_m + v_m \Gamma (\partial_i A^m) - 2\nu (\partial_k \partial_i A^m) \partial_k v_m.$$

The equation (19) is used for the first term and the equation (14) for the second term. One obtains

$$\Gamma w_i = f_i - (\partial_i u_j) w_j + 2\nu \left\{ (\partial_i A^m) C_{r,q;m} \partial_q v_r - (\partial_k \partial_i A^m) \partial_k v_m \right\}.$$

The proof ends by showing that the term in braces vanishes because of the identity

$$(\partial_i A^m) C_{r,q;m} = \partial_q \partial_i A^r.$$

An approach to the Euler equations based entirely on a variable w ([17], [21]) is well-known. The function w has the same curl as u, $\omega = \nabla \times u = \nabla \times w$. In the case of zero viscosity and no forcing, the local helicity $w \cdot \omega$ is conserved $D_t(w \cdot \omega) = 0$; this is easily checked using the fact that the vorticity obeys the "tangent" equation $D_t \omega = (\nabla u) \omega$ and the inviscid, unforced form of (29). The same proof verifies the Kelvin circulation theorem

$$\frac{d}{dt} \oint_{\gamma(t)} w \cdot dX = 0$$

on loops $\gamma(t)$ advected by the flow of u. Although obviously related, the two variables v and w have very different analytical merits. While the growth of w is difficult to control, in the inviscid case v does not grow at all, and in the viscous case its growth is determined by the magnitude of C which starts from zero. This is why we emphasize v as the primary variable and consider w a derived variable.

6 Gauge Invariance

The viscous equations display a gauge invariance. The numerical merits of different gauges for the Kuzmin-Oseledets approach in the zero viscosity case are described in ([23]). Consider a scalar function ϕ. If one transforms $v \mapsto \tilde{v} = v + \nabla_A \phi$ and $n \mapsto \tilde{n} = n + \phi$ then u remains unchanged in (1): $u \mapsto u$. The requirement that $\nabla_E \cdot u = 0$ does not specify this arbitrary ϕ.

Assume now that the scalar ϕ is advected passively by u and diffuses with diffusivity ν:

$$\Gamma \phi = 0.$$

Then, in view of (16), if v solves (18) then

$$\tilde{v} = v + \nabla_A \phi$$

also solves (18). If n solves (27) then

$$\tilde{n} = n + \phi$$

also solves (27). If w solves the equation (29) then

$$\tilde{w} = w + \nabla_E \phi$$

also solves (29). If ϕ solves

$$\Gamma \phi = -S$$

then

$$\tilde{w} = w + \nabla^E \phi$$

solves

$$\Gamma \tilde{w} + (\nabla u)^* \tilde{w} + \nabla^E S = 0.$$

This allows for an incompressible gauge $\nabla^E \cdot \tilde{w} = 0$. Replacing u by $(I - \alpha^2 \Delta)^{-1} u$ in the cotangent equation in such a gauge one obtains isotropic alpha models ([2], [15]). A gauge of the cotangent equation in the inviscid case hase been used for computations in the case of non-homogeneous boundary conditions in ([8]).

The vector fields obtained by taking the Eulerian gradient of passive scalars are homogeneous solutions of (29). The vector fields obtained by taking the Eulerian-Lagrangian gradient of passive scalars, $\nabla_A \phi$ are homogeneous solutions of (18). This can be used to show that if one chooses an initial datum for v that differs from u_0 by the gradient of an arbitrary function ϕ_0 there is no change in the evolution of u.

Proposition 4. *Let each of two functions v_j, $j = 1, 2$ solve the system*

$$\Gamma(u_j, \nabla)v_j = 2\nu C_j \nabla v_j + Q_j^* f$$

with periodic boundary conditions, coupled with

$$\Gamma(u_j, \nabla)A_j = 0$$

with periodic boundary conditions for $\ell_j = A_j - x$. Assume that the initial data for A_j are the same, $\ell_j(x, 0) = 0$. Assume that each velocity is determined from its corresponding virtual velocity by the rule

$$u_j = \mathbf{P}\left((\nabla A_j)^* v_j\right).$$

Assume, moreover, that at time $t = 0$ the virtual velocities differ by a gradient

$$\mathbf{P}v_1 = \mathbf{P}v_2 = u_0.$$

Then, as long as one of the solutions v_j is smooth one has

$$u_1(x, t) = u_2(x, t), \quad A_1(x, t) = A_2(x, t)$$

The same kind of result can be proved for (29) using the Eulerian gauge invariance.

7 Virtual vorticity and a Cauchy formula

The Eulerian derivatives of the velocity u can be related to the Eulerian-Lagrangian derivatives of v. Let us define

$$\omega(x, t) = \nabla_E \times u(x, t) \tag{30}$$

and

$$\zeta(x, t) = \nabla_A \times v(x, t). \tag{31}$$

These are the Eulerian curl of u and, respectively the Eulerian-Lagrangian curl of v. ζ is related to the anti-symmetric part of the Eulerian-Lagrangian gradient of v by the familiar formulae

$$\nabla_A^i v_m - \nabla_A^m v_i = \epsilon_{imp}\zeta_p, \quad \zeta_p = \frac{1}{2}\epsilon_{imp}\left(\nabla_A^i v_m - \nabla_A^m v_i\right)$$

and similar relations hold for ω. Differentiating (1) and using (6) one obtains

$$\frac{\partial u_j}{\partial x^i} = \mathbf{P}_{jl}\left(Det\left[\zeta; \frac{\partial A}{\partial x^i}; \frac{\partial A}{\partial x^l}\right]\right). \tag{32}$$

We will take the antisymmetric part; in order to ease the calculation we will use the mechanics notation

$$u_{j,i} = \frac{\partial u_j}{\partial x^i} = \nabla_E^i(u_j)$$

The detailed form of (32) is

$$u_{j,i} = \frac{1}{2}\left(A^m_{,j}A^p_{,i} - A^m_{,i}A^p_{,j}\right)\epsilon_{pmr}\zeta_r - n_{,ij} + \partial_j(A^m_{,i}v_m).$$

The last term equals $w_{i,j}$ where w is defined in (28). So

$$u_{j,i} - w_{i,j} = \frac{1}{2}\left(A^m_{,j}A^p_{,i} - A^m_{,i}A^p_{,j}\right)\epsilon_{pmr}\zeta_r - n_{,ij}$$

Taking the anti-symmetric part and using the fact that $u_{j,i} - u_{i,j} = w_{j,i} - w_{i,j}$ we obtain

$$\omega_q = \frac{1}{2}\epsilon_{qij}\left(Det\left[\zeta; \frac{\partial A}{\partial x^i}; \frac{\partial A}{\partial x^j}\right]\right) \tag{33}$$

Because of the linear algebra identity

$$((\nabla A)^{-1}\zeta)_q = (Det(\nabla A))^{-1}\,\frac{\epsilon_{qij}}{2}\left(Det\left[\zeta; \frac{\partial A}{\partial x^i}; \frac{\partial A}{\partial x^j}\right]\right)$$

one has

$$\omega = (Det(\nabla A))\,(\nabla A)^{-1}\zeta. \tag{34}$$

These relations are a generalization of the Cauchy formula to the viscous situation. In the absence of viscosity, ζ is just the initial vorticity composed with A. The quadratic expression in ∇A is just the inverse of ∇A due to the fact that ∇A has determinant equal to 1; in the viscous case the determinant is no longer 1 but this form of the Cauchy formula survives. In two-dimensions (33, 34) become

$$\omega = (Det(\nabla A))\,\zeta. \tag{35}$$

A consequence of (33) or (34) is the identity

$$\omega \cdot \nabla_E = (Det(\nabla A))\,(\zeta \cdot \nabla_A) \tag{36}$$

that generalizes the corresponding inviscid identity ([4]). These identities hold in the forced case also. One can prove by direct computation that determinant of ∇A obeys

$$(\partial_t + u \cdot \nabla_E)\,(Det(\nabla A)) = \nu\,(Det(\nabla A))\left\{C_{i,k;s}C_{s,k;i} + \nabla_k^E(C_{m,k;m})\right\}, \tag{37}$$

or, equivalently

$$\Gamma(\log(Det(\nabla A))) = \nu\left\{C_{i,k;s}C_{s,k;i}\right\}. \tag{38}$$

Note also that

$$\nabla_k^E \log\,(Det(\nabla A)) = C_{m,k;m} \tag{39}$$

and consequently, if $\nu = 0$ then $C_{m,k;m} = 0$ must hold.

The next task is to derive the evolution of ζ. We start with (18) and apply the Eulerian-Lagrangian curl. We use the notation

$$v_{i;j} = \nabla_A^j v_i$$

(thus for instance $C_{m,k;i} = \{A^m_{,k}\}_{;i}$) Applying ∇_A^j to (19) with $f = 0$, and using (16) we obtain

$$\Gamma v_{i;j} = 2\nu\left(C_{m,k;i}v_{m,k}\right)_{;j} + 2\nu C_{m,k;j}\nabla_k^E v_{i;m}. \tag{40}$$

Multiplying by ϵ_{qji} and using the fact that

$$(C_{m,k;i})_{;j} = (C_{m,k;j})_{;i}$$

we deduce

$$\Gamma\zeta_q = 2\nu C_{m,k;j}\nabla_E^k(\epsilon_{qji}v_{i;m}) + 2\nu C_{m,k;i}\epsilon_{qji}\nabla_E^k(v_{m;j})$$

$$+2\nu C_{m,k;i}\epsilon_{qji}\left(\nabla_A^j\nabla_E^k - \nabla_E^k\nabla_A^j\right)v_m.$$

Now we write

$$v_{i;m} = \frac{1}{2}(v_{i;m} - v_{m;i}) + \frac{1}{2}(v_{i;m} + v_{m;i})$$

and substitute in the first two terms above. The symmetric part cancels, the anti-symmetric part is related to ζ. We obtain

$$\Gamma\zeta_q = \nu C_{m,k;j}\nabla_E^k\left(\epsilon_{qji}\epsilon_{rmi}\zeta_r\right) + \nu C_{m,k;i}\nabla_E^k\left(\epsilon_{qji}\epsilon_{rjm}\zeta_r\right) +$$
$$+2\nu C_{m,k;i}\epsilon_{qji}\left(\nabla_A^j\nabla_E^k - \nabla_E^k\nabla_A^j\right)v_m.$$

Using now the commutation relation (7) and the rule of contraction of two ϵ_{ijk} tensors we get the equation

$$\Gamma\zeta_q = 2\nu C_{m,k;m}\nabla_E^k\zeta_q - 2\nu C_{q,k;j}\nabla_E^k\zeta_j + \nu C_{m,k;i}C_{r,k;j}\epsilon_{qji}\epsilon_{rmp}\zeta_p. \tag{41}$$

When $\nu = 0$ we recover the fact that $\Gamma\zeta = 0$, but, more importantly, ζ obeys a linear dissipative equation with $C_{m,k;i}$ as coefficients. Using just the Schwartz inequality pointwise we deduce

$$\Gamma|\zeta|^2 + \nu|\nabla_E\zeta|^2 \le 17\nu|C|^2|\zeta|^2 \tag{42}$$

where

$$|C|^2 = C_{m,k;i}C_{m,k;i}, \quad |\zeta|^2 = \zeta_q\zeta_q$$

are squares of Euclidean norms.

8 K-bounds

We are going to describe here bounds that are based solely on the kinetic energy balance in the Navier-Stokes equation (([6]) and references therein). These are very important, as they are the only unconditional bounds that are known for arbitrary time intervals. We call them kinetic energy bounds or in short, K-bounds. We start with the most important, the energy balance itself. From the Navier-Stokes equation one obtains the bound

$$\int |u(x,t)|^2 dx + \nu\int_{t_0}^t\int |\nabla u(x,s)|^2 dxds \le K_0 \tag{43}$$

with

$$K_0 = \min\{k_0; k_1\} \tag{44}$$

where

$$k_0 = 2\int |u(x,t_0)|^2 dx + 3(t-t_0)\int_{t_0}^t\int |f(x,s)|^2 dxds \tag{45}$$

and

$$k_1 = \int |u(x,t_0)|^2 dx + \frac{1}{\nu}\int_{t_0}^{t}\int |\Delta^{-\frac{1}{2}} f(x,s)|^2 dx ds \tag{46}$$

Note that we have not normalized the volume of the domain. The prefactors are not optimal. The energy balance holds for all solutions of the Navier-Stokes equations. We took an arbitrary starting time t_0. The bound K_0 is a nondecreasing function of $t - t_0$. We will use this fact tacitly below. In order to give a physical interpretation to this general bound it is useful to denote by

$$\epsilon(s) = \nu L^{-3} \int |\nabla u(x,s)|^2 dx$$

the volume average of the instantaneous energy dissipation rate, by

$$E(t) = \frac{1}{2L^3}\int |u(x,t)|^2 dx$$

the volume average of the kinetic energy; for any time dependent function $g(s)$, we write

$$\langle g(\cdot)\rangle_t = \frac{1}{t-t_0}\int_{t_0}^{t} g(s) ds$$

for the time average. We also write

$$F^2 = \left\langle L^{-3}\int |f(x,\cdot)|^2 dx \right\rangle_t,$$

$$G^2 = \left\langle L^{-3}\int |\Delta^{-\frac{1}{2}} f(x,\cdot)|^2 dx \right\rangle_t$$

and define the forcing length scale by

$$L_f^2 = \frac{G^2}{F^2}.$$

Then (43) implies

$$2E(t) + (t-t_0)\langle \epsilon(\cdot)\rangle_t \le 4E(t_0) + (t-t_0)F^2 \min\left\{\frac{L_f^2}{\nu}; 3(t-t_0)\right\}. \tag{47}$$

After a long enough time

$$t - t_0 \geq \frac{L_f^2}{3\nu},$$

the kinetic energy grows at most linearly in time

$$E(t) \leq 2E(t_0) + F^2 \left(\frac{(t - t_0)L_f^2}{\nu} \right).$$

The long time for the average dissipation rate is bounded

$$\limsup_{t \to \infty} \langle \epsilon(\cdot) \rangle_t \leq \frac{F^2 L_f^2}{\nu} = \epsilon_B. \tag{48}$$

These bounds are uniform in the size L of the period which we assume to be much larger than L_f. If the size of the period is allowed to enter the calculations then the kinetic energy is bounded by

$$E(t) \leq L^2 \frac{L_f^2 F_*^2}{\nu^2} + \left(E(t_0) - L^2 \frac{L_f^2 F_*^2}{\nu^2} \right) e^{-\frac{\nu(t-t_0)}{L^2}}$$

where

$$L_f^2 F_*^2 = \sup_t L^{-3} \int |(-\Delta)^{-\frac{1}{2}} f(x,t)|^2 dx.$$

This means that for much longer times

$$t - t_0 \geq \frac{L^2}{\nu}$$

the kinetic energy saturates to a value that depends on the large scale. But the bound (47) that is independent of L is always valid; it can be written in terms of

$$B = 4E(t_0) + (t - t_0)\epsilon_B \tag{49}$$

as

$$E(t) + (t - t_0) \langle \epsilon \rangle_t \leq B. \tag{50}$$

A useful K-bound is

$$\int_{t_0}^{t} \|u(\cdot, s)\|_{L^\infty(dx)} ds \leq K_\infty \tag{51}$$

The constant K_∞ has dimensions of length and depends on the initial kinetic energy, viscosity, body forces and time. The bound follows by interpolation from ([13]) and is derived in Appendix A together with the formula

$$K_\infty = C\left\{\frac{K_0}{\nu^2} + \sqrt{\nu(t-t_0)} + \frac{t-t_0}{\nu^2}\int_{t_0}^t \|f(\cdot,s)\|_{L^2}^2 ds\right\}. \tag{52}$$

The displacement ℓ satisfies certain K-bounds that follow from the bounds above and (12). We mention here

$$\|\ell(\cdot,t)\|_{L^\infty(dx)} \le \int_{t_0}^t \|u(\cdot,s)\|_{L^\infty(dx)} ds \le K_\infty, \tag{53}$$

The inequality (53) follows from (12) by multiplying with $\ell|\ell|^{2(m-1)}$, integrating,

$$\frac{1}{2m}\frac{d}{dt}\int |\ell(x,t)|^{2m}dx + \nu\int |\nabla\ell(x,t)|^2|\ell(x,t)|^{2(m-1)}dx +$$
$$+\nu\frac{m-1}{2}\int |\nabla|\ell(x,t)|^2|^2|\ell(x,t)|^{2(m-2)}dx +$$
$$+\int u(x,t)\cdot\ell(x,t)|\ell(x,t)|^{2(m-1)}dx \le 0, \tag{54}$$

and then ignoring the viscous terms, using Hölder's inequality in the last term, multiplying by m, taking the m-th root, integrating in time and then letting $m \to \infty$.

The case $m = 1$ gives

$$\frac{d}{2dt}\int |\ell(x,t)|^2 dx + \nu\int |\nabla\ell(x,t)|^2 dx \le \sqrt{K_0}\sqrt{\int |\ell(x,t)|^2 dx}$$

and consequently, we obtain by integration from $t_0 = 0$

$$\sqrt{\int |\ell(x,t)|^2 dx} \le t\sqrt{K_0}, \tag{55}$$

and then, using (55) we deduce the inequality

$$\int_0^t\int |\nabla\ell(x,s)|^2 dx ds \le \frac{K_0 t^2}{2\nu}. \tag{56}$$

Now we multiply (12) by $-\Delta\ell$, integrate by parts, use Schwartz's inequality to write

$$\frac{d}{dt}\int|\nabla\ell(x,t)|^2dx+\nu\int|\Delta\ell(x,t)|^2dx\le\sqrt{\int|\nabla u(x,t)|^2dx}\sqrt{\int|\nabla\ell(x,t)|^2dx}$$

$$-2\int \mathrm{T}race\left\{(\nabla\ell(x,t))(\nabla u(x,t))(\nabla\ell(x,t))^*\right\}dx$$

and then use the elementary inequality

$$\left(\int|\nabla\ell(x,t)|^4dx\right)^{\frac{1}{2}}\le C\|\ell(\cdot,t)\|_{L^\infty}\left(\int|\Delta\ell(x,t)|^2dx\right)^{\frac{1}{2}},$$

in conjunction with the Hölder inequality and (53) to deduce

$$\frac{d}{dt}\int|\nabla\ell(x,t)|^2dx+\nu\int|\Delta\ell(x,t)|^2dx\le$$

$$\sqrt{\int|\nabla\ell(x,t)|^2dx}\sqrt{\int|\nabla u(x,t)|^2dx}+C\frac{K_\infty^2}{\nu}\int|\nabla u(x,t)|^2dx.$$

We obtain, after integration and use of (43, 56)

$$\int|\nabla\ell(x,t)|^2dx+\nu\int_0^t\int|\Delta\ell(x,s)|^2dxds\le C\left(\frac{K_0t}{\nu}+\frac{K_\infty^2K_0}{\nu^2}\right).\tag{57}$$

Recalling the bound (49, 50) on kinetic energy we have:

Theorem 1. *Assume that the vector valued function ℓ obeys (12) and assume that the velocity $u(x,t)$ is a solution of the Navier-Stokes equations (or, more generally, that it is a divergence-free periodic function that satisfies the bounds (43) and (51)). Then ℓ satisfies the inequality (53) together with*

$$\frac{1}{L^3}\int|\ell(x,t)|^2dx\le(4E_0+t\epsilon_B)t^2,\tag{58}$$

$$\frac{1}{L^3t}\int_0^t\int|\nabla\ell(x,s)|^2dxds\le\frac{Bt}{2\nu},\tag{59}$$

and

$$\int |\nabla \ell(x,t)|^2 \frac{dx}{L^3} + \nu \int_0^t \int |\Delta \ell(x,s)|^2 \frac{dx}{L^3} ds \le C\left(\frac{Bt}{\nu} + \frac{K_\infty^2 B}{\nu^2}\right). \tag{60}$$

In these inequalities

$$E_0 = \frac{1}{2L^3}\int |u(x,0)|^2 dx,$$
$$B = 4E_0 + t\epsilon_B$$

and ϵ_B *is given in (48).*

A pair of points, $a = A(x,t)$, $b = A(y,t)$ situated at time $t = 0$ at distance $\delta_0 = |a - b|$ become separated by $\delta_t = |x - y|$ at time t. From the triangle inequality it follows that

$$(\delta_t)^2 \le 3|\ell(x,t)|^2 + 3|\ell(y,t)|^2 + 3(\delta_0)^2. \tag{61}$$

The displacement can be used in this manner to bound pair dispersion. Let us consider the pair dispersion

$$\langle \delta_t^2 \rangle = L^{-6} \int\int_{\{(x,y); |A(x,t)-A(y,t)| \le \delta_0\}} |x-y|^2 dxdy. \tag{62}$$

Using the triangle inequality (61) in (58) we obtain

Theorem 2. *Consider periodic solutions of the Navier-Stokes equation with large period L, and assume that the body forces have* L_f *finite. Then the pair dispersion obeys*

$$\langle \delta_t^2 \rangle \le 3\delta_0^2 + 24tE_0t^2 + 6\epsilon_B t^3. \tag{63}$$

Comment The bound ϵ_B does not depend on the size of box. In many physically realistic situations one injects energy at the boundary; in that case one can find ϵ_B independently of viscosity ([7]), without any assumptions. Use of the ODE $\frac{dX}{dt} = u(X,t)$ requires information about the gradient ∇A and produces worse bounds. The bound above is reminiscent of the Richardson pair dispersion law of fully developed turbulence ([20], [9]). The pair dispersion law states that the separation δ between fluid particles obeys

$$\langle |\delta|^2 \rangle \sim \epsilon t^3$$

where ϵ is the rate of dissipation of energy and t is time. This is supposed to hold in an inertial range, in statistical steady flux of energy, for times t that are neither too big nor too small and for unspecified initial separations. The "law" can be guessed by dimensional analysis by requiring the answer to depend solely on time and ϵ or can be derived using formally the Hölder exponent $1/3$ for velocity. A rigorous mathematical derivation from the Navier-Stokes equations is not available: one is faced with the difficulty that the prediction seems to require both non-Lipschitz, Hölder continuous velocities and a well defined notion of Lagrangian particle paths. Laboratory Lagrangian experiments have only recently begun to be capable of performing precise Lagrangian measurements and a quantitative confirmation of the Richardson law is still not definitive ([19]).

9 ϵ-bounds

This section is devoted to bounds on higher order derivatives of ℓ. These bounds require assumptions. We are going to apply the Laplacian to (12), multiply by $\Delta\ell$ and integrate. We obtain

$$\frac{1}{2}\frac{d}{dt}\int|\Delta\ell(x,t)|^2dx + \nu\int|\nabla\Delta\ell(x,t)|^2dx =$$

$$\int\partial_k u(x,t)\cdot\partial_k\Delta\ell(x,t)dx + I \tag{64}$$

where

$$I = \int\partial_k(u(x,t)\cdot\nabla\ell(x,t))\cdot\partial_k\Delta\ell(x,t)dx.$$

Now

$$I = \int(\partial_k u)\cdot\nabla\ell(x,t)\cdot\partial_k\Delta\ell(x,t)dx + II$$

where

$$II = \int u(x,t)\cdot\nabla(\partial_k\ell(x,t))\cdot\Delta\partial_k\ell(x,t)dx$$

and, integrating by parts

$$II = -\int\partial_l u(x,t)\cdot\nabla(\partial_k\ell(x,t))\cdot\partial_l\partial_k\ell(x,t)dx$$

and then again

$$II = \int \partial_l u(x,t) \cdot \nabla \partial_l \partial_k \ell(x,t) \cdot (\partial_k \ell(x,t)) dx$$

and so

$$I = \int \partial_l u_i(x,t) \partial_k \ell_j(x,t) \left\{ \partial_i \partial_k + \delta_{ik} \Delta \right\} \partial_l \ell_j(x,t) dx$$

Putting things together we get

$$|I| \le C \|\nabla \ell(\cdot,t)\|_{L^\infty} \|\nabla u(\cdot,t)\|_{L^2} \|\nabla \Delta \ell(\cdot,t)\|_{L^2}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\int |\Delta \ell(x,t)|^2 dx + \nu \int |\nabla \Delta \ell(x,t)|^2 dx \le$$

$$\frac{C}{\nu}\int |\nabla u(x,t)|^2 dx + C \|\nabla \ell(\cdot,t)\|_{L^\infty} \|\nabla u(\cdot,t)\|_{L^2} \|\nabla \Delta \ell(\cdot,t)\|_{L^2} \tag{65}$$

Now we use an interpolation inequality that is valid for periodic functions with zero mean and implies that

$$\|\nabla \ell\|_{L^\infty} \le c \|\Delta \ell\|_{L^2}^{\frac{1}{2}} \|\nabla \Delta \ell\|_{L^2}^{\frac{1}{2}}$$

Using this inequality we obtain

$$\frac{d}{dt}\int |\Delta \ell(x,t)|^2 dx + \nu \int |\nabla \Delta \ell(x,t)|^2 dx \le$$

$$\frac{C}{\nu}\int |\nabla u(x,t)|^2 dx + C\nu^{-3} \|\nabla u(\cdot,t)\|_{L^2}^4 \|\Delta \ell(\cdot,t)\|_{L^2}^2 \tag{66}$$

Therefore we deduce

$$\int |\Delta \ell(x,t)|^2 \frac{dx}{L^3} \le c\frac{B}{\nu^2} \exp\left\{ \frac{cL^6}{\nu^5} \int_0^t \epsilon^2(s) ds \right\} \tag{67}$$

where

$$\epsilon(s) = \nu L^{-3} \int |\nabla u(x,s)|^2 dx \tag{68}$$

is the instantaneous energy dissipation.

Proposition 5. *If ℓ solves (12) with periodic boundary conditions on a time interval $t \in [0,T]$ and if the integral*

$$\int_0^T \epsilon^2(s)ds$$

is finite, then

$$\int |\Delta\ell(x,t)|^2 \frac{dx}{L^3} + \nu \int_0^t \int |\nabla\Delta\ell(x,s)|^2 \frac{dx}{L^3} \le c\frac{B}{\nu^2} exp\left\{\frac{cL^6}{\nu^5}\int_0^t \epsilon^2(s)ds\right\}$$

holds for all $0 \le t \le T$.

10 Bounds for the virtual velocity

We prove here the assertion that v does not grow too much as long as the L^3 norm of C is not too large. We recall that v solves (19)

$$\Gamma v_i = 2\nu C_{m,k;i}\partial_k v_m + Q_{ji} f_j. \tag{69}$$

We multiply by $v_i|v|^{2(m-1)}$ and integrate:

$$\begin{aligned} &\frac{1}{2m}\frac{d}{dt}\int |v(x,t)|^{2m}dx + \nu\int |\nabla v(x,t)|^2|v(x,t)|^{2(m-1)}dx + \\ &+\nu\frac{m-1}{2}\int |\nabla|v(x,t)|^2|^2|v(x,t)|^{2(m-2)}dx = \\ &= 2\nu\int C_{m,k;i}(x,t)(\partial_k v_m(x,t))v_i(x,t)|v(x,t)|^{2(m-1)}dx + \\ &+\int Q_{ji}(x,t)f_j(x,t)v_i(x,t)|v(x,t)|^{2(m-1)}dx. \end{aligned} \tag{70}$$

We bound

$$2\nu\left|\int C_{m,k;i}(x,t)(\partial_k v_m(x,t))v_i(x,t)|v(x,t)|^{2(m-1)}dx\right| \le$$

$$\nu\int |\nabla v(x,t)|^2|v(x,t)|^{2(m-1)}dx + \nu\int |C(x,t)|^2|v(x,t)|^{2m}dx$$

where

$$|C(x,t)|^2 = \sum_{m,k,i} |C_{m,k;i}(x,t)|^2, \tag{71}$$

and we bound

$$\left| \int Q_{ji}(x,t) f_j(x,t) v_i(x,t) |v(x,t)|^{2(m-1)} dx \right| \leq$$

$$\left\{ \int |g(x,t)|^{2m} dx \right\}^{\frac{1}{2m}} \left\{ \int |v(x,t)|^{2m} dx \right\}^{\frac{2m-1}{2m}}$$

where

$$g_i(x,t) = Q_{ji}(x,t) f_j(x,t). \tag{72}$$

The inequality obtained is

$$\frac{d}{dt} \int |v(x,t)|^{2m} dx + \nu m(m-1) \int |\nabla |v(x,t)|^2|^2 |v(x,t)|^{2(m-2)} dx \leq$$

$$\leq 2m\nu \int |C(x,t)|^2 |v(x,t)|^{2m} dx +$$

$$+ 2m \left\{ \int |g(x,t)|^{2m} dx \right\}^{\frac{1}{2m}} \left\{ \int |v(x,t)|^{2m} dx \right\}^{\frac{2m-1}{2m}} \tag{73}$$

Let us consider for any $m \geq 1$ the quantity

$$q(x,t) = |v(x,t)|^m.$$

The inequality (73) implies

$$\frac{d}{dt} \int (q(x,t))^2 + 4\nu(1 - \frac{1}{m}) \int |\nabla q(x,t)|^2 \leq 2m\nu \int |C(x,t)|^2 (q(x,t))^2 dx +$$

$$+ 2m \left\{ \int |g(x,t)|^{2m} dx \right\}^{\frac{1}{2m}} \left\{ \int (q(x,t))^2 dx \right\}^{\frac{2m-1}{2m}}$$

Using the well-known Morrey-Sobolev inequality

$$\left\{ \int (q(x))^6 dx \right\}^{\frac{1}{3}} \leq C_0 \left\{ \int |\nabla q(x)|^2 dx + L^{-2} \int (q(x))^2 dx \right\}$$

and Hölder's inequality we deduce

$$\frac{d}{dt}\int (q(x,t))^2 + 4\nu(1-\frac{1}{m})\int |\nabla q(x,t)|^2 \le$$

$$\le 2m\nu C_0 \left\{\int |C(x,t)|^3 dx\right\}^{\frac{2}{3}} \left\{\int |\nabla q(x,t)|^2 dx + L^{-2}\int (q(x,t))^2 dx\right\}$$

$$+2m\left\{\int |g(x,t)|^{2m} dx\right\}^{\frac{1}{2m}} \left\{\int (q(x,t))^2 dx\right\}^{\frac{2m-1}{2m}}$$

Assume that, on the time interval $t \in [0,\tau]$, $C(x,t)$ obeys the smallness condition

$$\left\{\int |C(x,t)|^3 dx\right\}^{\frac{1}{3}} \le \sqrt{\frac{2(m-1)}{C_0 m^2}} \tag{74}$$

for $m > 1$ or

$$\left\{\int |C(x,t)|^3 dx\right\}^{\frac{1}{3}} \le \sqrt{\frac{1}{4C_0}} \tag{75}$$

for $m = 1$. Then

$$\frac{d}{dt}\|v(\cdot,t)\|_{L^{2m}} \le \frac{\nu(m-1)}{2m^2L^2}\|v(\cdot,t)\|_{L^{2m}} + \|g(\cdot,t)\|_{L^{2m}}$$

for $t \in [0,\tau]$ and consequently

$$\|v(\cdot,t)\|_{L^{2m}} \le \|v_0\|_{L^{2m}} e^{\frac{\nu(m-1)t}{2m^2L^2}} + \int_0^t \|g(\cdot,s)\|_{L^{2m}} \tag{76}$$

holds on the same time interval.

11 Bounds for the virtual vorticity

We take here the body forces equal to zero, and start directly with the pointwise inequality (42). Multiplying by $m|\zeta|^{2(m-1)}$ and integrating we obtain, as above

$$\frac{d}{dt}\int (q(x,t))^2 + 4\nu(1-\frac{1}{m})\int |\nabla q(x,t)|^2 \le 17m\nu\int |C(x,t)|^2 (q(x,t))^2 dx$$

for

$$q(x,t) = |\zeta(x,t)|^m.$$

Using the same Morrey-Sobolev inequality we deduce

$$\frac{d}{dt}\int (q(x,t))^2 + 4\nu(1-\frac{1}{m})\int |\nabla q(x,t)|^2 \le$$

$$\le 17m\nu C_0\left\{\int |C(x,t)|^3 dx\right\}^{\frac{2}{3}}\left\{\int |\nabla q(x,t)|^2 dx + L^{-2}\int (q(x,t))^2 dx\right\}$$

When $m = 1$ the coefficient in front of the gradient is ν, not zero. Assume that, on the time interval $t \in [0,\tau]$, the L^3 norm of the commutator coefficients C obey the smallness condition

$$\left\{\int |C(x,t)|^3 dx\right\}^{\frac{1}{3}} \le \sqrt{\frac{2(m-1)}{17C_0m^2}} \tag{77}$$

if $m > 1$. For $m = 1$ we use

$$\left\{\int |C(x,t)|^3 dx\right\}^{\frac{1}{3}} \le \sqrt{\frac{1}{34C_0}}. \tag{78}$$

The the inequalities above imply

$$\|\zeta(\cdot,t)\|_{L^{2m}} \le \|\omega_0\|_{L^{2m}} e^{c_m\nu L^{-2}t} \tag{79}$$

with c_m an appropriate constant. Note that a bound on $\nabla\zeta$ is also implied by the same calculation. Also, if $\nabla\zeta$ exceeds ζ by much (i.e. if small scales develop in ζ) then ζ decreases dramatically.

12 Appendix A

In this appendix we prove the inequality (51) and derive the explicit expression for K_∞. The calculation is based on ([13]). All constants C are non-dimensional and may change from line to line. Solutions u of the Navier-Stokes equations obey the differential inequality

$$\frac{d}{ds}\int |\nabla u(x,s)|^2 dx + \nu\int |\Delta u(x,s)|^2 dx$$

$$\leq \frac{C}{\nu^3}\left(\int |\nabla u(x,s)|^2 dx\right)^3 + \frac{C}{\nu}\int |f(x,s)|^2 dx.$$

The idea of ([13]) was to divide by an appropriate quantity to make use of the balance (43). The quantity is

$$(G(s))^2 = \left(\gamma^2 + \int |\nabla u(x,s)|^2 dx\right)^2$$

where γ is a positive constant that does not depend on s and will be specified later. Dividing by $(G(s))^2$, integrating in time from t_0 to t and using (43) one obtains

$$\int_{t_0}^t \|\Delta u(\cdot,s)\|_{L^2}^2 (G(s))^{-2} ds \leq$$

$$C\left(\frac{K_0}{\nu^5} + \frac{1}{\nu\gamma^2} + \frac{1}{\nu^2\gamma^4}\int_{t_0}^t \|f(\cdot,s)\|^2 ds\right).$$

The three dimensional Sobolev embedding-interpolation inequality for periodic mean-zero functions

$$\|u\|_{L^\infty} \leq C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}$$

is elementary. From it we deduce

$$\|u(\cdot,s)\|_{L^\infty} \leq C\|\nabla u(\cdot,s)\|_{L^2}^{\frac{1}{2}}(G(s))^{\frac{1}{2}}\left[\|\Delta u(\cdot,s)\|_{L^2}G(s)^{-1}\right]^{\frac{1}{2}}$$

Integrating in time, using the Hölder inequality, the inequality (43) and the inequalities above we deduce

$$\int_{t_0}^t \|u(\cdot,s)\|_{L^\infty} \leq Cr$$

where the length $r = r(\gamma, t, \nu, K_0)$ is given in terms of six length scales

$$\frac{K_0}{\nu^2} = r_0, \quad \frac{\nu^2}{\gamma^2} = r_1, \quad \frac{(t-t_0)\gamma^2}{\nu} = r_2,$$

$$(\gamma(t-t_0))^{\frac{2}{3}} = r_3, \quad \frac{t-t_0}{\nu^2}\int_{t_0}^t \|f(\cdot,s)\|_{L^2}^2 ds = r_4$$

and

$$r_5 = \sqrt{\nu(t-t_0)}.$$

The expression for r is

$$r = r_0 + (r_0)^{\frac{3}{4}}(r_1)^{\frac{1}{4}} + (r_0)^{\frac{1}{2}}(r_2)^{\frac{1}{2}} + (r_0)^{\frac{1}{4}}(r_3)^{\frac{3}{4}} +$$

$$(r_0)^{\frac{1}{4}}(r_4)^{\frac{1}{4}}(r_5)^{\frac{1}{2}} + (r_0)^{\frac{3}{4}}(r_4)^{\frac{1}{4}}\left(\frac{r_1}{r_2}\right)^{\frac{1}{4}}$$

The choice

$$\gamma^4 = \frac{\nu^3}{t - t_0}$$

entrains

$$r_1 = r_2 = r_3 = r_5$$

reducing thus the number of length scales to three, the energy viscous length scale r_0, the diffusive length scale r_5 and the force length scale r_4. The bound becomes

$$K_\infty = C(r_0 + r_4 + r_5)$$

i.e. (52).

13 Appendix B

We prove her the commutation relation (16). We take an arbitrary function g and compute $[\Gamma, L_i g])$ where $\Gamma = \Gamma_\nu(u, \nabla)$ and $L_i = \nabla_A^i$. We use first (10):

$$[\Gamma, L_i g] = \Gamma\left(Q_{ji}\partial_j g\right) - Q_{ji}\partial_j \Gamma g =$$

$$\Gamma(Q_{ji})\partial_j g + Q_{ji}\Gamma\partial_j g - 2\nu\partial_k(Q_{ji})\partial_k\partial_j g - Q_{ji}\partial_j\Gamma g =$$

(commuting in the last term ∂_j and Γ)

$$\Gamma(Q_{ji})\partial_j g - 2\nu\partial_k(Q_{ji})\partial_k\partial_j g - Q_{ji}\partial_j(u_k)\partial_k g =$$

(changing names of dummy indices in the last term)

$$\left(\Gamma(Q_{ji}) - Q_{ki}\partial_k(u_j)\right)\partial_j g - 2\nu\partial_k(Q_{ji})\partial_k\partial_j g =$$

(using (15))

$$2\nu Q_{jp}(\partial_l\partial_k A_p)(\partial_k Q_{li})\partial_j g - 2\nu\partial_k(Q_{ji})\partial_k\partial_j g =$$

(using the definition (5) of ∇_A

$$2\nu(\partial_l\partial_k A_p)(\partial_k Q_{li})(L_p g) - 2\nu\partial_k(Q_{ji})\partial_k\partial_j g =$$

(renaming dummy indices in the last expression)

$$2\nu(\partial_k(Q_{li}))\left\{(\partial_l\partial_k A_p)(L_p g) - \partial_l\partial_k g\right\} =$$

(using (6) in the last expression)

$$2\nu(\partial_k(Q_{li}))\left\{(\partial_l\partial_k A_p)(L_p g) - \partial_k\left(\partial_l(A_p)(L_p g)\right)\right\} =$$

(carrying out the differentiation in the last term and cancelling)

$$-2\nu(\partial_k(Q_{li}))\partial_l(A_p)\partial_k(L_p(g)) =$$

(using the differential consequence of the fact that Q and ∇A are inverses of each other)

$$2\nu Q_{li}(\partial_k\partial_l(A_p))\partial_k(L_p g) =$$

(using the definition (5) of ∇_A)

$$2\nu L_i(\partial_k(A_p))\partial_k(L_p g) =$$

(using the definition (8) of $C_{m,k;i}$)

$$2\nu C_{p,k;i}\partial_k(L_p g),$$

and that concludes the proof. We proceed now to prove (17). We start with (14)

$$\Gamma(\partial_k A_m) = -(\partial_k u_j)(\partial_j A_m)$$

and apply L_i:

$$L_i(\Gamma(\partial_k A_m)) = -L_i\left\{(\partial_k u_j)(\partial_j A_m)\right\}.$$

Using the commutation relation (16) and the definition (8) we get

$$\Gamma(C_{m,k;i}) = -L_i\left\{(\partial_k u_j)(\partial_j A_m)\right\} + 2\nu C_{p,l;i}\partial_l L_p(\partial_k A_m).$$

Using the fact that L_i is a derivation in the first term and the definition (8) in the last term we conclude that

$$\Gamma(C_{m,k;i}) = -(\partial_k u_j)C_{m,j;i} - (\partial_j A_m)(L_i(\partial_k u_j)) + 2\nu C_{p,l;i}(\partial_l C_{m,k;p})$$

which is (17). We compute now the formal adjoint of ∇^i_A

$$(\nabla^i_A)^* g = -\partial_j(Q_{ji}g) = -\nabla^i_A(g) - (\partial_j(Q_{ji}))g$$

(with (6))

$$(\nabla^i_A)^* g = -\nabla^i_A(g) - \{(\partial_j A_p) L_p(Q_{ji})\} g =$$

(using the fact that Q is the inverse of ∇A)

$$(\nabla^i_A)^* g = -\nabla^i_A(g) + Q_{ji} C_{p,j;p} g.$$

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