The Euler Equations and Non-Local Conservative Riccati Equations

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The purpose of this brief note is to present an infinite dimensional family of exact solutions of the incompressible three-dimensional Euler equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

The solutions we present have infinite kinetic energy and blow up in finite time. Blow up of other similar infinite energy solutions of Euler equations has been proved before ([1], [2]). The particular type of solution we will describe was proposed in ([3]). The Eulerian-Lagrangian approach we take ([4]) is not restricted to this particular case, but exact integration of the equations is. We will consider a two dimensional basic square Q of side L. The particular form of the solutions ([3]) is

$$\mathbf{u}(x, y, z, t) = (u(x, y, t), z\gamma(x, y, t)) \tag{2}$$

where the scalar valued function γ is periodic in both spatial variables with period L and the two dimensional vector $u(x, y, t) = (u_1(x, y, t), u_2(x, y, t))$ is also periodic with the same period. The associated two dimensional curl is

$$\omega(x, y, t) = \frac{\partial u_2(x, y, t)}{\partial x} - \frac{\partial u_1(x, y, t)}{\partial y}. \tag{3}$$

This represents the vertical (third) component of the vorticity $\nabla \times \mathbf{u}$ of the Euler system and, using the ansatz (2) it follows from the familiar three dimensional vorticity equation that the equation

$$\partial_t \omega + u \cdot \nabla \omega = \gamma \omega \tag{4}$$

should be satisfied for the recipe to succeed. On the other hand, one can easily check that the vertical component of the velocity $z\gamma(x, y, t)$ solves the vertical component of the Euler equations if γ solves the non-local Riccati equation

$$\frac{\partial \gamma}{\partial t} + u \cdot \nabla \gamma = -\gamma^2 + I(t) \tag{5}$$

where $I(t)$ is a function that depends on time only. The divergence-free condition for $\mathbf{u}$ becomes

$$\nabla \cdot u = -\gamma. \tag{6}$$

Because of the spatial periodicity of u one must make sure that

$$\int_Q \gamma(x, t) dx = 0 \tag{7}$$

holds throughout the evolution. This can be done provided the function $I(t)$ is given by

$$I(t) = \frac{2}{|Q|} \int_Q \gamma^2(x, t) dx \tag{8}$$

where

$$|Q| = \int_Q dx = L^2.$$

The velocity is determined from ω and γ using a stream function $\psi(x, y, t)$ and a potential $h(x, y, t)$ by

$$u = \nabla^{\perp} \psi + \nabla h \tag{9}$$

$$-\Delta h = \gamma, \tag{10}$$

$$-\Delta \psi = \omega \tag{11}$$

with periodic boundary conditions.

The ansatz $\mathbf{u}(x, y, z, t) = (u(x, y, t), z\gamma(x, y, t))$ associates to solutions of the system (4, 5, 8, 9, 10, 11) in $d = 2$ velocities $\mathbf{u}$ that obey the incompressible three dimensional Euler equations ([3], [5]). The divergence condition (6) follows from (9, 10). The compatibility condition $\int_Q \omega dx = 0$ is maintained throughout the evolution because of $(4, 6)$. We consider initial data

$$\gamma(x, y, 0) = \gamma_0(x, y), \quad \omega(x, y, 0) = \omega_0(x, y) \tag{12}$$

that are smooth and have mean zero $\int_Q \gamma_0 dx = \int_Q \omega_0 dx = 0$. The solutions of the system above have local existence and the velocity is as smooth as the initial data are, as long as $\int_0^T \sup_x |\gamma(x, t)| dt$ is finite (the result can be proved ([6]) following the idea of the proof of the well-known result ([7])). We will consider the characteristics

$$\frac{dX}{dt} = u(X, t) \tag{13}$$

and denoting $X(a, t)$ the characteristic that starts at $t = 0$ from a, $X(a, 0) = a$, we note that, prior to blow up the map $a \mapsto X(a, t)$ is one-to-one and onto as a map from $\mathbf{T}^2 = \mathbf{R}^2/L\mathbf{Z}^2$ to itself. The injectivity follows from the uniqueness of solutions of ordinary differential equations. The surjectivity can be proved by reversing time on characteristics, which can be done as long as the velocity is smooth. Our result is an explicit formula for γ on characteristics

$$\gamma(x, t) = \alpha(\tau(t)) \left\{ \frac{\gamma_0(A(x, t))}{1 + \tau(t)\gamma_0(A(x, t))} - \overline{\phi}(\tau(t)) \right\} \tag{14}$$

where $A(x, t)$ is the inverse of $X(a, t)$ (the "back-to-labels" map) and the functions $\tau(t)$, $\alpha(\tau)$ and $\overline{\phi}(\tau)$ are computed from the initial datum γ_0. More precisely we show

Theorem 1 *Consider the nonlocal conservative Riccati system (4, 5, 8, 9, 10, 11). There exist smooth, mean zero initial data for which the solution becomes infinite in finite time. Both the maximum and the minimum values of the solution γ diverge, to plus infinity and respectively to negative infinity at the blow up time. There is no initial datum for which only the minimum diverges. The general solution is given on characteristics in terms of the initial data $\gamma(x, 0) = \gamma_0(x)$ by (see (24))*

$$\gamma(X(a, t), t) = \alpha(\tau(t)) \left(\frac{\gamma_0(a)}{1 + \tau(t)\gamma_0(a)} - \overline{\phi}(\tau(t)) \right)$$

where

$$\overline{\phi}(\tau) = \left\{ \int_Q \frac{\gamma_0(a)}{(1+\tau\gamma_0(a))^2} da \right\} \left\{ \int_Q \frac{1}{1+\tau\gamma_0(a)} da \right\}^{-1},$$

$$\alpha(\tau) = \left\{ \frac{1}{|Q|} \int_Q \frac{1}{1+\tau\gamma_0(a)} da \right\}^{-2}$$

and

$$\frac{d\tau}{dt} = \alpha(\tau), \quad \tau(0) = 0.$$

The function $\tau(t)$ *can be obtained from*

$$t = \left(\frac{1}{|Q|}\right)^2 \int_Q \int_Q \frac{1}{\gamma_0(a) - \gamma_0(b)} \log\left(\frac{1+\tau\gamma_0(a)}{1+\tau\gamma_0(b)}\right) dadb.$$

The Jacobian $J(a,t) = \mathrm{Det}\left\{\frac{\partial X(a,t)}{\partial a}\right\}$ *is given by*

$$J(a,t) = \frac{1}{1+\tau(t)\gamma_0(a)} \left\{ \frac{1}{|Q|} \int_Q \frac{da}{1+\tau(t)\gamma_0(a)} \right\}^{-1}$$

The moments of γ *are given by*

$$\int_Q (\gamma(x,t))^p dx = (\alpha(\tau))^p \int_Q \left\{ \frac{\gamma_0(a)}{1+\tau(t)\gamma_0(a)} - \overline{\phi}(\tau(t)) \right\}^p J(a,t) da.$$

The blow up time $t = T_*$ *is given by*

$$T_* = \frac{1}{|Q|^2} \int\int \frac{1}{\gamma_0(a) - \gamma_0(b)} \log\left(\frac{\gamma_0(a) - m_0}{\gamma_0(b) - m_0}\right) dadb$$

where

$$m_0 = \min_Q \gamma_0(a) < 0.$$

We note that the curl ω plays a secondary role in this calculation and in the blow up. Indeed, the same formula and blow up occurs if $\omega_0 = 0$, or if the curl ω was smooth and computed in a different fashion than via (4).

1 Solving on characteristics

We will solve now the nonlocal Riccati equation on characteristics. We start with an auxiliary problem. Let ϕ solve

$$\partial_\tau \phi + v \cdot \nabla \phi = -\phi^2$$

together with

$$\nabla \cdot v(x, \tau) = -\phi(x, \tau) + \frac{1}{|Q|} \int_Q \phi(x, \tau) dx.$$

We will consider initial data that are smooth, periodic and have zero mean,

$$\int_Q \phi_0(x) dx = 0.$$

We will also assume that the curl $\zeta = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$ obeys

$$\partial_\tau \zeta + v \cdot \nabla \zeta = \left(\phi - \frac{3}{|Q|} \int_Q \phi(x, \tau) dx \right) \zeta.$$

Passing to characteristics

$$\frac{dY}{d\tau} = v(Y, \tau) \tag{15}$$

we integrate and obtain

$$\phi(Y(a, \tau), \tau) = \frac{\phi_0(a)}{1 + \tau \phi_0(a)}$$

valid as long

$$\inf_{a \in Q} (1 + \tau \phi_0(a)) > 0.$$

We need to compute

$$\overline{\phi}(\tau) = \frac{1}{|Q|} \int_Q \phi(x, \tau) dx.$$

The Jacobian

$$J(a, \tau) = Det \left\{ \frac{\partial Y}{\partial a} \right\}$$

obeys

$$\frac{dJ}{d\tau} = -h(a,\tau)J(a,\tau)$$

where

$$h(a,\tau) = \phi(Y(a,\tau),\tau) - \overline{\phi}(\tau).$$

Initially the Jacobian equals to one, so

$$J(a,\tau) = e^{-\int_0^\tau h(a,s)ds}.$$

So

$$J(a,\tau) = e^{\int_0^\tau \overline{\phi}(s)ds} \exp\left(-\int_0^\tau \frac{d}{ds}\log(1+s\phi_0(a))ds\right)$$

and thus

$$J(a,\tau) = e^{\int_0^\tau \overline{\phi}(s)ds}\frac{1}{1+\tau\phi_0(a)}.$$

The map $a \mapsto Y(a,\tau)$ is one and onto. The change of variables formula gives

$$\int_Q \phi(x,\tau)dx = \int_Q \phi(Y(a,\tau),t)J(a,\tau)da$$

and therefore

$$\overline{\phi}(\tau) = e^{\int_0^\tau \overline{\phi}(s)ds}\frac{1}{|Q|}\int_Q \frac{\phi_0(a)}{(1+\tau\phi_0(a))^2}da. \tag{16}$$

Consequently

$$\frac{d}{d\tau}e^{-\int_0^\tau \overline{\phi}(s)ds} = \frac{d}{d\tau}\frac{1}{|Q|}\int_Q \frac{1}{1+\tau\phi_0(a)}da.$$

Because both sides at $\tau = 0$ equal one, we have

$$e^{-\int_0^\tau \overline{\phi}(s)ds} = \frac{1}{|Q|}\int_Q \frac{1}{1+\tau\phi_0(a)}da \tag{17}$$

and, using (16),

$$\overline{\phi}(\tau) = \left\{\int_Q \frac{\phi_0(a)}{(1+\tau\phi_0(a))^2}da\right\}\left\{\int_Q \frac{1}{1+\tau\phi_0(a)}da\right\}^{-1} \tag{18}$$

Note that the function $\delta(x,\tau) = \phi(x,\tau) - \overline{\phi}(\tau)$ obeys

$$\frac{\partial\delta}{\partial\tau} + v\cdot\nabla\delta = -\delta^2 + 2\frac{1}{|Q|}\int_Q \delta^2 dx - 2\overline{\phi}\delta.$$

We consider now the function

$$\sigma(x,\tau) = e^{2\int_0^\tau \overline{\phi}(s)ds}\delta(x,\tau)$$

and the velocity

$$U(x,\tau) = e^{2\int_0^\tau \overline{\phi}(s)ds}v(x,\tau).$$

Multiplying the equation of δ by $e^{4\int_0^\tau \overline{\phi}(s)ds}$ we obtain

$$e^{2\int_0^\tau \overline{\phi}(s)ds}\frac{\partial\sigma}{\partial\tau} + U\cdot\nabla\sigma = -\sigma^2 + \frac{2}{|Q|}\int\sigma^2 dx.$$

Note that

$$\nabla\cdot U = -\sigma.$$

Now we change the time scale. We define a new time t by the equation

$$\frac{dt}{d\tau} = e^{-2\int_0^\tau \overline{\phi}(s)ds}, \tag{19}$$

$t(0) = 0$, and new variables

$$\gamma(x,t) = \sigma(x,\tau)$$

and

$$u(x,t) = U(x,\tau).$$

Now γ solves the nonlocal conservative Riccati equation

$$\frac{\partial\gamma}{\partial t} + u\cdot\nabla\gamma = -\gamma^2 + \frac{2}{|Q|}\int\gamma^2 dx \tag{20}$$

with periodic boundary conditions,

$$u = (-\Delta)^{-1}\left[\nabla^\perp\omega + \nabla\gamma\right] \tag{21}$$

and

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = \gamma\omega \tag{22}$$

The initial data are given by

$$\gamma_0(x) = \delta_0(x) = \phi_0(x).$$

Using (17) and integrating the equation (19) we see that the time change is given by the formula

$$t = \left(\frac{1}{|Q|}\right)^2 \int_Q \int_Q \frac{1}{\phi_0(a) - \phi_0(b)} \log \frac{1 + \tau\phi_0(a)}{1 + \tau\phi_0(b)} dadb \tag{23}$$

Note that the characteristic system

$$\frac{dX}{dt} = u(X, t)$$

is solved by

$$X(a, t) = Y(a, \tau)$$

where Y solves the system (15). This implies the formula

$$\gamma(X(a, t), t) = \alpha(\tau) \left(\frac{\phi_0(a)}{1 + \tau\phi_0(a)} - \overline{\phi}(\tau)\right) \tag{24}$$

with

$$\alpha(\tau) = e^{2 \int_0^\tau \overline{\phi}(s) ds}. \tag{25}$$

In view of (17), (18), (23), we have obtained a complete description of the general solution in terms of the initial data.

2 Blow up

Consider an initial smooth function $\gamma_0(a) = \phi_0(a)$ and assume that it has mean zero and that its minimum is $m_0 < 0$. As it is evident from the explicit formula the blow up time for $\phi(Y(a, \tau), \tau)$ is

$$\tau_* = -\frac{1}{m_0}$$

and $\phi(Y(a, \tau), \tau)$ diverges to negative infinity for some a, and not at all for others. This of course does not necessarily mean that γ blows up in the same fashion. Let us discuss the simplest case, in which the minimum is attained at a finite number of locations a_0, and near these locations, the function ϕ_0 has non-vanishing second derivatives, so that locally

$$\phi_0(a) \geq m_0 + C|a - a_0|^2$$

for $0 \le |a - a_0| \le r$. Then it follows that the integral

$$\frac{1}{|Q|}\int \frac{da}{\epsilon^2 + \phi_0(a) - m_0}$$

behaves like

$$\frac{1}{|Q|}\int \frac{da}{\epsilon^2 + \phi_0(a) - m_0} \sim \log\left\{\sqrt{1 + \left(\frac{Cr}{\epsilon}\right)^2}\right\}$$

for small ϵ. Taking

$$\epsilon^2 = \frac{1}{\tau} - \frac{1}{\tau_*}$$

we deduce that, for these kinds of initial data

$$e^{-\int_0^\tau \overline{\phi}(s)ds} \sim \log\left\{\sqrt{1 + \frac{C}{\tau_* - \tau}}\right\}.$$

For the same kind of functions and small $(\tau_* - \tau)$, the integral

$$\frac{1}{|Q|}\int_Q \frac{\phi_0(a)}{(1 + \tau\phi_0(a))^2} da \sim -\frac{C}{\tau_* - \tau}$$

and $t(\tau)$ has a finite limit $t \to T_*$ as $\tau \to \tau_*$. The average $\overline{\phi}(\tau)$ diverges to negative infinity,

$$\overline{\phi}(\tau) \sim -\frac{C}{\tau_* - \tau}\left[\log\left\{\sqrt{1 + \frac{C}{\tau_* - \tau}}\right\}\right]^{-1}.$$

The prefactor α becomes vanishingly small

$$\alpha(\tau) \sim (\log(\tau_* - \tau))^{-2}$$

and (24) becomes

$$\gamma(X(a,t),t) \sim (\log(\tau_* - \tau))^{-2}\left(\frac{\phi_0(a)}{1 + \tau\phi_0(a)} - \overline{\phi}(\tau)\right).$$

If the label is chosen so that $\phi_0(a) > 0$ then the first term in the brackets does not blow up and γ diverges to plus infinity. If the label is chosen at the minimum, or nearby, then the first term in the brackets dominates and the blow up is to negative infinity, as expected from the ODE. From the equation (19)

$$(\alpha(\tau))^{-1}d\tau = dt$$

it follows that

$$T_* - t \sim (\tau_* - \tau)\left(1 + \log\left(\frac{1}{\tau_* - \tau}\right)\right)^2$$

and the asymptotic behavior of the blow up in t follows from the one in τ. We end by addressing a question that was at some point raised by numerical simulations: can there be a one-sided blow up? From the representation (24) of the solution it follows that

$$M(t) \geq -\overline{\phi}(\tau)e^{2\int_0^\tau \overline{\phi}(s)ds}$$

holds for $M(t) = \sup_x \gamma(x,t)$. If one would assume that, up to the putative blow up

$$M(t) \leq C$$

with some fixed constant C, then it would follow that

$$-\frac{d}{d\tau}e^{2\int_0^\tau \overline{\phi}(s)ds} \leq 2C$$

and integrating between τ and τ_* that

$$e^{2\int_0^\tau \overline{\phi}(s)ds} \leq 2C(\tau_* - \tau).$$

This in turn would imply that $T_* = \infty$ and therefore no blow up for γ can occur in finite t. So the answer is that for no initial datum can there exist a one sided blow up for γ.
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