Near identity transformations for the Navier-Stokes equations

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1 Introduction

Ordinary incompressible Newtonian fluids are described by the Navier-Stokes equations. These equations have been used by engineers and physicists with a great deal of success and the range of their validity and applicability is well established. Together with other fundamental systems like the Schrödinger and Maxwell equations, these equations are among the most important equations of mathematical physics. Nevertheless, their mathematical theory is incomplete and requires cut-offs. The present state of knowledge is such that different approximations seem to be useful for different purposes. The mathematical questions of existence and regularity for incompressible fluid equations have been discussed in many books and review articles (for instance [1], [21], [35], [47], [10], [16], [36], [43], [46] and many more). In this work I describe some results reflecting research concerning diffusive-Lagrangian aspects of the Navier-Stokes equations [12], [14]. There are two distinct classes of approximations of the Navier-Stokes equations that we consider. In one class the energy dissipation is treated exactly, but the vorticity equation is not exact. This class contains the Galerkin approximations [21], [47] and mollified equations [3], [35] (see (11) below). The other class treats the vorticity equation exactly but the energy dissipation is approximated. This is the class of vortex methods [6] (see (12) below) and their generalizations. This class is related by a change of variables to a class of filtered approximations (13) of the formulation [33], [39]; the models [31], [4] are a subclass of these. The Navier-Stokes equations and their various approximations can be described in terms of near identity transformations. These are diffusive particle path transformations of physical space that start from the identity. The active velocity is obtained from the diffusive path transformation and a virtual velocity using the Weber formula. The active vorticity is computed from the diffusive path transformation and a virtual vorticity using a Cauchy formula. The path transformation and the virtual fields are computed in Eulerian coordinates ("laboratory frame"). In the absence of kinematic viscosity, both the virtual velocity and the virtual vorticity are passively transported ("frozen in") by the flow. In the presence of viscosity, these fields obey diffusion equations with coefficients that are proportional to the kinematic viscosity and are derived from the diffusive transformations. The diffusive path transformations are used for short time intervals, as long as the transformations do not stray too much from the identity. The duration of these intervals is determined by the requirement of invertibility of the gradient map. If and when the viscosity-induced change in the Jacobian reaches a pre-assigned level, one stops, and one restarts the calculation from the identity transformation, using as initial virtual field the previously computed active field.

2 Energy Dissipation

An incompressible fluid of constant density and temperature can be described in terms of the fluid velocity $u(x,t)$ and pressure $p(x,t)$, functions of Eulerian (laboratory) coordinates x, representing position and t, representing time. The Navier-Stokes equations are an expression of Newton's second law, and in the absence of external sources of energy they are

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \tag{1}$$

coupled with the constraint of incompressibility

$$\nabla \cdot u = 0. \tag{2}$$

In the initial value problem, a velocity

$$u(x,0) = u_0 \tag{3}$$

is given at $t = 0$. The coefficient $\nu > 0$ is the kinematic viscosity. In an idealized situation the velocity is defined for all $x \in \mathbf{R}^3$ and vanishes at infinity. The vorticity equation is obtained by taking the curl of the Navier-Stokes equations:

$$\partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = \omega \cdot \nabla u \tag{4}$$

where

$$\omega = \nabla \times u \tag{5}$$

is the vorticity. If ω is divergence-free one may invert this relation: Defining a stream-vector ψ that satisfies $-\Delta \psi = \omega$ using the Newtonian potential and then taking its curl, one obtains the familiar Biot-Savart law

$$u(x) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{1}{|x-y|^3} (x-y) \times \omega(y) dy. \tag{6}$$

If the initial velocity vanishes then $u(x,t) = 0$, $p(x,t) = 0$ solve the equations. Moreover, if u_0 is close to $u_0 = 0$ then the solution $u(x,t)$, $p(x,t)$ exists for all time, is smooth and converges to 0. The open question in this situation concerns the behavior of the solution for large initial data. The Navier-Stokes equations are a parabolic regularization of the Euler equations (obtained by setting $\nu = 0$). Although the viscous term is important, for the study of large data one needs to consider properties of the Euler equations. The Navier-Stokes equations conserve momentum (integral of velocity in the present setting). The total kinetic energy

$$\frac{1}{2} \int_{\mathbf{R}^3} |u(x,t)|^2 dx = K(t) \tag{7}$$

is dissipated by viscosity

$$\frac{1}{2} \int_{\mathbf{R}^3} |u(x,t)|^2 dx + \nu \int_{t_0}^{t} \int_{\mathbf{R}^3} |\nabla u(x,s)|^2 dx ds \le \frac{1}{2} \int_{\mathbf{R}^3} |u(x,t_0)|^2 dx. \tag{8}$$

The dissipation of kinetic energy is the strongest source of quantitative information about the Navier-Stokes equations that is presently known for all solutions. This dissipation is used to construct Leray weak solutions with finite kinetic energy that exist for all time, $u \in L^\infty(dt; L^2(dx))$, $\nabla u \in L^2(dt \otimes dx)$

[34]. This class of solutions is very wide. The solutions have partial regularity [3] but are not known to be smooth. The uniqueness of the Leray weak solutions is not known. Neveretheless, immediately after inception, at positive times arbitrarily close to the initial time, the Leray solutions have square integrable gradients and the solutions become smooth for an interval of time. The solution is then uniquely determined and remains smooth for an interval of time whose duration is bounded below by a non-zero constant. The issue is whether the smooth behavior continues for all time. The simplest self-similar blow up ansatz of Leray has been ruled out [37], [48]. The most important task is to obtain good a priori bounds for smooth solutions of the Navier-Stokes equations. If one has good bounds then the smoothness and uniqueness of the solution can be shown to persist [43], [44], [47], [21], [35]. In situations in which such bounds are not available, the study of solutions of the Navier-Stokes equations needs to be pursued by considering long-lived approximate solutions. The advantage of dealing with approximations, besides practicality, is conceptual simplicity: one may formulate sufficient conditions for global regularity quantitatively, in terms of the approximate solutions. If one devises approximations and obtains uniform bounds for them, then, by removing the approximation, one obtains rigorous bounds for weak solutions of the Navier-Stokes equations that are valid for all time. For instance one can prove:

Theorem 1. *Let u_0 be a function in $L^2(\mathbf{R}^3)$, that satifies the divergence-free condition $\nabla \cdot u_0 = 0$ in the sense of distributions. Let $T > 0$ be arbitrary. There exists a Leray weak solution $(u(x,t),(p(x,t))$ of the Navier-Stokes equations that is defined for $t \in [0,T]$, satisfies*

$$\frac{1}{2}\int_{\mathbf{R}^3} |u(x,t)|^2 dx + \nu \int_{t_0}^{t}\int_{\mathbf{R}^3} |\nabla u(x,s)|^2 dxds \le \frac{1}{2}\int_{\mathbf{R}^3} |u(x,t_0)|^2 dx$$

for all $t \ge t_0$ and $t_0 \in I \subset [0,T]$, where I is a set of full measure $|I| = T$. The initial time belongs to it, $0 \in I$, in other words $t_0 = 0$ is allowed. In addition, the solution satisfies

$$\int_0^T \|u(\cdot,s)\|_{L^\infty} ds \le K_\infty$$

with K_∞ a length scale determined mainly by the initial kinetic energy and viscosity:

$$K_\infty = C\left\{\nu^{-2}\int_{\mathbf{R}^3} |u_0(x)|^2 dx + \sqrt{\nu T}\right\}.$$

If the initial vorticity $\omega_0 = \nabla \times u_0$ is in L^1, $\int_{\mathbf{R}^3} |\omega_0| dx < \infty$ then it remains bounded in L^1 and moreover

$$\int_{\mathbf{R}^3} |\omega(x,t)| dx + \frac{1}{2\nu}\int_{\mathbf{R}^3} |u(x,t)|^2 dx \le \int_{\mathbf{R}^3} |\omega(x,t_0)| dx + \frac{1}{2\nu}\int_{\mathbf{R}^3} |u(x,t_0)|^2 dx$$

for $t \geq t_0$, $t_0 \in I$. If the initial data is in H^1 and the initial Reynolds number is small then the solution is infinitely differentiable for positive time and converges to 0. More precisely, if

$$R_0 = \frac{1}{\nu}\left(\int_{\mathbf{R}^3} |u_0(x)|^2 dx\right)^{\frac{1}{4}} \left(\int_{\mathbf{R}^3} |\nabla u_0(x)|^2 dx\right)^{\frac{1}{4}} < 2\pi^{-\frac{1}{2}} 3^{-\frac{3}{4}}$$

then the solution exists for all $t > 0$, belongs to $\mathcal{C}^\infty(\mathbf{R}^3)$ and converges to 0.

This theorem combines the bound in [8] that was proved using a version of the retarded mollification approximation procedure of [3] with the result of [30] that was proved in the space-periodic case using Galerkin approximations. The last statement about smooth solutions is proved by studying the evolution of the product of energy and enstrophy. The specific constant (about .495) comes from the fact that, for divergence-free, zero-mean functions $\|u\|_{L^\infty} \le \sqrt{4\pi}\sqrt{\|\nabla u\|_{L^2}}\sqrt{\|\Delta u\|_{L^2}}$ (see (22)) below. If the initial Reynolds number is small then it stays small and its rate of dissipation is a well-known quantity that controls global existence (see (23) below). Both the mollification approximation and the Galerkin truncation approximation procedure respect the energy dissipation inequality (8) exactly, but they introduce errors in the vorticity equation. The (not retarded) mollification equation is described below. One defines a mollified u by

$$[u]_\delta = \int_{\mathbf{R}^3} \delta^{-3} J\left(\frac{x-y}{\delta}\right) u(y) dy = J_\delta(-i\nabla)u. \tag{9}$$

Here $\delta > 0$ and the positive kernel J is normalized $\int_{\mathbf{R}^3} J(x)dx = 1$, smooth and decays sufficiently fast at infinity. Two canonical examples of such J are the Poisson kernel $J(x) = \pi^{-2}(1+|x|^2)^{-2}$ and the Gaussian $J(x) = (2\pi)^{-3/2}e^{-|x|^2/2}$. The Fourier transforms of J, $\widehat{J}(\xi) = e^{-|\xi|}$ and, respectively $\widehat{J}(\xi) = exp\left(-\frac{|\xi|^2}{2}\right)$, are non-negative, vanish at the origin, decay rapidly and are bounded above by 1. Because of the fact that at the Fourier transform level one has

$$\widehat{[u]_\delta}(\xi) = \widehat{J}(\delta\xi)\widehat{u}(\xi), \tag{10}$$

the operator of convolution with J_δ, $[u]_\delta = J_\delta(-i\nabla)u$ is a classical smoothing approximation of the identity. The mollified equation is

$$\partial_t u + [u] \cdot \nabla u - \nu \Delta u + \nabla p = 0 \tag{11}$$

together with $\nabla \cdot u = 0$. Here $[u] = [u]_\delta$ is computed by applying the mollifier at each instance of time. This nonlinear partial differential equation has global solutions for arbitrary divergence-free initial data $u_0 \in L^2$. The solutions are smooth on $(0,T] \times \mathbf{R}^3$ and, moreover the energy inequality (8) is valid for any $t_0 \in [0,T]$, $t \geq t_0$. The vorticity of the mollified equation does not obey exactly (4). By contrast, classical vortex methods [6] respect the structure of the vorticity equation (4) but do not obey exactly the energy dissipation inequality (8). In this paper we call vortex methods the equations

$$\partial_t \omega + [u] \cdot \omega - \nu \Delta \omega = \omega \cdot \nabla [u] \tag{12}$$

with u calculated from ω using the Biot-Savart law (6), and $[u] = [u]_\delta$ computed from u using the mollifier (9). Both equation and solutions depend on δ but we will keep notation light by dropping the reference to this dependence: $\omega = \omega_\delta$, $[u] = [u_\delta]_\delta$. These vortex methods may also be described by using an auxilliary variable w. One considers the equation

$$\partial_t w + [u] \cdot \nabla w - \nu \Delta w + (\nabla [u])^* \, w = 0. \tag{13}$$

(M^* means the transposed matrix). A direct calculation verifies that the curl of w , $\nabla \times w$ obeys the equation (12), as does ω. This calculation uses only the fact that $[u]$ is divergence-free. The system formed by the equation (13) coupled with

$$[u] = J_\delta(-i\nabla)\mathbf{P}(w) \tag{14}$$

is equivalent to (12, 6, 9). Here $\mathbf{P}$,

$$\mathbf{P}_{jl} = \delta_{jl} - \partial_j \Delta^{-1} \partial_l \tag{15}$$

is the Leray-Hodge projector on divergence-free vectors. The initial w is required to satisfy $\mathbf{P}w_0 = u_0$. At fixed positive δ the solution is smooth and global. If $\nu = 0$ then these systems have a Kelvin circulation theorem: the integral $\oint_\gamma w \cdot dx$ is conserved along closed paths γ that are transported by the flow of $[u]$. (In contrast, the mollified equations do not have a Kelvin circulation theorem). The energy dissipation principle for the vortex method is

$$\frac{1}{2}\int_{\mathbf{R}^3} u(x,t)\cdot[u](x,t)dx + \nu\int_{t_0}^{t}\int_{\mathbf{R}^3} \mathbf{tr}\left\{(\nabla u)(x,s)(\nabla[u](x,s))^*\right\}dxds \le$$

$$\frac{1}{2}\int_{\mathbf{R}^3} u(x,t_0)\cdot[u(x,t_0)]dx \tag{16}$$

This is obtained by taking the scalar product of (12) with $[\psi]$ where $u = \nabla\times\psi$. One can obtain the energy dissipation principle also by taking the scalar product of (13) with $[u]$. One uses the fact that $J_\delta(-i\nabla)$ is a scalar operator (multiple of the identity as a matrix, i.e. acts separately on each component of a vector) that commutes with differentiation. Then the cancellation of the nonlinearity follows from the divergence free condition. The energy dissipation principle gives strong control on the mollified quantities (or weak control on the unmollified ones):

$$\frac{1}{2}\int_{\mathbf{R}^3} u(x,t)\cdot[u](x,t)dx = \frac{1}{2}\int_{\mathbf{R}^3}\left(\widehat{J}(\delta\xi)\right)^{-1}\left|\widehat{[u]}(\xi,t)\right|^2 d\xi \tag{17}$$

and

$$\int_{\mathbf{R}^3} \mathbf{tr}\left\{(\nabla u)(x,s)(\nabla[u](x,s))^*\right\}dx = \int_{\mathbf{R}^3}\left(\widehat{J}(\delta\xi)\right)^{-1}|\xi|^2\left|\widehat{[u]}(\xi,t)\right|^2 d\xi \tag{18}$$

Because $\widehat{J}^{-1}$ is a positive function that grows exponentially at infinity, the inequality implies real analytic control on $[u]$:

$$\frac{1}{2}\int_{\mathbf{R}^3}\left|(J_\delta(-i\nabla))^{-\frac{1}{2}}([u])(x,t)\right|^2 dx +$$

$$\int_0^t\int_{\mathbf{R}^3}\left|\nabla(J_\delta(-i\nabla))^{-\frac{1}{2}}([u])(x,s)\right|^2 dxds \le \frac{1}{2}\int_{\mathbf{R}^3}|u_0(x)|^2 dx \tag{19}$$

One needs to bear in mind, however, that this is a weaker bound than the bound provided by the energy dissipation (8) for the mollified equation (11), where $(J_\delta(-i\nabla))^{-1}[u]$ is bounded in L^2.

3 Uniform bounds

The energy dissipation principle (8) holds exactly for the mollified equation (11) and has a counterpart for the vortex method (12) in (16, 19). These are uniform inequalities, in the sense that the coefficients are δ-independent and the right hand sides are bounded uniformly for all $\delta > 0$. Most uniform bounds are inherited by the solution of the Navier-Stokes equations by passage to limit. Some uniform bounds for the equation (11) can be summarized as follows:

Theorem 2. *Let u_0 be a square-integrable, divergence-free function. Let $\delta > 0$. Then there exits a unique solution (u, p) of (11) defined for all $t > 0$. The solution is real analytic for positive times. The limit $\lim_{t\to 0} u(x,t) = u_0(x)$ holds in a weak sense in L^2. The energy inequality (8) holds for any $0 \le t_0 \le t$. The uniform bound*

$$\int_0^T \|u(\cdot, s)\|_{L^\infty} ds \le K_\infty$$

holds with

$$K_\infty = C\left\{\nu^{-2}\int_{\mathbf{R}^3} |u_0(x)|^2 dx + \sqrt{\nu T}\right\}$$

and C a universal constant, independent of δ. If the initial vorticity $\omega_0 = \nabla \times u_0$ is in L^1, $\int_{\mathbf{R}^3} |\omega_0| dx < \infty$ then it remains bounded in L^1 and moreover

$$\int_{\mathbf{R}^3} |\omega(x,t)|dx + \frac{1}{2\nu}\int_{\mathbf{R}^3} |u(x,t)|^2 dx \le \int_{\mathbf{R}^3} |\omega(x,t_0)|dx + \frac{1}{2\nu}\int_{\mathbf{R}^3} |u(x,t_0)|^2 dx$$

for all $t \ge t_0$. In addition the vorticity direction

$$\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}$$

defined in the region $\{x | |\omega(x,t)| > 0\}$ satisfies

$$\int_0^t \int_{\{x||\omega(x,s)|>0\}} |\omega(x,s)| \left|\nabla \xi(x,s)\right|^2 dx ds \le \frac{1}{2}\nu^{-2}\int_{\mathbf{R}^3} |u_0(x)|^2 dx.$$

The proof of this result starts with the vorticity equation for the mollified equation:

$$\left(\partial_t + [u]\cdot\nabla - \Delta\right)\omega_i + \epsilon_{ijk}\partial_j([u]_l)(\partial_l u_k) = 0. \tag{20}$$

Here ϵ_{ijk} is the signature of the permutation $(1,2,3) \mapsto (i,j,k)$ and repeated indices are summed. Multiplying scalarly by ξ one obtains

$$(\partial_t + [u]\cdot\nabla - \Delta)\,|\omega| + \nu|\omega||\nabla\xi|^2 + Det(\xi, \nabla[u_l], \partial_l u) = 0$$

where $Det(a,b,c)$ is the determinant of the matrix formed by the three vectors a, b, c. Integrating in space and using the energy dissipation one can deduce the bounds for ω in L^1 and the bound on the direction ξ. For the bound on u in L^∞ one uses the enstrophy differential inequality

$$\frac{d}{2dt}\int_{\mathbf{R}^3}|\omega(x,t)|^2 + \nu\int_{\mathbf{R}^3}|\nabla\omega(x,t)|^2dx \le$$

$$\sqrt{4\pi}\left(\int_{\mathbf{R}^3}|\omega(x,t)|^2dx\right)^{\frac{3}{4}}\left(\int_{\mathbf{R}^3}|\nabla\omega(x,t)|^2dx\right)^{\frac{3}{4}}. \tag{21}$$

This is obtained from the vorticity equation (20) above by multiplication by ω, integration by parts, and use of bound (see (27) below)

$$\|u\|_{L^\infty} \le \sqrt{4\pi}\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}. \tag{22}$$

Then one employs the idea of [30]: one divides by $(c^2 + \int_{\mathbf{R}^3}|\omega(x,t)|^2dx)^2$, and integrates in time using the energy principle, ($c > 0$ is a constant). One obtains a bound for

$$\int_0^t\left\{\int_{\mathbf{R}^3}|\nabla\omega(x,s)|^2dx\right\}^{\frac{1}{3}}ds$$

in terms of the initial data. The L^∞ bound follows from interpolation (22), the bound above, and the energy principle. We omit further details.

4 Non-uniform bounds

If u is a smooth solution of the Navier-Stokes equations and if

$$\int_0^T\left(\int_{\mathbf{R}^3}|\omega(x,t)|^2dx\right)^2dt \le D^3 < \infty \tag{23}$$

then one can bound any derivative of u on $(0, T]$ in terms of the initial data, viscosity, T and D (see [21], and references therein). Similarly, if one has a bound

$$\int_0^T \|u(\cdot, t)\|^2_{L^\infty(dx)} dt \leq B < \infty \tag{24}$$

then one can bound any derivative of u on $(0, T]$ in terms of the initial data, viscosity, T and B. The quantities D and B have same dimensional count as viscosity (units of length squared per time). If one has a regularization that respects the energy dissipation and one has uniform bounds for the corresponding quantities then one can prove global existence of smooth solutions. If any of the two conditions is met then the solution is real analytic for positive times. Consider the mollified equation (11) at $\delta > 0$. Assuming for instance (23) one obtains bounds for $\sup_{t\leq T} \int_{\mathbf{R}^3} |\omega(x,t)|^2 dx$ and for $\int_0^T \int_{\mathbf{R}^3} |\nabla\omega(x,t)|^2 dxdt$ in terms of initial data, D and T, directly from (21). The interpolation (22) then produces a bound for B. Vice-versa, if one has the assumption (24) then one does not uses interpolation when one derives the enstrophy inequality from (20); rather, one integrates by parts to reveal u and one uses directly the assumption about $\|u\|_{L^\infty}$ to deduce a uniform bound for the maximum enstrophy in the time interval $[0, T]$:

$$\sup_{t\leq T} \int_{\mathbf{R}^3} |\omega(x,t)|^2 dx \leq \mathcal{E} < \infty \tag{25}$$

This allows to bound D. In either case, the number $\mathcal{E}$ depends on the numbers D (respectively B) of assumptions (23) (respectively (24)). By increasing $\mathcal{E}$, if necessary, we may assume, without loss of generality the condition

$$\rho = \mathcal{E}^2 T \nu^{-3} > 1. \tag{26}$$

This condition reflects the fact that we are not pursuing decay estimates. If no assumption is made then $\mathcal{E}$ depends on $\delta > 0$. Once the enstrophy is bounded in time, higher derivatives are bounded using the Gevrey-class method of [28].

Theorem 3. *Let $\delta > 0$. Consider solutions of (11) with initial data $u_0 \in L^2$, $\omega_0 \in L^2$. Assume that one of the inequalities (23) or (24) holds on the interval of time $[0, T]$. Then there exists a constant $c_0 \in (0, 1)$ depending* *only on the number $\rho = \rho(\mathcal{E}, \nu, T)$ of (26), so that*

$$\sup_{t_0 \le t \le T} \int_{\mathbf{R}^3} e^{2\lambda|\xi|} |\widehat{\omega}(\xi, t)|^2 d\xi \le 2\mathcal{E}$$

holds for all $0 < t_0 \le T$. Here $\lambda = \sqrt{\nu T} \min\left\{\frac{t_0}{T}; c_0\right\}$. If $\mathcal{E}$ is uniform in δ as $\delta \to 0$, then the solution of the Navier-Stokes equations with initial data u_0 is real-analytic and obeys the bound above.

Using the fact that $u = \nabla \times \psi$ with divergence-free stream vector ψ one sees easily that

$$|\widehat{u}(\xi)| \le \frac{1}{|\xi|} |\widehat{\omega}(\xi)|$$

holds pointwise. It is elementary then to check that

$$\int_{\mathbf{R}^3} e^{\lambda|\xi|} |\widehat{u}(\xi)| d\xi \le \sqrt{4\pi} \left\{ \int_{\mathbf{R}^3} e^{2\lambda|\xi|} |\widehat{\omega}(\xi)|^2 d\xi \right\}^{\frac{1}{4}} \left\{ \int_{\mathbf{R}^3} e^{2\lambda|\xi|} |\xi|^2 |\widehat{\omega}(\xi)|^2 d\xi \right\}^{\frac{1}{4}} \tag{27}$$

holds for any positive λ. Using the Fourier transform of (11) and the inequality above one may follow closely the idea of [28]. One considers the quantity

$$y(t) = \int_{\mathbf{R}^3} e^{2v(t-s)|\xi|} \, |\widehat{\omega}(\xi, t)|^2 \, d\xi$$

with $v = \sqrt{\frac{\nu}{T}}$ and $0 \le s \le t \le T$ and derives a differential inequality of the form

$$\frac{dy}{dt} \le c \left(\frac{y}{\nu}\right)^3 + c\frac{v^2}{\nu} y$$

with absolute constant c. At $t = s$ one has by construction $y(s) \le \mathcal{E}$. The differential inequality guarantees that y does not exceed $2\mathcal{E}$ on a time interval $s \le t \le s + 2c_0T$ with c_0 a small non-dimensional constant (proportional to $\rho^{-1} = \nu^3 T^{-1} \mathcal{E}^{-2}$). The time step $2c_0T$ is uniform because of the assumption that $\mathcal{E}$ is finite. One starts from $s = 0$. The differential inequality implies a non-trivial Gevrey-class bound for the second half of the first time interval $[0, 2c_0T]$. One now sets $s = c_0T$ and makes another step of duration $2c_0T$. At each step the second half of the time interval yields a nontrivial bound, and because the initial point s is advanced by a half step one covers all of $[c_0T, T]$. The pre-factor in the exponential bound is uniformly bounded below by $2c_0 vT$. If $t_0 \le c_0 T$ we can advance $2t_0$ at a time, and obtain similarly a uniform bound on $[t_0, T]$, with exponent pre-factor bounded below by $2t_0 v$. This result implies that the velocity is real analytic. One may extend u to a complex domain $z = x + iy$ by setting

$$u(x+iy,t) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{i(x+iy)\cdot\xi} \widehat{u}(\xi,t) d\xi$$

and, in view of (27), the integral is absolutely convergent for $|y| \le \lambda$, $t \ge t_0$. Note also that, at fixed $\delta > 0$ one has a finite, δ-dependent bound on $\mathcal{E}$, and therefore the solutions of the mollified equations are real-analytic. Likewise, if one allows for δ dependent bounds, then the vortex method also can be shown to have real-analytic solutions. In this case, however, because the energy principle is not strong enough one starts from the assumption (25):

Theorem 4. *Let the initial vorticity ω_0 belong to L^2. Consider, for $\delta > 0$ the solution of (12) and assume that (25) holds on a time interval $[0,T]$. Then there exists a constant $c_0 \in (0,1)$ depending only on the number $\rho = \rho(\mathcal{E}, \nu, T)$ of (26), so that*

$$\sup_{t_0 \le t \le T} \int_{\mathbf{R}^3} e^{2\lambda|\xi|} |\widehat{\omega}(\xi,t)|^2 d\xi \le 2\mathcal{E}$$

holds for all $0 < t_0 \le T$. Here $\lambda = \sqrt{\nu T} \min\left\{\frac{t_0}{T}; c_0\right\}$. If $\mathcal{E}$ is uniform in δ as $\delta \to 0$, then the solution of the Navier-Stokes equations (4) with initial data ω_0 is real-analytic and obeys the bound above.

The proof follows the same ideas as the proof of the corresponding result for the mollified equation (11).

5 Euler equations

The three dimensional Euler equations

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot u = 0 \tag{28}$$

are locally well-posed [26], [32]. They conserve kinetic energy (if the solutions are smooth enough [38], [27], [18]). Such smooth solutions can be interpreted [1] as geodesic paths on an infinite dimensional group of transformations.

Despite of energy conservation, gradients of solutions may grow [42]. The vorticity $\omega = \nabla \times u$ obeys

$$(\partial_t + u \cdot \nabla)\,\omega = \omega \cdot \nabla u. \tag{29}$$

Because of the quadratic nature of this equation and the fact that the strain matrix

$$S = \frac{1}{2}\left\{(\nabla u)) + (\nabla u)^*\right\} \tag{30}$$

is related to the vorticity by a linear classical singular Calderon-Zygmund integral, it was suggested [36] that blow up of the vorticity might occur in finite time. This problem is open, despite much research [5], [7], [11], [22], [29], [45]. The blow up cannot occur unless the time integral of the maximum modulus of vorticity diverges [2]. The vorticity magnitude obeys

$$(\partial_t + u \cdot \nabla)\,|\omega| = \alpha|\omega|, \tag{31}$$

where the logarithmic material stretching rate α can be represented [9] as

$$\alpha(x,t) = \frac{3}{4\pi}P.V.\int D(\widehat{y}, \xi(x-y,t), \xi(x,t))|\omega(x-y,t)|\frac{dy}{|y|^3}. \tag{32}$$

Here $\widehat{y} = \frac{y}{|y|}$,

$$D(\widehat{y}, \xi(x-y,t), \xi(x,t)) = (\widehat{y} \cdot \xi(x,t))\, Det(\widehat{y}, \xi(x-y,t), \xi(x,t)) \tag{33}$$

and

$$\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}. \tag{34}$$

In the two dimensional case $\xi = (0,0,1)$ and so $\alpha = 0$. In three dimensions, if the vorticity direction is well-behaved locally in regions of high vorticity, then there is a geometric depletion of nonlinearity. More precisely, if $\xi(x,t) \times \xi(x-y,t)$ vanishes in a quantitatively controlled fashion as $y \to 0$, (for instance $|\xi(x-y,t) \times \xi(y,t)| \leq k|y|$), then α can be bounded in terms of less singular integrals (for instance in terms of velocity instead of vorticity). This observation suggests a correlation between vorticity growth and the geometry of vortex tubes. Such a correlation has been observed in numerical studies and was exploited to prove conditional results regarding blow up for the Euler equations, [9], [20], the Navier-Stokes equations [19], and the quasi-geostrophic model [25]. The quasi-geostrophic model [9], [17], [23], [24], [40] is an example of an active scalar. Active scalars are advection-diffusion evolution equations for scalar quantities advected by an incompressible velocity they create: the velocity is obtained from the scalar using a fixed, time-independent formula:

$$\left(\partial_t + u\cdot\nabla - \kappa\Delta\right) q = 0 \quad u = U[q]. \tag{35}$$

The Euler equations themselves are an active vector system [12], [13]:

$$\left(\partial_t + u\cdot\nabla\right) A = 0, \tag{36}$$

with

$$u = W[A] \tag{37}$$

the Weber formula [43]

$$u(x,t) = \mathbf{P}\left((\nabla A)^* v\right). \tag{38}$$

Here $\mathbf{P}$ is the Leray-Hodge projector on divergence-free functions, and

$$v(x,t) = u_0(A(x,t)) \tag{39}$$

is a solution of

$$\left(\partial_t + u\cdot\nabla\right) v = 0 \tag{40}$$

The initial data for A is the identity

$$A(x,0) = x, \tag{41}$$

and the initial datum for v is u_0. Thus $A(x,t) = a$ is the inverse of the Lagrangian path $a \mapsto X(a,t)$. The familiar Cauchy formula is in this language

$$\omega_q = \frac{1}{2}\epsilon_{qij}\left(Det\left[\zeta; \frac{\partial A}{\partial x_i}; \frac{\partial A}{\partial x_j}\right]\right). \tag{42}$$

Here

$$\zeta(x,t) = \omega_0(A(x,t)) \tag{43}$$

is the solution of

$$(\partial_t + u \cdot \nabla)\,\zeta = 0 \tag{44}$$

with initial datum $\zeta(x, 0) = \omega_0(x)$.

One may use a near-identity approach to the incompressible Euler equations: One solves (36) for short time, as long as $\nabla A - I$ is not too large. Then one stops and resets $u = W[A]$ in place of u_0, sets $A = x$, adjusts the clock, and starts again. This approach allows one to interpret the condition [2] of absence of blow up in terms of ∇A : if

$$\int_0^T \|\nabla A(\cdot, t)\|_{L^\infty}^2 dt < \infty$$

then the solution of the Euler equations is smooth [12].

6 Diffusive Lagrangian transformations

The central object in the Lagrangian description of fluids is the Lagrangian path transformation $a \mapsto X(a, t)$; $x = X(a, t)$ represents the position at time t of the fluid particle that started at $t = 0$ from a. At time $t = 0$ the transformation is the identity, $X(a, 0) = a$. An Eulerian-Lagrangian formulation of the Navier-Stokes equations [14] parallels the active vector formulation of the Euler equations [12]. In order to unify the exposition we associate to a given divergence-free velocity $u(x, t)$ the operator

$$\partial_t + u \cdot \nabla - \nu\Delta = \Gamma_\nu(u, \nabla). \tag{45}$$

Using $\Gamma_\nu(u, \nabla)$ we associate to any divergence-free, time dependent velocity field u a transformation $x \mapsto A(x, t)$ that obeys

$$\Gamma_\nu(u, \nabla)A = 0, \tag{46}$$

with initial data

$$A(x, 0) = x \tag{47}$$

Boundary conditions are imposed by considering the displacement vector

$$\ell(x, t) = A(x, t) - x \tag{48}$$

that joins the Eulerian position x to the the diffusive label A. This is required to vanish at infinity, and one can think of A as being computed by solving

$$(\partial_t + u \cdot \nabla - \nu\Delta)\, \ell + u = 0 \tag{49}$$

with initial data

$$\ell(x, 0) = 0. \tag{50}$$

If u is the solution of the Euler equation and $\nu = 0$ in the equations (46, 49) then the map A is the inverse of the particle trajectory map $a \mapsto x = X(a,t)$. In the presence of viscosity this map obeys a diffusive equation, departing thus from its conventional interpretation as inverse of particle trajectories. Nevertheless, continuing the analogy with the inviscid situation, one uses the map $x \mapsto A(x,t)$ to pull back the Lagrangian differentiation with respect to particle position and write it in Eulerian coordinates. This Eulerian-Lagrangian derivative is given by

$$\nabla^A = Q^* \nabla \tag{51}$$

where

$$Q(x,t) = (\nabla A(x,t))^{-1}, \tag{52}$$

that is

$$\nabla_i^A = Q_{ji}\partial_j. \tag{53}$$

In the case $\nu = 0$ the invertibility of ∇A follows from incompressibility; in the diffusive case the determinant of ∇A does not remain identically equal to one as time passes. This imposes a constraint on the time of integration. We consider a small non-dimensional parameter $g > 0$ and work with the constraint

$$\sup_{0 \le t \le \tau} \sup_{x \in \mathbf{R}^3} |\nabla \ell(x,t)| \le g. \tag{54}$$

With this constraint satisfied we can guarantee the invertibility of ∇A. In order to describe the dynamics and their relationship to the Eulerian dynamics one needs to consider second derivatives of A. These influence the dynamics because commutators between Eulerian-Lagrangian and Eulerian derivatives do not vanish, in general:

$$\left[\nabla_i^A, \nabla_k\right] = C_{k;i}^m \nabla_m^A. \tag{55}$$

The coefficients $C_{k;i}^m$ are given by

$$C_{k;i}^m = \{\nabla_i^A(\partial_k \ell_m)\}. \tag{56}$$

Note that

$$C_{k;i}^m = Q_{ji}\partial_j\partial_k A^m = \nabla_i^A(\partial_k A^m) = [\nabla_i^A, \partial_k]A^m.$$

These commutator coefficients are related to the Christoffel coefficients of the usual flat connection in $\mathbf{R}^3$ computed using the change of variables $a = A(x,t)$. With this change of variables, a straight line in x, $x(s) = ms + b$ becomes the label path $a(s) = A(x(s),t)$ and the geodesic equation $\frac{d^2x}{ds^2} = 0$ becomes $\frac{d^2a^m}{ds^2} + \Gamma_{ij}^m \frac{da^i}{ds}\frac{da^j}{ds} = 0$ with

$$\Gamma_{ij}^m = -C_{k;j}^m Q_{ki}.$$

The simple geometry of $\mathbf{R}^3$ is hidden behind a complicated transformation, but the transformation is the main object of study. The coefficients $C_{k;i}^m$ (but not u) enter the commutation relation between the Eulerian-Lagrangian label derivative and $\Gamma_\nu(u,\nabla)$:

$$\left[\Gamma_\nu(u,\nabla), \nabla_i^A\right] = 2\nu C_{k;i}^m \partial_k \nabla_m^A \tag{57}$$

This commutation relation is the viscous counterpart of the inviscid commutation of time and label derivatives.

In the inviscid case the map A is the main active ingredient in the dynamics. The Weber formula (38) [43] computes the velocity at time t directly from the gradient of A using a passively advected velocity v (39). In the viscous case, the Weber formula

$$u = \mathbf{P}\left((\nabla A)^* v\right) \tag{58}$$

can still be used but $v(x,t)$ is no longer passive. Instead of (40) v obeys

$$\Gamma_\nu(u,\nabla)v = 2\nu C\nabla v, \tag{59}$$

that is

$$\Gamma_\nu(u,\nabla)v_i = 2\nu C^m_{k;i}\partial_k v_m \tag{60}$$

with initial data

$$v(x,0) = u_0(x). \tag{61}$$

The equations (46), (59) together with the Weber formula (58) are equivalent to the Navier-Stokes equations [14]:

Theorem 5. *Let A, v and u solve the system (46, 59, 58). Then u obeys the incompressible Navier-Stokes equations,*

$$\partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = 0, \quad \nabla\cdot u = 0.$$

We describe now kinematic consequences of (46). One starts with a smooth arbitrary incompressible velocity field u, one computes A using (46), then one computes the inverse matrix $Q = (\nabla A)^{-1}$ (at least for a short time), and then Eulerian-Lagrangian derivatives ∇^A and coefficients C. One then can evolve a vector v from an initial datum solving (59), and one can compute its Eulerian-Lagrangian curl $\zeta = \nabla^A\times v$. The resulting equations [14], [15] are summarized below.

Theorem 6. *Let u be an arbitrary divergence-free function and associate to it a map A solving (46) and a vector field v solving (59). Then w defined by*

$$w_i = (\partial_i A^m)v_m \tag{62}$$

obeys the cotangent equation

$$\Gamma_\nu(u,\nabla)w + (\nabla u)^* w = 0. \tag{63}$$

The Eulerian curl of w, $\nabla\times w$ obeys the equation

$$\Gamma_\nu(u,\nabla)\left(\nabla\times w\right) = (\nabla u)(\nabla\times w). \tag{64}$$

The Eulerian-Lagrangian curl of v, $\zeta = \nabla^A\times v$, obeys

$$\Gamma_\nu(u,\nabla)\zeta_q = 2\nu C^m_{k;m}\partial_k\zeta_q - 2\nu C^q_{k;j}\partial_k\zeta_j + \nu C^m_{k;i}C^r_{k;j}\epsilon_{qji}\epsilon_{rmp}\zeta_p. \tag{65}$$

The Eulerian curl of w, $\nabla \times w$ and Eulerian-Lagrangian curl of v, $\zeta = \nabla^A \times v$ are related by the formula

$$(\nabla_E \times w)_q = \frac{1}{2}\epsilon_{qij}\left(Det\left[\zeta; \frac{\partial A}{\partial x_i}; \frac{\partial A}{\partial x_j}\right]\right) \tag{66}$$

The determinant of ∇A obeys

$$\Gamma_\nu(u, \nabla)\left(\log Det(\nabla A)\right) = \nu\left\{C^i_{k;s}C^s_{k;i}\right\}. \tag{67}$$

These considerations apply to arbitrary u without having to impose the equation of state (58). When $\nu = 0$ the cotangent equation (63) appears in Hamiltonian formalisms [33], [39] for the Euler equation in various gauges. The numerical merits of these have been analyzed critically [41]. When u is related to w by a filtered Weber formula, a gauge of the cotangent equation appears as a model of formally "averaged" Euler equation [31] and, in a viscous case, as a model of Reynolds' equation [4].

Note that, when $\nu = 0$, the equation (65) is just the pure advection equation $(\partial_t + u \cdot \nabla)\,\zeta = 0$. If $\nu > 0$, ζ obeys a linear dissipative equation with coefficients $C^m_{k;i}$. Using Schwartz inequalities only, one obtains

$$\Gamma_\nu(u, \nabla)|\zeta(x,t)|^2 + \nu|\nabla\zeta(x,t)|^2 \leq 17\nu|C(x,t)|^2|\zeta(x,t)|^2 \tag{68}$$

where

$$|C(x,t)|^2 = C^m_{k;i}(x,t)C^m_{k;i}(x,t), \quad |\zeta(x,t)|^2 = \zeta_q(x,t)\zeta_q(x,t)$$

are squares of Euclidean norms.

The evolution of the coefficients $C^m_{k;i}$ defined in (56) is given by

$$\Gamma_\nu(u, \nabla)\left(C^m_{k;i}\right) = -(\partial_l A^m)\nabla^A_i(\partial_k(u_l))$$

$$-(\partial_k(u_l))C^m_{l;i} + 2\nu C^j_{l;i} \cdot \partial_l\left(C^m_{k;j}\right). \tag{69}$$

At time $t = 0$ the coefficients vanish, $C^m_{k;i}(x,0) = 0$. Note that the linear equation (64) is identical to the nonlinear vorticity equation (4). It is the relation (58) that decides whether or not we are solving the Navier-Stokes equation; if u is a solution of the Navier-Stokes equations then (58) means $u = \mathbf{P}w$ and consequently $\nabla \times u = \nabla \times w$ implies that the Eulerian curl of w is the fluid's vorticity. The formula (66) relating the vorticity to ζ is then a viscous counterpart of the Cauchy formula.

Theorem 7. *If u solves the Navier-Stokes equation, A solves (46), and v solves (59) then the Eulerian curl of u, $\omega = \nabla \times u$ is related to the Eulerian-Lagrangian curl of v, $\zeta = \nabla^A \times v$ by the Cauchy formula*

$$\omega_q = \frac{1}{2}\epsilon_{qij}\left(Det\left[\zeta; \frac{\partial A}{\partial x_i}; \frac{\partial A}{\partial x_j}\right]\right). \tag{70}$$

Because of the linear algebra identity

$$((\nabla A)^{-1}\zeta)_q = (Det(\nabla A))^{-1}\frac{\epsilon_{qij}}{2}\left(Det\left[\zeta; \frac{\partial A}{\partial x_i}; \frac{\partial A}{\partial x_j}\right]\right)$$

one has

$$\omega = (Det(\nabla A))\,(\nabla A)^{-1}\zeta. \tag{71}$$

In two-dimensions (70, 71) become

$$\omega = (Det(\nabla A))\,\zeta, \tag{72}$$

reflecting the fact that, for $\nu = 0$, $\omega = \zeta$ in that case. A consequence of (70) or (71) is the identity

$$\omega \cdot \nabla_E = (Det(\nabla A))\,(\zeta \cdot \nabla_A) \tag{73}$$

that generalizes the corresponding inviscid identity. Let us consider the expression

$$\mathcal{C}(q, M) = (DetM)M^{-1}q \tag{74}$$

defined for any pair (q, M), where $q \in \mathbf{R}^3$, $M \in GL(\mathbf{R}^3)$ are, respectively, a vector and an invertible matrix. This expression, underlying the Cauchy formula, is linear in q and quadratic in M,

$$\mathcal{C}(q, M)_k = \frac{1}{2}\epsilon_{ijk}Det\left(M_{\cdot,i}, M_{\cdot,j}, q\right). \tag{75}$$

The quadratic expression in the right hand side is defined for any matrix M. It is easy to check that

$$\mathcal{C}(q, MN) = \mathcal{C}\left(\mathcal{C}(q, M), N\right) \tag{76}$$

and

$$\mathcal{C}(q, \mathbf{I}) = q \tag{77}$$

hold, so $\mathcal{C}$ describes an action of $GL(\mathbf{R}^3)$ in R^3. A third property follows from the explicit quadratic expression (75)

$$\mathcal{C}(q, \mathbf{1} + N) = (1 + Tr(N))q - Nq + \mathcal{C}(q, N) \tag{78}$$

Here N is any matrix, and the meaning of $\mathcal{C}(q, N)$ is given by (75). If we consider, instead of vectors q and matrices M, vector valued functions $q(x, t)$ and matrix valued $M(x, t)$ and use the same formula

$$\mathcal{C}(q, M)(x, t) = \mathcal{C}(q(x, t), M(x, t)) = (Det(M(x, t))\,(M(x, t))^{-1}\,q(x, t) \tag{79}$$

then the properties (76, 77, 78) as well as (75) obviously still hold.

Denote

$$\epsilon(s) = \nu \int |\nabla u(x, s)|^2 dx$$

the total instantaneous energy dissipation rate, and

$$K(t) = \frac{1}{2} \int |u(x, t)|^2 dx$$

the total kinetic energy. We consider functions $u(x, t)$ that satisfy

$$K(t) + \int_0^t \epsilon(s) ds \le K_0, \tag{80}$$

and

$$\int_0^t \|u(\cdot, s)\|_{L^\infty(dx)} ds \le K_\infty \tag{81}$$

For solutions of the Navier-Stokes equations and for solutions of the mollified equation (11) the constants K_0, K_∞ depend on the initial kinetic energy, viscosity and time only. For vortex methods, however, these constants depend on the cut-off scale δ. The displacement ℓ satisfies certain bounds that follow from the bounds above and (49).

Theorem 8. *Assume that the vector valued function ℓ obeys (49) for $t \in [t_1, T]$ and $\ell(\cdot, t_1) = 0$, for some $t_1 \geq 0$. Assume that the velocity $u(x,t)$ is a divergence-free function that satisfies the bounds (80) and (81) on the time interval $[0, T]$. Then ℓ satisfies the inequality*

$$\|\ell(\cdot, t)\|_{L^\infty} \leq K_\infty \tag{82}$$

together with

$$\int |\ell(x,t)|^2 dx \leq 2K_0(t-t_1)^2, \tag{83}$$

$$\int_{t_1}^{t} \int |\nabla \ell(x,s)|^2 dx ds \leq \frac{K_0(t-t_1)^2}{\nu}, \tag{84}$$

and

$$\int |\nabla \ell(x,t)|^2 dx + \nu \int_{t_1}^{t} \int |\Delta \ell(x,s)|^2 dx ds \ \leq C_1 \left(\frac{K_0(t-t_1)}{\nu} + \frac{K_\infty^2 E_0}{\nu^2} \right). \tag{85}$$

Let us consider the analytic norms

$$\|u\|_{\{A,\lambda,p\}} = \left\{ \int e^{p\lambda|\xi|} |\widehat{u}(\xi)|^p d\xi \right\}^{\frac{1}{p}} \tag{86}$$

We will use $p = 1$ and $p = 2$. One can prove

Theorem 9. *Let the vector valued function ℓ solve (49) for $t \geq t_1 \geq 0$ with $\ell(\cdot, t_1) = 0$. Assume that a velocity $u(x,t)$ defined for $t \geq t_0$, $t_0 \leq t_1$ is a divergence-free function that satisfies*

$$\sup_{t_0 \leq t \leq T} \|u\|_{\{A,r,1\}} \leq U_r. \tag{87}$$

Then there exists an absolute constant (a pure number) c such that

$$\|\ell(\cdot, t)\|_{\{A,r,1\}} \leq (t - t_1) U_r e^{c\frac{(t-t_1)U_r^2}{\nu}} \tag{88}$$

holds for $t \geq t_1$.

Note that (27) reads

$$\|u\|_{\{A,\lambda,1\}} \leq \sqrt{4\pi}\|\omega\|^{\frac{1}{2}}_{\{A,\lambda,2\}}\|\nabla\omega\|^{\frac{1}{2}}_{\{A,\lambda,2\}} \tag{89}$$

Note also that, if $r < \lambda$ then

$$\|\nabla\omega\|_{\{A,r,2\}} \leq \frac{1}{e(\lambda - r)}\|\omega\|_{\{A,\lambda,2\}} \tag{90}$$

and

$$\|\nabla\omega\|_{\{A,r,1\}} \leq \frac{1}{e(\lambda - r)}\|\omega\|_{\{A,\lambda,1\}} \tag{91}$$

hold. Combining Theorem 3 or Theorem 4 with the preceding result we obtain therefore

Theorem 10. *Consider solutions of the mollified equations (11) or of the vortex method (12) associated to a filter (9). Assume the initial data are divergence-free and belong to* H^1, $u_0 \in L^2$, $\nabla \times u_0 = \omega_0 \in L^2$. *Consider* $0 \leq t \leq T$ *and let* $\mathcal{E}$ *denote a bound for the enstrophy on the time interval* $[0, T]$

$$\sup_{0\leq t\leq T}\int_{\mathbf{R}^3}|\omega(x,t)|^2dx \leq \mathcal{E}.$$

Consider an arbitrary transience time $0 < t_0 < T$ *and the length scale*

$$\lambda = \sqrt{\frac{\nu}{T}}\min\left\{t_0;\ c_1\nu^3\mathcal{E}^{-2}\right\}$$

with c_1 *a certain absolute constant. Then the velocity* u *obeys the bound (87) for any* $r < \lambda$, *with* U_r *given by*

$$U_r = c_2(\lambda - r)^{-\frac{1}{2}}\mathcal{E}^{\frac{1}{2}} \tag{92}$$

and $c_2 = 2\sqrt{\frac{2\pi}{e}}$. *Consequently, for any* $t_1 \geq t_0$, *the solution* ℓ *of the equation (49) with initial data* $\ell(\cdot, t_1) = 0$ *obeys the bound (88) and, for arbitrary* $r_1 < r < \lambda$,

$$\|\nabla\ell(\cdot,t)\|_{\{A,r_1,1\}} \leq e^{-1}(r - r_1)^{-1}(t - t_1)U_re^{c\frac{(t-t_1)U_r^2}{\nu}} \tag{93}$$

holds for $t \geq t_1$.

These bounds depend on the cut-off scale δ of the filter only through the bound on enstrophy $\mathcal{E}$. In particular, if the enstrophy is bounded uniformly for small δ on a time interval then the results above apply for the Navier-Stokes equations on that time interval. If we measure length in units of $\sqrt{\nu T}$ and time in units of T then the enstrophy bound for $u(x,t) = \sqrt{\frac{\nu}{T}}\tilde{u}\left(\frac{x}{\sqrt{\nu T}}, \frac{t}{T}\right)$ becomes

$$\sup_{0\le s\le 1} \|\nabla\tilde{u}(\cdot,s)\|_{L^2} \le G$$

with G the non-dimensional number given by

$$G^2 = \nu^{-\frac{3}{2}}T^{\frac{1}{2}}\mathcal{E} \tag{94}$$

Note that the number ρ of (26) is just $\rho = G^4$. In these units G is the only solution dependent parameter that we do not control. In terms of G and in these units, the definition of λ becomes

$$\frac{\lambda}{\sqrt{\nu T}} = \tilde{\lambda} = \min\{s_0;\ c_1G^{-4}\}, \tag{95}$$

with $s_0 \in (0,1)$ arbitrary. The bound (92), with for $0 < \tilde{r} < \tilde{\lambda}$, $r = \sqrt{\nu T}\tilde{r}$, becomes

$$\sqrt{\frac{T}{\nu}}U_r = \tilde{U}_{\tilde{r}} \le c_2(\tilde{\lambda} - \tilde{r})^{-\frac{1}{2}}G. \tag{96}$$

For fixed s_0 and large G we may take $\tilde{\lambda} \sim G^{-4}$ so that if we consider $\tilde{r} = (1-\gamma)\tilde{\lambda}$ with $0 < \gamma < 1/4$ we deduce

$$\tilde{U}_{\tilde{r}} \le c_3G^3 \tag{97}$$

Choosing $r_1 = (1-\gamma)r$ in (93) we deduce from (93)

$$\|\nabla\ell(\cdot,t)\|_{\{A;r_1,1\}} \le g \tag{98}$$

for $t_0 \le t_1 \le t \le t_1 + \tau T$ with

$$\tau = c_4gG^{-7}. \tag{99}$$

Choosing $r_2 = (1-\gamma)r_1$ we deduce that

$$\|\nabla\nabla\ell(\cdot,t)\|_{\{A;r_2,1\}} \le c_5(\nu T)^{-\frac{1}{2}}G^4g \tag{100}$$

holds on the same time interval.

7 Conclusions

The viscous Navier-Stokes equations and their approximations can be described using diffusive, near-identity transformations. The velocity is obtained from the near-identity transformation using a Weber formula and a virtual velocity. The vorticity is obtained from the near-identity transformation using a Cauchy formula and a virtual vorticity. The virtual velocity and the virtual vorticity obey diffusive equations, which reduce to passive advection formally, if the viscosity is zero. Apart from being proportional to the viscosity, the coefficients of these diffusion equations involve second derivatives of the near identity transformation and are related to the Christoffel coefficients. If and when the near-identity transformation departs excessively from the identity, one resets the calculation. Lower bounds on the minimum time between two successive resettings are given in terms of the maximum enstrophy.

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