Nonlinear Fokker-Planck Navier-Stokes Systems

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September 27, 2005

Abstract

We consider Navier-Stokes equations coupled to nonlinear Fokker-Planck equations describing the probability distribution of particles interacting with fluids. We describe relations determining the coefficients of the stresses added in the fluid by the particles. These relations link the added stresses to the kinematic effect of the fluid's velocity on particles and to the inter-particle interaction potential. In equations of type I, where the added stresses depend linearly on the particle distribution density, energy balance requires a response potential. In equations of type II, where the added stresses depend quadratically on the particle distribution, energy balance can be achieved without a dynamic response potential. In unforced energetically balanced equations, all the steady solutions have fluid at rest and particle distributions obeying an uncoupled Onsager equation. Systems of equations of type II have global smooth solutions if inertia is neglected.

Key Words: Fokker-Planck equations, Navier-Stokes equations, Smoluchowski equations, microscopic inclusions.
AMS classification numbers:35Q30, 82C31,76A05.

1 Introduction

Fluids with complex rheological properties are of great scientific interest and have a rich phenomenology. Their mathematical description remains challenging. With the exception of ([20]) most of the progress is rather recent. Global existence of weak solutions, with mollyfied velocities and linear Fokker-Planck equations with additional boundary conditions has been obtained in ([1]). Global existence for shear flow Hookean dumbell models was proved in ([11]). The local existence of various systems has been obtained ([5], [12], [17]). Global existence for small data for linear Fokker-Planck coupled with Navier-Stokes equations was obtained in [14]. The recent work ([19]) is the only global regularity result for large data that I am aware of. It applies to the case of coupled linear Fokker-Planck and Stokes system. The study of time asymptotics is also in its early development stage. High intensity asymptotics for uncoupled Smoluchowski equations have been studied in ([2], [6], [15], [16]). The long time effects of shear in Doi-Smoluchowski equations have been investigated in ([7], [8]). The long time asymptotics of coupled systems using entropy methods has been studied in ([13]).

In this paper we consider a complex fluid with microscopic inclusions. The fluid is governed by the incompressible Navier-Stokes equation, and the microscopic insertions influence the fluid through an added macroscropic stress. The fluid's velocity $u(x,t)$ obeys thus the three dimensional Navier-Stokes equations

$$\left\{\begin{array}{c} \nabla_x\cdot u = 0,\\ \partial_t u + u\cdot\nabla_x u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma + F \end{array}\right. \tag{1}$$

Derivatives with respect to coordinates $x\in\mathbb{R}^3$ will be indicated by a subscript x. $F(x,t)$ are given smooth body forces, $\nu > 0$ is the kinematic viscosity. The expression $\mathrm{div}_x\sigma$,

$$\mathrm{div}_x\sigma = \nabla_x\cdot\sigma = \sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j},$$

represents the forces due to the presence of microscopic insertions: the tensor $\sigma_{ij}(x,t)$ is the additional stress in the fluid. The insertions are objects parameterized by a microscopic variable m which belongs to a connected smooth Riemannian manifold M of dimension d. In the case of rod-like particles, M is the unit sphere in three dimensions, and $m\in\mathbb{S}^2\subset\mathbb{R}^3$ represents the director of the rods. More complicated particles require more degrees of freedom for the configuration space M. For instance, articulated rods with several articulations, require a phase space which is a product of spheres.

We will use local coordinates $m = m(\phi)$ with $\phi = (\varphi_1, \dots, \varphi_d) \in \mathbb{R}^d$. In the paper we use the fact that a smooth Riemannian metric g_{ij} exists on M. As it is customary g^{ij} will denote the inverse of g_{ij} ([9]). Compactness of M will also be assumed, for the sake of simplicity, although a non-compact manifold could be allowed as well. (In the case of non-compact M some of the assumptions on the coefficients would have to be augmented and some arguments would have to be modified, mostly technical changes). Derivatives with respect to the microscopic variable are designated by the subscript g:

$$\nabla_g h = (\partial_{\varphi_1} h, \dots \partial_{\varphi_d} h).$$

When f, h are scalars and V is a $(0,1)$ tensor on M we define

$$\mathrm{div}_g(Vf) = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^{d} \partial_{\varphi_\alpha} \left(\sqrt{g} g^{\alpha\beta} V_\beta f \right)$$

and recall that the volume element on M is locally

$$dm = \sqrt{g} d\phi,$$

the Laplace-Beltrami operator is

$$\Delta_g h = \mathrm{div}_g \nabla_g h$$

and

$$\int\limits_M h \mathrm{div}_g(Vf) dm = - \int\limits_M f(V \cdot \nabla_g h) dm$$

holds for smooth functions where $V \cdot \nabla_g h = g^{\alpha\beta} V_\beta \frac{\partial h}{\partial \varphi_\alpha}$. We will attempt to distinguish between Greek alphabet indices $\alpha, \beta, \dots$ running from 1 to d and related to the microscopic variables m, and Roman alphabet indices $i, j, \dots$ related to the spatial variable x and running from 1 to 3.

One of the fundamental assumptions of the current literature on the subject is that the added stresses $\sigma(x,t)$ do not depend explicitly on the microscopic variable m. Consequently, the velocity $u = u(x,t)$ does not depend on the variable m.

The microscopic insertions at time t and macroscopic place x are described by the probability $f(x,m,t)dm$. The suspension stress tensor is then given by an expansion

$$\sigma(x,t) = \sigma^{(1)}(x,t) + \sigma^{(2)}(x,t) \tag{2}$$

where

$$\sigma_{ij}^{(1)}(x,t) = \int_M \gamma_{ij}^{(1)}(m) f(x,m,t)dm \tag{3}$$

and

$$\sigma_{ij}^{(2)}(x,t) = \int_M \int_M \gamma_{ij}^{(2)}(m,n) f(x,m,t) f(x,n,t) dmdn. \tag{4}$$

This, and more general expansions for σ are encountered in the polymer literature ([4]). The structure coefficients in the expansion, $\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}$ are smooth, time independent, x independent, and do not depend on f. In this work we will use models in which the particles interact through potentials that depend linearly and nonlocally on the particle density distribution f ([18]). Because of this, an expansion (2) with only two terms is sufficient. One could consider interaction potentials that depend nonlinearly on f and with them, higher nonlinear dependence of σ on f, as well as explicit dependence of σ on $\nabla_x u$. In the dilute cases, when the interaction of particles is modelled by an f-independent (zeroth power) expression, as in the dumbell, FENE and polynomial force models, then the customary expression for σ is linear in f, and is given for instance, for rod-like particles by

$$\sigma_{ij}^{(1)}(x,t) = \frac{kT}{4\pi} \int_{\mathbb{S}^2} \left(m_i m_j - \frac{\delta_{ij}}{3} \right) f(x,m,t)dm \tag{5}$$

where kT is an energy scale associated to the microscopic suspension.

Modelling the added stresses is a complicated task. The purpose of this work is twofold. First, we discuss models from an energy principle point of view, and we reveal necessary relationships dictated by energy requirements. We distinguish between the two types of relationship between σ and f. We refer to the case in which $\sigma^{(2)} = 0$, so that σ depends linearly on f as "equations of type I", and to the case in which $\sigma^{(2)} \neq 0$ as "equations of type II". The second purpose of the article is to prove a global existence result for equations of type II.

The evolution of the density f is governed by a nonlinear Fokker-Planck equation

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g (Gf) = \epsilon \Delta_g f \tag{6}$$

Here $\epsilon \geq 0$ is an inverse time scale associated to diffusion of the microscopic particles. The tensor G is made of two parts

$$G = \nabla_g U + W, \tag{7}$$

The $(0,1)$ tensor field W is obtained from the macroscopic gradient of velocity in a linear smooth fashion, given locally as

$$W(x,m,t) = (W_\alpha(x,m,t))_{\alpha=1,\dots,d} = \left(\sum_{i,j=1}^{3} c_\alpha^{ij}(m)\frac{\partial u_i}{\partial x_j}(x,t)\right)_{\alpha=1,\dots,d.} \tag{8}$$

The smooth coefficients $c_\alpha^{ij}(m)$ do not depend on the solution, time or x and, like the coefficients $\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}$ are a constitutive part of the model.

In the case of rod-like particles W is the projection of $\nabla_x u(x,t)m$ onto the tangent plane to $\mathbb{S}^2$ at m:

$$W(x,m,t) = (\nabla_x u(x,t))m - ((\nabla_x u(x,t))m \cdot m)m. \tag{9}$$

This determines the coefficients $c_\alpha^{ij}(m)$ for this case. The field W represents the rotation and stretching of microscopic insertions due to macroscopic flow; in jargon, this is termed a *macro-micro* interaction. In the same vein, σ is a *micro-macro* interaction. There are two types of potentials. The nonlocal microscopic interaction potential is ([18])

$$(\mathcal{K}f)(x,m,t) = \int_M K(m,q)f(x,q,t)dq. \tag{10}$$

Here $K(m,q)$ is a smooth, time independent, x independent, symmetric function $K : M \times M \to \mathbb{R}$. This is yet another constitutive parameter in the system; the potential $\mathcal{K}f$ represents a *micro-micro* interaction.

The model for the total potential U depends on equation type. In order to have an energy principle, even in the absence of interparticle interactions ($K = 0$), it turns out that the coefficients $\gamma_{ij}^{(1)}$ are determined by the coefficients c_α^{ij}. Therefore, if the *macro-micro* effect W is imposed by kinematic physical considerations, (as is the case in (9)), then a linear constitutive form $\sigma^{(1)}$ of the added stresses is dictated by energetics requirements alone, irrespective of particle interaction potentials. In order to restore an energy balance for equations of type I, in the presence of interparticle interactions, the total potential U is given by

$$U(x,m,t) = \frac{1}{\tau}\left((\mathcal{K}f)(x,m,t) + \delta V(x,m,t)\right) \tag{11}$$

where τ is a time scale associated with the microscopic interactions and the term δV obeys the equation

$$\partial_t \delta V + u \cdot \nabla_x \delta V + W \cdot \nabla_g \delta V = -W \cdot \nabla_g(\mathcal{K}f). \tag{12}$$

This potential is a *macro-micro* term, and is *evolutionary*: It arises in response to the *micro-micro* interactions, and plays an absolutely crucial role in the energetics of type I equations. Because of (11) and (12) it follows that

$$\partial_t U + u \cdot \nabla_x U + W \cdot \nabla_g U = \frac{1}{\tau} D_t(\mathcal{K} f). \tag{13}$$

This shows that the potential U evolves incorporating both macroscopic and microscopic effects. We denote a^{-1}

$$\epsilon\tau = a^{-1}$$

the nondimensional ratio of microscopic time scales and denote by V

$$V = \mathcal{K} f + \delta V \tag{14}$$

the nondimensional total potential, so that $\frac{U}{\epsilon} = aV$.

In the case of type II equations one can satisfy the energetic requirements in the presence of interparticle interactions by relating the coefficients $\gamma^{(2)}$ to K.

2 Energetics For Type I Equations

We start with a concrete example: rod-like particles with $M = \mathbb{S}^2$ with $\sigma = \sigma^{(1)}$ given in (5). Then the system has a Lyapunov functional, the energy

$$E = \frac{1}{2} \int |u(x,t)|^2 dx + \mathcal{F} \tag{15}$$

with free energy $\mathcal{F}$ given by

$$\mathcal{F} = \frac{kT}{20\pi} \int \int\limits_{\mathbb{S}^2} \left\{ f \log f - a \left(\delta V + \frac{1}{2} f \mathcal{K} f \right) \right\} dm dx. \tag{16}$$

Theorem 1 *Let $(u, f, \delta V)$ be a smooth solution of the system (1, 6, 12) with constitutive relations (5, 7, 9, 10, 11) and $F = 0$. Then*

$$\frac{d}{dt} E(t) = -\nu \int |\nabla_x u(x,t)|^2 dx - \frac{kT\epsilon}{20\pi} \int \int\limits_{\mathbb{S}^2} f \left| \nabla_g \left(\log f - aV \right) \right|^2 dm dx \tag{17}$$

Proof. The identity

$$\mathrm{div}_g W(x,m,t) = -5(\nabla_x u(x,t))m\cdot m \tag{18}$$

follows from the definition (9). Using the definition (5), it follows that

$$\int\int_{\mathbb{S}^2} f(x,m,t)\mathrm{div}_g W(x,m,t)dmdx = \frac{20\pi}{kT}\int u(x,t)\cdot\nabla_x\sigma(x,t)dx \tag{19}$$

Multiplying the equation (1) by u, integrating by parts and using the identity (19) we obtain the Navier-Stokes energy balance

$$\frac{d}{2dt}\int |u|^2dx + \nu\int|\nabla_x u(x,t)|^2\,dx = \frac{kT}{20\pi}\int\int_{\mathbb{S}^2}(f\mathrm{div}_g W)dmdx \tag{20}$$

Let us denote now

$$D_t = \partial_t + u\cdot\nabla_x$$

and

$$\mathcal{E}(t) = \int\int_{\mathbb{S}^2}\left(\epsilon f\log f - \frac{1}{\tau}f\delta V - \frac{1}{2\tau}f\mathcal{K}f\right)dmdx. \tag{21}$$

Note that

$$\mathcal{F} = \frac{kT}{20\pi\epsilon}\mathcal{E} \tag{22}$$

Using the fact that $\int\int_{\mathbb{S}^2} D_t f dmdx = 0$, the fact that $\nabla_x\cdot u = 0$, the relation (11), the fact that $\mathcal{K}$ is a symmetric operator and that D_t commutes with $\mathcal{K}$ we have that

$$\frac{d}{dt}\mathcal{E} = \int\int_{\mathbb{S}^2}\left\{(D_t f)\left(\epsilon\log f - U\right) - fD_tU + \frac{1}{\tau}fD_t\mathcal{K}f\right\}dmdx$$

and thus, in view of (13)

$$\frac{d}{dt}\mathcal{E} = \int\int_{\mathbb{S}^2}\left\{(D_t f)\left(\epsilon\log f - U\right) + fW\cdot\nabla_g U\right\}dmdx.$$

The equation (6) has the structure

$$D_t f = \mathrm{div}_g(f(\nabla_g(\epsilon\log f - U) - W)).$$

Using it, we have

$$\frac{d}{dt}\mathcal{E} = -\int\int_{\mathbb{S}^2} f\left|\nabla_g(\epsilon \log f - U)\right|^2 dmdx +$$

$$+\int\int_{\mathbb{S}^2} \{\mathrm{div}_g(UfW) - \epsilon\mathrm{div}_g(fW)\log f\}\, dmdx$$

Therefore

$$\frac{d}{dt}\mathcal{E} = -\int\int_{\mathbb{S}^2} f\left|\nabla_g(\epsilon \log f - U)\right|^2 dmdx - \epsilon\int\int_{\mathbb{S}^2}(f\mathrm{div}_g W)dmdx \tag{23}$$

Multiplying by $\frac{kT}{20\pi\epsilon}$ and using $\frac{U}{\epsilon} = aV$ we obtain

$$\frac{d}{dt}\mathcal{F} = -\frac{kT\epsilon}{20\pi}\int\int_{\mathbb{S}^2} f\left|\nabla_g\left(\log f - aV\right)\right|^2 dmdx - \frac{kT}{20\pi}\int\int_{\mathbb{S}^2}(f\mathrm{div}_g W)dmdx \tag{24}$$

Adding to (20) we finish the proof.

Note that when $F = 0$ the only steady solutions have no flow, ($u = 0$ or any constant, by Galilean invariance), and solve Onsager's equation ([18], [2])

$$\log f = aV - \log Z \tag{25}$$

with Z a constant serving as normalizing factor. Let us use the calculation above as a guide to understand the energetics for more general type I equations. Assume thus that $\sigma = \sigma^{(1)}$ with $\sigma^{(1)}$ given in (3). Then the Navier-Stokes energy balance is, after one integration by parts,

$$\frac{d}{2dt}\int |u|^2 dx + \nu\int |\nabla_x u|^2 dx =$$

$$-\int\int_M \left(\sum_{i,j=1}^{3}\frac{\partial u_i}{\partial x_j}(x,t)\gamma_{ij}^{(1)}(m)\right) f(x,m,t)dxdm. \tag{26}$$

This balance occurs no matter what is the equation for f. Let us assume first that the equation for f is the simplest possible, with only a *macro-micro* interaction W,

$$D_t f = \epsilon\Delta_g f - \mathrm{div}_g\left(Wf\right)$$

with W given by (8). Then the evolution of the entropy is

$$\frac{d}{dt}\int\int_M f\log f dmdx = -\epsilon\int\int_M |\nabla_g f|^2 f^{-1}dmdx - \int\int_M (\mathrm{div}_g W) f dmdx$$

Because

$$\mathrm{div}_g W = \sum_{ij=1}^{3} \mathrm{div}_g(c^{ij})\frac{\partial u_i}{\partial x_j} \tag{27}$$

we see that the system has a Lyapunov functional of the form

$$\text{kinetic energy} + \lambda\,\text{entropy}$$

if and only if there exist constants λ and μ so that

$$\gamma_{ij}^{(1)} = -\lambda \mathrm{div}_g c^{ij} + \mu\delta_{ij} \tag{28}$$

holds. This happens in the case of rigid rods where (9) produces the balance (18) which is of the form above. Thus, energy requirements impose that the coefficients of (8) determine those of (3). Once the relation (28) has been established then the general type I calculation is very similar to the the concrete example and gives:

Theorem 2 *Type I equations (1, 6, 12), with constitutive equations (2, 3, 7, 8, 10, 11), with $F = 0$, $\sigma^{(2)} = 0$ and (28), have a Lyapunov functional*

$$E = \frac{1}{2}\int |u|^2 dx + \lambda\int\int_M \left\{ f\log f - a\left(\delta V + \frac{1}{2}\mathcal{K}f\right) f\right\} dmdx. \tag{29}$$

If $u, f, \delta V$ is a smooth solution then

$$\frac{dE}{dt} = -\nu\int |\nabla_x u|^2 dx - \lambda\epsilon\int\int_M f\,|\nabla_g(\log f - aV)|^2\, dmdx \tag{30}$$

holds. If the smooth solution is time independent, then $u = 0$, $\delta V = \delta V_{|t=0}$ and f solves the Onsager equation

$$\log f = aV + \log(Z^{-1}) \tag{31}$$

with an appropriate normalizing constant Z.

Proof. We form the integral $\mathcal{E}$ as in (21):

$$\mathcal{E}(t) = \int \int_M \left(\epsilon f \log f - \frac{1}{\tau} f \delta V - \frac{1}{2\tau} f \mathcal{K} f \right) dm dx. \tag{32}$$

The calculation leading to (23) can be repeated verbatim and leads to

$$\frac{d}{dt}\mathcal{E} = - \int \int_M f \left| \nabla_g (\epsilon \log f - U) \right|^2 dm dx - \epsilon \int \int_M (f \mathrm{div}_g W) dm dx. \tag{33}$$

Using (27), (28) and the fact that $(\nabla_x u)$ is traceless we have

$$-\lambda f \mathrm{div}_g W = f \sum_{ij=1}^{3} \gamma_{ij}^{(1)}(m) \frac{\partial u_i}{\partial x_j}. \tag{34}$$

We obtain (30) by multiplying (33) by $\lambda \epsilon^{-1}$ and adding to (26).

3 Energetics For Type II Equations

In this case the energy balance in the Navier-Stokes equations is modified because of the presence of the terms in $\sigma^{(2)}$. The balance (26) becomes

$$\frac{d}{2dt} \int |u|^2 dx + \nu \int |\nabla_x u|^2 dx =$$

$$- \int \int_M \left(\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}(x,t) \gamma_{ij}^{(1)}(m) \right) f(x,m,t) dx dm$$

$$- \int \int_M \int_M \left(\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}(x,t) \gamma_{ij}^{(2)}(m,n) \right) f(x,m,t) f(x,n,t) dx dm dn \tag{35}$$

This balance holds in a type II equation, irrespective of the form of the equation obeyed by f. If this equation is of the form

$$D_t f = \epsilon \mathrm{div}_g \left(f \nabla_g \left(\log f - a \mathcal{K} f \right) \right) - \mathrm{div}_g (W f) \tag{36}$$

(i.e. $\delta V = 0$), then one can maintain energy balance if (28) holds, and if there exists a constant η so that

$$\gamma_{ij}^{(2)}(m,n) = -a \lambda c_{\alpha}^{ij}(m) g^{\alpha\beta}(m) \partial_\beta K(m,n) + \eta \delta_{ij} \tag{37}$$

Theorem 3 *Type II equations (1, 36) with constitutive equations (2, 3, 4, 8, 10), with $F = 0$, (28, 37) have a Lyapunov functional*

$$E(t) = \frac{1}{2}\int |u|^2 dx + \lambda \int\int_M \left\{ f\log f - \frac{a}{2}(\mathcal{K}f)f \right\} dxdm. \tag{38}$$

If (u, f) is a smooth solution then

$$\frac{dE}{dt} = -\nu \int |\nabla_x u|^2 dx - \lambda\epsilon \int\int_M f \left|\nabla_g (\log f - a\mathcal{K}f)\right|^2 dmdx. \tag{39}$$

If the smooth solution is time independent, then $u = 0$ and f solves the Onsager equation

$$\log f = a\mathcal{K}f + \log(Z^{-1}) \tag{40}$$

with an appropriate normalizing constant Z.

Proof. Using (36) we have

$$\frac{d}{dt}\int\int_M \left\{ f\log f - \frac{a}{2}(\mathcal{K}f)f \right\} dmdx = -\epsilon \int\int_M f \left|\nabla_g(\log f - a\mathcal{K}f)\right|^2 dmdx$$

$$+ \int\int_M \left\{ aW \cdot \nabla_g (\mathcal{K}f) - \mathrm{div}_g W \right\} f dmdx \tag{41}$$

The condition (37) implies that

$$\lambda a W \cdot \nabla_g (\mathcal{K}f) = -\sum_{ij=1}^{3} \frac{\partial u_i}{\partial x_j} \int_M \gamma_{ij}^{(2)}(m,n) f(x,n,t) dn \tag{42}$$

Multiplying (41) by λ, adding to (35) and using (42) and (34) finishes the proof.

Remark. We note that, for type II equations, knowledge of the *micro-micro* potential K and of the nature of the *macro-micro* effect (8) i.e., the coefficients c_α^{ij}, coupled with energetic balance, determine the coefficients $\gamma_{ij}^{(1)}$ and $\gamma_{ij}^{(2)}$.

4 A Global Regularity Result For Type II Equations

We consider a nonlinear Fokker-Planck system

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(Wf) + \frac{1}{\tau}\mathrm{div}_g(f\nabla_g(\mathcal{K}f)) = \epsilon\Delta_g f \tag{43}$$

where W is given in (8) and $\mathcal{K}$ in (10). We use the time scales τ and ϵ^{-1} separately, so that we can state a theorem that allows for the limit cases $\epsilon = 0$, $\tau = \infty$. The velocity is related to f via the Stokes equations:

$$-\nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma + F, \quad \nabla_x \cdot u = 0. \tag{44}$$

The added stresses are given by the type II relation (2, 3, 4). We take periodic boundary conditions for the Stokes equations.

Theorem 4 *Let $\epsilon \geq 0$, $\tau \in (0,\infty]$. If the initial distribution $f(x,m,0)$ is smooth, positive and normalized,*

$$\int_M f_0(x,m)dm = 1,$$

and the initial velocity is smooth, then the system (43), (44), with constitutive relations (8), (10), (2, 3, 4) has global smooth solutions.

This theorem was proved in ([19]) for $\epsilon > 0, \tau = \infty$, $M = \mathbb{S}^2$ using the nonlinear dissipative structure of the Fokker-Planck equation. Our proof does not use this dissipation.

Proof. We consider the $L^2(M)$ selfadjoint pseudodifferential operator ([10])

$$R = (-\Delta_g + \mathbf{I})^{-\frac{s}{2}} \tag{45}$$

with $s > \frac{d}{2} + 1$. We will use the following properties of R:

$$[R, \nabla_x] = 0, \tag{46}$$

$$R\nabla_g : L^1(M) \to L^2(M) \quad \text{is bounded,} \tag{47}$$

$$R\nabla_g : L^2(M) \to L^\infty(M) \quad \text{is bounded,} \tag{48}$$

$$[\nabla_g c, R^{-1}] : H^s(M) \to L^2(M) \quad \text{is bounded,} \tag{49}$$

for any smooth function $c : M \to \mathbb{R}$, and

$$R : L^2(M) \to H^s(M) \quad \text{is bounded.} \tag{50}$$

We differentiate (43) with respect to x, apply R, multiply by $R\nabla_x f$ and integrate on M. Let us denote by

$$N(x,t)^2 = \int_M |R\nabla_x f(x,m,t)|^2 \, dm \tag{51}$$

the square of the L^2 norm of $R\nabla_x f$ on the unit sphere. The following lemma is the main tool for regularity:

Lemma 1 *Let $u(x,t)$ be a smooth, divergence-free function and let f solve (43). There exists an absolute constant $c > 0$ (depending only on dimensions of space, the coefficients c_α^{ij} and M, but not on u, f, ϵ, τ) so that*

$$(\partial_t + u \cdot \nabla_x)\, N \le c(|\nabla_x u| + \frac{1}{\tau})N + c|\nabla_x \nabla_x u| \tag{52}$$

holds pointwise in (x,t).

Proof of Lemma 1. The normalization

$$\int_M f(x,m,t)dm = 1$$

and smoothness of $f(x,t)$ are easy to prove, and we will omit the proof. The evolution equation of N is

$$\frac{1}{2}\left(\partial_t + u \cdot \nabla_x\right) N^2 = -D + I + II + III + IV \tag{53}$$

with

$$D = \epsilon \int_M |\nabla_g R \nabla_x f|^2 \, dm \tag{54}$$

$$I = -\frac{\partial u_j}{\partial x_k} \int_M \left(R\frac{\partial f}{\partial x_j}\right)\left(R\frac{\partial f}{\partial x_k}\right) dm \tag{55}$$

$$II = -\sum_{\alpha=1}^{2}(\nabla_x \frac{\partial u_i}{\partial x_j})\int_M (R\mathrm{div}_g(c_\alpha^{ij} f))(\nabla_x Rf)dm, \tag{56}$$

$$III = -\sum_{\alpha=1}^{2}\frac{\partial u_i}{\partial x_j}\int_M (R\mathrm{div}_g(c_\alpha^{ij}\nabla_x f))(R\nabla_x f)dm, \tag{57}$$

and

$$IV = -\frac{1}{\tau}\int_M R\mathrm{div}_g(\nabla_x\{f\nabla_g(\mathcal{K}f)\})R\nabla_x f dm. \tag{58}$$

Now we start estimating terms. $D \geq 0$ will be discarded. Clearly

$$|I| \leq c|\nabla_x u|N^2. \tag{59}$$

In order to bound II we use (47) to bound

$$\|R\nabla_g(c_\alpha^{ij} f)\|_{L^2(M)} \leq c\|f\|_{L^1(M)} = c$$

so that we have

$$|II| \leq c\,|\nabla_x\nabla_x u|\,N. \tag{60}$$

In order to bound III we need to use the commutator carefully. We start by writing

$$R\mathrm{div}_g(c_\alpha^{ij}\nabla_x f) = R\mathrm{div}_g(c_\alpha^{ij}R^{-1}R\nabla_x f) =$$
$$\mathrm{div}_g(c_\alpha^{ij}R\nabla_x f) + \left[R\mathrm{div}_g c_\alpha^{ij}, R^{-1}\right]R\nabla_x f.$$

The second term obeys

$$\|\left[R\mathrm{div}_g c_\alpha^{ij}, R^{-1}\right]R\nabla_x f\|_{L^2(M)} \leq cN$$

because, in view of (49) and (50) one has that

$$\left[R\mathrm{div}_g c_\alpha^{ij}, R^{-1}\right] : L^2(M) \to L^2(M) \quad \text{is bounded.}$$

The first term needs to be integrated against $R\nabla_x f$ and integration by parts gives

$$\int_M (\mathrm{div}_g(c_\alpha^{ij}R\nabla_x f))R\nabla_x f dm = \frac{1}{2}\int_M (\mathrm{div}_g c_\alpha^{ij})|R\nabla_x f|^2 dm.$$

We obtain thus

$$|III| \leq c\,|\nabla_x u|\,N^2 \tag{61}$$

The term IV is split in two, $IV = A + B$

$$A = -\frac{1}{\tau}\int\limits_M R\mathrm{div}_g(\{(\nabla_x f)\nabla_g(\mathcal{K}f)\})R\nabla_x f dm \tag{62}$$

and

$$B = -\frac{1}{\tau}\int\limits_M R\mathrm{div}_g(\{f\nabla_g(\mathcal{K}\nabla_x f)\})R\nabla_x f dm. \tag{63}$$

The $(0,1)$ tensor $\Phi(x,m,t) = (\nabla_g \mathcal{K}f)(x,m,t)$ is smooth in m for fixed x,t and

$$\|\Phi(x,\cdot,t)\|_{W^{s,\infty}(M)} \le c_s$$

holds for any s, with c_s depending only on the kernel K. We write the term A

$$\begin{array}{rl} A = & -\frac{1}{\tau}\int\limits_M R\mathrm{div}_g(\{(\nabla_x f)\Phi\})R\nabla_x f dm \\ = & \frac{1}{\tau}\int\limits_M R^{-1}(R\nabla_x f)\{\Phi\cdot\nabla_g R^2\nabla_x f))dm \\ = & -\frac{1}{2\tau}\int\limits_M \mathrm{div}_g\{\Phi\}\left|R\nabla_x f\right|^2 dm + \frac{1}{\tau}\int\limits_M (R\nabla_x f)\left[R^{-1},\Phi\nabla_g\right]R(R\nabla_x f)dm \end{array}$$

In view of (49), (50), the operator

$$\left[R^{-1},\Phi\nabla_g\right]R : L^2(M) \to L^2(M)$$

is bounded with norm bounded by an a priori constant. It follows that

$$|A| \le \frac{c}{\tau}N^2(x,t)$$

holds. The term B is easier to bound, because

$$(\mathcal{K}\nabla_x f)(x,m,t) = \int\limits_M R^{-1}K(m,n)R\nabla_x f(x,n,t)dn$$

and thus

$$\|(\nabla_g\mathcal{K}\nabla_x f)(x,\cdot,t)\|_{L^\infty(M)} \le cN(x,t).$$

Using (47) it follows that

$$|B| \le \frac{c}{\tau}N^2(x,t)$$

and consequently

$$|IV| \le \frac{c}{\tau} N^2(x,t). \tag{64}$$

Putting together the inequalities (59), (60), (61) and (64) we finish the proof of the lemma.

We return to the proof of the theorem. The Stokes system (44) is elliptic, and it is well known that

$$\|\nabla u\|_{L^\infty(dx)} \le C\|\sigma\|_{L^\infty(dx)} \left\{1 + \log_+ \|\nabla_x \sigma\|_{L^q(dx)}\right\} + \|F\|_{L^q(dx)} \tag{65}$$

holds for $q > 3$. Also

$$\|\nabla_x \nabla_x u\|_{L^q(dx)} \le C \left\{\|\nabla_x \sigma\|_{L^q(dx)} + \|F\|_{L^q(dx)}\right\} \tag{66}$$

holds for any $1 < q < \infty$. Now it follows from (2, 3, 4) that

$$|\sigma(x,t)| \le c \tag{67}$$

holds with a constant that depends only on the coefficients $\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}$. Differentiating in x (3) and (4) it follows that

$$|\nabla_x \sigma(x,t)| \le cN(x,t) \tag{68}$$

holds with a constant c that depends only on the smooth coefficients $\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}$. Indeed, in the case of $\sigma^{(2)}$ for example

$$\nabla_x \sigma^{(2)}(x,t) = \int_M \int_M \left(R_m^{-1}\gamma^{(2)}(m,n)\right)\left(R_m \nabla_x f(x,m,t)\right) f(x,n,t)\,dmdn +$$

$$+ \int_M \int_M \left(R_n^{-1}\gamma^{(2)}(m,n)\right)\left(R_n \nabla_x f(x,n,t)\right) f(x,m,t)\,dmdn$$

where we used R_m and R_n to denote R acting in the m variables or n variables. The inequality (68) follows from the smoothness of the coefficients γ and the positivity and normalization of f. Now the proof follows along the classical lines: we prove easily a local existence and smoothness result of the system. We deduce from (52, 65, 66, 67, 68) that, for $q > 3$, the norm

$$y(t) = \|N(\cdot,t)\|_{L^q(dx)}$$

obeys

$$\frac{dy}{dt} \le c(1 + \log_+ y)y + Cy + M$$

with constant c, C, M. This proves that this norm is bounded apriori by a finite function of time, for all time. Once this is achieved, higher derivatives are controlled inductively. We omit further details.

5 Conclusions

Type I nonlinear Fokker-Planck equations (6) coupled with Navier-Stokes equations (1) are equations in which the added stresses σ depend in a linear fashion (3) on the density of particles. Such systems are energetically balanced if (28) holds and if the dynamical response potential δV obeying (12) is included in the system. Type II nonlinear Fokker-Planck equations (43) coupled with Navier-Stokes equations are equations in which the added stresses (2, 3, 4) depend quadratically on the density of particles. Such systems are energetically balanced if both (28) and (37) hold. Energetically balanced means in both situations that the natural total energy of the system is dissipated in the absence of external or boundary forcing. Steady solutions of unforced energetically balanced systems have necessarily the fluid at rest, and solve the steady uncoupled Onsager equation for the particle distribution. The energy balance is sufficient to determine the coefficients of the stresses added by the microscopic insertions in the fluid. This provides a guiding principle for modelling that is independent of closure strategies.

Type II equations have global smooth solutions if inertia is neglected, so that the fluid's equation is Stokes' equation (44). In the coupled system (43, 44) the particle distribution density $f = f(x, m, t)$ obeys a transport equation $D_t f + \mathrm{div}_g(Wf) \sim 0$ if we neglect both particle diffusion and interparticle potential interaction. The field W is proportional to $\nabla_x u$ (8), the fluid's physical space gradient. The Stokes equation with periodic boundary conditions imposes a balance $\nabla_x u = H\sigma$ with H a singular integral operator of Calderon-Zygmund type. Therefore the equation for f appears close to $D_t f + \mathrm{div}_g((H(\sigma)f)) \sim 0$. The constitutive relations for σ are of the type $\sigma = \gamma^{(1)}(f) + \gamma^{(2)}(f \otimes f)$. Substituting, we see that the equation for f has the appearance of a nonlocal inviscid Burgers equation. Why does it not shock, then? The important facts for global regularity are: the smoothing properties in microscopic variables of the transformation $f \mapsto \sigma$, and the tensorial nature of (8) where $\nabla_x u$ acts on particles multiplicatively. The presence of inertia is difficult to handle but some two dimensional and filtered three-dimensional type II equations have global smooth solutions ([3]).

Acknowledgment. Research partially supported by NSF-DMS grant 0504213.

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