Singular, weak and absent: solutions of the Euler equations

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We will describe necessary and sufficient conditions for blowup and discuss weak solutions for the incompressible Euler equations. We will also describe a result concerning anomalous dissipation of energy.

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I. INTRODUCTION

The incompressible Euler equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p \;=\; 0, \qquad (1)$$

$$\nabla\cdot\boldsymbol{u} \;=\; 0. \qquad (2)$$

We will discuss the case of $x \in \boldsymbol{R}^3$ and require that the velocity decay at infinity fast enough. The curl of u, $\omega = \nabla\times u$ obeys the quadratic vorticity equation,

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u}\cdot\nabla)\omega = (\omega\cdot\nabla)\boldsymbol{u}. \qquad (3)$$

The right hand side of this equation equals $S\omega$ where $S = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ and S can be expressed as a principal-value singular integral

$$(S(x,t))_{ij} = P.V.\int K_{ijk}(\widehat{y})\omega_k(x-y,t)\frac{dy}{|y|^3} \qquad (4)$$

with

$$K_{ijk}(\widehat{y}) = \frac{3}{8\pi}\left(\epsilon_{ipk}\widehat{y}_j + \epsilon_{jpk}\widehat{y}_i\right)\widehat{y}_p \qquad (5)$$

and $\widehat{y} = \frac{y}{|y|}$. The integral operator $\omega \mapsto S$ is of classical Calderon-Zygmund type. This means that the equation of evolution for ω is quadratic nonlinear nonlocal. There exist equations of this type that exhibit blowup, the formation of finite time singularities from smooth and localized initial data.

II. CONDITIONS FOR THE ABSENCE OF BLOWUP

Assuming that the initial data $\boldsymbol{u}_0$ is smooth enough, the Beale-Kato-Majda criterion [1] states that if the time integral of the spatial maximum of vorticity is finite, i.e. if

$$\int_0^T \left(\sup_x |\omega(x,t)|\right) dt < \infty \qquad (6)$$

then the solution is smooth on the time interval $[0,T]$.

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A. A necessary criterion based on the direction of vorticity

The evolution of $|\omega|$ is given by

$$\left(\partial_t + u\cdot\nabla\right)|\omega| = \alpha|\omega| \qquad (7)$$

with

$$\alpha = (\nabla u)\xi\cdot\xi = S\xi\cdot\xi, \qquad (8)$$

$$\xi = \frac{\omega}{|\omega|} \qquad (9)$$

and it turns out [2] that

$$\alpha(x,t) = \frac{3}{4\pi}P.V.\int_{\mathbf{R}^3} D(\widehat{y},\xi(x-y,t),\xi(x,t))|\omega(x-y,t)|\frac{dy}{|y|^3} \qquad (10)$$

with

$$D(e_1,e_2,e_3) = (e_1\cdot e_3)\det(e_1,e_2,e_3). \qquad (11)$$

Clearly, if ξ does not vary in space then $\alpha = 0$; this situation is encouneterd in two space dimensions. In general

$$\begin{array}{c}|D(\widehat{y},\xi(x-y,t),\xi(x,t))| \le \\ |\xi(x-y,t)\times\xi(x,t)| = |\sin\phi|\end{array} \qquad (12)$$

where ϕ is the angle between the unit vortex line tangent vectors $\xi(x-y,t)$ and $\xi(x,t)$. Some degree of smoothness of the bundle of vortex lines near a potential singularity may result in averting blowup [3]. For simplicity, we'll discuss Lipschitz continuous cases, although Hölder continuous cases may be analyzed in a similar fashion. We distinguish between the sine-Lipschitz case (i.e. $\sin\phi$ is locally Lipschitz), when the vortex lines are at worst locally osculating anti-parallel

$$|\xi(x-y,t)\times\xi(x,t)| \le C_a|y|, \quad \text{for } |y| \le r(t) \qquad (13)$$

and the Lipschitz case (i.e. ξ is locally Lipschitz) when the vortex lines are at worst locally osculating parallel

$$|\xi(x-y,t)-\xi(x,t)| \le C_p|y|, \quad \text{for } |y| \le r(t) \qquad (14)$$

Clearly, Lipschitz implies sine-Lipschitz because

$$|\xi(x-y,t)-\xi(x,t)| \ge |\xi(x-y,t)\times\xi(x,t)|\,. \qquad (15)$$

but the other implication is not true in general. In order to analyze the depletion effect due to organized vortex line structure we take a fixed $\rho > 0$, consider $r < \rho$, we take a smooth function $0 \le \chi \le 1$ compactly supported in the unit ball in $\boldsymbol{R}^3$ and define an inner rate of strain S^r as

$$(S^r(x,t))_{ij} = P.V. \int \chi\left(\frac{y}{r}\right) K_{ijk}(\widehat{y})\omega_k(x-y)\frac{dy}{|y|^3}. \tag{16}$$

Similarly, we define an outer rate of strain S_ρ as

$$(S_\rho(x,t))_{ij} = P.V. \int \left(1-\chi\left(\frac{y}{\rho}\right)\right) K_{ijk}(\widehat{y})\omega_k(x-y)\frac{dy}{|y|^3} \tag{17}$$

and an intermediate rate of strain as

$$(S_r^\rho(x,t))_{ij} = P.V. \int \left(\chi\left(\frac{y}{\rho}\right)-\chi\left(\frac{y}{r}\right)\right) K_{ijk}(\widehat{y})\omega_k(x-y)\frac{dy}{|y|^3}. \tag{18}$$

This yields a decomposition

$$S = S^r + S_r^\rho + S_\rho \tag{19}$$

Using (8) we have a corresponding decomposition of the stretching factor:

$$\alpha(x,t) = \alpha^r(x,t) + \alpha_r^\rho(x,t) + \alpha_\rho(x,t). \tag{20}$$

For instance, the inner stretching factor is

$$\alpha^r(x,t) = \frac{3}{4\pi} P.V. \int \chi\left(\frac{y}{r}\right) D(\widehat{y}, \xi(x-y,t)\ , \xi(x,t))|\omega(x-y,t)|\frac{dy}{|y|^3}. \tag{21}$$

Now let us make some specific assumptions about the blowup. These are not exhaustive, but exemplify the method of [3] in sligtly different circumstances. Our statements will be for the time interval $[0,T)$ and we should think of this being a short time before the blowup, by adjusting $t=0$ to be just before the suspected blowup time. We assume that there exists a point in space (without loss of generality, this can be $x=0$) so that the vorticity is going to blow up at $t=T$ somewhere in the neighborhood $B_t = \{x | |x| < r(t)\}$ of this point. We do not assume that the vorticity is small outside this region, nor do we assume that the velocity is bounded.

Blowup assumption **A**: We assume that there exists one vortex line that is sine-Lipschitz and stays in B_t, that is,

$$(A) \begin{cases} \exists q \in B_0 \text{ such that } x = X(q,t) \in B_t \text{ for } t \in [0,T), \\ (13) \text{ holds for } x = X(q,t),\ |y| \le r(t), \\ \exists c,\ 0 < c \le 1, \text{ such that} \\ |\omega(X(q,t),t)| \ge c \sup_{z\in \boldsymbol{R}^3} |\omega(z,t)|. \end{cases}$$

Here $X(q,t)$ is the Lagrangian trajectory with initial label q. The assumption is thus that there exists one trajectory carrying a fraction of the maximum vorticity and which has a coherent sine-Lipschitz vortex line field near it at each instance of time, short time before blowup.

From (13) and (21) we obtain with $x = X(q,t)$, for $r \le r(t)$

$$|\alpha^r(x,t)| \le rC_a \sup_{z\in\boldsymbol{R}^3} |\omega(z,t)|. \tag{22}$$

For the intermediate stretching factor we obtain from (18) and one integration by parts that

$$|\alpha_r^\rho(x,t)| \le c\frac{U(x,t)}{r} \tag{23}$$

with

$$U(x,t) = \sup_{|x-z|\le\rho} |u(z,t)|. \tag{24}$$

The outer stretching factor is bounded

$$|\alpha_\rho(x,t)| \le c\rho^{-\frac{3}{2}}\|u_0\|_{L^2}. \tag{25}$$

Denoting

$$U(t) = \sup_x U(x,t), \quad \Omega(t) = \sup_{z\in\boldsymbol{R}^3} |\omega(z,t)| \tag{26}$$

we can prove using only the Biot-Savart law [4] and the conservation of kinetic energy that

$$U(t) \le c\|\boldsymbol{u}_0\|_{L^2}^{\frac{2}{5}}\Omega(t)^{\frac{3}{5}} \tag{27}$$

holds for $t < T$. This is done by splitting the Biot-Savart integral in an inner integral, where we use $\Omega(t)$, and an outer integral, where we integrate by parts and use $\|u\|_{L^2}$. The inequality (27) then follows by choosing the optimal splitting in order to minimize the bound. Putting together the inequalities (22), (23), (25) and using

$$|\omega(X(q,t),t)| \le |\omega_0(q)| \exp \int_0^t |\alpha(X(q,s),s)|ds \tag{28}$$

we see that if

$$\int_0^T \inf_{r\le r(t)} \left\{\frac{U(t)}{r} + rC_a\Omega(t)\right\} dt < \infty \tag{29}$$

then no blowup occurs. For example, let us make the assumption that

$$(T-t)\Omega(t) \le C \tag{30}$$

holds with some constant C. If $r(t) \sim (T-t)^a$, then we have two possibilities. If $a < \frac{1}{5}$ then we may choose $r = \sqrt{\frac{U(t)}{C\Omega(t)}}$ for $T-t$ small, optimizing in (29), and using (27); we see then that no blowup may occur. If $a \ge \frac{1}{5}$, then we have to take $r = r(t)$ in (29) and in that case no blow up occurs if $a < \frac{2}{5}$.

Thus, if the blow up assumption **A** holds and also (30) is valid then $a \ge \frac{2}{5}$ is necessary for blowup. That means that in order for blow up to occur, the vortex lines must become incoherent at distances that are rapidly vanishing.

This kind of argument can yield more restrictive results if more information about the geometry of the vortical region is provided.

There are many results giving criteria for the absence of blowup. In [5] it is shown that simple one-scale selfsimilar blowup is impossible. Absence of squirt singularities is proved in [6]. In [7] a detailed analysis was carried out based on a number of assumptions concerning the geometry of vortex lines and the magnitude of velocity.

B. A sufficient criterion based on the pressure Hessian

Let

$$\Pi(x,t) = \left(\frac{\partial^2 p}{\partial x_i\,\partial x_j}\right) \tag{31}$$

and consider

$$Q(t) = \{x \mid \Pi(x,t) > 0\} \tag{32}$$

the region where Π is positive definite. (Note that non-degenerate local minima of $p(x,t)$ are in $Q(t)$.) Then the following is sufficient for blowup:

$$(B)\begin{cases} \exists a \in Q(0), \ \text{ such that } X(a,t) \in Q(t), \ \forall t \in [0,T] \\ \omega_0(b) = 0, \ \text{for } |b-a| \text{ small enough,} \\ T\rho(S_0)(a) > 3 \\ \text{where } \rho(S_0) = \ \text{ is the spectral radius of } S_0. \end{cases}$$

The idea of the proof was used in [8] to prove blowup for distorted Euler equations. We consider the equation obeyed by the rate of strain matrix,

$$D_t S + S^2 + \Pi - \frac{|\omega|^2}{4} P_\omega^\perp = 0 \tag{33}$$

where $D_t = \partial_t + \boldsymbol{u}\cdot\nabla$ and $P_\omega^\perp$ is the matrix that projects a vector onto the plane perpendicular on the direction of ω. The proof of the result is by contradiction. We assume that the solution is smooth up to time T. Then one can find a smooth function ϕ_0 with small support so that

$$\begin{cases} \int_{\boldsymbol{R}^3} |\phi_0(a)|^2 da = 1, \\ \int_{\boldsymbol{R}^3} S_0(a)\phi_0(a)\cdot\phi_0(a) da < 0, \\ T\left|\int_{\boldsymbol{R}^3} S_0(a)\phi_0(a)\cdot\phi_0(a) da\right| > 1, \\ \phi_0|\omega_0|^2 = 0, \end{cases} \tag{34}$$

and also, if we solve

$$D_t\phi = 0, \quad \phi(a,0) = \phi_0(a) \tag{35}$$

then

$$supp\phi(t) \subset Q(t) \quad \text{ holds, for } 0 \le t \le T. \tag{36}$$

We take

$$y(t) = \int S(x,t)\phi(x,t)\cdot\phi(x,t) dx \tag{37}$$

This blows up before T:

$$\frac{d}{dt}y + y^2 \le 0 \tag{38}$$

because

$$|\omega(x,t)|^2 |\phi(x,t)| = 0, \tag{39}$$

$$\int_{\boldsymbol{R}^3} |\phi(x,t)|^2 dx = 1 \tag{40}$$

and Cauchy-Schwarz

$$\int_{\boldsymbol{R}^3} |S\phi|^2 dx \ge y^2(t). \tag{41}$$

III. WEAK SOLUTIONS

The Navier-Stokes equations have global weak solutions in a natural space [9]. The same cannot be said about the Euler equations, but can be said about the surface quasi-geostrophic equations ([10], [11]). A general methodology for the construction of useful weak solutions does not exist, but the steps are usually: good approximation, integration by parts, weak continuity. Minimal requirements for weak solutions for the Euler equations are that they should be given by a weakly continuous function of time $\boldsymbol{u}(t)$ with values in the space of locally L^2 functions (uniformly, a technical requirement)

$$u \in C_w[0,T; L^2_{loc,u}]$$

such that, for every divergence-free compactly supported smooth function φ

$$\begin{array}{c} \int u(t)\cdot\varphi dx - \int u_0\cdot\varphi dx = \\ \int_0^t \int \text{Trace}\left[(u\otimes u)(\nabla\varphi)\right] dx ds \end{array}$$

holds. The surface quasi-geostrophic equation (QG henceforth)

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \\ u = R^\perp\theta \end{cases} \tag{42}$$

has served as a didactic model for 3D Euler equations [2], [12]. Here $R = (\nabla)\Lambda^{-1}$ are Riesz operators and $\Lambda = (-\Delta)^{1/2}$ is the Zygmund operator. The equations are in two spatial dimensions and θ is a scalar. Analogous to vortex lines, the iso-θ lines are material, and the "vorticity" equation

$$\partial_t\left(\nabla^\perp\theta\right) + u\cdot\nabla\left(\nabla^\perp\theta\right) = \left(\nabla^\perp\theta\right)\cdot\nabla u \tag{43}$$

has the same stretching term as (3). A criterion like the Beale-Kato-Majda criterion is valid, and the geometric depletion of nonlinearity via the direction of "vorticity" takes place as well. In order to understand why these equations have weak solutions, the easiest route is via Fourier series in the periodic case.

A. Weak solutions for QG

For periodic $\theta = \sum_{j\in \boldsymbol{Z}^2} \widehat{\theta}(j)e^{i(j\cdot x)}$, the equation (42) is equivalent to an infinite sequence of ordinary differential equations:

$$\frac{d}{dt}\widehat{\theta}(l) = \sum_{j+k=l} \left(j^{\perp}\cdot k\right)|j|^{-1}\widehat{\theta}(j)\widehat{\theta}(k) \tag{44}$$

Using the fact that $j^{\perp}\cdot k$ is antisymmetric in j,k while the sum is over a symmetric set of vectorial indices and $\widehat{\theta}(j)\widehat{\theta}(k)$ is symmetric in j,k, it follows that

$$\frac{d}{dt}\widehat{\theta}(l) = \sum_{j+k=l} \gamma^l_{j,k}\widehat{\theta}(j)\widehat{\theta}(k) \tag{45}$$

where

$$\gamma^l_{j,k} = \frac{1}{2}(j^{\perp}\cdot k)\left(\frac{1}{|j|}-\frac{1}{|k|}\right) \tag{46}$$

Now clearly

$$\left|\gamma^l_{j,k}\right| \le \frac{|l|^2}{\max\{|j|\,,|k|\}} \tag{47}$$

Consequently

$$\begin{array}{c}\|(-\Delta)^{-1}\left[B(\theta_1,\theta_1)-B(\theta_2,\theta_2)\right]\|_w \le \\ C\left(\|\theta_1\|_{L^2}+\|\theta_2\|_{L^2}\right)\times \\ \left\{\|\theta_1-\theta_2\|_w\left(1+\log_+\|\theta_1-\theta_2\|_w\right)\right\}\end{array} \tag{48}$$

where the weak norm $\|\theta\|_w = \sup_{j\in\boldsymbol{Z}^2}|\widehat{\theta}(j)|$ We see that the nonlinearity is weakly quasi-Lipschitz, with loss of two derivatives. The loss of derivatives does not impede existence theory for weak solutions. It does however prevent a proof of uniqueness of these weak solutions, and that is still open. The inequality allows a simple strategy of proof of existence of weak solutions. Any approximation procedure that gives long lived solutions and respects the conservation law $\theta\in L^2$ can be used. Then passing to a weakly convergent subsequence we obtain the fact that the weak limit solves weakly the equation, because of the inequality (48) and the strong convergence in $\|\ \|_w$ that follows from weak L^2 convergence. Although the QG equation is two dimensional, the reason for the property that allowed the global weak solutions is structural, not dimensional.

B. Littlewood-Paley decomposition and Euler equations

The Littlewood-Paley decomposition is a useful tool. For functions that are sufficiently well behaved at infinity it is enough to look at the so called inhomogeneous decomposition:

$$u = \sum_{j=-1}^{\infty}\Delta_j(u) \tag{49}$$

The operators Δ_j are defined using the Fourier transform $\mathcal{F}$ and have the properties

$$\begin{array}{c}\operatorname{supp}\mathcal{F}(\Delta_j(u)) \subset \{\xi\,;\,|\xi|\in 2^j[\frac{1}{2},\frac{5}{4}]\}\\ \Delta_j\Delta_k \neq 0 \Rightarrow |j-k|\le 1,\\ \left(\Delta_j+\Delta_{j+1}+\Delta_{j+2}\right)\Delta_{j+1} = \Delta_{j+1}\\ \Delta_j\left(S_{k-2}(u)\Delta_k(v)\right)\neq 0 \Rightarrow k\in[j-2,j+2]\\ \text{where } S_k(u) = \sum_{j=-1}^{k}\Delta_j(u).\end{array}$$

Specifically,

$$\Delta_j = \Psi_j(D) = \Psi_0(2^{-j}D), \quad \Delta_{-1}u = \Phi_{-1}(D)u.$$

where Φ_{-1} is radial, nonincreasing, C^∞ and

$$\begin{cases}\Phi_{-1}=1,\ 0\le r\le a\\ \Phi_{-1}=0,\ r\ge b\\ 0<a<b<1\end{cases}$$

$$\Psi_0(r) = \Phi_{-1}(r/2)-\Phi_{-1}(r), \quad \Psi_j(r) = \Psi_0(2^{-j}r).$$

$$(\Psi(D)u)(x) = (2\pi)^{-n}\int_{\boldsymbol{R}^n} e^{i(x\cdot\xi)}\Psi(\xi)\widehat{u}(\xi)d\xi$$

$\widehat{u}(\xi) = \mathcal{F}(u)(\xi) = \int_{\boldsymbol{R}^n} e^{-i(x\cdot\xi)}u(x)dx$ and $a<b<\frac{4}{3}a$ (For instance $a=1/2,\ b=5/8$ works.)

The Littlewood decomposition can be used to define inhomogeneous Besov spaces

$$\|u\|_{B^s_{p,q}} = \left\|\left\{2^{sj}\|\Delta_j(u)\|_{L^p}\right\}_j\right\|_{\ell^q(\boldsymbol{N})}.$$

and the space $B^s_{p,c(\boldsymbol{N})}$ which is the closed subspace of $B^s_{p,\infty}$ formed with functions such that

$$\lim_{j\to\infty}2^{sj}\|\Delta_j(u)\|_{L^p} = 0.$$

In $B^s_{p,q}$, s counts the number of derivatives, p refers to the L^p space and q is an interpolation index.

C. Euler weak solutions: main difficulty

The nonlinearity in the Euler equations is

$$B(u,v) = \boldsymbol{P}(u\cdot\nabla v) = \Lambda\boldsymbol{H}(u,v) \tag{50}$$

with $\boldsymbol{P}$ the Leray-Hodge projection on divergence-free functions and

$$\left[\boldsymbol{H}(u,v)\right]_i = R_j(u_jv_i)+R_i(R_kR_l(u_kv_l)), \tag{51}$$

and $R_k = \partial_k\Lambda^{-1}$ Riesz transforms. Applying Δ_q we have

$$\Delta_q(B(u,v)) = C_q(u,v)+I_q(u,v) \tag{52}$$

where

$$C_q(u,v) = \sum_{p\ge q-2,\ |p-p'|\le 2}\Delta_q(\Lambda\boldsymbol{H}(\Delta_pu,\Delta_{p'}v)) \tag{53}$$

and

$$I_q(u,v) = \sum_{j=-2}^{2} [\Delta_q \Lambda \boldsymbol{H}(S_{q+j-2}u, \Delta_{q+j}v) + \Delta_q \Lambda \boldsymbol{H}(S_{q+j-2}v, \Delta_{q+j}u)\ ] \tag{54}$$

is essentially the Bony paraproduct [13]. For L^2 weak solutions it would be desirable to have a bound of the type

$$\|\Lambda^{-M}(B(u_1,u_1) - B(u_2,u_2))\|_w \leq C\|u_1 - u_2\|_w^a [\|u_1\|_{L^2} + \|u_2\|_{L^2}\|]^{2-a} \tag{55}$$

with $a > 0$ and $\|f\|_w$ a weak enough norm so that weak convergence in L^2 implies, after localization, strong convergence in the w norm. The number M could be as large as needed. An inequality (55) is true for $I(u,v)$ but not for $C(u,v)$. On the other hand, if one wishes weak solutions with positive derivative exponents, for instance weak solutions in $B^{\frac{1}{3}}_{3,q}$, then $C(u,v)$ has good continuity properties, and $I(u,v)$ does not [14]. The terms I_q, if retained alone, would produce a leaky Galerkin approximation

$$\frac{\partial \Delta_q(u)}{\partial t} = I_q(u,u),$$

and the terms $C_q(u,u)$ an ill-formed shell model

$$\frac{\partial \Delta_q(u)}{\partial t} = C_q(u,u).$$

A description of the regularity of some shell models is given in [15].

D. The Onsager Conjecture

Although weak solutions with positive smoothness have not been proven to exist (see [16], [17] for examples of weak solutions), the subject is important because of the relation to turbulence. The Onsager conjecture [18], [19] asserts that kinetic energy is conserved for solutions in C^s with $s > \frac{1}{3}$ and dissipated for rougher solutions, in particular in $C^{\frac{1}{3}}$. The paper [20] proves that if weak solutions belong to $L^3[0,T; B^s_{3,\infty}]$ with $s > \frac{1}{3}$ then they conserve kinetic energy. The paper [21] extended this to spaces in which the fractional derivative D^s (2^{js} in the Littlewood-Paley decomposition) is replaced with any function of $f(D)$ such that $f(D)D^{-\frac{1}{3}} \to \infty$ as $D \to \infty$. This actually follows also from the proof in [20]. More recently, it was shown [14] that weak solutions of the 3D Euler equations in $u \in L^3([0,T], B^{1/3}_{3,c(\boldsymbol{N})})$ conserve kinetic energy. On the other hand, there exist functions in $B^{\frac{1}{3}}_{3\infty}$ that are divergence-free and do not conserve energy in the sense to be made more precise below. Consider the flux

$$\Pi_N := \int_{\boldsymbol{R}^3} \text{Trace } [S_N(u \otimes u)\nabla S_N(u)]\, dx. \tag{56}$$

This is the (formal) time derivative

$$\Pi_N = \frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{R}^3} |S_N(u(t))|^2\, dx$$

of the energy contained in $S_N(u)$ when u solves the Euler equation. There exist functions in $B^{1/3}_{3,\infty}$ that are divergence-free and obey $\liminf_{N\to\infty} |\Pi_N| > 0$. On the other hand, if $u \in B^{\frac{1}{3}}_{3,c(\boldsymbol{N})}$ then $\limsup_{N\to\infty} |\Pi_N| = 0$. More specifically, if we let

$$K(j) = \begin{cases} 2^{\frac{2j}{3}}, & j \leq 0; \\ 2^{-\frac{4j}{3}}, & j > 0, \end{cases}$$

and

$$d_j = 2^{j/3}\|\Delta_j(u)\|_3, \text{ for } j \geq -1, \quad d_j = 0 \text{ for } j < -1$$
$$d^2 = \{d_j^2\}_j$$

If $u \in L^2$ then it can be shown that

$$|\Pi_N| \leq C(K * d^2)^{3/2}(N) \tag{57}$$

where $*$ means convolution of sequences. Consequently, of course

$$\limsup_{N\to\infty} |\Pi_N| \leq \limsup_{N\to\infty} d_N^3, \tag{58}$$

but moreover, a strong localization of the flux results from (57).

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