Euler Equations, Navier-Stokes Equations and Turbulence

C.I.M.E. Lectures, 2003.

Peter Constantin, version 2/21/2004

1 Introduction

In 2004 the mathematical world will mark 120 years since the advent of turbulence theory ([80]). In his 1884 paper Reynolds introduced the decomposition of turbulent flow into mean and fluctuation and derived the equations that describe the interaction between them. The Reynolds equations are still a riddle. They are based on the Navier-Stokes equations, which are a still a mystery. The Navier-Stokes equations are a viscous regularization of the Euler equations, which are still an enigma. Turbulence is a riddle wrapped in a mystery inside an enigma ([11]).

Crucial for the determination of the mean in the Reynolds equation are Reynolds stresses, which are second order moments of fluctuation. The fluctuation requires information about small scales. In order to be able to compute at high Reynolds numbers, in state-of-the-art engineering practice, these small scales are replaced by sub-grid models. "Que de choses il faut ignorer pour 'agir' ! " sighed Paul Valéry ([88]). ("How many things must one ignore in order to 'act' ! ") The effect of small scales on large scales is the riddle in the Reynolds equations. In 1941 Kolmogorov ([65]) ushered in the idea of universality of the statistical properties of small scales. This is a statement about the asymptotics: long time averages, followed by the infinite Reynolds number limit. This brings us to the mystery in the Navier-Stokes equations: the infinite time behavior at finite but larger and larger Reynolds numbers. The small Reynolds number behavior is trivial (or "direct", to use the words of Reynolds himself). Ruelle and Takens suggested in 1971 that deterministic chaos emerges at larger Reynolds numbers ([83]). The route to chaos itself was suggested to be universal by Feigenbaum ([49]). Foias and Prodi discussed finite dimensional determinism in the Navier-Stokes equations already in 1967 ([55]), four years after the seminal work of Lorenz ([68]). The dynamics have indeed finite dimensional character if one confines oneself to flows in bounded regions in two dimensions ([2], [31], [32], [34], [56], [69]). In three dimensions, however, the long time statistics question is muddied by the blow up problem. Leray ([67]) showed that there exist global solutions, but such solutions may develop singularities. Do such singularities exist? And if they do, are they relevant to turbulence? The velocities observed in turbulent flows on Earth are bounded. If one accepts this as a physical assumption, then, invoking classical results of Serrin ([84]), one concludes that Navier-Stokes singularities, if they exist at all, are not relevant to turbulence. The experimental evidence, so far, is of a strictly positive energy dissipation rate $0 < \epsilon = \langle \nu |\nabla u|^2 \rangle$, at high Reynolds numbers. This is consistent with large gradients of velocity. The gradients of velocity intensify in vortical activity. This activity consists of three mechanisms: stretching, folding and reconnection of vortices. The stretching and folding are inviscid mechanisms, associated with the underlying incompressible Euler equations. The reconnection is the change of topology of the vortex field, and it is not allowed in smooth solutions of the Euler equations. This brings us to the enigma of the Euler equations, and it is here where it is fit we start.

2 Euler Equations

The Euler equations of incompressible fluid mechanics present some of the most serious challenges for the analyst. The equations are

$$D_t u + \nabla p = 0 \tag{1}$$

with $\nabla \cdot u = 0$. The function $u = u(x,t)$ is the velocity of an ideal fluid at the point x in space at the moment t in time. The fluid is assumed to have unit density. The velocity is a three-component vector, and x lies in three dimensional Euclidean space. The requirement that $\nabla \cdot u = 0$ reflects the incompressibility of the fluid. The material derivative (or time derivative along particle trajectories) associated to the velocity u is

$$D_t = D_t(u, \nabla) = \partial_t + u \cdot \nabla. \tag{2}$$

The acceleration of a particle passing through x at time t is $D_t u$. The Euler equations are an expression of Newton's second law, $F = ma$, in the form $-\nabla p = D_t u$. Thus, the only forces present in the ideal incompressible Euler equations are the internal forces at work keeping the fluid incompressible. These forces are not local: the pressure obeys

$$-\Delta p = \nabla \cdot (u \cdot \nabla u) = \mathrm{T}r\left\{(\nabla u)^2\right\} = \partial_i \partial_j (u_i u_j).$$

If one knows the behavior of the pressure at boundaries then the pressure satisfies a nonlocal functional relation of the type $p = F([u \otimes u])$. For instance, in the whole space, and with decaying boundary conditions

$$p = R_i R_j (u_i u_j)$$

where $R_i = \partial_i (-\Delta)^{-\frac{1}{2}}$ are Riesz transforms. (We always sum repeated indices, unless we specify otherwise. The pressure is defined up to a time dependent constant; in the expression above we have made a choice of zero average pressure.)

Differentiating the Euler equations one obtains:

$$D_t U + U^2 + \mathrm{T}r\left\{(R \otimes R)\, U^2\right\} = 0$$

where $U = (\nabla u)$ is the matrix of derivatives. We used the specific expression for p written above for the whole space with decaying boundary conditions.

This equation is quadratic and it suggests the possibly of singularities in finite time, by analogy with the ODE $\frac{d}{dt}U + U^2 = 0$. In fact, the distorted Euler equation

$$\partial_t U + U^2 + \mathrm{Tr}\left\{(R \otimes R)\, U^2\right\} = 0$$

does indeed blow up ([15]). The incompressibility constraint $\mathrm{T}rU = 0$ is respected by the distorted Euler equation. However, the difference between the Eulerian time derivative ∂_t and the Lagrangian time derivative D_t is significant. One may ask whether true solutions of the Euler equations do blow up. The answer is yes, if one allows the solutions to have infinite kinetic energy. We will give an example in Section Three. The blow up is likely due to the infinite supply of energy, coming from infinity. The physical question of finite time local blow up is different, and perhaps even has a different answer.

In order to analyze nonlinear PDEs with physical significance one must take advantage of the basic invariances and conservation laws associated to the equation. When properly understood, the reasons behind the conservation laws show the way to useful cancellations.

Smooth solutions of the Euler equations conserve total kinetic energy, helicity and circulation. The total kinetic energy is proportional to the L^2 norm of velocity. This is conserved for smooth flows. The Onsager conjecture ([72], [48]) states that this conservation occurs if and only if the solutions are smoother than the velocities supporting the Kolmogorov theory, (Hölder continuous of exponent 1/3). The "if" part was proved ([28]). The "only if" part is difficult: there is no known notion of weak solutions dissipating energy but with Hölder continuous velocities. The work of Robert ([81]) and weak formulations of Brenier and of Shnirelman are relevant to this question ([85], [6]).

In order to describe the helicity and circulation we need to talk about vorticity and about particle paths. The Euler equations are formally equivalent to the requirement that two first order differential operators commute:

$$[D_t, \Omega] = 0.$$

The first operator $D_t = \partial_t + u \cdot \nabla$ is the material derivative associated to the trajectories of u. The second operator

$$\Omega = \omega(x,t) \cdot \nabla$$

is differentiation along vortex lines, the lines tangent to the vorticity field ω. The commutation means that vortex lines are carried by the flow of u, and is equivalent to the equation

$$D_t\omega = \omega \cdot \nabla u. \tag{3}$$

This is a quadratic equation because ω and u are related, $\omega = \nabla \times u$. If boundary conditions for the divergence-free ω are known (periodic or decay at infinity cases) then one can use the Biot-Savart law

$$u = \mathcal{K}_{3DE} * \omega = \nabla \times (-\Delta)^{-1}\omega \tag{4}$$

coupled with (3) as an equivalent formulation of the Euler equations, the vorticity formulation used in the numerical vortex methods of Chorin ([13], [14]). The helicity is

$$h = u \cdot \omega.$$

The Lagrangian particle maps are

$$a \mapsto X(a,t), \quad X(a,0) = a.$$

For fixed a, the trajectories of u obey

$$\frac{dX}{dt} = u(X,t).$$

The incompressibility condition implies

$$det\,(\nabla_a X) = 1.$$

The Euler equations can be described ([63], [1]) formally as Euler-Lagrange equations resulting from the stationarity of the action

$$\int_a^b \int |u(x,t)|^2\, dxdt$$

with

$$u(x,t) = \frac{\partial X}{\partial t}(A(x,t),t)$$

and with fixed end values at $t = a, b$ and

$$A(x,t) = X^{-1}(x,t).$$

Helicity integrals ([71])

$$\int_T h(x,t)dx = c$$

are constants of motion, for any vortex tube T (a time evolving region whose boundary is at each point parallel to the vorticity, $\omega \cdot N = 0$ where N is the normal to ∂T at $x \in \partial T$.) The constants c have to do with the topological complexity of the flow.

Davydov, and Zakharov and Kuznetsov ([42], [93]) have formulated the incompressible Euler equations as a Hamiltonian system in infinite dimensions in Clebsch variables. These are a pair of active scalars θ, φ which are constant on particle paths,

$$D_t\varphi = D_t\theta = 0$$

and also determine the velocity via

$$u^i(x,t) = \theta(x,t)\frac{\partial\varphi(x,t)}{\partial x_i} - \frac{\partial n(x,t)}{\partial x_i}.$$

The helicity constants vanish identically for flows which admit a Clebsch variables representation. Indeed, for such flows the helicity is the divergence of a field that is parallel to the vorticity $h = -\nabla \cdot (n\omega)$. This implies that not all flows admit a Clebsch variables representation. But if one uses more variables, then one can represent all flows. This is done using the Weber formula ([90]) which we derive briefly below.

In Lagrangian variables the Euler equations are

$$\frac{\partial^2 X^j(a,t)}{\partial^2 t} = -\frac{\partial p(X(a,t),t)}{\partial x_j}. \tag{5}$$

Multiplying this by $\frac{\partial X^j(a,t)}{\partial a_i}$ we obtain

$$\frac{\partial^2 X^j(a,t)}{\partial t^2}\frac{\partial X^j(a,t)}{\partial a_i} = -\frac{\partial \tilde{p}(a,t)}{\partial a_i}$$

where $\tilde{p}(a,t) = p(X(a,t),t)$. Forcing out a time derivative in the left-hand side we obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial X^j(a,t)}{\partial t}\frac{\partial X^j(a,t)}{\partial a_i}\right] = -\frac{\partial \tilde{q}(a,t)}{\partial a_i}$$

with

$$\tilde{q}(a,t) = \tilde{p}(a,t) - \frac{1}{2}\left|\frac{\partial X(a,t)}{\partial t}\right|^2$$

Integrating in time, fixing the label a we obtain:

$$\frac{\partial X^j(a,t)}{\partial t}\frac{\partial X^j(a,t)}{\partial a_i} = u^i_{(0)}(a) - \frac{\partial \tilde{n}(a,t)}{\partial a_i}$$

with

$$\tilde{n}(a,t) = \int_0^t \tilde{q}(a,s)ds$$

where

$$u_{(0)}(a) = \frac{\partial X(a,0)}{\partial t}$$

is the initial velocity. We have thus:

$$(\nabla_a X)^* \partial_t X = u_{(0)}(a) - \nabla_a \tilde{n}.$$

where we denote M^* the transpose of the matrix M.
Multiplying by $[(\nabla_a X(a,t))^*]^{-1}$ and reading at $a = A(x,t)$ with

$$A(x,t) = X^{-1}(x,t)$$

we obtain the Weber formula

$$u^i(x,t) = \left(u^j_{(0)}(A(x,t))\right)\frac{\partial A^j(x,t)}{\partial x_i} - \frac{\partial n(x,t)}{\partial x_i}.$$

This relationship, together with boundary conditions and the divergence-free requirement can be written as

$$u = W[A,v] = \mathbf{P}\left\{(\nabla A)^* v\right\} \tag{6}$$

where $\mathbf{P}$ is the corresponding projector on divergence-free functions and v is the virtual velocity

$$v = u_{(0)} \circ A.$$

We will consider the cases of periodic boundary conditions or whole space. Then

$$\mathbf{P} = I + R \otimes R$$

holds, with R the Riesz transforms. This procedure turns A into an *active scalar system*

$$\begin{cases} D_tA = 0, \\ D_tv = 0, \\ u = W[A, v]. \end{cases} \tag{7}$$

Active scalars ([17]) are solutions of the passive scalar equation $D_t\theta = 0$ which determine the velocity through a time independent, possibly non-local equation of state $u = U[\theta]$.

Conversely, and quite generally: Start with two families of labels and virtual velocities

$$A = A(x, t, \lambda), \quad v(x, t, \lambda)$$

depending on a parameter λ such that

$$D_tA = D_tv = 0$$

with $D_t = \partial_t + u \cdot \nabla_x$. Assume that u can be reconstructed from A, v via a generalized Weber formula

$$u(x, t) = \int \nabla_x A(x, t, \lambda)v(x, t, \lambda)d\mu(\lambda) - \nabla_x n$$

with some function n, and some measure $d\mu$. Then u solves the Euler equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla \pi = 0$$

where

$$\pi = D_tn + \frac{1}{2}|u|^2.$$

Indeed, using the kinematic commutation relation

$$D_t\nabla_x f = \nabla_x D_t f - (\nabla_x u)^*\nabla_x f$$

and differentiating the generalized Weber formula we obtain:

$$D_tu = D_t(\int \nabla_x Avd\mu - \nabla_x n) =$$

$$-\int ((\nabla_x u)^*\nabla_x A)vd\mu - \nabla_x(D_tn) + (\nabla_x u)^*\nabla n =$$

$$-\nabla_x(D_tn) - (\nabla_x u)^* \left[\int (\nabla_x A)vd\mu - \nabla_x n\right] =$$

$$-\nabla_x(D_t n) - (\nabla_x u)^* u = -\nabla_x(\pi).$$

The circulation is the loop integral

$$C_\gamma = \oint_\gamma u \cdot dx$$

and the conservation of circulation is the statement that

$$\frac{d}{dt} C_{\gamma(t)} = 0$$

for all loops carried by the flow. This follows from the Weber formula because

$$u^j(X(a,t)) \frac{\partial X^j}{\partial a_i} = u^i_{(0)}(a) - \frac{\partial \tilde{n}(a,t)}{\partial a_i}.$$

The important thing here is that the right hand side is the sum of a time independent function of labels and a label gradient. Viceversa, the above formula follows from the conservation of circulation. The Weber formula is equivalent thus to the conservation of circulation.

Differentiating the Weber formula, one obtains

$$\frac{\partial u^i}{\partial x_j} = \mathbf{P}_{ik} \left(Det \left[\frac{\partial A}{\partial x_j}; \frac{\partial A}{\partial x_k}; \omega_{(0)}(A) \right] \right).$$

Here we used the notation

$$\omega_{(0)} = \nabla \times u_{(0)}.$$

Taking the antisymmetric part one obtains the Cauchy formula:

$$\omega_i = \frac{1}{2} \epsilon_{ijk} \left(Det \left[\frac{\partial A}{\partial x_j}; \frac{\partial A}{\partial x_k}; \omega_{(0)}(A) \right] \right).$$

which we write as

$$\omega = \mathcal{C}[\nabla A, \zeta] \tag{8}$$

with ζ the Cauchy invariant

$$\zeta(x,t) = \omega_{(0)} \circ A.$$

Therefore the active scalar system

$$\begin{cases} D_t A = 0, \\ D_t \zeta = 0, \\ u = \nabla \times (-\Delta)^{-1} \left(\mathcal{C}[\nabla A, \zeta] \right) \end{cases} \tag{9}$$

is an equivalent formulation of the Euler equations, in terms of the Cauchy invariant ζ. The purely Lagrangian formulation (5) of the Euler equations is in phrased in terms of independent label variables (or ideal markers) a, t, except that the pressure is obtained by solving a Poisson equation in Eulerian independent variables x, t. The rest of PDE formulations of the Euler equations described above were: the Eulerian velocity formulation (1), the Eulerian vorticity formulation (3), the Eulerian-Lagrangian virtual velocity formulation (7) and the Eulerian-Lagrangian Cauchy invariant formulation (9). The Eulerian-Lagrangian equations are written in Eulerian coordinates x, t, in what physicists call "laboratory frame". The physical meaning of the dependent variables is Lagrangian.

The classical local existence results for Euler equations can be proved in either purely Lagrangian formulation ([45]), in Eulerian formulation ([70]) or in Eulerian-Lagrangian formulation ([19]). For instance one has

Theorem 1 *([19]) Let $\alpha > 0$, and let u_0 be a divergence free $C^{1,\alpha}$ periodic function of three variables. There exists a time interval $[0, T]$ and a unique $C([0, T]; C^{1,\alpha})$ spatially periodic function $\ell(x, t)$ such that*

$$A(x, t) = x + \ell(x, t)$$

solves the active scalar system formulation of the Euler equations,

$$\frac{\partial A}{\partial t} + u \cdot \nabla A = 0,$$

$$u = \mathbf{P}\left\{(\nabla A(x, t))^* u_0(A(x, t))\right\}$$

with initial datum $A(x, 0) = x$.

A similar result holds in the whole space, with decay requirements for the vorticity. As an application, let us consider rotating three dimensional incompressible Euler equations

$$\partial_t u + u \cdot \nabla u + \nabla \pi + 2\Omega e_3 \times u = 0.$$

The Weber formula for relative velocity is:

$$u(x, t) = \mathbf{P}(\partial_i A^m(x, t) u_0^m(A(x, t), t))$$

$$+\Omega \mathbf{P}\left\{(\widehat{z}; A(x, t), \partial_i A(x, t)) - (\widehat{z}; x; e_i)\right\}.$$

Here Ω is the constant angular velocity (not $\omega \cdot \nabla$), and e_i form the canonical basis of $\mathbf{R}^3$. We consider the Lagrangian paths $X(a,t)$ associated to the relative velocity u, and their inverses $A(x,t) = X^{-1}(x,t)$, obeying

$$(\partial_t + u \cdot \nabla) A = 0.$$

As a consequence of the Cauchy formula for the total vorticity $\omega + 2\Omega e_3$ one can prove that the direct Lagrangian displacement

$$\lambda(a,t) = X(a,t) - a$$

obeys a time independent differential equation. The Cauchy formula for the total vorticity (the vorticity in a non-rotating frame) follows from differentiation of the Weber formula above and is the same as in the non-rotating case

$$\omega + 2\Omega e_3 = \mathcal{C}[\nabla A, \zeta + 2\Omega e_3]$$

Composing with X the right hand side is

$$\mathcal{C}[\nabla A, \zeta + 2\Omega e_3] \circ X = \left(\omega_{(0)} + 2\Omega e_3\right) \cdot \nabla_a X.$$

Rearranging the Cauchy formula we obtain

$$\partial_{a_3} \lambda(a,t) + \frac{1}{2}\rho_0(a)\xi_0(a) \cdot \nabla_a \lambda(a,t) =$$

$$= \frac{1}{2}\left(\rho_t(a)\xi(a,t) - \rho_0(a)\xi(a,0)\right)$$

where

$$\rho_t(a) = \frac{|\omega(X(a,t),t)|}{\Omega}$$

is the local Rossby number and $\xi = \frac{\omega}{|\omega|}$ is the unit vector of relative vorticity direction. This fact explains directly ($\partial_{a_3}\lambda = O(\rho)$) the fact that strong rotation inhibits vertical transport ([24]). In particular, one can prove rather easily

Theorem 2 *Let $u_0 \in H^s(\mathbf{T}^3)$, $s > \frac{5}{2}$ and let $T > 0$ be small enough . For each Ω, consider the inverse and direct Lagrangian displacements*

$$\ell(x,t) = A(x,t) - x$$

and

$$\lambda(a_1, a_2, a_3, t) = X(a_1, a_2, a_3, t) - (a_1, a_2, a_3).$$

They obey

$$\|\partial_{x_3}\ell(\cdot, t)\|_{L^\infty(dx)} \le C\rho$$

and

$$\|\partial_{a_3}\lambda(\cdot, t)\|_{L^\infty(da)} \le C\rho$$

with $\rho = \Omega^{-1}\sup_{0\le t\le T}\|\omega(\cdot, t)\|_{L^\infty(dx)}$.

Let $\Omega_j \to \infty$ *be an arbitrary sequence and let* $X^j(a_1, a_2, a_3, t)$ *denote the Lagrangian paths associated to* Ω_j. *Then, there exists a subsequence (denoted for convenience by the same letter* j*) an invertible map* $X(a_1, a_2, a_3, t)$*, and a periodic function of two variables* $\lambda(a_1, a_2, t)$ *such that*

$$\lim_{j\to\infty} X^j(a_1, a_2, a_3, t) = X(a_1, a_2, a_3, t)$$

holds uniformly in a, t *and*

$$X(a_1, a_2, a_3) = (a_1, a_2, a_3) + \lambda(a_1, a_2, t)$$

This represents a nonlinear Taylor-Proudman theorem in the presence of inertia. It also implies, at positive Rossby number, that the vertical transport, in a relative vorticity turnover time is of the order of the local Rossby number. The nonlinear Taylor Proudman statement and derivation of effective equations are usually addressed via analysis of resonant interactions ([46], [3]).

3 An Infinite Energy Blow Up Example

A classical stagnation point ansatz for the solution of the Euler equations was shown to lead to blow up in three dimensions by J.T. Stuart ([87]), and two dimensions by Childress, Iearly, Spiegel and Young ([12]). A new ansatz, and numerics for the singularity are due to Gibbon and Ohkitani ([76]). The ansatz requires infinite energy, separation of variables, and demands the three dimensional velocity in the form

$$\mathbf{u}(x, y, z, t) = (u(x, y, t), z\gamma(x, y, t))$$

with $\gamma(x,y,t)$ and $u(x,y,t) = (u_1(x,y,t), u_2(x,y,t))$ periodic functions of two variables with fundamental domain Q. The divergence-free condition for $\mathbf{u}$ becomes the two dimensional

$$\nabla \cdot u = -\gamma.$$

The Euler equations respect this ansatz and the dynamics reduce to a pair of equations, one for the vorticity in the vertical direction

$$\omega_3(x,y,t) = \frac{\partial u_2(x,y,t)}{\partial x} - \frac{\partial u_1(x,y,t)}{\partial y},$$

and one for the variable γ which represents the vertical derivative of the vertical component of velocity. The velocity is recovered using constitutive equations for a stream function ψ and a potential h. This entire system is

$$\begin{cases} \partial_t \omega_3 + u \cdot \nabla \omega_3 = \gamma \omega_3. \\ \frac{\partial \gamma}{\partial t} + u \cdot \nabla \gamma = -\gamma^2 + \frac{2}{|Q|} \int_Q \gamma^2(x,t) dx \\ u = \nabla^{\perp} \psi + \nabla h \\ -\Delta h = \gamma, \\ -\Delta \psi = \omega_3. \end{cases}$$

The two dynamical equations for (ω_3, γ) coupled with the constitutive equations for u form the nonlocal Riccati system. This blows up from all nontrivial initial data:

Theorem 3 *([18]) Consider the nonlocal conservative Riccati system. For any smooth, mean zero initial data $\gamma_0 \neq 0$, ω_0, the solution becomes infinite in finite time. Both the maximum and the minimum values of the component γ of the solution diverge to plus infinity and respectively to negative infinity at the blow up time.*

One can determine explicitly the blow up time and the form of γ on characteristics, without having to actually integrate the characteristic equations, which may be rather difficult. The solution is given on characteristics $X(a,t)$ in terms of the initial data $\gamma(x,0) = \gamma_0(x)$ by

$$\gamma(X(a,t),t) = \alpha(\tau(t)) \left(\frac{\gamma_0(a)}{1 + \tau(t)\gamma_0(a)} - \overline{\phi}(\tau(t)) \right)$$

where

$$\overline{\phi}(\tau) = \left\{\int_Q \frac{\gamma_0(a)}{(1+\tau\gamma_0(a))^2}da\right\}\left\{\int_Q \frac{1}{1+\tau\gamma_0(a)}da\right\}^{-1},$$

$$\alpha(\tau) = \left\{\frac{1}{|Q|}\int_Q \frac{1}{1+\tau\gamma_0(a)}da\right\}^{-2}$$

and

$$\frac{d\tau}{dt} = \alpha(\tau), \quad \tau(0) = 0.$$

The function $\tau(t)$ can also be obtained implicitly from

$$t = \left(\frac{1}{|Q|}\right)^2 \int_Q\int_Q \frac{1}{\gamma_0(a)-\gamma_0(b)} \log\left(\frac{1+\tau\gamma_0(a)}{1+\tau\gamma_0(b)}\right) dadb.$$

The blow up time $t = T_*$ is given by

$$T_* = \frac{1}{|Q|^2}\int\int \frac{1}{\gamma_0(a)-\gamma_0(b)} \log\left(\frac{\gamma_0(a)-m_0}{\gamma_0(b)-m_0}\right) dadb$$

where

$$m_0 = \min_Q \gamma_0(a) < 0.$$

The Jacobian $J(a,t) = \mathrm{Det}\left\{\frac{\partial X(a,t)}{\partial a}\right\}$ is given by

$$J(a,t) = \frac{1}{1+\tau(t)\gamma_0(a)}\left\{\frac{1}{|Q|}\int_Q \frac{da}{1+\tau(t)\gamma_0(a)}\right\}^{-1}$$

Because the Jacobian can be found explicitly, one can get some of the Eulerian information as well: integrals of powers of γ can be computed. The moments of γ are given by

$$\int_Q (\gamma(x,t))^p dx =$$

$$(\alpha(\tau))^p \int_Q \left\{\frac{\gamma_0(a)}{1+\tau(t)\gamma_0(a)} - \overline{\phi}(\tau(t))\right\}^p J(a,t)da.$$

The proof of these facts is based on several auxilliary constructions. Let ϕ solve

$$\partial_\tau \phi + v\cdot\nabla\phi = -\phi^2$$

together with

$$\nabla \cdot v(x, \tau) = -\phi(x, \tau) + \frac{1}{|Q|}\int_Q \phi(x, \tau)dx.$$

We take the curl $\zeta = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$ and demand that

$$\partial_\tau \zeta + v \cdot \nabla \zeta = \left(\phi - \frac{3}{|Q|}\int_Q \phi(x, \tau)dx\right)\zeta.$$

Note that this is a linear equation when ϕ is known, which allows a bootstrap argument to guarantee that the construction does not breakdown before the blow up time derived below. Passing to characteristics

$$\frac{dY}{d\tau} = v(Y, \tau)$$

we integrate and obtain

$$\phi(Y(a, \tau), \tau) = \frac{\phi_0(a)}{1 + \tau\phi_0(a)}$$

valid as long

$$\inf_{a \in Q}(1 + \tau\phi_0(a)) > 0.$$

We need to compute

$$\overline{\phi}(\tau) = \frac{1}{|Q|}\int_Q \phi(x, \tau)dx.$$

The Jacobian

$$J(a, \tau) = Det\left\{\frac{\partial Y}{\partial a}\right\}$$

obeys

$$\frac{dJ}{d\tau} = -h(a, \tau)J(a, \tau)$$

where

$$h(a, \tau) = \phi(Y(a, \tau), \tau) - \overline{\phi}(\tau).$$

Initially the Jacobian equals to one, so

$$J(a, \tau) = e^{-\int_0^\tau h(a,s)ds}.$$

Consequently

$$J(a,\tau) = e^{\int_0^\tau \overline{\phi}(s)ds} \exp\left(-\int_0^\tau \frac{d}{ds}\log(1+s\phi_0(a))ds\right)$$

and thus

$$J(a,\tau) = e^{\int_0^\tau \overline{\phi}(s)ds}\frac{1}{1+\tau\phi_0(a)}.$$

The map $a \mapsto Y(a,\tau)$ is one and onto. Changing variables one has

$$\int_Q \phi(x,\tau)dx = \int_Q \phi(Y(a,\tau),t)J(a,\tau)da$$

and therefore

$$\overline{\phi}(\tau) = e^{\int_0^\tau \overline{\phi}(s)ds}\frac{1}{|Q|}\int_Q \frac{\phi_0(a)}{(1+\tau\phi_0(a))^2}da.$$

Consequently

$$\frac{d}{d\tau}e^{-\int_0^\tau \overline{\phi}(s)ds} = \frac{d}{d\tau}\frac{1}{|Q|}\int_Q \frac{1}{1+\tau\phi_0(a)}da.$$

Because both sides at $\tau = 0$ equal one, we have

$$e^{-\int_0^\tau \overline{\phi}(s)ds} = \frac{1}{|Q|}\int_Q \frac{1}{1+\tau\phi_0(a)}da$$

and

$$\overline{\phi}(\tau) = \left\{\int_Q \frac{\phi_0(a)}{(1+\tau\phi_0(a))^2}da\right\}\left\{\int_Q \frac{1}{1+\tau\phi_0(a)}da\right\}^{-1}.$$

Note that the function $\delta(x,\tau) = \phi(x,\tau) - \overline{\phi}(\tau)$obeys

$$\frac{\partial\delta}{\partial\tau} + v\cdot\nabla\delta = -\delta^2 + 2\frac{1}{|Q|}\int_Q \delta^2 dx - 2\overline{\phi}\delta.$$

We consider now the function

$$\sigma(x,\tau) = e^{2\int_0^\tau \overline{\phi}(s)ds}\delta(x,\tau)$$

and the velocity

$$U(x,\tau) = e^{2\int_0^\tau \overline{\phi}(s)ds}v(x,\tau).$$

Multiplying the equation of δ by $e^{4\int_0^\tau \overline{\phi}(s)ds}$ we obtain

$$e^{2\int_0^\tau \overline{\phi}(s)ds}\frac{\partial\sigma}{\partial\tau} + U\cdot\nabla\sigma = -\sigma^2 + \frac{2}{|Q|}\int \sigma^2 dx.$$

Note that

$$\nabla \cdot U = -\sigma.$$

Now we change the time scale. We define a new time t by the equation

$$\frac{dt}{d\tau} = e^{-2\int_0^\tau \overline{\phi}(s)ds},$$

$t(0) = 0$, and new variables

$$\gamma(x,t) = \sigma(x,\tau)$$

and

$$u(x,t) = U(x,\tau).$$

Now γ solves the nonlocal conservative Riccati equation

$$\frac{\partial \gamma}{\partial t} + u \cdot \nabla \gamma = -\gamma^2 + \frac{2}{|Q|}\int \gamma^2 dx$$

with periodic boundary conditions,

$$u = (-\Delta)^{-1}\left[\nabla^\perp \omega + \nabla \gamma\right]$$

and

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \gamma \omega$$

The initial data are

$$\gamma_0(x) = \delta_0(x) = \phi_0(x)$$

and

$$t = \left(\frac{1}{|Q|}\right)^2 \int_Q \int_Q \frac{1}{\phi_0(a) - \phi_0(b)} \log \frac{1 + \tau\phi_0(a)}{1 + \tau\phi_0(b)} da db.$$

Note that the characteristic system

$$\frac{dX}{dt} = u(X,t)$$

is solved by

$$X(a,t) = Y(a,\tau)$$

This implies the relationship

$$\gamma(X(a,t),t) = \alpha(\tau)\left(\frac{\phi_0(a)}{1 + \tau\phi_0(a)} - \overline{\phi}(\tau)\right)$$

$$\alpha(\tau) = e^{2\int_0^\tau \overline{\phi}(s)ds}.$$

If the initial smooth function $\gamma_0(a) = \phi_0(a)$, of mean zero has minimum $m_0 < 0$, then the blow up time is

$$\tau_* = -\frac{1}{m_0}$$

Lets consider now a simple example, in order to determine the blow up asymptotics. Let ϕ_0 attain the minimum m_0 at a_0, and assume locally that

$$\phi_0(a) \geq m_0 + C|a - a_0|^2$$

for $0 \leq |a - a_0| \leq r$. Then it follows that the integral

$$\frac{1}{|Q|}\int \frac{da}{\epsilon^2 + \phi_0(a) - m_0}$$

behaves like

$$\frac{1}{|Q|}\int \frac{da}{\epsilon^2 + \phi_0(a) - m_0} \sim \log\left\{\sqrt{1 + \left(\frac{Cr}{\epsilon}\right)^2}\right\}$$

for small ϵ. Taking

$$\epsilon^2 = \frac{1}{\tau} - \frac{1}{\tau_*}$$

it follows that

$$e^{-\int_0^\tau \overline{\phi}(s)ds} \sim \log\left\{\sqrt{1 + \frac{C}{\tau_* - \tau}}\right\}$$

and for small $(\tau_* - \tau)$

$$\frac{1}{|Q|}\int_Q \frac{\phi_0(a)}{(1 + \tau\phi_0(a))^2}da \sim -\frac{C}{\tau_* - \tau}$$

and thus $t(\tau)$ has a finite limit $t \to T_*$ as $\tau \to \tau_*$. The average $\overline{\phi}(\tau)$ diverges to negative infinity,

$$\overline{\phi}(\tau) \sim -\frac{C}{\tau_* - \tau}\left[\log\left\{\sqrt{1 + \frac{C}{\tau_* - \tau}}\right\}\right]^{-1}.$$

The prefactor α becomes vanishingly small

$$\alpha(\tau) \sim (\log(\tau_* - \tau))^{-2}$$

and so

$$\gamma(X(a,t),t) \sim (\log(\tau_* - \tau))^{-2} \left(\frac{\phi_0(a)}{1 + \tau\phi_0(a)} - \overline{\phi}(\tau) \right).$$

If $\phi_0(a) > 0$ then the first term in the brackets does not blow up and γ diverges to plus infinity. If the label is chosen at the minimum, or nearby, then the first term in the brackets dominates and the blow up is to negative infinity, as expected from the ODE. From

$$(\alpha(\tau))^{-1} d\tau = dt$$

it follows that the asymptotic behavior of the blow up in t follows from the one in τ:

$$T_* - t \sim (\tau_* - \tau) \left(1 + \log \left(\frac{1}{\tau_* - \tau} \right) \right)^2$$

4 Navier-Stokes Equations

The Navier-Stokes equations are

$$D_\nu u + \nabla p = 0, \tag{10}$$

together with the incompressibility condition $\nabla \cdot u = 0$. The operator D_ν

$$D_\nu = D_\nu(u, \nabla) = \partial_t + u \cdot \nabla - \nu\Delta \tag{11}$$

describes advection with velocity u and diffusion with kinematic viscosity $\nu > 0$. When $\nu = 0$ we recover formally the Euler equations (1), and $D_{\nu|\nu=0} = D_t$. The vorticity $\omega = \nabla \times u$ obeys an equation similar to (3):

$$D_\nu \omega = \omega \cdot \nabla u. \tag{12}$$

The Eulerian-Lagrangian equations (7) and (9) have also viscous counterparts ([20]). The equation corresponding to (7) is

$$\left\{ \begin{array}{c} D_\nu A = 0, \\ D_\nu v = 2\nu C \nabla v, \\ u = W[A, v] \end{array} \right. \tag{13}$$

The $u = W[A, v]$ is the Weber formula (6), same as in the case of $\nu = 0$. The right hand side of (13) is given terms of the connection coefficients

$$C^m_{k;i} = \left((\nabla A)^{-1}\right)_{ji} (\partial_j \partial_k A^m).$$

The detailed form of virtual velocity equation in (13) is

$$D_\nu v_i = 2\nu C^m_{k;i} \partial_k v_m.$$

The connection coefficients are related to the Christoffel coefficients of the flat Riemannian connection in $\mathbf{R}^3$ computed using the change of variables $a = A(x, t)$:

$$C^m_{k;i}(x, t) = -\Gamma^m_{ji}(A(x, t)) \frac{\partial A^j(x, t)}{\partial x_k}$$

The equation $D_\nu(u, \nabla)A = 0$ describes advection *and diffusion* of labels. Use of traditional ($D_t A = 0$) Lagrangian variables when $\nu > 0$ would introduce third order derivatives of A in the viscous evolution of the Cauchy invariant, making the equations ill posed: the passive characteristics of u are not enough to reconstruct the dynamics.

The diffusion of labels is a consequence of the physically natural idea of adding Brownian motion to the Lagrangian flow. Indeed, if $u(X(a, t), t)$ is known, and if

$$dX(a, t) = u(X(a, t), t)dt + \sqrt{2\nu} dW(t), \ X(a, 0) = a,$$

with $W(t)$ standard independent Brownian motions in each component, and if

$$Prob\{X(a, t) \in dx\} = \rho(x, t; a)dx$$

then the expected value of the back to labels map

$$A(x, t) = \int \rho(x, t; a) a da$$

solves

$$D_\nu(u, \nabla)A = 0.$$

In addition to being well posed, the Eulerian-Lagrangian viscous equations are capable of describing vortex reconnection. We associate to the virtual velocity v the Eulerian-Lagrangian curl of v

$$\zeta = \nabla^A \times v \tag{14}$$

where

$$\nabla_i^A = \left((\nabla A)^{-1}\right)_{ji} \partial_j$$

is the pull back of the Eulerian gradient. The viscous analogue of the Eulerian-Lagrangian Cauchy invariant active scalar system (9) is

$$\left\{\begin{array}{c} D_\nu A = 0, \\ D_\nu \zeta^q = 2\nu G_p^{qk} \partial_k \zeta^p + \nu T_p^q \zeta^p, \\ u = \nabla \times (-\Delta)^{-1} \left(\mathcal{C}[\nabla A, \zeta]\right) \end{array}\right. \tag{15}$$

The Cauchy transformation

$$\mathcal{C}[\nabla A, \zeta] = (det(\nabla A))(\nabla A)^{-1}\zeta.$$

is the same as the one used in the Euler equations, (8). The specific form of the two terms on the right hand side of the Cauchy invariant's evolution are

$$G_p^{qk} = \delta_p^q C_{k;m}^m - C_{k;p}^q, \tag{16}$$

and

$$T_p^q = \epsilon_{qji} \epsilon_{rmp} C_{k;i}^m C_{k;j}^r. \tag{17}$$

The system (13) is equivalent to the Navier-Stokes system. When $\nu = 0$ the system reduces to (7). The system (15) is equivalent to the Navier-Stokes system, and reduces to (9) when $\nu = 0$.

The pair (A, v) formed by the diffusive inverse Lagrangian map and the virtual velocity are akin to charts in a manifold. They are a convenient representation of the dynamics of u for some time. When the representation becomes inconvenient, then one has to change the chart. This may (and will) happen if ∇A becomes non-invertible. Likewise, the pair (A, ζ) formed with the "back-to-labels" map A and the diffusive Cauchy invariant ζ are convenient charts. In order to clarify this statement let us introduce the terminology of "group expansion" for the procedure of resetting. More precisely, the group expansion for (13) is defined as follows. Given a time interval $[0, T]$ we consider resetting times

$$0 = t_0 < t_1 < \ldots < t_n \ldots \leq T.$$

On each interval $[t_i, t_{i+1}]$, $i = 0, \ldots$ we solve the system (13):

$$\left\{\begin{array}{c} D_\nu(u, \nabla)A = 0, \\ D_\nu(u, \nabla)v = 2\nu C\nabla v, \\ u = \mathbf{P}\left((\nabla A)^* v\right). \end{array}\right.$$

with resetting values

$$\begin{cases} A(x,t_i) = x, \\ v(x,t_i+0) = ((\nabla A)^* v)(x,t_i-0). \end{cases}$$

We require the strong resetting criterion that $\nabla\ell = (\nabla A) - \mathbf{I}$ must be smaller than a preassigned value ϵ in an analytic norm: $\exists\lambda$ such that for all $i \geq 1$ and all $t \in [t_i, t_{i+1}]$ one has

$$\int e^{\lambda|k|} \left|\widehat{\ell}(k)\right| dk \leq \epsilon < 1.$$

If there exists N such that $T = \sum_{i=0}^{N}(t_{i+1} - t_i)$ then we say that the group expansion *converges* on $[0,T]$. A group expansion of (15) is defined similarly. The resetting conditions are

$$\begin{cases} A(x,t_i) = x, \\ \zeta(x,t_i+0) = \mathcal{C}[(\nabla A))(x,t_i-0), \zeta(x,t_i-0)]. \end{cases}$$

The strong analytic resetting criterion is the same. The first interval of time $[0,t_1)$ is special. The initial value for v is u_0 (the initial datum for the Navier-Stokes solution), and the initial value for ζ is ω_0, the corresponding vorticity. The local time existence is used to guarantee invertibility of the matrix ∇A on $[0,t_1)$ and Gevrey regularity ([57]) to pass from moderately smooth initial data to Gevrey class regular solutions. Note that the resetting data are such that both u and ω are time continuous.

Theorem 4 *([21]) Let $u_0 \in H^1(\mathbf{R}^3)$ be divergence-free. Let $T > 0$. Assume that the solution of the Navier-Stokes equations with initial datum u_0 obeys $\sup_{0\leq t\leq T} \|\omega(\cdot,t)\|_{L^2(dx)} < \infty$. Then there exists $\lambda > 0$ so that, for any $\epsilon > 0$, there exists $\tau > 0$ such that both group expansions converge on $[0,T]$ and the resetting intervals can be chosen to have any length up to τ, $t_{i+1} - t_i \in [0,\tau]$. The velocity u, solution of the Navier-Stokes equation with initial datum u_0, obeys the Weber formula (6). The vorticity $\omega = \nabla \times u$ obeys the Cauchy formula (8).*

Conversely, if one group expansion converges, then so does the other, using the same resetting times. The Weber and Cauchy formulas apply and reconstruct the solution of the Navier-Stokes equation. The enstrophy is bounded $\sup_{0\leq t\leq T} \|\omega(\cdot,t)\|_{L^2(dx)} < \infty$, and the Navier-Stokes solution is smooth.

The quantity λ can be estimated explicitly in terms of the bound of enstrophy, time T, and kinematic viscosity ν. Then τ is proportional to ϵ, with a coefficient of proportionality that depends on the bound on enstrophy, time T and ν. The converse statement, that if the group expansion converges, then the enstrophy is bounded, follows from the fact that there are finitely many resettings. Indeed, the Cauchy formula and the near identity bound on ∇A imply a doubling condition on the enstrophy on each interval. It is well-known that the condition regarding the boundedness of the enstrophy implies regularity of the Navier-Stokes solution. Our definition of convergent group expansion is very demanding, and it is justified by the fact that once the enstrophy is bounded, one could mathematically demand analytic norms. But the physical resetting criterion is the invertibility of the matrix ∇A. The Euler equations require no resetting as long as the solution is smooth. The Navier-Stokes equations, at least numerically, require numerous and frequent resettings. There is a deep connection between these resetting times and vortex reconnection ([74], [75]). In the Euler equation, as long as the solution is smooth, the Cauchy invariant obeys $\zeta(x,t) = \omega_{(0)}(A(x,t))$ with $\omega_{(0)} = \omega_0$, the initial vorticity. The topology of vortex lines is frozen in time. In the Navier-Stokes system the topology changes. This is the phenomenon of vortex reconnection. There is ample numerical and physical evidence for this phenomenon. In the more complex, but similar case of magneto-hydrodynamics, magnetic reconnection occurs, and has powerful physical implications. Vortex reconnection is a dynamical dissipative process. The solutions of the Navier-Stokes equations obey a space time average bound ([16], [21])

$$\int_0^T \int_{\mathbf{R}^3} |\omega(x,t)| \left|\nabla_x \left(\frac{\omega(x,t)}{|\omega(x,t)|}\right)\right|^2 dxdt \leq \frac{1}{2}\nu^{-2} \int_{\mathbf{R}^3} |u_0(x,t)|^2 dx.$$

This bound is consistent with the numerically observed fact that the region of high vorticity is made up of relatively straight vortex filaments (low curvature of vortex lines) separated by distances that vanish with viscosity. The process by which this separation is achieved is vortex reconnection. When vortex lines are locally aligned, a geometric depletion of nonlinearity occurs, and the local production of enstrophy drops. Actually, the Navier-Stokes equations have global smooth solutions if the vorticity direction field $\frac{\omega}{|\omega|}$ is Lipschitz continuous ([29]) in regions of high vorticity. So, vortex reconnection is a regularizing mechanism.

In the case of the Navier-Stokes system the virtual velocity and the Cauchy invariant in an expansion can be computed using the back to labels map A, but unlike the case of the Euler equations, they are no longer frozen in time: they diffuse. Let us recall that, using the smooth change of variables $a = A(x,t)$ (at each fixed time t) we compute the Euclidean Riemannian metric by

$$g^{ij}(a,t) = (\partial_k A^i)(\partial_k A^j)(x,t) \tag{18}$$

The equations for the virtual velocity and for the Cauchy invariant can be solved by following the path A, i.e., by seeking

$$\begin{aligned} v(x,t) &= \upsilon(A(x,t),t), \\ \zeta(x,t) &= \xi(A(x,t),t) \end{aligned} \tag{19}$$

The equations for υ and ξ become purely diffusive. Using $D_\nu A = 0$, the operator D_ν becomes

$$D_\nu(f \circ A) = \Big((\partial_t - \nu g^{ij}\partial_i\partial_j)f\Big) \circ A \tag{20}$$

The equation for υ follows from (13):

$$\partial_t \upsilon_i = \nu g^{mn}\partial_m\partial_n \upsilon_i - 2\nu V_i^{mj}\partial_m \upsilon_j \tag{21}$$

with

$$V_i^{mj} = g^{mk}\Gamma_{ik}^j$$

The derivatives are with respect to the Cartesian coordinates a. The equation reduces to $\partial_t \upsilon = 0$ when $\nu = 0$, and in that case we recover $\upsilon = u_{(0)}$, the time independent initial velocity. For $\nu > 0$ the system is parabolic and well posed. The equation for ξ follows from (15):

$$\partial_t \xi^q = \nu g^{ij}\partial_i\partial_j \xi^q + 2\nu W_n^{qk}\partial_k \xi^n + \nu T_p^q \xi^p \tag{22}$$

with

$$\begin{cases} W_n^{qk} = -\delta_n^q g^{kr}\Gamma_{rp}^p + g^{kp}\Gamma_{pn}^q, \\ T_p^q = \epsilon_{qji}\epsilon_{rmp}\Gamma_{\alpha j}^r \Gamma_{\beta i}^m g^{\alpha\beta} \end{cases}$$

Again, when $\nu = 0$ this reduces to the invariance $\partial_t \xi = 0$. But in the presence of ν this is a parabolic system. Both the Cauchy invariant and the virtual velocity equations start out looking like the heat equation, because

$g^{mn}(a,0) = \delta^{mn}$ and $\Gamma^i_{jk}(a,0) = 0$. The equation for the determinant of ∇A is

$$D_\nu \left(\log(det(\nabla A)\right) = \nu \left\{ C^i_{k;s} C^s_{k;i} \right\} \tag{23}$$

The initial datum vanishes. When $\nu = 0$ we recover conservation of incompressibility. In the case $\nu > 0$ the inverse time scale in the right hand side of this equation is significant for reconnection. Because the equation has a maximum priciple it follows that

$$det(\nabla A)(x,t) \geq exp\left\{ -\nu \int_{t_i}^t \sup_x \left\{ C^i_{k;s} C^s_{k;i} \right\} d\sigma \right\}$$

Considering

$$g = det(g_{ij}) \tag{24}$$

where g_{ij} is the inverse of g^{ij} and observing that

$$g(A(x,t)) = (det(\nabla A))^{-2}$$

we deduce that the equation (23) becomes

$$\partial_t(\log(\sqrt{g})) = \nu g^{ij} \partial_i \partial_j \log(\sqrt{g}) - \nu g^{\alpha\beta} \Gamma^m_{\alpha p} \Gamma^p_{\beta m} \tag{25}$$

The initial datum is zero, the equation is parabolic, has a maximum priciple and is driven by the last term. The form (25) of the equation (23) has the same interpretation: the connection coefficients define an inverse length scale associated to A. The corresponding inverse time scale

$$\nu \left\{ C^i_{k;s} C^s_{k;i} \right\} = \nu \left\{ g^{mn} \Gamma^i_{ms} \Gamma^s_{ni} \right\} \circ A$$

decides the time interval of validity of the chart A, and the time to reconnection. Let us denote

$$\begin{aligned} L &= \log(\sqrt{g}) \\ V^i &= g^{jk} \Gamma^i_{jk}, \\ F &= g^{jk} \Gamma^i_{js} \Gamma^s_{ki}. \end{aligned}$$

The equation (25) for L can be written as

$$\partial_t L = \nu g^{ij} \partial_i \partial_j L - \nu F.$$

We will make use now of a few well-known facts from Riemannian geometry. The first fact is

$$\partial_j \log(\sqrt{g}) = \Gamma^m_{jm}. \tag{26}$$

The second fact is the vanishing of the curvature of the Euclidean Riemannian connection in $\mathbf{R}^3$

$$\partial_k\Gamma^i_{ql} + \Gamma^i_{pk}\Gamma^p_{ql} = \partial_l\Gamma^i_{qk} + \Gamma^i_{pl}\Gamma^p_{qk}. \tag{27}$$

The third fact is that the connection is compatible with the metric

$$\partial_i g^{jk} + \Gamma^j_{ip}g^{pk} + \Gamma^k_{ip}g^{jp} = 0. \tag{28}$$

Taking the sum $i = l$ in (27) we deduce that

$$\Gamma^i_{pk}\Gamma^p_{qi} = \partial_i\Gamma^i_{qk} - \partial_k\Gamma^i_{qi} + \Gamma^i_{pi}\Gamma^p_{qk}.$$

Using (26) and multiplying by g^{kq} we deduce

$$F = -g^{ij}\partial_i\partial_j L + V^j\partial_j L + g^{\alpha\beta}\partial_i\Gamma^i_{\alpha\beta}.$$

But, using (28) we deduce

$$g^{\alpha\beta}\partial_i\Gamma^i_{\alpha\beta} = 2F - V^i\partial_i L + \mathrm{d}ivV$$

with

$$\mathrm{d}ivV = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}V^i\right).$$

Therefore

$$g^{ij}\partial_i\partial_j L - F = \mathrm{div}V, \tag{29}$$

and we obtained an alternative form of (25):

$$\partial_t\sqrt{g} = \nu\partial_i\left(\sqrt{g}g^{\alpha\beta}\Gamma^i_{\alpha\beta}\right). \tag{30}$$

Let us introduce now

$$f^q = \frac{1}{\sqrt{g}}\xi^q. \tag{31}$$

Writing $\xi = \sqrt{g}f$ in (22) and using (30) it follows that

$$\partial_t f^q = \nu g^{\alpha\beta}\partial_\alpha\partial_\beta f^q + \nu D^{qk}_n\partial_k f^n + \nu F^q_p f^p \tag{32}$$

with

$$D^{qk}_n = 2g^{kp}\Gamma^q_{pn}$$

and

$$F^q_p = T^q_p + 2g^{kj}\Gamma^q_{jp}\partial_k L + \delta^q_p g^{\alpha\beta}\left(\Gamma^i_{\alpha j}\Gamma^j_{\beta i} - \Gamma^i_{\alpha i}\Gamma^j_{\beta j}\right).$$

In the derivation of the last expression we used the fact that $\sqrt{g} = e^L$ and the equation (29). Let us note here what happens in two spatial dimensions. In that case $A^3 = x^3$, $\Gamma_{ij}^3 = 0$, and $f^1 = f^2 = 0$. Then the free term F_3^3 in (32) vanishes because T_3^3 cancels the δ_3^3 term. The rest of the terms involve Γ_{ij}^3 and vanish. The equation becomes thus the scalar equation

$$\partial_t f^3 = \nu g^{\alpha\beta}\partial_\alpha\partial_\beta f^3$$

which means that

$$\omega^3 = f^3 \circ A.$$

Because the rest of components are zero, this means that in two dimensions $\omega = f \circ A$. This represents the solution of the two dimensional vorticity equation

$$D_\nu \omega = 0$$

as it is readily seen from (20). The three dimensional situation is more complicated, f obeys the nontrivial well posed parabolic system (32), and the Cauchy formula in terms of f reads

$$\omega = (\nabla A)^{-1}(f \circ A). \tag{33}$$

The function $f \circ A = (det(\nabla A))\,\zeta$ deserves just as much the name "Cauchy invariant" as does $\zeta = \nabla^A \times v$. In the inviscid case these functions coincide, of course. The Eulerian evolution equation of $\tilde{\zeta} = (det(\nabla A))\zeta$ is similar to (15). There are two more important objects to consider, in the viscous context: circulation, and helicity. We note that the Weber formula implies that

$$udx - vdA = -dn$$

and therefore

$$\oint_{\gamma\circ A} udx = \oint_\gamma vda \tag{34}$$

holds for any closed loop γ. Similarly, in view of the Cauchy formula one has that the helicity obeys

$$h = u \cdot \omega = v \cdot \tilde{\zeta} - \nabla \cdot (n\omega)$$

So, if $T \circ A$ is a vortex tube, then

$$\int_{T\circ A} hdx = \int_{T\circ A} v \cdot \tilde{\zeta}dx = \int_T v \cdot \xi da. \tag{35}$$

The metric coefficients g^{ij} determine the connection coefficients, as it is well known. But they do not determine their own evolution as they change under the Navier-Stokes equations. (The evolution equation of g^{ij} involves ∇u and ∇A). It is therefore remarkable that the virtual velocity, Cauchy invariant and volume element evolve according to equations that do not involve explicitly the velocity, once one computes in a diffusive Lagrangian frame. This justifies the following terminology: we will say that a function f is *diffusively Lagrangian* under the Navier-Stokes flow if $f \circ A$ obeys an evolution equation with coefficients determined locally by the Euclidean Riemannian metric induced by the change of variables A. Thus, for instance, the metric itself is not diffusively Lagrangian. The previous calculations can be summarized thus:

Theorem 5 *The virtual velocity v, the Cauchy invariant ζ and the Jacobian determinant $det(\nabla A)$ associated to solutions of the Navier-Stokes equations are diffusively Lagrangian.*

5 Approximations

We will describe here approximations of the Navier-Stokes (and Euler) equations. These approximations are partial differential equations with globally smooth solutions. We'll consider a mollifier: an approximation of the identity obtained by convolution with a smooth function which decays enough at infinity, is positive and has integral equal to one. The mollified u is denoted $[u]$:

$$[u]_\delta = \delta^{-3} \int_{\mathbf{R}^3} J\left(\frac{x-y}{\delta}\right) u(y)dy = J_\delta(-i\nabla)u = J_\delta u = [u]$$

The length scale $\delta > 0$ is fixed in this section. In order to recover the original equations one must pass to the limit $\delta \to 0$. The first approximation concerns the Eulerian velocity formulation (10), and is due to Leray ([67]):

$$\partial_t u + [u] \cdot \nabla u - \nu \Delta u + \nabla p = 0 \tag{36}$$

together with $\nabla \cdot u = 0$. The Eulerian vorticity formulation (12) has an approximation which corresponds to Chorin's vortex methods ([13]):

$$\partial_t \omega + [u] \cdot \omega - \nu \Delta \omega = \omega \cdot \nabla [u] \tag{37}$$

with u calculated from ω using $\omega = \nabla \times u$. This relationship can be written as

$$[u] = J_\delta \left(\nabla \times (-\Delta)^{-1}\right) \omega.$$

The virtual velocity active scalar system (13) has an approximation

$$\left\{\begin{array}{c} D_\nu([u], \nabla)A = 0, \\ D_\nu([u], \nabla)v = 2\nu C \nabla v, \\ u = W[A, v]. \end{array}\right. \tag{38}$$

The relationship determining the advecting velocity is thus

$$[u] = J_\delta W[A, v]$$

with $W[A, v]$ the same Weber formula (6). The Cauchy invariant active scalar system (15) is approximated in a similar manner

$$\left\{\begin{array}{c} D_\nu([u], \nabla)A = 0, \\ D_\nu([u], \nabla)\zeta^q = 2\nu G_p^{qk} \partial_k \zeta^p + \nu T_p^q \zeta^p, \\ u = \nabla \times (-\Delta)^{-1} \left(\mathcal{C}[\nabla A, \zeta]\right) \end{array}\right. \tag{39}$$

The Cauchy formula (8) is the same, and the diffusive Lagrangian terms G_p^{qk} and T_p^q are given by the same expressions (16), (17) as in the Navier-Stokes case. In fact, the approximations of both group expansions are defined in exactly the same way: one respects the constitutive laws relating virtual velocity or Cauchy invariant to velocity, and the same resetting rules. One modifies the advecting velocity: D_ν is replaced by $D_\nu([u], \nabla)$.

All four approximations are done by mollifying, but they are not equivalent. The Leray approximate equation (36) has the same energy balance as the Navier-Stokes equation,

$$\frac{d}{2dt} \int |u|^2 dx + \nu \int |\nabla u|^2 dx = 0$$

but, when one sets $\nu = 0$ the circulation integrals are not conserved. The vorticity approximation conserves circulation, but has different energy structure (the integrals of $u \cdot [u]$ decay). The approximations of the active scalar systems provide convergent group expansions for the vorticity approximation, not the Leray approximation.

Theorem 6 *Consider $u_0 \in H^1(\mathbf{R}^3)$, divergence free, and let $T > 0$, $\nu > 0$, $\delta > 0$ be fixed. Consider a fixed smooth, normalized mollifier J. Then the group expansions of (38) and (39) converge on $[0, T]$. The Cauchy formula (8) gives the solution of the approximate vorticity equation (37). The virtual velocity, Cauchy invariant, and Jacobian determinant are diffusively Lagrangian: they solve the evolution equations (21, 22, 25 (or 30)) with coefficients determined locally from the Euclidean Riemannian metric, in coordinates $a = A(x, t)$ computed in the expansion.*

The proof starts by verifying that there exists $\tau > 0$ so that both expansions converge with resetting times $t_{i+1} - t_i \geq \tau$. This is straightforward, but rather technical. Then one considers the variable

$$w = (\nabla A)^* v.$$

It follows from (38) that w obeys the equation

$$D_\nu([u], \nabla)w + (\nabla[u])^* w = 0.$$

This equation is an approximation of yet another formulation of the Navier-Stokes equations ([66], [78], [82]) used in numerical simulations ([7], [8]), and related to the alpha model ([52], [59]). Taking the curl of this equation it follows that $b = \nabla \times w$ solves

$$D_\nu([u], \nabla)b = b \cdot \nabla[u]$$

The resetting conditions are such that both w and b are continuous in time. The Weber formula implies that the advecting velocity $[u]$ is given by $[u] = J_\delta \mathbf{P} w$, and hence it is continuous in time. It follows that the b solves the equation (37) for all $t \in [0, T]$. The initial datum for b is ω_0, so it follows that $b = \omega$ for all t. But the Cauchy formula holds for b, and that finishes the proof. The fact that the Cauchy formula holds for b is kinematic: Indeed, if A, v solve any virtual velocity system (38) with some smooth advecting velocity $[u]$, then the Eulerian curl b of $(\nabla A)^* v$ is related to the Lagrangian curl ζ of v by the Cauchy formula (8).

6 The QG Equation

Two dimensional fluid equations can be described as active scalars

$$(\partial_t + u \cdot \nabla)\theta + c\Lambda^\alpha \theta = f$$

with an incompressible velocity given in terms of a stream function

$$u = \nabla^{\perp}\psi = \begin{pmatrix} -\partial_2\psi \\ \partial_1\psi \end{pmatrix}$$

computed in terms of θ as

$$\psi = \Lambda^{-\beta}\theta.$$

The constant $c \geq 0$ and smooth time independent source term f are given. The operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined in the whole $\mathbf{R}^2$ or in $\mathbf{T}^2$. Interesting examples are $\beta = 2$ (usual hydrodynamical stream function) with $\alpha = 2$ (usual Laplacian dissipation), and $\beta = 1$, (surface quasigeostrophic equation, QG in short) with $\alpha = 1$ (critical dissipation). We describe briefly the quasigeostrophic equation. When $c = 0$, $f = 0$ the equation displays a number of interesting features shared with the three dimensional Euler equations ([17]). In particular, the blow up of solutions with smooth initial data is a difficult open problem ([36], [37], [40], [77]). There is additional structure: the QG equation has global weak solutions. This has been proved by Resnick in his thesis ([79]); a concise description of the idea can be found also in ([27]). The dissipative system ([9], [27], [39], [61], [91], [92]) with $c > 0$ and $\alpha \in [0, 2]$ has a maximum principle (again proved by Resnick, and again explained in ([27]). The critical dissipative QG equation ($c = 1$, $f = 0$, $\alpha = \beta = 1$) with smooth and localized initial data in $\mathbf{R}^2$ has the following properties: Weak solutions in $L^{\infty}(dt; L^2(dx) \cap L^{\infty}(dx)) \cap L^2(dt; H^{\frac{1}{2}})$ exist for all time. The L^{∞} norm is nonincreasing on solutions. The solutions are smooth (in a variety of spaces) and unique for short time. If the initial data is small in L^{∞} then the solution is smooth for all time and decays. The subcritical case ($\alpha > 1$, more dissipation, less difficulty) has global unique smooth solutions. The main open problems for the QG equations are: uniqueness of weak solutions, global regularity for large data for critical and supercritical dissipation.

A nice pointwise inequality for fractional derivatives has been discovered recently by A. Cordoba and D. Cordoba ([41]. The inequality provides another proof of the maximum principle, and has independent interest. I'll explain it briefly below. One starts with the Poisson kernel

$$P(z,t) = c_n \frac{t}{(|z|^2 + t^2)^{\frac{n+1}{2}}}$$

in $\mathbf{R}^n$ for $t \geq 0$. The constant c_n is normalizing:

$$\int_{\mathbf{R}^n} P(z,t)dz = 1.$$

Convolution with $P(\cdot, t)$ is a semigroup. The semigroup identity is

$$e^{-t\Lambda} f = \int\limits_{\mathbf{R}^n} P(z,t)\tau_z(f)dz$$

where

$$(\tau_z(f))(x) = f(x-z).$$

and Λ is the Zygmund operator,

$$\Lambda = (-\Delta)^{\frac{1}{2}}.$$

The quickest way to check the semigroup identity is by taking the Fourier transform, $\widehat{P}(\xi, t) = e^{-t|\xi|}$. If f is in the domain of the generator of a semigroup then

$$-\Lambda f = \lim_{t\downarrow 0} t^{-1}\left(e^{-t\Lambda} - I\right) f$$

and consequently

$$-\Lambda f = \lim_{t\downarrow 0} \int\limits_{\mathbf{R}^n} t^{-1} P(z,t)\delta_z(f)dz$$

where

$$\delta_z(f)(x) = f(x-z) - f(x).$$

If the function f is smooth enough, then this becomes

$$-\Lambda f = c_n \int\limits_{\mathbf{R}^n} \frac{1}{|z|^{n+1}}\delta_z(f)dz$$

and it makes sense as a singular integral. This was proved by A. Cordoba and D. Cordoba in ([41]) where they discovered and used the pointwise inequality

$$f\Lambda f \geq \frac{1}{2}\Lambda(f^2).$$

This inequality is the consequence of an identity.

Proposition 1 *For any two C_0^∞ functions f, g one has the pointwise identity*

$$\Lambda(fg) = f\Lambda g + g\Lambda f - I_2(f,g) \tag{40}$$

with I_2 defined by

$$I_2(f,g) = c_n \int\limits_{\mathbf{R}^n} \frac{1}{|z|^{n+1}}(\delta_z(f))(\delta_z(g))dz \tag{41}$$

The proof follows the calculation of Cordoba and Cordoba: Start with

$$(\delta_z(f))^2 = \delta_z(f^2) - 2f(\delta_z(f))$$

and integrate against $c_n|z|^{-n-1}dz$. One obtains

$$\Lambda(f^2) = 2f\Lambda f - I_2(f,f) \tag{42}$$

By polarization, i.e., by appliying (42) to f replaced by $f + \epsilon g$, then differentiating in ϵ and then setting $\epsilon = 0$, one deduces (40), and finsihes the proof of the proposition. We remark that one can also obtain higher order identities in the same manner. One starts with

$$(\delta_z f)^m = \sum_{j=1}^{m}(-1)^j \left(\begin{array}{c} m \\ j \end{array}\right) f^{m-j}\delta_z(f^j).$$

Integrating against $c_n|z|^{-n-1}dz$, one obtains the generalized identity

$$I_m(f,\dots,f) = \sum_{j=1}^{m}(-1)^{j-1} \left(\begin{array}{c} m \\ j \end{array}\right) f^{m-j}\Lambda(f^j) \tag{43}$$

where the multilinear nonlocal integral is

$$I_m(f,\dots,f) = c_n \int_{\mathbf{R}^n} \frac{1}{|z|^{n+1}} (\delta_z(f))^m \, dz. \tag{44}$$

In view of the definition of Besov spaces in terms of δ_z ([86]) it is clear that the multilinear operators I_m are well behaved in Besov spaces. Also, for m even $I_m(f,\dots,f) \ge 0$ pointwise. Thus, for instance

$$0 \le I_4(f,f,f,f) = 4f^3\Lambda f + 4f\Lambda f^3 - 6f^2\Lambda f^2 - \Lambda f^4$$

holds pointwise.

Let us also note here that $I_2(f,g)$ gives a quick proof of an extension ([64]) of a Moser calculus inequality of Kato and Ponce ([62]):

Proposition 2 *If $1 < p < \infty$, $1 < p_i < \infty$, $i = 1,2$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, then*

$$\|\Lambda(fg) - f\Lambda g - g\Lambda f\|_{L^p(\mathbf{R}^n)} \le$$
$$C\left\{\|f\|_{L^{p_1}(\mathbf{R}^n)}\|\Lambda f\|_{L^{p_1}(\mathbf{R}^n)}\|g\|_{L^{p_2}(\mathbf{R}^n)}\|\Lambda g\|_{L^{p_2}(\mathbf{R}^n)}\right\}^{\frac{1}{2}}$$

holds for all $f \in W^{1,p_1}$, $g \in W^{1,p_2}$.

The proof is straightforward: In view of (40) one needs to check the inequality for $I_2(f,g)$. We write

$$I_2(f,g) = J_r(f,g) + K_r(f,g)$$

with

$$J_r(f,g) = c_n \int_{|z|\le r} |z|^{-n-1}(\delta_z(f))(\delta_z(g))dz$$

and $K_r(f,g)$ the rest. We note that

$$\|J_r(f,g)\|_{L^p(\mathbf{R}^n)} \le Cr\|\Lambda f\|_{L^{p_1}(\mathbf{R}^n)}\|\Lambda g\|_{L^{p_2}(\mathbf{R}^n)}$$

and

$$\|K_r(f,g)\|_{L^p(\mathbf{R}^n)} \le Cr^{-1}\|f\|_{L^{p_1}(\mathbf{R}^n)}\|g\|_{L^{p_2}(\mathbf{R}^n)}$$

with C depending on n, p_1 and p_2 only. These inequalities follow from the elementary

$$\|\delta_z f\|_{L^p(\mathbf{R}^n)} \le |z|\|\nabla f\|_{L^p(\mathbf{R}^n)}$$

and the boundedness of Riesz transforms in L^p. Optimizing in r we obtain the desired result. It is quite obvious also that

$$\|J_r(f,g)\|_{L^p(\mathbf{R}^n)} \le Cr^{s_1+s_2-1}\|f\|_{B^{s_1,\infty}_{p_1}}\|g\|_{B^{s_2,\infty}_{p_2}}$$

holds for any $0 \le s_1 < 1$, $0 \le s_2 < 1$, $s_1 + s_2 > 1$. Here we used

$$\|f\|_{B^{s,\infty}_p} = \|f\|_{L^p(\mathbf{R}^n)} + \sup |z|^{-s}\|\delta_z f\|_{L^p(\mathbf{R}^n)}$$

Consequently

$$\|I_2(f,g)\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{B^{s_1,\infty}_{p_1}}\|g\|_{B^{s_2,\infty}_{p_2}} \tag{45}$$

holds.

Proposition 3 *Consider F, a convex C^2 function of one variable. Assume that the function f is smooth and bounded. Then*

$$F'(f)\Lambda f \ge \Lambda(F(f)) \tag{46}$$

holds pointwise. In particular, if $F(0) = 0$ and $f \in C_0^\infty$

$$\int_{\mathbf{R}^n} F'(f)\Lambda f dx \ge 0.$$

Indeed

$$F(\beta) - F(\alpha) - F'(\alpha)(\beta - \alpha) \geq 0$$

holds for all α, β. We substitute $\beta = f(x - z)$, $\alpha = f(x)$ and integrate against $c_n|z|^{-n-1}dz$. Note that $F(f(x))$ is bounded because F is continuous and thus bounded on the range of f. The inequality

$$D_{2m} = \int f^{2m-1}\Lambda f dx \geq 0$$

implies that the L^{2m} norms do not increase on solutions of the critical dissipative QG equation, and the maximum principle follows.

7 Dissipation and Spectra

Turbulence theory concerns itself with statistical properties of fluids. Some of the objects encountered in turbulence theory are the mean velocity $\langle u \rangle$, Eulerian velocity fluctuation $v = u - \langle u \rangle$, energy dissipation rate $\nu\langle|\nabla u|^2\rangle$, velocity correlation functions $\langle \delta_y v \otimes v \rangle$, velocity structure functions $\langle \delta_y v \otimes \delta_y v \rangle$, higher order structure functions, $\langle \delta_{y_1} v \otimes \cdots \delta_{y_m} v \rangle$. We use the notation $\delta_y u = u(x - y) - u(x)$. The operation $\langle \cdots \rangle$ is ensemble average, a functional integral. The Navier-Stokes equations represent the underlying dynamics. A mathematical framework related to the Navier-Stokes equations has been developed ([60], [51], [89]), but the mathematical advance has been slow. Not complexity but rather simplicity is the essence of the difficulty: turbulent flows obey nontrivial statistical laws. Among these laws, the law for wall bounded flows ([5]), and for scaling of Nusselt number with Rayleigh number in Rayleigh-Bénard turbulence in Helium ([73]) are major examples. A celebrated physical theoretical prediction concerning universality in turbulence, is due to Kolmogorov ([65],[58]). One of the simplest and most important question in turbulence is: how much energy is dissipated by the flow. This question is of major importance for engineering applications because the energy disspated by turbulence is transferred to objects immersed in it. Mathematically, the question is about the long time average of certain integrals, low order (first and second) moments of the velocity and velocity gradients in forced flows. There is a rigorous mathematical method ([25], [26], [35], [43]) to obtain bounds for these bulk quantities, and make contact with experiments in convection and shear dominated turbulence.

The Kolmogorov 2/3 law for homogeneous turbulence is

$$S_2(r) \sim (\epsilon r)^{\frac{2}{3}}$$

with

$$S_2(r) = \langle (\delta_r u)^2 \rangle$$

the second order longitudinal structure function and with

$$\epsilon = \nu \langle |\nabla u|^2 \rangle$$

the energy dissipation rate. The law is meant to hold asymptotically, for large Reynolds numbers, and for r in a range of scales, $[L, k_d^{-1}]$, called the inertial range. The energy dissipation rate ϵ per unit mass has dimension of energy per time, $cm^2 sec^{-3}$. The Kolmogorov law follows from dimensional analysis if one postulates that in the inertial range the (second order) statistics of the flow depend only on the parameter ϵ, because the typical longitudinal velocity fluctuations over a distance r should be an expression with units of $cmsec^{-1}$ and such an expression is $(\epsilon r)^{\frac{1}{3}}$. The energy spectrum for homogeneous turbulence is

$$E(k) = \frac{1}{2} \int\limits_{|\xi|=k} \langle |\widehat{u}(\xi, t)|^2 \rangle dS(\xi)$$

with $\widehat{u}$ the spatial Fourier transform. The Kolmogorov-Obukhov energy spectrum law is

$$E(k) \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

The asymptotic equality takes place for $k \in [k_0, k_d]$ where

$$k_d = \nu^{-\frac{3}{4}} \epsilon^{\frac{1}{4}}$$

is the dissipation wave number, and k_0 is the integral scale. The interval $[k_0, k_d]$ is the inertial range. The statements about asymptotic equality refer to the high Reynolds number limit, $k^{\frac{5}{3}} \epsilon^{-\frac{2}{3}} E(k) \to C$, as $Re \to \infty$. The Reynolds number is

$$Re = \frac{UL}{\nu} = \left\{ \langle |u|^2 \rangle \right\}^{\frac{1}{2}} (k_0 \nu)^{-1}$$

where $U^2 = \langle |u|^2 \rangle$, $L = k_0^{-1}$. The physical intuition is the following: energy is put into the system at large scales $L = k_0^{-1}$. This energy is transferred to small scales without loss, in a statistically selfsimilar manner. At scale k (in wave number space) the only allowed external parameter is ϵ. The energy spectrum has dimension of energy per wave number, because the energy spectrum is the integrand in the one dimensional integral

$$E = \int_0^\infty E(k)dk.$$

Thus $E(k)$ is measured in $cm^3 sec^{-2}$. The wave number is measured in cm^{-1}. A time scale formed with ϵ and k is $t_k = \epsilon^{-\frac{1}{3}} k^{-\frac{2}{3}}$. This is the time of transfer of energy at wave-number scale k. Using this time scale and the length scale k^{-1} one arrives at the expression $E(k) \sim k^{-3} \epsilon^{\frac{2}{3}} k^{\frac{4}{3}}$, the Kolmogorov-Obukhov spectrum. The spectrum can be derived also from the 2/3 law.

The dissipation of the energy occurs at a dissipation scale k_d. This is an inverse length scale formed using only the kinematic viscosity (measured in $cm^2 sec^{-1}$) and ϵ.

The fact that Laplacian dissipation and no other should be used to study turbulence acquires a physical justification: the Kolmogorov dissipation scale is based on the Laplacian, and it is observed experimentally quite convincingly.

Let us describe in more mathematical terms the issues. We start with the dissipation law. The Kolmogorov theory predicts that the energy dissipation is

$$\epsilon \sim \frac{U^3}{L}.$$

The first difficulty one encounters is due to the need to solve three dimensional Navier-Stokes equations. The second difficulty concerns the ensemble average: one needs homogeneity (translation invariance of the statistics) and bounded solutions. One way out of this is to take a set of bounded solutions and perform a well defined operation M that is manifestly translation invariant, normalized and positivity preserving ([30]). This gives upper bounds for energy dissipation, and structure functions of order one and two in the whole space. Assuming scaling $M((\delta_r u)^2) \sim r^s$ we proved in ([30]) that $s \geq 2/3$. In ([30]) the body forced were assumed to be uniformly bounded, but not uniformly square integrable. If the body forces are uniformly square integrable, for instance in bounded domains, then an idea of Foias ([50]) can be used to bound the dissipation without assuming that the velocities are bounded. An upper bound on dissipation in bounded domains, exploiting this idea is given in ([44]).

I will explain the idea below in the whole space, with square integrable forces. Consider solutions of Leray regularized solutions of Navier-Stokes equations in $\mathbf{R}^3$,

$$D_\nu([u],\nabla)u + \nabla p = f, \quad \nabla\cdot u = 0,$$

with smooth, time independent, deterministic divergence-free body forces f,

$$f(x) = \int_{\mathbf{R}^3} e^{2\pi i\xi\cdot x}\widehat{f}(\xi)d\xi$$

with $\widehat{f}$ supported in $|\xi|\le k_0$. We consider a long time average M_T.

$$\frac{1}{T}\int_0^T h(\cdot,t)dt = M_T(h)$$

We define the averaging procedure to be

$$\langle h(x,t)\rangle = k_0^3 \limsup_{T\to\infty} M_T \int_{\mathbf{R}^3} h(x,t)dx$$

We set

$$U^2 = \langle |u|^2\rangle,\ F^2 = \langle |f|^2\rangle,$$

and

$$L^{-1} = \|\nabla f\|_{L^\infty}F^{-1}.$$

Note that $L^{-1}\le 2\pi k_0$. The energy dissipation of the Leray regularized solution $\nu\langle|\nabla u|^2\rangle$ is bounded uniformly, independently of the regularization. The upper bound is:

$$\epsilon \le \frac{U^3}{L} + \sqrt{\epsilon}\sqrt{\nu}\frac{U}{L}$$

This implies, of course,

$$\epsilon \le \frac{U^3}{L} + \frac{\nu U^2}{4L^2} + \frac{\sqrt{\nu}U}{2L}\sqrt{\frac{\nu U^2}{L^2} + \frac{4U^3}{L}}$$

and consequently

$$\limsup_{Re\to\infty} \epsilon L U^{-3} \le 1.$$

For the proof one applies first M_T and obtains

$$f_i = \partial_j M_T([u]_j u_i) + \partial_i M_T p - \nu\Delta M_T u + M_T u_t$$

because $M_T f = f$. One then takes the scalar product with f.

$$\int_{\mathbf{R}^3} |f|^2 dx = -\int_{\mathbf{R}^3} (\partial_j f_i) M_T([u]_j u_i) dx$$

$$+\nu \int_{\mathbf{R}^3} \nabla f \cdot \nabla M_T(u) dx + \frac{1}{T}(\int_{\mathbf{R}^3} f \cdot (u(\cdot, T) - u(\cdot, 0)) dx.$$

Then we deduce

$$\|f\|_{L^2}^2 \leq \|\nabla f\|_{L^\infty} M_T(\|u\|_{L^2}^2) + \nu \int_{\mathbf{R}^3} |\nabla f| \, |\nabla M_T u| \, dx + O(\frac{1}{T}).$$

We did use the fact that the mollifier is normalized, so $\|[u]\|_{L^2}^2 \leq \|u\|_{L^2}^2$. But

$$|\nabla M_T u|^2 \leq M_T |\nabla u|^2$$

Multiplying by k_0^3, dividing by F it follows that:

$$F \leq \frac{1}{L} k_0^3 M_T(|u|^2) + \frac{\nu}{L}\sqrt{k_0^3 M_T \|\nabla u\|_{L^2}^2} + O(\frac{1}{T})$$

and letting $T \to \infty$

$$F \leq \frac{U^2}{L} + \sqrt{\nu}\sqrt{\epsilon}/L + O(\frac{1}{T}).$$

But

$$\epsilon \leq FU$$

follows immediatly from the energy balance, and that concludes the proof of the upper bound. We have therefore

Theorem 7 *Consider solutions of Leray's approximation*

$$D_\nu([u], \nabla)u + \nabla p = f, \ \nabla \cdot u = 0$$

in $\mathbf{R}^3$ *with divergence-free, time independent body forces with Fourier transform supported in* $|\xi| \leq k_0$. *Let*

$$\epsilon = \lim\sup_{T\to\infty} \frac{1}{T} \int_0^T \nu \|\nabla u(\cdot, t)\|^2 dt,$$

$$U^2 = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \|u(\cdot,t)\|^2 dt,$$

$$F^2 = \|f(\cdot)\|^2,$$

where $\|h\|^2 = k_0^3 \int |h|^2 dx$ *is normalized* L^2 *norm. Let also*

$$L^{-1} = \frac{\|\nabla f\|_{L^\infty}}{F}$$

define a length scale associated to the forcing and

$$Re = \frac{UL}{\nu}$$

be the Reynolds number. Then

$$\epsilon \le \frac{U^3}{L}\left(1 + Re^{-\frac{1}{2}} + \frac{3}{4}Re^{-1}\right)$$

holds, uniformly for all mollifiers, $L^2(\mathbf{R}^3)$ *initial data and* f *with the required properties.*

The normalization k_0^3 for volume is arbitrary: any other normalization works and does not change the inequality. The uniformity with respect to mollifiers implies that the result holds for suitable weak solutions of the Navier-Stokes equations. The problem of a lower bound is outstanding, and open. When one replaces $[u]$ by the Bessel potential $(I-\alpha^2\Delta)^{-1}u$ one obtains the Leray alpha model ([10]).

In order to describe in more detail the issues concerning the spectrum it is convenient to phrase them using a Littlewood-Paley decomposition of functions in $\mathbf{R}^d$. This employs a nonnegative, nonincreasing, radially symmetric function

$$\phi_{(0)}(k) = \phi_{(0)}(|k|)$$

with properties $\phi_{(0)}(k) = 1$, $k \le \frac{5}{8}k_0$, $\phi_{(0)}(k) = 0$, $k \ge \frac{3}{4}k_0$. The positive number k_0 is a wavenumber unit; it allows to make dimensionally correct statements. One sets

$$\phi_{(n)}(k) = \phi(2^{-n}k), \quad \psi_{(0)}(k) = \phi_{(1)}(k) - \phi_{(0)}(k)$$
$$\psi_{(n)}(k) = \psi_{(0)}(2^{-n}k), \quad n \in \mathbf{Z}.$$

The properties

$$\psi_{(n)}(k) = 1, \text{ for } k \in 2^n k_0[\tfrac{3}{4}, \tfrac{5}{4}],$$
$$\psi_{(n)}(k) = 0, \text{ for } k \notin 2^n k_0[\tfrac{5}{8}, \tfrac{3}{2}]$$

follow from construction. The Littlewood-Paley operators $S^{(m)}$ and Δ_n are multiplication in Fourier representation by $\phi_{(m)}(k)$ and, respectively by $\psi_{(n)}(k)$. For any $m \in \mathbf{Z}$, the Littlewood Paley decomposition of h is

$$h = S^{(m)}h + \sum_{n \geq m} h_{(n)}$$

and for mean zero function h that decay at infinity, $S^{(m)}h \to 0$ as $m \to -\infty$ and the Littlewood-Paley decomposition is:

$$h = \sum_{n=-\infty}^{\infty} h_{(n)}$$

with

$$h_{(n)} = \Delta_n h = \int_{\mathbf{R}^d} \Psi_{(n)}(y)(\delta_y h) dy.$$
$$\Psi_{(n)}(y) = \int e^{i2\pi y \cdot \xi} \psi_{(n)}(\xi) d\xi$$
$$\widehat{\Psi}_{(n)} = \psi_{(n)}, \quad (\delta_y h)(x) = h(x-y) - h(x).$$

Δ_n is a weighted sum of finite difference operators at scale $2^{-n}k_0^{-1}$ in physical space. For each fixed $k > 0$ at most three Δ_n do not vanish in their Fourier representation at ξ with $k = |\xi|$:

$$\widehat{h_{(n)}}(\xi) \neq 0 \Rightarrow n \in I_k = \left\{ [-1, 1] + \log_2 \left(\frac{k}{k_0} \right) \right\} \cap \mathbf{Z}.$$

Let us consider a slightly large set of indices

$$J_k = \left\{ [-2, 2] + \log_2 \left(\frac{k}{k_0} \right) \right\} \cap \mathbf{Z}.$$

If ξ is a wave number whose magnitude $|\xi|$ is comparable to k, $\frac{k}{2} \leq |\xi| \leq 2k$, then, if $u(x,t)$ is an L^2 valued function of t, one has for almost every ξ

$$\widehat{u}(\xi, t) = \sum_{n \in J_k} \widehat{u}_{(n)}(\xi, t)$$

because $I_{|\xi|} \subset J_k$. Consequently, because J_k has at most five elements,

$$\frac{2}{3k} \int_{\frac{k}{2}}^{2k} d\lambda \int_{|\xi|=\lambda} |\widehat{u}(\xi, t)|^2 \, dS(\xi) \leq \frac{10}{3k} \sum_{n \in J_k} \|u_{(n)}(\cdot, t)\|^2_{L^2(\mathbf{R}^d)}.$$

Viceversa, because the functions $\psi_{(n)}$, $n \in J_k$ are non-negative, bounded by 1, and supported in $[\frac{5}{32}k, 6k]$ one has also

$$\frac{1}{k}\sum_{n\in J_k}\|u_{(n)}(\cdot,t)\|^2_{L^2(\mathbf{R}^d)} \le \frac{5}{k}\int_{\frac{5k}{32}}^{6k}\left(\int_{|\xi|=\lambda}|\widehat{u}(\xi,t)|^2 dS(\xi)\right)d\lambda$$

Most of the experimental evidence on $E(k)$ is plotted on a log-log scale. So, for the purpose of estimating exponents in a power law, and in view of the above inequalities, we found it reasonable to consider ([22], [23], [38]) an average of the spectrum, defined as follows. One defines the Littlewood-Paley spectrum of the function $u(x,t)$ to be

$$E_{LP}(k) = \frac{1}{k}\sum_{n\in J_k}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|u_{(n)}(\cdot,t)\|^2_{L^2(\mathbf{R}^d)}dt.$$

From the definition and the considerations above it follows that

$$cE_{LP}(k) \le \frac{3}{2k}\int_{\frac{k}{2}}^{2k}E(\lambda)d\lambda \le CE_{LP}(k)$$

holds for all $k > 0$, with c, C positive constants depending only on the choice of Littlewood-Paley template function $\phi_{(0)}$. We will consider the Leray approximation again in $\mathbf{R}^3$. Then the components obey

$$D_\nu([u],\nabla)u_{(n)} + \nabla p_{(n)} = W_n + f_{(n)} \tag{47}$$

where $p_{(n)} = \Delta_n p$, are the Littlewood-Paley components of the pressure, $f_{(n)} = \Delta_n f$, are the components of the force and

$$W_n(x,t) =$$

$$\int_{\mathbf{R}^3}\Psi_{(n)}(y)\,\partial_{y_j}\left(\delta_y([u]_j)(x,t)\delta_y(u)(x,t)\right)dy.$$

The 4/5 law of isotropic homogeneous turbulence

$$S_3(r) \sim \epsilon r$$

where $S_3 = \langle(\delta_r u)^3\rangle$ is the third order longitudinal structure function. The 4/5 law would follow from the Navier-Stokes equations if assumptions of isotropic, homogeneous, stationary statistics could be applied. The natural mathematical assumption associated to the law is that

$$\widehat{\epsilon} = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \|u(\cdot,t)\|^3_{B_3^{\frac{1}{3},\infty}} dt < \infty.$$

The Besov space $L^3_{loc,unif}(dt)(B_3^{\frac{1}{3},\infty})$ is thus a natural space for a Kolmogorov theory. (It is also the natural space for the Onsager conjecture, ([28]). The uniform bound assumed above is used for the inequality

$$\limsup_{T\to\infty} \frac{k_0^3}{T}\int_0^T \int_{\mathbf{R}^3} |\delta_y u(x,t)|^3 dxdt \le \widehat{\epsilon}|y|$$

Armed with this inequality, one obtains a bound on the energy production in the Littlewood-Paley spectrum. Indeed,

$$W_n(x,t)u_{(n)}(x,t) = -\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \left\{\partial_{y_j}\Psi_{(n)}(y)\Psi_{(n)}(z)\right\}\left\{\delta_y[u_j]\delta_y u_i \delta_z u_i\right\}(x,t)dydz$$

and therefore

$$\langle W_n u_{(n)}\rangle \le C_\Psi \widehat{\epsilon}$$

with

$$C_\Psi = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} |y|^{\frac{2}{3}}|z|^{\frac{1}{3}}\Psi_{(0)}(y)\Psi_{(0)}(z)dydz$$

Considering $\widehat{f}(\xi)$ supported in $|\xi| \le k_0$ then, $f_{(n)} = 0$ for $n \ge 1$ and the balance

$$\epsilon_{(n)} = \langle \nu |u_{(n)}|^2\rangle = \langle W_n u_{(n)}\rangle$$

follows. This implies

Theorem 8 *Consider smooth, time independent divergence free forces with Fourier transform supported in $|\xi| \le k_0$. Consider solutions of the Leray approximation*

$$D_\nu([u],\nabla)u + \nabla p = f, \;\; \nabla\cdot u = 0,$$

with square integrable initial data. Then

$$\epsilon_{(n)} \le C_\Psi \widehat{\epsilon}$$

holds for $n \geq 1$. *Also*

$$E_{LP}(k) \leq \beta_a \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

holds for $k \in [ak_d, k_d]$, *with* $\frac{k_0}{k_d} \leq a \leq 1$ *and with*

$$\beta_a = C_{\Psi} a^{-\frac{4}{3}} \widehat{\epsilon}(\epsilon)^{-1}$$

Consequently

$$\frac{1}{k} \int_{\frac{k}{2}}^{2k} E(\lambda) d\lambda \leq \gamma_a \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

holds with $\gamma_a = C\beta_a$.

This result contains an unconditional statement. For each fixed mollifier J_δ, the Leray system has global smooth solutions and $\widehat{\epsilon}$ is finite. The bound on the spectrum depends on the mollifier through $\widehat{\epsilon}$. But even if we make the assumption that $\widehat{\epsilon}$ is bounded independently of δ, we still have a result only in a limited range of physical scales.

Similar results can be obtained for two dimensional turbulence. In two dimensions there is no need to consider approximations, because solutions are well behaved. But the range limitations are still there. The spectrum suggested for the direct cascade is the 2D Kraichnan spectrum:

$$E(k) = C\eta^{\frac{2}{3}} k^{-3}.$$

with $\eta = \langle \nu |\nabla \omega|^2 \rangle$, the rate of dissipation of enstrophy. The dissipative cutoff scale is the wave number k_η formed with ν and η:

$$k_\eta = \nu^{-\frac{1}{2}} \eta^{\frac{1}{6}}$$

Bounds on the spectrum for two dimensional turbulence have been addressed by Foias and collaborators ([33], [53], [54]). The result of ([22]) is

Theorem 9 *Consider two dimensional incompressible Navier-Stokes equations*

$$D_\nu u + \nabla p = f$$

with time independent, divergence-free forces whose Fourier transform is supported in $|\xi| \leq k_0$. *For any* $\frac{k_0}{k_\eta} \leq a \leq 1$ *there exists a constant* C_a *such that* *the Litlewood-Paley energy spectrum of solutions of two dimensional forced Navier-Stokes equations obeys the bound*

$$E_{LP}(k) \le C_a k^{-3}$$

for $k \in [ak_\eta, k_\eta]$. *Consequently*

$$\frac{1}{k}\int_{\frac{k}{2}}^{2k} E(\lambda)d\lambda \le \tilde{C}_a k^{-3}$$

holds with $\tilde{C}_a = CC_a$.

The range of physical space scales is again bounded. The mathematical reason for this limitation is the fact that one uses $-\nu\Delta$ and one has to let $\nu \to 0$. The technical tools we have at this moment do not allow us to rigorously obtain an effective nonzero eddy diffusivity to replace kinematic viscosity. When the model allows a non-vanishing positive linear operator then the scale limitation is no longer present. This is the case if one uses the forced surface quasigeostrophic model for the inverse cascade, appropriate for boundary forced rotating geophysical systems. One obtains ([23]) a spectrum consistent with the spectrum

$$E(k) \sim k^{-2}, \; k \le k_f.$$

obtained in Swinney's lab ([4]):

Theorem 10 *Consider the QG equation*

$$\partial_t\theta + u\cdot\nabla\theta + c\Lambda\theta = f$$

in $\mathbf{R}^2$, *with* $u = c_1 R^{\perp}\theta$, *with deterministic forcing* f *with Fourier transform supported in* $|\xi| \le k_f$, *and with* L^2 *initial data. Then*

$$E_{LP}(k) \le Ck^{-2}$$

holds for weak solutions solutions of QG, for all $k < k_f$.

References

[1] Arnold, V. I., Khesin, B. A., *Topological methods in Hydrodynamics*, Applied Mathematical Sciences, **125**, Springer-Verlag, New York.

[2] A.V. Babin, M.I. Visik, Attractors of partial differential equations and estimates of their dimension, Uspekhi Mat. Nauk **38** (1982), 133-187.

[3] A. Babin, A. Mahalov, B. Nicolaenko, Long time averaged Euler and Navier-Stokes equations for rotating fluids, in "Structure and nonlinear waves in fluids" (K. Kirchgassner and A. Mielke eds.,) World Scientific 1995, 145-157.

[4] C. Baroud, B. Plapp, Z-S. She, H. Swinney, Anomalous self-similarity in rotating turbulence, Phys. Rev. Lett. **88** (2002), 114501.

[5] G. I. Barenblatt, A. J. Chorin, V. M. Prostokishin, Self-similar intermediate structures in turbulent boundary layers at large Reynolds numbers, J. Fluid. Mech. **410** (2000), 263-283.

[6] Y. Brenier, Topics on hydrodynamics and area preserving maps,in Handbook of Mathematical fluid dynamics (S. Friedlander, D. Serre eds.,) Elsevier (2003), 55-86.

[7] T. Buttke, Lagrangian numerical methods which preserve the Hamiltonian structure of incompressible fluid flow, in "Vortex flows and related numerical methods", J.T. Beale, G. H. Cottet and S. Huberson (Eds), NATO ASI Series, Vol. 395, Kluwer, Norwell, 1993.

[8] T. Buttke and A. Chorin, Turbulence calculations in magnetization variables, Appl. Num. Math. **12** (1993), 47-54.

[9] D. Chae, J. Lee, Global well-posedness in the supercritical dissipative quasigeostrophic equation, CMP **233** (2003), 297-311.

[10] A. Cheskidov, D. Holm, E, Olson, E. Titi, On a Leray alpha model of turbulence, Proc. Roy. Soc. London, to appear (2004).

[11] Sir S. W. Churchill, *"I cannot forecast to you the action of Russia. It is a riddle wrapped in a mystery inside an enigma"*, Radio Broadcast, Oct. 1939.

[12] S. Childress, G. R. Ierley, E. A. Spiegel, W. R. Young, Blow -up of unsteady two-dimensional Euler and Navier-Stokes solutions having stagnation-point form, J. Fluid. Mech. **203** (1989), 1-22.

[13] A. Chorin, Numerical study of slightly viscous flow, J. Fluid. Mech **57** (1973), 785-796.

[14] A. Chorin, Vorticity and Turbulence, Apllied Mathematical Sciences **103**, Springer-Verlag.

[15] P. Constantin, Note on loss of regularity for solutions of the 3D incompressible and related equations, Commun. Math. Phys. **106** (1986), 311-325.

[16] P. Constantin, Navier-Stokes equations and area of interfaces, Commun. Math. Phys. **129** (1990), 241 - 266.

[17] P. Constantin, Geometric and analytic studies in turbulence, in Trends and Perspectives in Appl. Math., L Sirovich ed., Appl. Math. Sciences **100**, Springer-Verlag (1994).

[18] P. Constantin, The Euler Equations and Nonlocal Conservative Riccati Equations, Intern. Math. Res. Notes, **9** (2000), 455-465.

[19] P. Constantin, An Eulerian-Lagrangian approach for incompressible fluids: local theory, Journal of the AMS, **14** (2001), 263-278.

[20] P. Constantin, An Eulerian-Lagrangian approach to the Navier-Stokes equations, Commun. Math. Phys. **216** (2001), 663-686.

[21] P. Constantin, Near identity transformations for the Navier-Stokes equations, in *Handbook of Mathematical Fluid Dynamics*, Volume 2, S. Friedlander and D. Serre Edtrs, Elsevier (2003).

[22] P. Constantin, The Littlewood-Paley spectrum in 2D turbulence, Theor. Comp. Fluid Dyn.**9** (1997), 183-189.

[23] P. Constantin, Energy spectrum of quasi-geostrophic turbulence, Phys. Rev. Lett. **89** (October 2002) 18, 184501.

[24] P. Constantin, Transport in rotating fluids, Discrete and Continuous Dynamical systems, **10**, Spring 2004, to appear.

[25] P. Constantin, Ch. Doering, Variational bounds on energy dissipation in incompressible flows: II. Channel flow, Phys. Rev E **51** (1995), 3192-3198.

[26] P. Constantin and Ch. Doering, Heat transfer in convective turbulence, Nonlinearity **9** (1996) 1049-1060.

[27] P. Constantin, D. Cordoba and J. Wu, On the critical dissipative quasi-geostrophic equation, Indiana University Mathematics Journal, **50** (2001), 97-107.

[28] P. Constantin,W. E, E. S. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equations, Commun. Math. Phys., **165**(1994), 207-209.

[29] P. Constantin and C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations, Indiana Univ. Math. Journal, **42** (1993), 775.

[30] P. Constantin, C. Fefferman, Scaling exponents in fluid turbulence: some analytic results. Nonlinearity **7** (1994), 41-57.

[31] P. Constantin and C. Foias, Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equations, Commun. Pure apll. Math. **38** (1985), 1-27.

[32] P. Constantin, C. Foias, *Navier-Stokes Equations*, The University of Chicago Press, Chicago, 1988.

[33] P. Constantin, C. Foias and O. Manley, Effects of the forcing function spectrum on the energy spectrum in 2-D turbulence, Phys. Fluids, **6**, (1994), 427.

[34] P. Constantin, C. Foias, R. Temam, On the dimension of the attractors in two dimensional turbulence, Physica **D 30**, (1988), 284-296.

[35] P. Constantin, C. Hallstrom and V. Putkaradze, Heat transport in rotating convection, Physica **D 125**, (1999), 275-284.

[36] P. Constantin, A. Majda and E. Tabak, Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar, Nonlinearity, **7** (1994), 1495-1533.

[37] P. Constantin, Q. Nie, N. Schorghoffer, Nonsingular surface quasi-geostrophic flow, Phys. Letters A **241** (27 April 1998), 168-172.

[38] P. Constantin, Q. Nie and S. Tanveer, Bounds for second order structure functions and energy spectrum in turbulence Phys. Fluids **11** (1999), 2251-2256.

[39] P. Constantin and J. Wu, Behavior of solutions to 2D quasi-geostrophic equations, SIAM J. Math. Anal., **30** (1999) 937-948.

[40] D. Cordoba, Nonexistence of simple hyperbolic blow up for the quasi-geostrophic equation, Ann. of Math. **148** (1998), 1135-1152.

[41] A. Cordoba and D. Cordoba, A maximum principle applied to quasi-geostrophic equations, Comunn. Math. Phys, accepted 2003.

[42] B. I . Davydov, Dokl. Akad. Nauk. SSSR **2** (1949), 165.

[43] C. Doering and P. Constantin, Energy dissipation in shear driven turbulence, Phys.Rev.Lett. **69** (1992), 1648-1651.

[44] C. Doering and C. Foias, Energy dissipation in body-forced turbulence, Journal of Fluid Mechanics **467**, (2002), 289-306.

[45] D. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. of Math. **92** (1970), 102-163.

[46] P. Embid and A. Majda, Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity, Commun. PDE **21** (1996), 619-658.

[47] L. C. Evans and R. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press (1992).

[48] G. Eyink, Energy dissipation without viscosity in the ideal hydrodynamics, I. Fourier analysis and local energy transfer, Phys. **D 3-4** (1994), 222-240.

[49] M. Feigenbaum, The transition to aperiodic behavior in turbulent systems, Comm. Math. Phys. **77** (1980), 65-86.

[50] Foias, private communication (1999).

[51] C. Foias, Statistical study of Navier-Stokes equations, I and II, Rend. Sem. Univ. Padova, **48** (1973), 219-349; **49** (1973)

[52] C. Foias, D. Holm, E. Titi, The Navier-Stokes alpha model of turbulence, Physica **D 152** (2001), 505-519.

[53] C. Foias, M. Jolly, O. Manley, Kraichnan turbulence via finite time averages, preprint 2003.

[54] C. Foias, O. Manley, R. Rosa, R. Temam, *Navier-Stokes equations and turbulence*, Cambridge University Press, (2001).

[55] C. Foias and G. Prodi, Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2, Rend. Sem. Mat. Univ. Padova **39** (1967), 1-34.

[56] C. Foias, R. Temam, On the Hausdorff dimension of an attractor for the two dimensional Navier-Stokes equations, Phys. Lett. A **93**, (1983), 451-454.

[57] C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier-Stokes equations, J. Funct. Anal. **87** (1989), 359-369.

[58] U. Fisch, *Turbulence*, (1995) Cambridge University Press.

[59] D. Holm, J. Marsden, T. Ratiu, Euler-Poincare models of ideal fluids with nonlinear dispersion, Phys. Rev. Lett. **349** (1998), 4173-4177.

[60] E. Hopf, Statistical Hydrodynamics and functional calculus, J. Rat. Mech. Anal., **1** (1952), 87-123.

[61] Ning Ju, Existence and uniqueness of the solution to the dissipative 2D quasigeostrophic equation in Sobolev space, CMP, to appear.

[62] T. Kato, G. Ponce, Commutator estimates and Euler and Navier-Stokes equations, Commun. Pure Appl. Math **41** (1988), 891-907.

[63] Lord Kelvin, Maximum and minimum energy in vortex motion, *Mathematical and Physical Papers* IV, (1910), Cambridge University Press.

[64] C. Kenig, G. Ponce, L. Vega, Well-posedness of the initial value problem for the Korteweg - de Vries equation, J. Amer. Math. Soc., **4** (1991), 323-347.

[65] A.N. Kolmogorov, Local structure of turbulence in an incompressible fluid at very high Reynolds number, Dokl. Akad. Nauk. SSSR **30** (1941), 299-303.

[66] G. A.Kuzmin, Ideal incompressible hydrodynamics in terms of momentum density, Phys. Lett. A **96** (1983), 88-90.

[67] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Mathematics **63**(1934), 193-248.

[68] E. N. Lorenz, Deterministic nonperiodic flow, J. Atm. Sci. **20** (1963), 130-141.

[69] O. A. Ladyzhenskaya, On finite dimensionality of bounded invariant sets for the Navier-Stokes equations and some other dissipative systems, Zap. Nauchn. Sem. LOMI **115** (1982), 137-155.

[70] A. Majda and A. Bertozzi, Vorticity and incompressible flow, Cambridge Texts in Applied Mathematics, Cambridge University Press 2002.

[71] H. K. Moffatt, The degree of knottedness of tangled vortex lines, J. Fluid Mech., **35** (1969), 117-129.

[72] L. Onsager, Statistical Hydrodynamics, Nuovo Cimento **6**(2) (1949), 279-287.

[73] J.J. Niemela, L. Skrbek, K.R. Sreenivasan,and R.J. Donnelly, Turbulent convection at very high Rayleigh numbers, Nature, **404** (2000), 837-840.

[74] K. Ohkitani, P. Constantin, Numerical study of the Eulerian-Lagrangian formulation of the Navier-Stokes equations, Phys. Fluids **15-10** (2003), 3251-3254.

[75] K. Ohkitani, P. Constantin, Numerical study of the Eulerian-Lagrangian analysis of the Navier-Stokes turbulence, work in preparation.

[76] K. Ohkitani and J. Gibbon, Numerical study of singularity formation in a class of Euler and Navier-Stokes flows, Phys. Fluids **12** (2000), 3181.

[77] K. Ohkitani and M. Yamada, Inviscid and inviscid-limit behavior of a surface quasi-geostrophic flow, Phys. Fluids **9** (1997), 876.

[78] V. I. Oseledets, On a new way of writing the Navier-Stokes equation. The Hamiltonian formalism. Commun. Moscow Math. Soc (1988), Russ. Math. Surveys **44** (1989), 210-211.

[79] S. Resnick, Dynamical Problems in Nonlinear Advective Partial Differential equations, Ph.D. Thesis, University of Chicago, Chicago (1995).

[80] O. Reynolds, On the dynamical theory of incompressible viscous fluids and the determination of the criterion, Phil. Trans. Roy. Soc. London **186** (1884), 123-161.

[81] R. Robert, Statistical Hydrodynamics (Onsager revisited), in Handbook of Mathematical Fluid Dynamics, II, (S. Friedlander and D. Serre eds.,) Elsevier (2003), 3-54.

[82] P.H. Roberts, A Hamiltonian theory for weakly interacting vortices, Mathematica **19** (1972), 169-179.

[83] D. Ruelle, F. Takens, On the nature of turbulence, Comm. Math. Phys. **20** (1971), 167-192.

[84] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes Equations, Arch. Ratl. Mech. Anal **9**, (1962), 187-195.

[85] A. Shnirelman, Weak solutions of incompressible Euler equations, in Handbook of Mathematical Fluid Dynamics, (S. Friedlander, D. Serre eds.,) Elsevier (2003), 87-116.

[86] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press (1970).

[87] J.T. Stuart, Nonlinear Euler partial differential equations: Singularities in their solution, in *Applied Mathematics, Fluid Mechanics, Astrophysics (Cambridge MA 1987)*, World Sci., Singapore 1988, 81-95.

[88] Paul Valéry, Oeuvres II, Tel Quel I, Choses Tues, Bibl. de la Pléiade, NRF, Gallimard (1960), 503.

[89] M.I Vishik and A.V. Fursikov, *Mathematical problems of statistical hydrodynamics*, (1988) Kluwer, Dordrecht.

[90] W. Weber, Uber eine Transformation der hydrodynamischen Gleichungen, J. Reine Angew. Math. **68** (1868), 286-292.

[91] J. Wu, the quasigeostophic equation and its two regularizations, Commun. PDE **27**(2002), 1161-1181.

[92] J. Wu, Existence of solutions to the dissipative quasigeostrophic equations in Besov space, preprint 2003.

[93] V. E. Zakharov, E. A. Kuznetsov, Variational principle and canonical variables in magnetohydrodynamics, Doklady Akademii Nauk SSSR **194** (1970), 1288-1289.