Logarithmic Bounds for Infinite Prandtl Number Rotating Convection

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I. Introduction

Convection refers to fluid motion that is induced by buoyancy. In thermal convection buoyancy is due to temperature differences and one of the interesting questions is how much of the total heat transfer is due to convection. The natural measure of this quantity is the Nusselt number, N, and many experiments and numerical simulations have been performed to discern the relationship between N and the various parameters which describe the system. Much of this research has focused on the forcing parameter [1] - [6], although it has been observed that rotation plays a nontrivial role as well [7].

The standard mathematical description of a convective system in a rotating frame of reference is based on the rotating Boussinesq equations for Rayleigh-Bénard convection (see, for example, Chandrasekhar [8]). This is a system of equations coupling the three dimensional Navier-Stokes equations to a heat advection-diffusion equation. The parameters in this system are the Rayleigh number R which captures the forcing, and the Ekman number E which is inversly proportional to the rate of rotation. The only known rigorous upper bound for N at large values of the Rayleigh number is of the order $R^{\frac{1}{2}}$. This bound was first derived by Howard using variational methods [9]. More recently, a background method [10] has been used to obtain this bound as well [11]. This bound is also valid in the presence of rotation [12]. Experimental and numerical findings, however, indicate a bound of the form

$$N \sim R^q$$

where the reported values for q belong approximately to the interval $[\frac{2}{7}, \frac{1}{3}]$ for large R. The exponents 2/7 and 1/3 have been discussed by several authors [13] -[21].

A third parameter in the system is the Prandtl number, a parameter determined by the physical characteristics of the fluid. The Prandtl number is the ratio of the kinematic viscocity to the heat conduction coefficient. A simplified set of equations can be derived by taking the limit as the Prandtl number goes to infinity. These equations are easier to analyze than the Boussinesq equations; in particular one can prove global existence and uniqueness of smooth solutions ([12]). The known rigorous bounds for the rotating infinite Prandtl number system are the uniform bound

$$N \leq 1 + C_1 R^{\frac{2}{5}}$$

and the rotation dependent bound

$$N \leq 1 + C_2 E R^2$$

(with constants independent of E and R). The latter bound is most useful for strong rotation. These upper bounds were both obtained using the background field method [12, 22, 23, 24]. In the absence of rotation ($E = \infty$) a bound of the form

$$N \leq 1 + cR^{\frac{1}{3}}(1 + \log_+ R)^{\frac{2}{3}}$$

has been obtained [25]. The $\frac{1}{3}$ exponent is physical and close to the experimentally observed exponents. The goal of this paper is to provide a similar bound in the rotating case, allowing for finite values of E. As we shall see the correction due to rotation vanishes as $E \to \infty$, and we recover the above logarithmic bound even for rather strong rotation ($E \geq R^{-\frac{1}{6}}(\log R)^{-\frac{5}{6}}$). However as rotation is increased even further the logarithmic bound deteriorates, allowing for the observed increase of Nusselt number at intermediate rotation rates [7]. The $R^{\frac{2}{5}}$ bound may take over for a range of E. As $E \to 0$ the ER^2 bound takes over and accounts for the decrease of the Nusselt number due to very strong stratification.

The paper is organized as follows. In the next section we recall the equations, basic facts about the Nusselt number and some uniform estimates that hold for all Ekman numbers. In the third section we describe the method for bounding the heat flux and the results. The fourth section is devoted to proofs of the estimates of the non-rotation terms and the fifth section to the proofs of the estimates due to rotation.

II. Infinite Prandtl Number Equations

We begin with the equations of motion for infinite Prandtl number Rayleigh-Bénard convection in a rotating reference frame, where the Boussinesq approximation is used for the buoyancy force. These form a system of five equations for velocities (u, v, w), pressure p and temperature T in three spatial dimensions. The components of the velocity vector $\mathbf{u} = (u, v, w)$ satisfy the equations

$$-\Delta u - E^{-1}v + p_x = 0 \tag{1}$$
$$-\Delta v + E^{-1}u + p_y = 0 \tag{2}$$
$$-\Delta w + p_z = RT \tag{3}$$

and the divergence-free condition

$$u_x + v_y + w_z = 0. \tag{4}$$

The temperature, T, is advected according to the active scalar equation

$$(\partial_t + \mathbf{u} \cdot \nabla)T = \Delta T. \tag{5}$$

The two nondimensional parameters are the Rayleigh number, R, which describes the forcing due to the heat difference, and the Ekman number E which is inversely proportional to the rate of rotation.

We will consider a rectangular domain, with the vertical height scaled to 1 and the horizontal lengths scaled to the aspect ratio L. The horizontal independent variables (x, y) belong to a square $Q \subset \mathbf{R}^2$ of side length L. The vertical variable z belongs to the interval $[0, 1]$. The non-negative variable t represents time. For boundary conditions we will consider all the functions u, v, w, p, T periodic in x and y with period L. The velocity components u, v, and w vanish for $z = 0$ and 1 while the temperature T obeys $T = 0$ at $z = 1$ and $T = 1$ at $z = 0$. By taking a function $\tau(z)$ that satisfies $\tau(0) = 1$ and $\tau(1) = 0$, we will express the temperature as

$$T(x, y, z, t) = \tau(z) + \theta(x, y, z, t). \tag{6}$$

The role of τ is that of a convenient background which carries the inhomogeneous boundary conditions; thus θ obeys the same homogeneous boundary conditions as the velocity. The equation obeyed by θ is

$$(\partial_t + \mathbf{u} \cdot \nabla - \Delta)\, \theta = \tau'' - w\tau' \tag{7}$$

where we have used $\tau' = \frac{d\tau}{dz}$. We will use a normalized L^2 norm

$$\|f\|^2 = \frac{1}{L^2} \int_0^1 \int_0^L \int_0^L |f(x, y, z)|^2 \, dxdydz.$$

We denote by Δ_D^{-1} the inverse of the Laplacian with periodic-Dirichlet boundary conditions and the Laplacian in the horizontal directions x and y is denoted by Δ_h. We will use $< \cdot >$ for the long time average,

$$\langle f \rangle = \limsup_{t \to \infty} \frac{1}{t} \int_0^t f(s) ds.$$

The total heat transport is quantified by the Nusselt number which is defined in terms of a long time average of the vertical heat flux:

$$N = 1 + \left\langle \int_0^1 b(z)\, dz \right\rangle \tag{8}$$

where

$$b(z) = \frac{1}{L^2} \int_0^L \int_0^L w(x, y, z) T(x, y, z) \, dx \, dy. \tag{9}$$

we note that by using (6) the quantity $b(z)$ can be written

$$b(z) = \frac{1}{L^2} \int_0^L \int_0^L w(x,y,z)\theta(x,y,z)\, dx\, dy. \tag{10}$$

One can verify that the Nusselt number is also expressed as

$$N = \left\langle \|\nabla T\|^2 \right\rangle. \tag{11}$$

From the velocity equations it follows that

$$\|\nabla \mathbf{u}\|^2 = \frac{R}{L^2} \int_0^L \int_0^L \int_0^1 w(x,y,z,t)T(x,y,z,t)\, dxdydz \tag{12}$$

holds at each instant of time and thus

$$\left\langle \|\nabla \mathbf{u}\|^2 \right\rangle = R(N-1). \tag{13}$$

The temperature equation obeys a maximum principle so that

$$0 \le T \le 1$$

holds pointwise in space and time and consequently from (12) it follows that

$$\|\nabla \mathbf{u}\|^2 \le R^2 \tag{14}$$

at each instance of time. One can easily derive from the rotating infinite Prandtl number system (see, for example [23]) two coupled equations for the vertical velocity w and the vertical component of vorticity $\zeta = v_x - u_y$:

$$\Delta^2 w - E^{-1}\zeta_z = -R\Delta_h T \tag{15}$$

$$-\Delta\zeta - E^{-1}w_z = 0. \tag{16}$$

Multiplying the first equation by w, the second one by ζ, adding and integrating we deduce that

$$\|\Delta w\|^2 + 2\|\nabla \zeta\|^2 \le R^2 \tag{17}$$

holds pointwise in time. We used the fact, due to incompressibility, that w together with w_z and ζ vanish at the vertical boundaries.

III. Bounding the Heat Flux

From the definition of the Nusselt number (11), given in terms of the temperature, we can derive an equivalent expression in terms of the background profile and fluctuations using (6) to replace T with its decomposition into τ and θ in the quantity $|\nabla T|^2$. This gives us the expression

$$N = \langle \|\nabla\theta|^2\rangle + \int_0^1 (\tau')^2\, dz + 2\left\langle \int_0^L \int_0^L \int_0^1 \theta_z \tau'\, dzdydx \right\rangle .$$

The last term may be replaced by multiplying the evolution equation for θ (7) by θ and integrating. Upon taking a long time average, we have

$$\left\langle \int_0^L \int_0^L \int_0^1 \tau'\theta_z\, dzdydx \right\rangle = \langle \|\nabla\theta\|^2\rangle - \left\langle \int_0^L \int_0^L \int_0^1 w\tau'\, dzdydx \right\rangle$$

where we have made use of the boundary conditions and the incompressibility condition. Combining these, we have the following form for the Nusselt number:

$$N + \langle \|\nabla\theta\|^2\rangle = 2\left\langle -\int_0^1 \tau'(z) b(z) dz \right\rangle + \int_0^1 \left(\tau'(z)\right)^2 dz. \tag{18}$$

Let us now write

$$b(z,t) = \frac{1}{L^2}\int_0^L \int_0^L \int_0^z \int_0^{z_1} w_{zz}(x,y,z_2,t)\theta(x,z)\, dx\, dy\, dz\, dz_2\, dz_1.$$

It follows that

$$|b(z,t)| \leq \frac{1}{2} z^2 \left(1 + \|\tau\|_{L^\infty}\right) \|w_{zz}\|_{L^\infty(dz;L^1(dx))}. \tag{19}$$

Restricting ourselves to bounded profiles, $\|\tau\|_{L^\infty} \leq 1$, and relaxing the sup-norm we have simply

$$|b(z,t)| \leq z^2 \|w_{zz}(\cdot,t)\|_{L^\infty}. \tag{20}$$

Consequently we obtain the inequality

$$N \leq \int_0^1 \left(\tau'(z)\right)^2 dz + 2\int_0^1 z^2 |\tau'(z)| \left\langle \|w_{zz}(\cdot,t)\|_{L^\infty}\right\rangle dz \tag{21}$$

Up to this point, the background profile τ has not been specified. We will choose for simplicity a smooth approximation of a profile concentrated in a boundary layer of width δ, for example $\tau(z) = 1 - z/\delta$ for $0 \le z \le \delta$ and 0 for $z > \delta$. We will assume thus that the function $\tau(z)$ obeys

$$|\tau'(z)| \le C\frac{1}{\delta}$$

$$|\tau''(z)| \le C\frac{1}{\delta^2}$$

$|\tau(z)| \le 1$ and $\tau'(z) = 0$ for $z > \delta$. We will adjust δ to optimize the bounds but we will require at least

$$\delta \ge \frac{C}{R^p}.$$

We will not attempt to optimize prefactors in this paper; we will simply denote Rayleigh and Ekman number independent constants by C. The power p is not specified (this assumption will only be used inside a logarithmic bound). Before optimizing in δ we deduce from (21) the inequality

$$N \le \frac{C}{\delta} + C\delta^2 \left\langle \|w_{zz}(\cdot, t)\|_{L^\infty} \right\rangle \tag{22}$$

We will use now the two equations (15) and (16) to derive a single expression for w_{zz}, the quantity relevant to calculations of the heat flux. Noting that ζ vanishes on the vertical boundaries, solving for ζ in the second equation and substituting into the first equation, we obtain

$$\Delta^2 w + E^{-2}(\partial_z \Delta_D^{-1} \partial_z) w = -R\Delta_h T. \tag{23}$$

Moving the rotation term to the right hand side and applying the inverse bilaplacian, we deduce

$$w = -R(\Delta_{DN}^2)^{-1}\Delta_h T - E^{-2}(\Delta_{DN}^2)^{-1}(\partial_z \Delta_D^{-1} \partial_z) w. \tag{24}$$

Here $(\Delta_{DN}^2)^{-1}$ is the inverse bilaplacian with homogeneous Dirichlet and Neumann boundary conditions. Notice that $\Delta_h T = \Delta_h \theta$ since the background temperature profile τ depends on z only. Taking two z derivatives then gives

$$w_{zz} = -RB_1\theta - E^{-2}B_2 w \tag{25}$$

where

$$B_1 = \partial_{zz}(\Delta_{DN}^2)^{-1}\Delta_h \tag{26}$$

and

$$B_2 = \partial_{zz}(\Delta_{DN}^2)^{-1}(\partial_z\Delta_D^{-1}\partial_z). \tag{27}$$

We will estimate the quantity of interest to us $\|w_{zz}\|_{L^\infty}$ using the decomposition above. Obviously

$$\langle\|w_{zz}(\cdot,t)\|_{L^\infty}\rangle \leq R\,\langle\|B_1\theta(\cdot,t)\|_{L^\infty}\rangle + E^{-2}\,\langle\|B_2 w(\cdot,t)\|_{L^\infty}\rangle \tag{28}$$

holds.

In the following sections, we prove the two key estimates

$$\langle\|B_1\theta(\cdot,t)\|_{L^\infty}\rangle \leq C\left\{1 + C\log_+ R\right\}^2 \tag{29}$$

and

$$\langle\|B_2 w(\cdot,t)\|_{L^\infty}\rangle \leq C\sqrt{R(N-1)} \tag{30}$$

Using these two inequalities, and the combination of (28) and (22), we can optimize with respect to δ we obtain our main result:

Theorem 1. *There exists a constant C such that the Nusselt number for the infinite Prandtl number equation with rotation is bounded by*

$$N \leq 1 + CR^{\frac{1}{3}}\left\{1 + \log_+ R\right\}^{\frac{2}{3}}$$

when

$$E \geq R^{-\frac{1}{6}}\{1 + \log_+ R\}^{-\frac{5}{6}}.$$

When $E \leq R^{-\frac{1}{6}}\{1 + \log_+ R\}^{-\frac{5}{6}}$ then the Nusselt number obeys

$$N \leq CE^{-\frac{4}{5}}R^{\frac{1}{5}}$$

Indeed, using the bounds (29) and (30) together with (28) in (22) and optimizing with respect to δ we obtain

$$N \leq 1 + C\left\{R(1 + \log_+ R)^2\right\}^{1/3} + CE^{-\frac{2}{3}}R^{\frac{1}{6}}(N-1)^{\frac{1}{6}} \tag{31}$$

which implies the statement of the theorem. From inequality (31) and the previously obtained bounds

$$N \leq 1 + CR^{\frac{2}{5}}$$

$$N \leq 1 + CER^{2}$$

the following picture emerges. For rotations ranging from very weak to rather strong, $\left(E \geq R^{-\frac{1}{6}}\{1+\log_{+} R\}^{-\frac{5}{6}}\right)$, the bound $R^{\frac{1}{3}}\left\{1+\log_{+} R\right\}^{\frac{2}{3}}$ applies. For stronger rotation, $R^{-\frac{1}{4}} \leq E \leq R^{-\frac{1}{6}}\{1+\log_{+} R\}^{-\frac{5}{6}}$, the bound $N \leq 1 + CE^{-\frac{4}{5}}R^{\frac{1}{5}}$ is optimal. For stronger rotation yet, $R^{-2} \leq E \leq R^{-\frac{1}{4}}$, the bound $N \leq 1 + CR^{\frac{2}{5}}$ operates, and finally at exceedingly large rotation $E \leq R^{-2}$ the Nusselt number becomes bounded and then identically one. If instead of varying rotation at fixed Rayleigh numbers one varies the Rayleigh numbers and fixes the Ekman number, then the logarithmic one third power law bound emerges for any fixed rotation, no matter how strong, provided the Rayleigh number is high enough.

IV. Singular integrals and the B_1 term

In this section, we outline the estimates and results for the nonrotating case. Consider the operator

$$B_1 = \frac{\partial^2}{\partial z^2}(\Delta_{DN}^2)^{-1}\Delta_h$$

where $w = (\Delta_{DN}^2)^{-1} f$ is the solution of

$$\Delta^2 w = f$$

with horizontally periodic and vertically Dirichlet and Neumann boundary conditions $w = w' = 0$. Logarithmic L^∞ estimates for B_1 were obtained in [25]. They are recalled in the following

Theorem 2. *For any $\alpha \in (0,1)$ there exists a positive constant C_α such that every Hölder continuous function θ that is horizontally periodic and vanishes at the vertical boundaries satisfies*

$$\|B_1\theta\|_{L^\infty} \leq C_\alpha \|\theta\|_{L^\infty}\left\{1 + \log(1 + \|\theta\|_{C^{0,\alpha}})\right\}^2. \tag{32}$$

The spatial $C^{0,\alpha}$ norm is defined as

$$\|\theta\|_{C^{0,\alpha}} = \sup_{X=(x,y,z)\in Q\times[0,1]} |\theta(X,t)| + \sup_{X\neq Y} \frac{|\theta(X,t)-\theta(Y,t)|}{|X-Y|^{\alpha}}.$$

The proof decomposes $B_1\theta$ into the sum

$$B_1\theta = (I - B_3 + B_4 + B_5)B_3\theta$$

where

$$B_3(\theta) = (\Delta_D)^{-1}\,\Delta_h\theta.$$

B_3 is an integral operator with kernel K given by

$$B_3(\theta)(x,y,z) = \\ L^{-2}\int_0^L\int_0^L\int_0^1 K(x-\xi, y-\eta, z, \zeta)\,(\theta(\xi,\eta,\zeta)-\theta(x,y,z))\,d\xi\,d\eta\,d\zeta. \quad (33)$$

B_4 and B_5 are singular layer integral operators with kernels that are singular at the boundary. The operator B_4 can be written as

$$B_4(\theta)(x,y,z) = \\ L^{-2}\int_0^L\int_0^L\int_0^1 J(x-\xi, y-\eta, z, \zeta)\,(\theta(\xi,\eta,\zeta)-\theta(\xi,\eta,1))\,d\xi\,d\eta\,d\zeta \quad (34)$$

and

$$B_5(\theta)(x,y,z) = \\ L^{-2}\int_0^L\int_0^L\int_0^1 S(x-\xi, y-\eta, z, \zeta)\,(\theta(\xi,\eta,\zeta)-\theta(\xi,\eta,0))\,d\xi\,d\eta\,d\zeta \quad (35)$$

for any continuous function θ that obeys the homogeneous boundary conditions (so that $\theta(\xi,\eta,0)=\theta(\xi,\eta,1)=0$). It was shown in [25] that there exist constants such that

$$|K(x-\xi, y-\eta, z, \zeta)| \le C\left(|x-\xi|^2 + |y-\eta|^2 + |z-\zeta|^2\right)^{-\frac{3}{2}} \quad (36)$$

$$|J(x-\xi, y-\eta, z, \zeta)| \le C\left(|x-\xi|^2 + |y-\eta|^2 + |1-\zeta|^2\right)^{-\frac{3}{2}}, \quad (37)$$

$$|S(x-\xi, y-\eta, z, \zeta)| \le C\left(|x-\xi|^2 + |y-\eta|^2 + |\zeta|^2\right)^{-\frac{3}{2}}. \qquad (38)$$

Once these inequalities are established it is not difficult to derive for all B_j, $j = 3, 4, 5$ the estimates

$$\|B_j\theta\|_{L^\infty} \le C_\alpha \|\theta\|_{L^\infty}\left[1 + \log(1 + \|\theta\|_{C^{0,\alpha}})\right] \qquad (39)$$

for which the bound in (32) follows by composition. We will make now contact with the dynamical evolution of θ given by (7) by establishing two inequalities. The first,

$$\|\nabla\theta\|_{L^4}^2 \le C\|\theta\|_{L^\infty}\|\Delta\theta\|_{L^2},$$

is obtained by integration by parts and hold for all functions that are smooth enough and obey the homogeneous boundary conditions. The second inequality,

$$\frac{1}{L^2}\int_0^L\int_0^L |w(x,y,z,t)|^2 \le z\|\nabla\mathbf{u}(\cdot,t)\|^2,$$

follows from the boundary conditions, the fundamental theorem of calculus and the Schwartz inequality. Multiplying (7) by $-\Delta\theta$ and integrating one obtains, after using these last two inequalities,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \|\Delta\theta\|^2 \le C\|\nabla\mathbf{u}\|^2\left\{1 + \int_0^1 \left[(\tau''(z))^2 + z(\tau'(z))^2\right] dz\right\}. \qquad (40)$$

Now using the bound on $\|\nabla\mathbf{u}\|$, (14), and taking a long time average we see that there exists a positive constant C such that

$$\left\langle\|\Delta\theta\|^2\right\rangle \le CR^2\left\{1 + \int_0^1 \left[(\tau''(z))^2 + z(\tau'(z))^2\right] dz\right\}. \qquad (41)$$

By Sobolev embedding it follows that averages of squares of spatial $C^{0,\alpha}$ norms of θ are bounded by the same right hand side:

$$\left\langle\|\theta\|_{C^{0,\alpha}}^2\right\rangle \le CR^2\left\{1 + \int_0^1 \left[(\tau''(z))^2 + z(\tau'(z))^2\right] dz\right\}. \qquad (42)$$

Taking long time averages in the estimate (32) and using the concavity of the logarithm and the bound (42) we deduce the bound

$$\langle\|B_1\theta(\cdot,t)\|_{L^\infty}\rangle \le$$
$$C\left\{1 + \log\left[1 + CR^2\left\{1 + \int_0^1 \left[(\tau''(z))^2 + z(\tau'(z))^2\right] dz\right\}\right]\right\}^2 \qquad (43)$$

Using the general conditions on τ that make the integrals of gradients of τ not larger than powers of R we obtain (29).

V. Estimates for the Rotation Term

The goal of this section is to derive inequality (30), the estimate which appears in the rotating term. This is done using the bound on $\|\nabla \mathbf{u}\|$ (13), the lemma below, and taking long time averages.

Lemma. *For the operator B_2 defined by (27), there exists a constant C such that*

$$\|B_2 w\|_{L^\infty}^2 \leq C\|w_z\|^2. \tag{44}$$

To prove the lemma we will use Sobolev embedding to obtain pointwise bounds from bounds in H^2; in other words we will use

$$\|B_2 w\|_{L^\infty} \leq C\|(1-\Delta)B_2 w\|. \tag{45}$$

By showing that

$$\|B_2 w\| \leq \frac{1}{2}\|w_z\| \tag{46}$$

and that

$$\|\Delta B_2 w\| \leq \frac{\sqrt{2}+1}{\sqrt{2}}\|w_z\| \tag{47}$$

the lemma will follow. We derive first the inequality (46). Recalling that B_2 is defined as $[\partial_{zz}(\Delta_{DN}^2)^{-1}\partial_z\Delta_D^{-1}\partial_z]$, it is clear that the inequality follows from a corresponding bound of the norm of the operator $\left[\partial_{zz}(\Delta_{DN}^2)^{-1}\partial_z\Delta_D^{-1}\right]$ in L^2. We accomplish this by showing that $\partial_{zz}(\Delta_{DN}^2)^{-1}$ and $\partial_z\Delta_D^{-1}$ are both bounded in L^2. For the first of these, let ϕ be the solution of the bilaplacian equation $\Delta_{DN}^2\phi = f$. Multiplying this equation by ϕ and integrating over the whole domain gives

$$(\Delta^2\phi, \phi) = (f, \phi) \tag{48}$$

where

$$(f,g) = \frac{1}{L^2}\int_0^1\int_0^L\int_0^L f(x,y,z)g(x,y,z)dxdydz.$$

Expressing the bilaplacian as

$$\Delta^2 = \partial_{zzzz} + 2\partial_{zz}\Delta_h + \Delta_h^2, \tag{49}$$

it follows after integrating by parts that

$$(\Delta^2\phi, \phi) = \|\phi_{zz}\|^2 + 2\|\phi_{xz}\|^2 + 2\|\phi_{yz}\|^2 + \|\Delta_h\phi\|^2. \tag{50}$$

The boundary terms obtained by integrating by parts all vanish because of the boundary conditions. Equations (48), (50) imply that

$$\|\phi_{zz}\|^2 \leq \|f\|\|\phi\|. \tag{51}$$

We now note from the the fundamental theorem of calculus applied twice and the boundary conditions that

$$\|\phi\| \leq \frac{1}{\sqrt{2}}\|\phi_{zz}\| \tag{52}$$

and therefore from (51) we have

$$\|\phi_{zz}\| \leq \frac{1}{\sqrt{2}}\|f\|.$$

Since ϕ is by definition the solution to the bilaplacian equation, we can rewrite this inequality as

$$\|\partial_{zz}(\Delta_{DN}^2)^{-1}f\| \leq \frac{1}{\sqrt{2}}\|f\|. \tag{53}$$

We bound the operator $\partial_z\Delta^{-1}$ in the same way. Let ψ represent the solution to the Poisson equation $\Delta\psi = f$ with Dirichlet boundary conditions. Multiplying by ψ and integrating over the domain yields

$$\|\partial_z\Delta_D^{-1}f\| \leq \frac{1}{\sqrt{2}}\|f\|. \tag{54}$$

Now the L^2 bounds given by equations (53) and (54) can be used to obtain the estimate (46) on the operator B_2.

For (47) we need to show that

$$\|\Delta\partial_{zz}(\Delta_{DN}^2)^{-1}\partial_z\Delta_D^{-1}\partial_z w\| \leq \frac{\sqrt{2}+1}{\sqrt{2}}\|\partial_z w\|. \tag{55}$$

Noticing that $\partial_z \Delta_D^{-1}$ may be written

$$\Delta\Delta(\Delta_{DN}^2)^{-1}\partial_z\Delta_D^{-1}$$

and by expressing $\Delta = \partial_{zz} + \Delta_h$, we obtain the following form for the operator in (55):

$$\Delta\partial_{zz}(\Delta_{DN}^2)^{-1}\partial_z\Delta_D^{-1} = \left[I - \partial_{zz}\Delta_h(\Delta_{DN}^2)^{-1} - \Delta_h^2(\Delta_{DN}^2)^{-1}\right]\partial_z\Delta_D^{-1}. \qquad (56)$$

We have already shown that $\partial_z\Delta_D^{-1}$ is bounded in L^2, so we need only concern ourselves with the other two operators. Let φ be the solution to the bilaplacian equation $\Delta^2\varphi = f$ with Dirichlet and Neumann boundary conditions. Multiplying by $\Delta_h^2\varphi$ and integrating over the domain, we obtain

$$\left(\Delta^2\varphi, \Delta_h^2\varphi\right) = \left(f, \Delta_h^2\varphi\right) \qquad (57)$$

Noting that we can separate the bilaplacian into vertical and horizontal derivatives, we have

$$\left(\Delta^2\varphi, \Delta_h^2\varphi\right) = \left(\partial_{zzzz}\varphi, \Delta_h^2\varphi\right) + 2\left(\Delta_h\partial_{zz}\varphi, \Delta_h^2\varphi\right) + \|\Delta_h^2\varphi\|^2. \qquad (58)$$

Integrating by parts, the first term gives

$$\left(\partial_{zzzz}\varphi, \Delta_h^2\varphi\right) = \|\partial_{zz}\Delta_h\varphi\|^2.$$

The boundary terms disappear due to boundary conditions. Similarly, the second term in (58) becomes, after integrating by parts,

$$\left(\Delta_h\partial_{zz}\varphi, \Delta_h^2\varphi\right) = \|\Delta_h\partial_{xz}\varphi\|^2 + \|\Delta_h\partial_{yz}\varphi\|^2.$$

Again, because of the boundary conditions, the boundary terms vanish. Equations (57) and (58) together with the Schwartz inequality, yield

$$\|\partial_{zz}\Delta_h\varphi\|^2 + 2\|\Delta_h\partial_{xz}\varphi\|^2 + 2\|\Delta_h\partial_{yz}\varphi\|^2 + \frac{1}{2}\|\Delta_h^2\varphi\|^2 \leq \frac{1}{2}\|f\|^2.$$

This inequality implies both that

$$\|\partial_{zz}\Delta_h(\Delta_{DN}^2)^{-1}w\|^2 \leq \frac{1}{2}\|f\|^2$$

and also that

$$\|\Delta_h^2(\Delta_{DN}^2)^{-1}w\|^2 \leq \|f\|^2.$$

Now by using (56) we obtain the estimate stated in (55) and proof of the lemma is completed.

VI. Discussion

For infinite Prandtl number convection without rotation, there exists a rigorous upper bound on the heat transfer which is of the order $R^{1/3}(\log R)^{2/3}$. In the presence of rotation, however, a low order perturbation to the bilaplacian operator is introduced. This has the effect of an additional term in the upper bound for the heat transfer, as seen in (31). As the rotation is increased the bound deteriorates slowly but holds as long as $E \geq R^{-1/6}(\log R)^{-5/6}$. For a region $R^{-1/4} \leq E \leq R^{-1/6}(\log R)^{-5/6}$ a bound of the type $N \leq E^{-4/5}R^{1/5}$ is the best known bound, for stronger rotation $R^{-2} \leq E \leq R^{-1/4}$ the uniform bound $N \leq R^{2/5}$ applies and if rotation is increased further the Nusselt number becomes bounded and then equal to one. On the other hand, suppose the rotation is arbitrary but fixed and the Rayleigh number is increased; for sufficiently large Rayleigh numbers the logarithmic $R^{1/3}(\log R)^{2/3}$ bound applies.

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