Filtered Viscous Fluid Equations

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December 13, 2000

1 Introduction

The Eulerian description of an incompressible viscous fluid of constant density and temperature is concerned with the fluid velocity $u(x,t)$ and pressure $p(x,t)$ recorded at fixed positions $x \in \mathbf{R}^n$, $n = 3$ as functions of time t. The solutions of the incompressible, inviscid Euler equations can be thought of as geodesic paths on an infinite dimensional group of transformations ([1]). This is done using the Lagrangian description. In this description the basic object is a transformation $a \mapsto X(a,t)$ that represents the position $x = X(a,t)$ at time t of the fluid particle that started at $t = 0$ from a. At time $t = 0$ the transformation is the identity, $X(a,0) = a$. An Eulerian-Lagrangian formulation of the Euler equations ([8]) can be written in terms of the inverse map $A(x,t) = X^{-1}(x,t)$ as

$$(\partial_t + u \cdot \nabla) A = 0, \quad u = W[A]$$

where $W[A]$ is the Weber formula ([28], see (1) below). This form of the classical Euler equations is similar to the active scalar equations of ([11], [12]).

The viscous Navier-Stokes equations admit an Eulerian-Lagrangian formulation in terms of an appropriate diffusive "back-to-labels"map A and a virtual velocity v ([9]). The term "virtual" refers to fields that, in the absence of viscosity, are time independent functions of the Lagrangian labels. (Thus, if viscosity is absent $v(x,t) = u_0(A(x,t))$.) The presence viscosity does produce a dynamical change in these fields, and they do not remain passive.

Using the Eulerian-Lagrangian approach one can prove bounds that hold for all time for the diffusive map $A(x,t)$, its Eulerian gradient $\nabla A(x,t)$ and even its second derivatives $\nabla\nabla A(x,t)$. The approach to the Navier-Stokes equations based on the variables A and v also affords a distinction between the stretching of Eulerian line elements that occurs also in the zero viscosity case, and the viscosity induced changes in the virtual fields. For short times and smooth flows the virtual velocity v and the virtual vorticity ζ are nearly conserved, and in dissipative events, they are expected to decay. A Cauchy formula for the viscous Navier-Stokes equation and an evolution equation for the virtual vorticity, in which the stretching term is absent are derived in ([9]). The Cauchy formula expresses the Eulerian vorticity in terms of the diffusive map A and the virtual vorticity, in the exact same manner as in the Euler equations. The difference is that the virtual vorticity is no longer a frozen function of A that does not change in time. The virtual vorticity may play a useful role in the study of vortex reconnection. Important coefficients C involving second order derivatives of A arise when one computes the commutator between the Eulerian gradient and the Lagrangian gradient. These coefficients evolve in time, starting from zero, and enter as basic coefficients in the equations obeyed by virtual velocity, virtual vorticity and in the viscous commutator between the temporal advection-diffusion derivative and the spatial Eulerian-Lagrangian label derivatives.

In this paper we show that these considerations apply to a large class of model equations, that include the Navier-Stokes equations as a limiting case. These models can be characterized as Navier-Stokes equations filtered in a manner that preserves exactly the vorticity equation ([6]).

2 Active Scalars, Active Vectors and the Weber formula

An Eulerian-Lagrangian description of the Euler equations has been used in ([7], [8]) for local existence results and constraints on blow-up. Local existence for three dimensional Euler equations is of course classical ([14]). The paper ([7]) shows that the Beale-Kato-Majda result ([2]) can be interpreted as stating that a finite time singularity in the three dimensional Euler equations implies an incompressible shock in the inverse of the Lagrangian particle map. The Eulerian-Lagrangian formulation of the Navier-Stokes equations ([9]) starts with an Eulerian velocity $u(x,t)$, a three-component vector u_i, $i = 1, 2, 3$ that is a function of three Eulerian space coordinates x and time t. The Eulerian velocity $u(x,t)$ is written as

$$u_i = (\partial_i A^m) v_m - \partial_i n. \tag{1}$$

Repeated indices are summed, ∂_i is derivative in the i-th Euclidean direction. The incompressibility

$$\nabla \cdot u = 0 \tag{2}$$

imposed in the ansatz (1) results in the equation

$$\Delta n = \nabla \cdot (\nabla A)^* v) \,. \tag{3}$$

Substituting in (1) one obtains

$$u = \mathbf{P}\left((\nabla A)^* v\right). \tag{4}$$

The notation $(\nabla A)^*$ means the transpose of the matrix ∇A and $\mathbf{P}$ is the Leray-Hodge projector on divergence-free functions with matrix elements

$$\mathbf{P}_{jl} = \delta_{jl} - \partial_j \Delta^{-1} \partial_l. \tag{5}$$

In the absence of viscosity, A is the inverse of the particle trajectory map $a \mapsto x = X(a,t)$. In the presence of viscosity we required this map to obey a diffusive equation. Note that this is not the conventional Lagrangian particle picture. In the absence of viscosity, $v(x,t)$ is the initial velocity composed with the back-to-labels map; in this case (1) is the Weber formula ([28]) that has been used in numerical and theoretical studies ([16], [17], [21]). We refer to v as the "virtual velocity". We associate to a given divergence-free velocity $u(x,t)$ the operator

$$\partial_t + u \cdot \nabla - \nu \Delta = \Gamma_\nu(u, \nabla). \tag{6}$$

We write ∂_t for time derivative. The coefficient $\nu > 0$ is the kinematic viscosity of the fluid. When applied to a vector or a matrix, Γ acts as a diagonal operator, i.e. on each component separately. The diffusive back-to-labels map A is required to obey

$$\Gamma_\nu(u, \nabla) A = 0. \tag{7}$$

By (7) we express therefore the advection *and diffusion* of A. The maximum principle holds. The vector

$$\ell(x,t) = A(x,t) - x \tag{8}$$

is referred to as "displacement". The equation obeyed by ℓ

$$\Gamma_\nu(u,\nabla)\ell + u = 0 \tag{9}$$

is obviously equivalent to (7). We discuss periodic boundary conditions

$$\ell(x + Le^j, t) = \ell(x,t),$$

where e^j is the unit vector in the j-th direction. It is important to note that the initial data for the displacement is zero:

$$\ell(x,0) = 0. \tag{10}$$

The virtual velocity is required to obey

$$\Gamma_\nu(u,\nabla)v = 2\nu C\nabla v + Q^* f, \tag{11}$$

or, more precisely

$$\Gamma_\nu(u,\nabla)v_i = 2\nu C_{k;i}^m \partial_k v_m + Q_i^j f_j. \tag{12}$$

The vector $f = f(x,t)$ represents the body forces. The boundary conditions are also periodic

$$v(x + Le^j, t) = v(x,t)$$

and the initial data are, for instance

$$v(x,0) = u_0(x). \tag{13}$$

The coefficients $C_{k;i}^m$ are derived from A:

$$C_{k;i}^m = (\nabla A)_{ji}^{-1}(\partial_j\partial_k A^m).$$

The reason for the equations (7), (11) is ([9])

Proposition 1. *Let u be given by (4) and assume that the displacement solves (9) and that the virtual velocity solves (11). Then u obeys the incompressible Navier-Stokes equations,*

$$\partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = f, \quad \nabla\cdot u = 0.$$

In the inviscid case the system

$$(\partial_t + u \cdot \nabla) A = 0; \quad u = W[A]$$

is an active vector system. The term refers to the fact that the vectors A obey a pure advection equation and determine their own velocity by a time independent equation of state. The term active scalar was used in ([11], [12]) and other publications to mean scalar solutions of pure advection equations that determine their own velocity by a time independent equation of state. The term was coined as a twist on "pasive scalars", solutions of pure advection equations with prescribed velocities. In the active vector and scalar equations the non-local term enters in a multiplicative fashion (in the equation of state $u = W[A]$) and therefore a maximum principle holds. In the usual Eulerian formulation of the Euler and Navier-Stokes equation in terms of the velocity, the non-local term, the pressure, enters additively and a maximum principle for the velocity is not available. A maximum principle for the velocity would imply regularity for the Navier-Stokes equations.

In ([9]) we derived various bounds for the displacement, its first and second derivatives, virtual velocities and virtual vorticity. The bounds use the fact that the active vector formulation has a maximum principle. The connection with the Kuzmin-Oseledets approach ([22], ([25], [29], [3], [15] [19], [23], [5], [27], [26], [13]) is the following: the variable w in that approach (impulse, velicity, magnetization...) is obtained by applying the transposed Eulerian gradient of A to v, the virtual velocity.

Proposition 2. *Let u be an arbitrary spatially periodic smooth function and assume that a displacement ℓ solves the equation (9) and a virtual velocity v obeys the equation (11) with periodic boundary conditions and with C computed using $A = x + \ell$. Then w defined by*

$$w_i = (\partial_i A^m) v_m \tag{14}$$

obeys the equation

$$\Gamma_\nu(u, \nabla) w + (\nabla u)^* w = f. \tag{15}$$

The paper ([27]) is particularly useful for the assessment of the situation from a computational point of view. The variables w are not convenient for computations because the presence of the gradient ∇u in the w equation is responsible for rapid growth of errors. In contrast, the equations for A and v have no such stretching term, and consequently, their numerical evolution will be stable.

3 Filtered viscous fluid equations

We consider the equation

$$\Gamma_\nu(u, \nabla)w + (\nabla u)^* w = 0. \tag{16}$$

This equation, coupled with the Weber relation

$$u = \mathbf{P}w \tag{17}$$

is equivalent to the Navier-Stokes equation. Instead of using the relation (17) we approximate the Navier-Stokes equation by applying a filter

$$u = J_\delta \mathbf{P} w. \tag{18}$$

The approximation of identity J_δ is a Fourier multiplier,

$$(J_\delta(q))(x) = \sum_{k\in\mathbf{Z}^3,\, k\neq 0} j(\delta|k|) q_k e^{\frac{2\pi}{L} i x\cdot k} \tag{19}$$

where

$$q(x) = \sum_{k\in\mathbf{Z}^3,\, k\neq 0} q_k e^{\frac{2\pi}{L} i x\cdot k}.$$

The symbol j is real, strictly positive, decreasing fast to zero at infinity and normalized, $j(0) = 1$. We will use

$$j(\lambda) = e^{-\lambda}. \tag{20}$$

We consider now the equation (16) together with (18). This is a closed nonlinear system. Note that the Eulerian curl of w, $\xi = \nabla^E \times w$ satisfies

$$\Gamma_\nu(u, \nabla)\xi = \xi \cdot \nabla u, \tag{21}$$

that is the exact same equation as the vorticity equation, except that here the velocity is filtered. Thus, any inviscid filtered Kuzmin-Oseledets equation can be seen as a particular vortex method, such as the vortex blob method of Chorin ([6]). In ([24]) the authors show that a certain choice of the blob smoothing in inviscid two dimensional equations leads to second grade non-Newtonian fluid equations, or "averaged" Euler equations.

The three dimensional filtered viscous fluid system has smooth solutions. The starting point of the proof is the energy balance

$$\int (\partial_t w \cdot u)dx + \nu \int (\nabla w \cdot \nabla u)dx = 0. \tag{22}$$

This balance follows from (16) using the fact that u is divergence-free and the identity

$$u_j(\partial_j w_i)u_i + (\partial_i u_j)w_j u_i = \partial_j(u_j(w\cdot u))$$

that holds pointwise. Now the fact that J is symmetric and that it commutes with $\mathbf{P}$ implies that

$$\frac{d}{2dt}\int w\cdot u dx = \int \partial_t w \cdot u.$$

Using (22) it follows that

$$\frac{d}{2dt}\int (w\cdot u)dx + \nu\int(\nabla w\cdot\nabla u)dx = 0.$$

Theorem 1. *Let u_0 be a divergence-free real periodic function*

$$u_0(x) = \sum_{k\in\mathbf{Z}\, k\neq 0} u_k(0)e^{\frac{2\pi}{L}ix\cdot k}$$

belonging to the space G_{δ_0} of divergence-free functions having $J_{\delta_0}^{-1}\in L^2(dx)$:

$$\sum_{k\neq 0} e^{2\delta_0|k|}|u_k(0)|^2 < \infty$$

Let $\delta < \delta_0$ and let $\nu > 0$, $T > 0$ be given. Then the solution of the equation

$$\Gamma_\nu(u,\nabla)w + (\nabla u)^* w = 0$$

with

$$u = J_\delta \mathbf{P} w$$

and initial datum

$$w_0 = J_\delta^{-1}u_0$$

exists for all $t\in[0,T]$. Moreover,

$$\frac{1}{2}\|J_\delta^{-\frac{1}{2}}u(\cdot,t)\|^2_{L^2(dx)} + \nu\int_0^t \|\nabla J_\delta^{-\frac{1}{2}}u(\cdot,s)\|^2_{L^2(dx)}ds \le \frac{1}{2}\|J_\delta^{-\frac{1}{2}}u_0\|^2_{L^2(dx)}.$$

Note that $J_\delta^{-\frac{1}{2}} = J_{-\frac{\delta}{2}}$ is a Fourier multiplier and that the spaces G_δ are Gevrey classes. The uniform inequality in the theorem is thus a very strong analytic bound, with coefficients that do not deteriorate as $\delta \to 0$. A proof of the theorem can be constructed using Galerkin approximations: One uses the projector

$$(\mathbf{P}_N w)(x) = \sum_{k\neq 0, |k|\leq N} w_k e^{\frac{2\pi}{L} i x\cdot k}$$

to devise the truncated equations

$$\partial_t w_N - \nu\Delta w_N + \mathbf{P}_N \left\{u_N \cdot \nabla w_N + (\nabla u_N)^* w_N\right\} = 0$$

with

$$u_N = (J_\delta \mathbf{P} w_N)$$

and with initial data

$$w_N(0) = \mathbf{P}_N \left(J_\delta^{-1}\mathbf{P} u_0\right)$$

One gathers the uniform bound

$$\frac{1}{2}\|J_\delta^{-\frac{1}{2}} u_N(\cdot, t)\|^2_{L^2(dx)} + \nu \int_0^t \|\nabla J_\delta^{-\frac{1}{2}} u_N(\cdot, s)\|^2_{L^2(dx)} ds \leq \frac{1}{2}\|J_\delta^{-\frac{1}{2}} u_0\|^2_{L^2(dx)}.$$

that comes from the same cancellation in the equation for $\int (\partial_t w_N \cdot u_N) dx$ as before. Also one uses the straightforward bound

$$\|\nabla u_N\|_{L^\infty} \leq b_\delta \|\nabla J_\delta^{-\frac{1}{2}} u_N\|_{L^2(dx)}$$

where the N independent constant is given by

$$b_\delta = \sqrt{\sum_{k\in\mathbf{Z}^3,\, k\neq 0} e^{-\delta|k|}}$$

Then one can start gathering bounds on w_N directly using standard energy methods. One obtains bounds that uniform in N:

$$\sup_{t\leq T}(\|w_N(\cdot, t)\|^2_{L^2} + L^2\|\nabla w_N\|^2_{L^2}) +$$

$$\nu\int_0^T \left\{\|\nabla w_N(\cdot, t)\|^2_{L^2(dx)} + L^2\|\Delta w_N(\cdot, t)\|^2_{L^2(dx)}\right\} dt \leq G(\delta, u_0, \nu, T)$$

Finally one passes to the limit $N \to \infty$.

Remark. Different filters lead to the isotropic averaged equations of [4] and [20].

4 The Cauchy formula

Let u be *any* given smooth incompressible velocity field defined on a time interval $[0, T]$. We consider an associated displacement that solves the equation

$$\Gamma_\nu(u, \nabla)\ell + u = 0 \tag{23}$$

with initial datum equal to zero. We associate to ℓ the corresponding map $A = x + \ell$. The map $x \mapsto A(x,t)$ can be used to define the analogue of Lagrangian differentiation with respect to initial particle position, in Eulerian coordinates

$$\nabla^A = Q^* \nabla^E. \tag{24}$$

Here ∇^E is usual Euclidean derivative and

$$Q(x,t) = (\nabla A(x,t))^{-1}. \tag{25}$$

The detailed expression of the Eulerian-Lagrangian ∇_A is

$$\nabla_i^A = Q_i^j \partial_j \tag{26}$$

The Eulerian spatial derivatives can be expressed in terms of the Eulerian-Lagrangian derivatives via

$$\nabla_i^E = (\partial_i A^m)\, \nabla_m^A \tag{27}$$

While the commutators $[\nabla_i^E, \nabla_k^E] = 0$, $\left[\nabla_i^A, \nabla_k^A\right] = 0$ vanish, the cross-commutators between Eulerian-Lagrangian and Eulerian derivatives do not vanish, in general:

$$\left[\nabla_i^A, \nabla_k^E\right] = C_{k;i}^m \nabla_m^A. \tag{28}$$

The coefficients $C_{k;i}^m$ are given by

$$C_{k;i}^m = \nabla_i^A(\partial_k \ell^m). \tag{29}$$

Note that

$$C_{k;i}^m = Q_i^j \partial_j \partial_k A^m = \nabla_i^A(\nabla_k^E A^m) = [\nabla_i^A, \nabla_k^E]A^m.$$

The commutator coefficients C are related to the Christoffel coefficients Γ_{ij}^m of the trivial flat connection in $\mathbf{R}^3$ computed at $a = A(x,t)$ by the formula

$$\Gamma_{ij}^m = -Q_j^k C_{k;i}^m.$$

The matrix $\nabla A(x,t)$ is invertible. The equation obeyed by ∇A follows from (23)

$$\Gamma(\nabla A) + (\nabla A)(\nabla u) = 0. \tag{30}$$

The product $(\nabla A)(\nabla u)$ is matrix product in the order indicated. The inverse matrix $Q = (\nabla A)^{-1}$ obeys

$$\Gamma Q = (\nabla u)Q + 2\nu Q \partial_k(\nabla A)\partial_k Q. \tag{31}$$

The commutator coefficients $C_{k;i}^m$ enter also in the commutation relation between the Eulerian-Lagrangian label derivative and $\Gamma_\nu(u,\nabla)$:

$$\left[\Gamma, \nabla_i^A\right] = 2\nu C_{k;i}^m \nabla_k^E \nabla_m^A \tag{32}$$

The evolution of the coefficients $C_{k;i}^m$ defined in (29) can be computed using (30) and (32):

$$\Gamma\left(C_{k;i}^m\right) = -(\partial_l A^m)\nabla_i^A(\partial_k(u^l))$$

$$-(\partial_k(u_l))C_{l;i}^m + 2\nu C_{l;i}^j \cdot \partial_l\left(C_{k;j}^m\right). \tag{33}$$

Note that the initial datum vanishes. The determinant of ∇A obeys

$$\Gamma_\nu(u,\nabla)\left(\log Det(\nabla A)\right) = \nu\left\{C_{k;s}^i C_{k;i}^s\right\}. \tag{34}$$

Note, again, that the initial datum vanishes. Also, note that the kinematic viscosity and commutator coefficients determine the extent of volume distortion. The commutator coefficients are used to define a virtual velocity v by solving the equation

$$\Gamma_\nu(u,\nabla)v_i = 2\nu C_{k;i}^m \nabla_k^E v_m. \tag{35}$$

The variable $w = (\nabla A)^* v$ satisfies the equation (16). Let us consider the Eulerian-Lagrangian curl of v:

$$\zeta = \nabla^A \times v. \tag{36}$$

This virtual vorticity ζ is related to the anti-symmetric part of the Eulerian-Lagrangian gradient of v by the familiar formulae

$$\nabla_i^A v_m - \nabla_m^A v_i = \epsilon_{imp}\zeta_p, \quad \zeta_p = \frac{1}{2}\epsilon_{imp}\left(\nabla_i^A v_m - \nabla_m^A v_i\right).$$

We will use the mechanics notation

$$w_{i,j} = \partial_j w_i = \nabla_j^E(w_i).$$

Differentiating $w_i = A^m_{,i} v_m$ we get

$$w_{i,j} = A^m_{,i} v_{m,j} + A^m_{,ij} v_m = A^m_{,i} v_{m,j} - A^m_{,j} v_{m,i} + w_{j,i}.$$

Using (27) we deduce

$$w_{i,j} - w_{j,i} = \frac{1}{2}\left(A^m_{,i}A^p_{,j} - A^m_{,j}A^p_{,i}\right)\epsilon_{pmr}\zeta_r. \tag{37}$$

The Eulerian curl of w, $\nabla^E \times w$ obeys therefore

$$\left(\nabla^E \times w\right)_q = \frac{1}{2}\epsilon_{qij}\left(Det\left[\zeta; \frac{\partial A}{\partial x^i}; \frac{\partial A}{\partial x^j}\right]\right) \tag{38}$$

Because of the linear algebra identity

$$((\nabla A)^{-1}\zeta)_q = (Det(\nabla A))^{-1}\,\frac{\epsilon_{qij}}{2}\left(Det\left[\zeta; \frac{\partial A}{\partial x^i}; \frac{\partial A}{\partial x^j}\right]\right)$$

one has

$$\nabla^E \times w = (Det(\nabla A))\,(\nabla A)^{-1}\zeta. \tag{39}$$

The relations (38, 39) are the analogue of the viscous Cauchy formula ([9]). In two-dimensions (38, 39) become

$$\left(\nabla^E\right)^{\perp} w = (Det(\nabla A))\,\zeta. \tag{40}$$

A consequence of (38) or (39) is the identity

$$\left(\nabla^E \times w\right)\cdot\nabla_E = (Det(\nabla A))\,(\zeta\cdot\nabla_A) \tag{41}$$

that generalizes the corresponding identity ([9]). These identities hold in the forced case also.

Let us consider, for any pair (q, M), where $q \in \mathbf{R}^3$, $M \in GL(\mathbf{R}^3)$ are, respectively, a vector and an invertible matrix, the expression

$$\mathcal{C}(q, M) = (DetM)M^{-1}q. \tag{42}$$

This expression, underlying the Cauchy formula, is linear in q and quadratic in M,

$$\mathcal{C}(q, M)_i = \frac{1}{2}\epsilon_{ijk} Det\left(M_{\cdot,i}, M_{\cdot,j}, q\right). \tag{43}$$

The quadratic expression in the right hand side is defined for any matrix M. It is easy to check that

$$\mathcal{C}(q, MN) = \mathcal{C}\left(\mathcal{C}(q, M), N\right) \tag{44}$$

and

$$\mathcal{C}(q, \mathbf{I}) = q \tag{45}$$

hold, so $\mathcal{C}$ describes an action of $GL(\mathbf{R}^3)$ in R^3. A third property follows from the explicit quadratic expression (43)

$$\mathcal{C}(q, \mathbf{1} + N) = (1 + Tr(N))q - Nq + \mathcal{C}(q, N) \tag{46}$$

Here N is any matrix, and the meaning of $\mathcal{C}(q, N)$ is given by (43). If we consider, instead of vectors q and matrices M, vector valued functions $q(x, t)$ and matrix valued $M(x, t)$ and use the same formula

$$\mathcal{C}(q, M)(x, t) = \mathcal{C}(q(x, t), M(x, t)) = (Det(M(x, t))\,(M(x, t))^{-1}\,q(x, t) \tag{47}$$

then the properties (44, 45, 46) as well as (43) obviously still hold.

We derive now the evolution of ζ. We start with (35) and apply the Eulerian-Lagrangian curl. We use the notation

$$v_{i;j} = \nabla^A_j v_i$$

(thus for instance $C^m_{k;i} = (A^m_{,k})_{;i}$) Applying ∇^A_j to (35), and using (32) we obtain

$$\Gamma_\nu(u, \nabla)v_{i;j} = 2\nu\left(C^m_{k;i}v_{m,k}\right)_{;j} + 2\nu C^m_{k;j}\nabla^E_k v_{i;m}. \tag{48}$$

Multiplying by ϵ_{qji} and using the fact that

$$(C^m_{k;i})_{;j} = (C^m_{k;j})_{;i}$$

we deduce

$$\Gamma_\nu(u,\nabla)\zeta_q = 2\nu C^m_{k;j}\nabla^E_k(\epsilon_{qji}v_{i;m}) + 2\nu C^m_{k;i}\epsilon_{qji}\nabla^E_k(v_{m;j})$$
$$+2\nu C^m_{k;i}\epsilon_{qji}\left(\nabla^A_j\nabla^E_k - \nabla^E_k\nabla^A_j\right)v_m.$$

Now we write

$$v_{i;m} = \frac{1}{2}(v_{i;m} - v_{m;i}) + \frac{1}{2}(v_{i;m} + v_{m;i})$$

and substitute in the first two terms above. The symmetric part cancels, the anti-symmetric part is related to ζ. We obtain

$$\Gamma\zeta_q = \nu C^m_{k;j}\nabla^E_k\left(\epsilon_{qji}\epsilon_{rmi}\zeta_r\right) + \nu C^m_{k;i}\nabla^E_k\left(\epsilon_{qji}\epsilon_{rjm}\zeta_r\right) +$$
$$+2\nu C^m_{k;i}\epsilon_{qji}\left(\nabla^A_j\nabla^E_k - \nabla^E_k\nabla^A_j\right)v_m.$$

Using now the commutation relation (28) and the rule of contraction of two ϵ_{ijk} tensors we get the equation

$$\Gamma\zeta_q = 2\nu C^m_{k;m}\nabla^E_k\zeta_q - 2\nu C^q_{k;j}\nabla^E_k\zeta_j + \nu C^m_{k;i}C^r_{k;j}\epsilon_{qji}\epsilon_{rmp}\zeta_p. \tag{49}$$

When $\nu = 0$ we obtain the fact that $\Gamma\zeta = 0$. Moreover, ζ obeys a linear dissipative equation with $C^m_{k;i}$ as coefficients. This equation and its derivation is the same as in ([9]) where u is a solution of the Navier-Stokes equation. Using the Schwartz inequality, we obtain pointwise

$$\Gamma|\zeta|^2 + \nu|\nabla_E\zeta|^2 \leq 17\nu|C|^2|\zeta|^2 \tag{50}$$

where

$$|C|^2 = C^m_{k;i}C^m_{k;i}, \quad |\zeta|^2 = \zeta_q\zeta_q$$

are squares of Euclidean norms. This shows that ζ decreases significantly if it develops large gradients.

All these considerations apply to arbitrary u without having to impose the equation of state (1). It is the relation (1) that decides whether or not we are solving the Navier-Stokes equation. If u is a solution of the Navier-Stokes equation then $u = \mathbf{P}w$ from (1), so that $\nabla^E \times u = \nabla^E \times w$ and then the Cauchy formula (38) relates the vorticity $\omega(x,t) = \nabla^E \times u(x,t)$ to ζ ([9]). All the filtered fluid equations of this kind obey a viscous Cauchy formula:

Theorem 2. *The solutions w of the system*

$$\Gamma_\nu(u,\nabla)w + (\nabla u)^* w = 0, \quad u = J_\delta \mathbf{P} w$$

are given by

$$w = (\nabla A)^* v$$

where $A = x + \ell$ solves

$$\Gamma_\nu(u,\nabla)A = 0$$

and v solves

$$\Gamma_\nu(u,\nabla)v = 2\nu C \nabla v.$$

The Eulerian curl of w, $\xi = \nabla^E \times w$ is related to the Eulerian-Lagrangian curl of v, $\zeta = \nabla^A \times v$ by the viscous Cauchy formula

$$\xi = Det(\nabla A)\,(\nabla A)^{-1}\,\zeta.$$

Acknowledgments. This work is a continuation of research started while the author was visiting Princeton University partially supported by an AIM fellowship. Part of this work was done at the Institute for Theoretical Physics in Santa Barbara, whose hospitality is gratefully acknowledged. This research is supported in part by NSF-DMS9802611. I thank the referee for helpful remarks.

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