On the critical dissipative quasi-geostrophic equation

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Abstract: The 2D quasi-geostrophic (QG) equation is a two dimensional model of the 3D incompressible Euler equations. When dissipation is included in the model then solutions always exist if the dissipation's wave number dependence is super-linear. Below this critical power the dissipation appears to be insufficient. For instance, it is not known if the critical dissipative QG equation has global smooth solutions for arbitrary large initial data. In this paper we prove existence and uniqueness of global classical solutions of the critical dissipative QG equation for initial data that have small L^∞ norm. The importance of an L^∞ smallness condition is due to the fact that L^∞ is a conserved norm for the non-dissipative QG equation and is non-increasing on all solutions of the dissipative QG., irrespective of size.

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1 Introduction

Do singularities develop in finite time in smooth solutions of unforced, incompressible 3D fluid equations? This challenging question remains yet unanswered. Lower dimensional model equations have been proposed and studied ([2],[9],[3],[5]) in an attempt to develop mathematical insight in this problem. The 2D quasi-geostrophic (QG) equation is one of these models. The dissipative QG equation is

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\Delta)^{\alpha}\theta = 0, \tag{1.1}$$

where $\alpha \in [0,1]$, $\kappa > 0$ is the dissipative coefficient, and the 2D velocity field $u = (u_1, u_2)$ is determined from θ by a stream function ψ via the auxiliary relations

$$(u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right), \qquad (-\Delta)^{\frac{1}{2}}\psi = \theta. \tag{1.2}$$

Equation (1.1) with $\alpha = 1/2$ is the critical dissipative QG. Criticality means that the dissipation balances nonlinearity when one takes into account the conservation laws.

In addition to its intrinsic mathematical interest, the equation (1.1) is relevant in the context of general quasi-geostrophic models of atmospheric and ocean fluid flow [10].

We are concerned here with global existence results for solutions of the initial-value problem (IVP) for equation (1.1) wherein

$$\theta(x, 0) = \theta_0(x) \tag{1.3}$$

is specified. We consider periodic boundary conditions with period box $\Omega = [0, 2\pi]^2$. Without loss of generality we may restrict the discussion to θ that obey for all time $(2\pi)^{-2}\int_{\Omega} \theta dx = 0$.

The issue of global existence for equation (1.1) is non-trivial ([4],[14]). If no smallness condition is imposed on the initial data, then the issue of global existence for arbitrary data is open. In this paper we show that if the L^{∞}-norm of the initial data is small then (1.1) possesses a global solution in the critical case $\alpha = 1/2$. The QG equations (dissipative or not) have global weak solutions for arbitrary L^2 initial data ([11], see Appendix B). The L^∞ norm condition is significant in view of the fact that the QG equations have a maximum principle ([11], see Appendix A) that ensures that for all time and initial data the L^∞ norm is non-increasing in time. If the initial data is smooth enough then the solution of the critical dissipative QG is unique, smooth and decays in time. These results and their proofs are presented in section 2. When $\alpha > 1/2$, the smallness assumption on the data is not needed for global existence of smooth solutions for equation (1.1). This can be proved using the same ideas as for the critical case. The theory of global existence and regularity in the sub-critical ($\alpha > 1/2$) case is thus in a satisfactory state. More details for the sub-critical case can be found in [14].

We establish now some of the notation. The Fourier transform of f is $\widehat{f}$

$$\widehat{f}(k) = \frac{1}{(2\pi)^2}\int_\Omega f(x)e^{-ik\cdot x}dx.$$

Λ is used to denote the operator $(-\Delta)^{\frac{1}{2}}$, defined at the Fourier level by

$$\widehat{\Lambda f}(k) = |k|\widehat{f}(k).$$

The relation in (1.2) can be identified as

$$u = \left(-\partial_{x_2}\Lambda^{-1}\theta,\, \partial_{x_1}\Lambda^{-1}\theta\right) = (-R_2\theta,\, R_1\theta) \quad \text{or} \quad \widehat{u}(j) = i\left(\widehat{j}\right)^\perp \widehat{\theta}(j), \tag{1.4}$$

where $i = \sqrt{-1}$, $\widehat{j} = \frac{j}{|j|}$ for $j \in \mathbf{Z}^2 \setminus \{0\}$, $(j_1, j_2)^\perp = (-j_2, j_1)$, and R_1 and R_2 are Riesz transforms ([12]). The spaces H^s are the familiar Sobolev spaces of functions having s derivatives in L^2.

2 Global existence

The initial value problem for the critical dissipative QG equation is

$$\begin{cases} \theta_t + u\cdot\nabla\theta + \kappa\Lambda\theta = 0, & (x,t)\in\Omega\times[0,\infty), \\ u = (u_1,u_2) = (-R_2\theta,\, R_1\theta), & (x,t)\in\Omega\times[0,\infty), \\ \theta(x,0) = \theta_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

where $\kappa > 0$ is a constant.

In this section we assume that the L^∞ norm of the initial data θ_0 is small. We establish that the IVP (2.1) has a global bounded solution in H^1. If the initial data is smoother (H^2) then the solution's norm in H^2 is non-increasing in time. The solution becomes real analytic at positive time and and decays exponentially. For the sub-critical case, no smallness assumption is necessary and these results hold for arbitrary data.

We start with an apriori estimate.

Theorem 2.1 *There exists a constant c_∞ such that for any $\theta_0 \in H^2 \cap \mathcal{C}^3$ with*

$$\|\theta_0\|_{L^\infty} \le c_\infty \kappa, \tag{2.2}$$

the classical solution θ of the IVP (2.1) satisfies

$$\|\theta(\cdot, t)\|_{H^2} \le \|\theta_0\|_{H^2} \tag{2.3}$$

for all $t \ge 0$.

Proof. Multiplying the first equation in (2.1) by $\Delta^2\theta$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int |\Delta\theta|^2\,dx + \kappa\int |(-\Delta)^{\frac{5}{4}}\theta|^2 dx = -\int \Delta^2\theta\,(u\cdot\nabla\theta)dx.$$

Further integration by parts gives

$$\int \Delta^2\theta\,(u\cdot\nabla\theta)dx = 2\int \nabla u\cdot(\nabla(\nabla\theta))\Delta\theta + \int(\Delta u\cdot\nabla\theta)\Delta\theta$$

By Hölder's inequality,

$$\left|\int \Delta^2\theta\,(u\cdot\nabla\theta)dx\right| \le C\left[\|\nabla u\|_{L^3}\|\Delta\theta\|_{L^3}^2 + \|\Delta u\|_{L^3}\|\nabla\theta\|_{L^3}\|\Delta\theta\|_{L^3}\right]$$

The Riesz transforms are bounded in L^p spaces, so

$$\|\Delta u\|_{L^3} \le C\|\Delta\theta\|_{L^3}, \qquad \|\nabla u\|_{L^3} \le C\|\nabla\theta\|_{L^3}.$$

The Gagliardo-Nirenberg inequalities

$$\|\nabla\theta\|_{L^3} \le C\|\theta\|_{L^\infty}^{\frac{7}{9}}\|(-\Delta)^{\frac{5}{4}}\theta\|_{L^2}^{\frac{2}{9}}, \qquad \|\Delta\theta\|_{L^3} \le C\|\theta\|_{L^\infty}^{\frac{1}{9}}\|(-\Delta)^{\frac{5}{4}}\theta\|_{L^2}^{\frac{8}{9}},$$

follow from classical ones ([6]) by complex interpolation (see also[8]). Using them we obtain

$$\left|\int (-\Delta)^2\theta(u\cdot\nabla\theta)dx\right| \le C\|\theta\|_{L^\infty}\|(-\Delta)^{\frac{5}{4}}\theta\|_{L^2}^2.$$

Collecting the above estimates, we have

$$\frac{1}{2}\frac{d}{dt}\int |\Delta\theta|^2\,dx + \kappa\int |(-\Delta)^{\frac{5}{4}}\theta|^2 dx \le C_\infty\|\theta\|_{L^\infty}\|(-\Delta)^{\frac{5}{4}}\theta\|_{L^2}^2.$$

It was proved in ([11]) (see the Appendix A) that θ satisfies the maximum principle

$$\|\theta(\cdot,t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty} \quad \text{for all } t \ge 0.$$

Taking $c_\infty = (C_\infty)^{-1}$ the bound (2.3) then follows from the smallness condition (2.2). This completes the proof of the theorem.

Theorem 2.2 *There exists a constant c_∞ (the same as in Theorem 2.1 above) so that for any $\theta_0 \in H^2$ with $\|\theta_0\|_{L^\infty} \le c_\infty\kappa$ the IVP (2.1) has a unique global solution θ satisfying*

$$\|\theta(\cdot,t)\|_{H^2} \le \|\theta_0\|_{H^2}$$

for any $t \ge 0$.

Proof. Let θ be the unique local solution on $[0,T_0]$ with T_0 depending on $\|\theta_0\|_{H^2}$ only (standard techniques can be applied to show that a unique local solution exists and depends continuously on initial data in H^2). By Theorem 2.1, θ satisfies

$$\|\theta(\cdot,t)\|_{H^2} \le \|\theta_0\|_{H^2}$$

for any $t \in [0,T_0]$. Therefore the local solution can be extended uniquely to $[0,2T_0]$ and the global solution is obtained by repeating this procedure.

Theorem 2.3 *Assume that the initial data $\theta_0 \in H^2$ satsifies the bound $\|\theta_0\|_{L^\infty} < c_\infty\kappa$. Then the solution of the IVP (2.1) decays exponentially*

$$\|\theta\|_{H^2}^2 \le \exp(-ct)\|\theta_0\|_{H^2}^2$$

for all $t \ge 0$. Here $c = 2(\kappa - c_\infty^{-1}\|\theta_0\|_{L^\infty})$, and c_∞ is the same as in the previous theorems.

The proof is a trivial consequence of the obvious Poincare inequality

$$\int_{\Omega}|(-\Delta)^{\frac{5}{4}}\theta|^2dx \geq \int_{\Omega}|(-\Delta)\theta|^2dx.$$

Theorem 2.4 *Assume that the initial data $\theta_0 \in H^2$ satsifies the bound $\|\theta_0\|_{L^\infty} < c_\infty\kappa$. Then there exists a time $t_0 > 0$ such that the solution of the IVP (2.1) is real analytic for $t \geq t_0$. More precisely, there exists an extension of θ, $\theta(z,t)$ that is an analytic function in the time-expanding strip $\Sigma_t \subset \mathbf{C}^2$,*

$$\Sigma_t = \{z = x + iy;\ x \in \Omega,\ |y| < \frac{1}{2}\kappa(t - t_0)\}$$

and obeys the inequality

$$|\theta(z,t)| \leq \frac{\kappa}{2}$$

uniformly for $z \in \Sigma_t$.

Proof. Let us consider t_0 to be the first time when

$$Y(t) = \sum_{j\in\mathbf{Z}^2\setminus\{0\}} |\widehat{\theta}(j,t)|$$

becomes smaller than $\kappa/4$:

$$Y(t_0) \leq \kappa/4.$$

In view of the preceding theorem and the elementary inequality

$$Y(t) \leq C\|\theta(t)\|_{H^2}$$

the existence of t_0 is guaranteed. Consider the function

$$y(t) = \sum_{j\in\mathbf{Z}^2\setminus\{0\}} |\widehat{\theta}(j,t)| \exp\left\{\frac{(t - t_0)\kappa|j|}{2}\right\}$$

and the function

$$z(t) = \sum_{j\in\mathbf{Z}^2\setminus\{0\}} |j||\widehat{\theta}(j,t)| \exp\left\{\frac{(t - t_0)\kappa|j|}{2}\right\}$$

Formally

$$\frac{dy}{dt} + z(t)(\frac{\kappa}{2} - y(t)) \leq 0$$

holds. This implies that the function $y(t)$ is non-increasing for $t \ge t_0$. The more rigorous proof requires one to take only a finite sum, and introduce an artificial power to avoid differentiating the modulus at zero:

$$y_{n,\epsilon}(t) = \sum_{j\in \mathbf{Z}^2\setminus\{0\},\,|j|\le n} |\widehat{\theta}(j,t)|^{1+\epsilon} \exp\left\{\frac{(t-t_0)\kappa|j|}{2}\right\}$$

This is now a differentiable function in time. One differentiates, and obtains

$$\frac{d}{dt} y_{n,\epsilon}(t) + \frac{\kappa}{2} z_{n,\epsilon}(t) \le \gamma_\epsilon y(t) z(t)$$

where

$$z_{n,\epsilon}(t) = \sum_{j\in \mathbf{Z}^2\setminus\{0\},\,|j|\le n} |j||\widehat{\theta}(j,t)|^{1+\epsilon} \exp\left\{\frac{(t-t_0)\kappa|j|}{2}\right\},$$

and $\gamma_\epsilon = (1+\epsilon)\|\theta_0\|_{L^2}^{\epsilon}$, and thus $\lim_{\epsilon\to 0}\gamma_\epsilon = 1$. One integrates from $t = t_0$ to t and passes to the limit $n \to \infty$, $\epsilon \to 0$. One obtains then

$$y(t) + \frac{\kappa}{2}\int_{t_0}^{t} z(s)ds \le \frac{\kappa}{4} + \int_{t_0}^{t} y(s)z(s)ds$$

Then, because $y(t_0) \le \frac{\kappa}{4}$ it follows that the set $\{t \ge t_0 |\ y(s) \le \frac{\kappa}{2}, \forall s, t_0 \le s \le t\}$ equals $[t_0, \infty)$. The completely rigorous proof requires a regularization of the equation so that $z(t)$ is guaranteed to be finite. After $t = t_0$ one may use Galerkin approximations for this purpose. The conclusion is that, for $t \ge t_0$ one has

$$\sum_{j\in \mathbf{Z}\setminus\{0\}} |\widehat{\theta}(j,t)| \exp\left\{\frac{(t-t_0)\kappa|j|}{2}\right\} \le \frac{\kappa}{2}$$

The analytic extension is

$$\theta(z,t) = \sum_{j\in \mathbf{Z}^2\setminus\{0\}} e^{ij\cdot z}\widehat{\theta}(j,t)$$

and the uniform convergence and bound in Σ_t follow. This completes the proof. The idea of using time dependent exponential weights was introduced in ([7]).

Initial data in H^1 are sufficient for a global existence results to hold.

Theorem 2.5 *There exists a constant d_∞ such that for any $\theta_0 \in H^1$ and $\|\theta_0\|_{L^\infty} \leq d_\infty \kappa$ there exists a weak solution of the QG equation satisfying*

$$\|\theta(\cdot, t)\|_{H^1} \leq \|\theta_0\|_{H^1}$$

for any $t \geq 0$.

Proof. It was proved in ([11]) that the IVP (2.1) with $\theta_0 \in L^2$ has a global weak solution $\theta(\cdot, t) \in L^2$ (see Appendix B). The L^2 weak solutions are constructed using a Galerkin approximation. The H^1 weak solutions can be constructed by solving approximate equations

$$\partial_t \theta + u_\delta \cdot \nabla \theta + \kappa \Lambda \theta = 0, \tag{2.4}$$

where

$$u_\delta = k_\delta * u = k_\delta * \left(R^\perp \theta\right)$$

and k_δ the periodic Poisson kernel in 2D given at the Fourier level by

$$\widehat{k_\delta}(\xi) = e^{-\delta|\xi|}$$

$\xi \in \mathbf{Z}^2$. The approximations have gobal smooth solutions for positive time, uniform bounds in $L^\infty(dt; L^2(dx))$ and converge weakly to solutions of the QG equation. In addition, and in contrast with Galerkin approximations, these approximations have monotonic non-increasing L^p norms of u. Multiplying the first equation in (2.4) by $\Delta\theta$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int |\nabla\theta|^2 dx + \kappa \int |\Lambda^{\frac{3}{2}}\theta|^2 dx \leq \int |(\nabla\theta) \cdot \nabla u_\delta \cdot (\nabla\theta)| dx.$$

By a similar argument as in the proof of Theorem 2.1, the term on the right hand side can be bounded as follows.

$$\int |(\nabla\theta) \cdot \nabla u_\delta \cdot (\nabla\theta)| dx \leq C\|\nabla\theta\|_{L^3}^2 \|\nabla u_\delta\|_{L^3} \leq C\|\nabla\theta\|_{L^3}^3 \leq C\|\theta\|_{L^\infty}\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2.$$

In addition to the boundedness of Riesz transforms in L^p spaces and an appropriate Gagliardo-Nirenberg inequality, we use here the fact that convolution with the Poisson kernel does not increase L^p norms. Therefore,

$$\frac{1}{2}\frac{d}{dt}\int |\nabla\theta|^2 dx + \kappa \int |\Lambda^{\frac{3}{2}}\theta|^2 dx \leq D_\infty \|\theta\|_{L^\infty}\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2.$$

Using $d_\infty = (D_\infty)^{-1}$ and the maximum principle we deduce from $\|\theta_0\|_{L^\infty} \le d_\infty$ and the inequality above that

$$\|\theta(\cdot,t)\|_{H^1} \le \|\theta_0\|_{H^1} \quad \text{and} \quad \int_0^t \int |\Lambda^{\frac{3}{2}}\theta|^2(x,\tau)dx\, d\tau < \infty \tag{2.5}$$

for any $t > 0$. The solutions obtained thus are relatively strong and the equation holds in time integral form in L^2.

For the sub-critical case $\alpha > \frac{1}{2}$ the maximum principle allows one to get a sublinear bound of the nonlinearity in terms of the dissipation and, consequently, the global existence result holds without any smallness assumption on $\|\theta\|_{L^\infty}$.

Theorem 2.6 *Let $\alpha > \frac{1}{2}$ and $\theta_0 \in H^2$. Then there exists a unique global solution θ solving the IVP (2.1). The solution is real analytic for positive time and decays exponentially to zero.*

3 Appendix A

In [11] it was shown that a solution to (2.1) for $\frac{1}{2} \le \alpha \le 1$ satisfies the following maximum principle

$$\|\theta(\cdot,t)\|_{L^p} \le \|\theta_0\|_{L^p} \quad \text{for } 1 < p \le \infty \quad \text{for all } t \ge 0.$$

Below we give a short description of the proof in the case $\alpha = \frac{1}{2}$.

Define $\tilde{\theta} = k_s * \theta$ where $\hat{k}_s(\xi) = e^{-s|\xi|}$. The Poisson kernel k_s is positive and has integral equal to one. $\tilde{\theta}$ satisfies

$$\frac{d\tilde{\theta}}{ds} + \Lambda\tilde{\theta} = 0$$

and

$$\frac{d\|\tilde{\theta}\|_{L^p}^p}{ds} + p\int |\tilde{\theta}|^{p-2}\tilde{\theta}\Lambda\tilde{\theta}dx = 0.$$

Integrating with respect to s we obtain

$$p\int_{s_1}^{s_2}\int |\tilde{\theta}|^{p-2}\tilde{\theta}\Lambda\tilde{\theta}dxds = \|\tilde{\theta}\|_{L^p}^p(s_1) - \|\tilde{\theta}\|_{L^p}^p(s_2)$$

Using the properties of the Poisson Kernel k_s

$$\|k_{s_1} * \theta\|_{L^p}^p - \|k_{s_2-s_1} * (k_{s_1} * \theta)\|_{L^p}^p \geq 0.$$

therefore

$$\int |\theta|^{p-2}\theta\Lambda\theta dx = lim_{s\to 0^+}\int |\tilde{\theta}|^{p-2}\tilde{\theta}\Lambda\tilde{\theta}dx \geq 0.$$

Hence, returning to the evolution equation for θ

$$\frac{d\|\theta\|_{L^p}^p}{dt} = -p\int |\theta||^{p-2}\theta\Lambda\theta dx \leq 0.$$

4 Appendix B

It was proved in ([11]) that the QG equations, dissipative or not, have global weak solutions in L^2. We present here a brief description of the main reason for this fact. We consider QG as an infinite system of ODEs:

$$\frac{d}{dt}\widehat{\theta}(l,t) + \kappa|l|^{2\alpha}\widehat{\theta}(l,t) = b_l(\theta,\theta) \tag{4.6}$$

with

$$b_l(\theta,\theta) = \sum_{j+k=l}\frac{1}{|j|}\left(j^\perp\cdot k\right)\widehat{\theta}(j,t)\widehat{\theta}(k,t) \tag{4.7}$$

Note that because of the perpendicularity

$$\left(j^\perp\cdot k\right) = \left(j^\perp\cdot l\right) = -\left(k^\perp\cdot l\right)$$

and because of symmetry considerations in the sum we can write

$$b_l(\theta,\theta) = \sum_{j+k=l}\gamma_{j,k}^l\widehat{\theta}(j,t)\widehat{\theta}(k,t) \tag{4.8}$$

with

$$\gamma_{j,k}^l = \frac{1}{2}\left(j^\perp\cdot l\right)\frac{|k|-|j|}{|j||k|}.$$

Note the inequality

$$\left|\gamma_{j,k}^l\right| \leq \frac{|l|^2}{2(\max|j|;\ |k|)}$$

Consider now the weak norm

$$\|\theta\|_w = \sup_{j\in\mathbf{Z}^2\setminus\{0\}} |\widehat{\theta}(j)|$$

The nonlinearity can be written as

$$B(\theta,\theta)(x) = \sum_{l\in\mathbf{Z}^2\setminus\{0\}} b_l(\theta,\theta)e^{il\cdot x}$$

The main observation is that the nonlinearity has a weak continuity property. Let θ_1 and θ_2 be in L^2. There exists a constant C so that

$$\|\Lambda^{-2}(B(\theta_1,\theta_1) - B(\theta_2,\theta_2))\|_w \leq$$
$$C\|\theta_1-\theta_2\|_w\left(1+\log\left(1+\|\theta_1-\theta_2\|_w^{-1}\right)\right)\left(\|\theta_1\|_{L^2}+\|\theta_2\|_{L^2}\right)$$

This is the key ingredient in the existence of weak solutions. The method of proof employs a Galerkin approximation. The weak continuity guarantees the fact that the weak limit of the approximations solves the weak formulation of the equation.

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